Algebraic entropy of a multi-term recurrence of the Hietarinta-Viallet type

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Abstract

We introduce a family of extensions of the Hietarinta-Viallet equation to a multi-term recurrence relation via a reduction from the coprimeness-preserving extension to the discrete KdV equation. The recurrence satisfies the irreducibility and the coprimeness property although it is nonintegrable in terms of an exponential degree growth. We derive the algebraic entropy of the recurrence by an elementary method of calculating the degree growth. The result includes the entropy of the original Hietarinta-Viallet equation.

1 Introduction

There has been a question of what is exactly the discrete integrability. Various attempts have been made to construct a reasonable definition of discrete integrability by analogy with that of continuous systems. An underlying idea is that the integrability of a discrete equation is closely related to the slow growth of certain quantities. One of the first criteria for discrete integrability is the singularity confinement test (SC test) [1], which was introduced as a discrete analogue of the Painlevé test for differential equations. The SC test asserts that if all the singularities of a discrete equation are resolved after a finite iterations: i.e., the information on the initial values are recovered, then the equation passes the test. The SC test has been successfully applied to discrete QRT mappings to discover numerous nonautonomous recurrences including the discrete Painlevé equations [2].

Another famous criterion uses the algebraic entropy [3]. The algebraic entropy of a discrete mapping is a non-negative scalar which is related to the degree growth of the iterated mapping. The algebraic entropy E of a mapping φ is defined as

$$E := \lim_{n \to \infty} \frac{\log(d_n)}{n},$$

where d_n is the degree of the *n*-th iterate deg φ^n of some initial condition. If a discrete equation has zero algebraic entropy the equation is decided to be integrable, otherwise when the entropy is positive the equation is a non-integrable mapping. In this article we hire the zero algebraic entropy condition as a working definition of discrete integrability. It has been discovered that a certain type of discrete equations has positive algebraic entropy while passing the SC test. The first example of this kind is the mapping by Hietarinta and Viallet [4]:

$$x_{n+1} = -x_{n-1} + x_n + \frac{a}{x_n^2},\tag{1}$$

where a is a nonzero constant. The algebraic entropy of (1) is derived to be $\log \frac{3+\sqrt{5}}{2}$ by an algebraic method [3], and by an algebro-geometric method (blowing ups and construction of a space of initial conditions) [5].

Now a lot of examples of confining equations (whose singularities are all confined) with positive algebraic entropies are known [6, 7, 8, 9], however they are all equations on a one-dimensional lattice. It has been a problem to find the singularity confining equations with exponential degree growth defined over a multi-dimensional lattice. Recently several such examples have been discovered using the notion of the coprimeness property. The coprimeness property was introduced as an algebraic re-interpretation of the singularity confinement test, originating from the idea that a common factor between two iterates corresponds to a common zero/pole. Let φ be a dynamical system of a variable x_h $(h \in L)$ where L is an integer lattice. Then φ has the coprimeness property if there exists a positive constant D such that arbitrary two iterates x_h and x_k are pairwise coprime over the field of rational functions of the initial variables, on condition that $dist(h, k) \geq D$, where we have introduced a non-trivial metric 'dist' on the lattice L. Roughly speaking the system satisfies the coprimeness property if any pair of iterates that stay far enough away from each other on the lattice is coprime. Many of the known integrable systems satisfy the coprimeness property [10, 11, 12]. Moreover, many non-integrable coprimeness-preserving extensions to the well-known integrable equations were discovered including the so-called CP discrete KdV equation, the CP discrete Toda equation and the CP Somos-4 sequence [9, 13, 14]. We shall call such equations as belonging to the "CP class" in this article.

Let us focus on the following two-dimensional CP class equation extended from the discrete KdV equation [13]:

$$x_{t,n} + x_{t-1,n-1} = \frac{a}{x_{t,n-1}^k} + \frac{b}{x_{t-1,n}^k}.$$
(2)

Here k is a positive even integer and $a, b \neq 0$. Note that if $k \geq 3$ is odd the equation does not pass the singularity confinement test. The transformation of variables corresponding to its singularity pattern is

$$x_{t,n} = \frac{f_{t,n} f_{t-1,n-1}}{f_{t-1,n}^k f_{t,n-1}^k}.$$

Equation (2) transforms into the following recurrence analogous to the tau-function form in the integrable cases:

$$f_{t,n} = \frac{-f_{t-2,n-2}f_{t-1,n}^k f_{t,n-1}^k + af_{t-1,n-1}^{k^2-1} f_{t,n-2}^{k^2} f_{t-1,n}^k f_{t-2,n-1}^k + bf_{t-1,n-1}^{k^2-1} f_{t-2,n}^{k^2-1} f_{t,n-1}^{k^2} f_{t,n-1}^k f_{t-1,n-2}^k}{f_{t-2,n-1}^k f_{t-1,n-2}^k}.$$
 (3)

The irreducibility and the coprimeness of (3) are first addressed in [13] and the complete proof is published in [20]. See Appendix for details. In this article, we study the following recurrence

$$x_m + x_{m-p-q} = \frac{a}{x_{m-q}^k} + \frac{b}{x_{m-p}^k} \qquad (k \in 2\mathbb{Z}_{>0}), \tag{4}$$

where $1 \le p < q$ are positive integers coprime with each other. Note that if $(p,q) = r \ r \ge 2$, then the iteration splits into r independent orbits on which the results in this article can be applied. The equation (4) is obtained as a reduction of (2) and can be considered as an extension of the Hietarinta-Viallet equation into a multi-term recurrence relation. The equation (4) is transformed into the "tau-function" form (5)

$$f_m f_{m-2p-q}^k f_{m-p-2q}^k = -f_{m-p}^k f_{m-q}^k f_{m-2p-2q}^k + a f_{m-p}^k f_{m-p-q}^{k^2-1} f_{m-2p-q}^{k^2} + b f_{m-q}^k f_{m-p-q}^{k^2-1} f_{m-2p}^{k^2} f_{m-p-2q}^k,$$
(5)

via the transformation (6):

$$x_m = \frac{f_m f_{m-p-q}}{f_{m-p}^k f_{m-q}^k}.$$
 (6)

It is proved that (5) also satisfies the Laurent, the irreducibility and the coprimeness properties as in Appendix.

2 Algebraic entropy

In this section we obtain the algebraic entropy of the equation (4). It is worth noting that since (4) is a multi-term recurrence, it is not easy to apply an algebro-geometric technique to obtain the space of initial conditions to derive the algebraic entropy. Thus we stick to elementary estimation of the growth of the degrees. From here on let us fix the set of initial variables of Equation (5) as $\boldsymbol{f} = \{f_{-2p-2q}, f_{-2p-2q+1}, ..., f_{-1}\}$. The initial variables of (4) corresponding to \boldsymbol{f} is denoted as $\boldsymbol{x} := \{x_{-p-q}, x_{-p-q+1}, ..., x_{-1}\}$. Let us denote by $\operatorname{Ord}(r)$ the degree of a rational function $r(\boldsymbol{x})$ with respect to \boldsymbol{x} : i.e., if we write r = f/g, $f, g \in \mathbb{Z}[\boldsymbol{x}, a, b]$, where f, g are pairwise coprime as polynomials of \boldsymbol{x} , then $\operatorname{Ord}(\boldsymbol{x}) := \max[\operatorname{deg}(f), \operatorname{deg}(g)]$. Let us denote by $\operatorname{Ord}_{x_s}(f)$ the degree of f with respect to x_s .

The main theorem of this article gives the algebraic entropy of (4).

Theorem 2.1

The algebraic entropy $E_{p,q}$ of the equation (4) for a positive even k is given by

$$E_{p,q} = \log \Lambda_{p,q},$$

where $\Lambda_{p,q}$ is the largest real root of

$$\lambda^{p+q} - k(\lambda^p + \lambda^q) + 1 = 0. \tag{7}$$

Now we shall prepare several propositions. Let us define two subsets of \boldsymbol{f} as $\boldsymbol{f}_a := \{f_{-2p-2q}, f_{-2p-2q+1}, ..., f_{-p-q-1}\}$ and $\boldsymbol{f}_b := \{f_{-p-q}, f_{-p-q+1}, ..., f_{-1}\}$. When we rewrite the iterate f_m (m = 0, 1, 2, ...) using $\boldsymbol{f}_a \cup \boldsymbol{x}$ or $\boldsymbol{f}_b \cup \boldsymbol{x}$ instead of \boldsymbol{f} , we obtain the following proposition:

Proposition 2.2

Each iterate f_m is expressed as

$$f_m(\boldsymbol{f}_a, \boldsymbol{x}) = u_m(\boldsymbol{f}_a)g_m(\boldsymbol{x}), \tag{8a}$$

$$f_m(\boldsymbol{f}_b, \boldsymbol{x}) = v_m(\boldsymbol{f}_b)h_m(\boldsymbol{x}). \tag{8b}$$

Here u_m , v_m are Laurent monomials that satisfy $u_m u_{m-p-q} = u_{m-p}^k u_{m-q}^k$ and $v_m v_{m-p-q} = v_{m-p}^k v_{m-q}^k$, and $g_m(\mathbf{x})$, $h_m(\mathbf{x})$ are irreducible Laurent polynomials that satisfy (5).

Proof First we study the case of $-2p - 2q \le m \le -1$. By transforming the variables in f_b into those in $f_a \cup x$ we have

$$f_{-p-q} = \frac{f_{-2p-q}^{k} f_{-p-2q}^{k}}{f_{-2p-2q}} x_{-p-q}, \quad f_{-p-q+1} = \frac{f_{-2p-q+1}^{k} f_{-p-2q+1}^{k}}{f_{-2p-2q+1}} x_{-p-q+1}, \cdots$$

$$f_{-q-1} = \frac{f_{-p-q-1}^{k} f_{-2q-1}^{k}}{f_{-p-2q-1}} x_{-q+1}, \quad f_{-q} = \frac{f_{-2q}^{k} f_{-2p-2q}^{k^{2}-1}}{f_{-2p-2q}^{k}} x_{-q} x_{-p-q}^{k}, \cdots$$

$$f_{-1} = \frac{f_{-p-1}^{k} f_{-q-1}^{k}}{f_{-p-q-1}} x_{-1}.$$

Thus $f_m (-2p - 2q \le m \le -1)$ is inductively expressed as the following form: $f_m = u_m(f_a)g_m(x)$ where u_m is a Laurent monomial and g_m is a monomial. Note that $u_m = f_m$, $g_m = 1$ $(-2p - 2q \le m \le -p - q - 1)$. From the transformation (6), we have $u_m u_{m-p-q} = u_{m-p}^k u_{m-q}^k$. Thus Proposition 2.2 for u_m, g_m with $m \le -1$ is proved. Let us inductively prove the statement for u_m and g_m with $m \ge -1$. Let us fix $r \ge -1$ and assume Proposition 2.2 for all $m \le r$: From the relations

$$\frac{u_{r+1-p}^{k}u_{r+1-q}^{k}u_{r+1-2p-2q}^{k}}{u_{r+1-2p-q}^{k}u_{r+1-p-2q}^{k}} = \frac{u_{r+1-p}^{k}u_{r+1-q}^{k}}{u_{r+1-p-q}},$$
$$\frac{u_{r+1-p}^{k}u_{r+1-p-q}^{k-1}u_{r+1-2p-q}^{k}u_{r+1-2p-q}^{k}}{u_{r+1-2p-q}^{k}u_{r+1-p-2q}^{k}} = \frac{u_{r+1-p}^{k}u_{r+1-q}^{k}}{u_{r+1-p-q}},$$
$$\frac{u_{r+1-q}^{k}u_{r+1-p-q}^{k-1}u_{r+1-p-2q}^{k}}{u_{r+1-2p-q}^{k}u_{r+1-p-2q}^{k}} = \frac{u_{r+1-q}^{k}u_{r+1-p}^{k}}{u_{r+1-p-q}},$$

we have

$$\begin{split} f_{r+1} &= \frac{1}{f_{r+1-2p-q}^k f_{r+1-p-2q}^k} \left(-f_{r+1-p}^k f_{r+1-q}^k f_{r+1-2p-2q}^k \right. \\ &\quad + a f_{r+1-p}^k f_{r+1-p-q}^{k^2-1} f_{r+1-2q}^{k^2} f_{r+1-2p-q}^k + b f_{r+1-q}^k f_{r+1-p-q}^{k^2-1} f_{r+1-2p}^{k^2} f_{r+1-p-2q}^k \right) \\ &= \frac{u_{r+1-q}^k u_{r+1-p}^k}{u_{r+1-p-q}} \frac{1}{g_{r+1-2p-q}^k g_{r+1-p-2q}^k} \left(-g_{r+1-p}^k g_{r+1-q}^k g_{r+1-2p-2q}^k + a g_{r+1-p}^k g_{r+1-p-q}^{k^2-1} g_{r+1-2p-q}^k g_{r+1-2p-q}^k + b g_{r+1-q}^k g_{r+1-p-q}^{k^2-1} g_{r+1-2p-2q}^k \right) \end{split}$$

Therefore we obtain

$$u_{r+1} = \frac{u_{r+1-q}^k u_{r+1-p}^k}{u_{r+1-p-q}},$$

and that g_{r+1} satisfies (5). The irreducibility of g_m follows from Theorem A.2.

The same argument applies to the case of $v_m(f_b)$ and $h_m(x)$.

Lemma 2.3

 $h_m(\mathbf{x})$ is a polynomial of x_{-p-q} , whose constant term is nonzero.

Lemma 2.3 is readily obtained by verifying $x_m \neq 0, \infty$ when we substitute $x_{-p-q} = 0$ into the initial variables of (4).

We shall give the lower bound for the algebraic entropy in Proposition 2.4.

Proposition 2.4

Let $\Lambda_{p,q}$ be the same as in Theorem 2.1. The algebraic entropy of Equation (4) satisfies

$$E_{p,q} \ge \log \Lambda_{p,q}.$$

Proof Recall that $x_m = \frac{h_m h_{m-p-q}}{h_{m-p}^k h_{m-q}^k}$, where $\{h_m\}$ are pairwise coprime irreducible Laurent polynomials. It is easy to show that h_m has the following unique factorization:

$$h_m = h_m^{(0)} h_m^{(1)}$$

where $h_m^{(0)}$ is a Laurent monomial of \boldsymbol{x} , and $h_m^{(1)}$ is a polynomial of \boldsymbol{x} that satisfies $h_m^{(1)}|_{x_j=0} \neq 0$ for every j. Let d_m be the degree of h_m with respect to x_{-p-q} . From Lemma 2.3, the term $h_m^{(0)}$ does not include x_{-p-q} . Thus

$$\operatorname{Ord}(x_m) \ge \operatorname{Ord}(h_m^{(1)} h_{m-p-q}^{(1)}) = d_m + d_{m-p-q}.$$

Let us define a degree $d_m^* := d_m \big|_{a=b=0}$ then $d_m \ge d_m^*$. Since we have $x_m = x_{m-p-q}$ for a = b = 0,

$$h_m = \frac{-h_{m-p}^k h_{m-q}^k h_{m-2p-2q}^k}{h_{m-2p-q}^k h_{m-p-2q}^k},$$

and thus d_m^* satisfies

$$d_m^* = k(d_{m-p}^* + d_{m-q}^*) - k(d_{m-2p-q}^* + d_{m-p-2q}^*) + d_{m-2p-2q}^*.$$

Here, initial data are $d_m = d_m^* = 0$ $(-2p - 2q + 1 \le m \le -1)$, $d_{-2p-2q} = d_{-2p-2q}^* = 1$. Therefore d_m^* grows as $d_m^* \sim \Lambda_{p,q}^m$, where $\Lambda_{p,q}$ is the largest real root of

$$\begin{split} \lambda^{2p+2q} &- k(\lambda^{2p+q} + \lambda^{q+2p}) + k(\lambda^p + \lambda^q) - 1 \\ &= (\lambda^{p+q} - 1) \left(\lambda^{p+q} - k(\lambda^p + \lambda^q) + 1 \right) = 0. \end{split}$$

Properties of $\Lambda_{p,q}$ will be discussed in Lemma A.12 in Appendix. Therefore using Lemma A.13 in Appendix, we can find a positive constant c such that $d_m^* \ge c\Lambda_{p,q}^m$. Thus we have

$$\operatorname{Ord}(x_m) \ge c\Lambda_{p,q}^m,$$

which readily derives $E_{p,q} \ge \log \Lambda_{p,q}$.

Next we obtain the upper bound for $E_{p,q}$, which is not as simple as the lower bound as is often the case with algebraic entropy of discrete equations.

Lemma 2.5

The term g_m is uniquely factorized as $g_m = g_m^{(0)} g_m^{(1)}$, where $g_m^{(0)}$ is a monic monomial, and $g_m^{(1)}$ is a polynomial satisfying $g_m^{(1)}(\boldsymbol{x})|_{\boldsymbol{x}_s=0} \neq 0$ for s = -1, -2, ..., -p-q. Let $\alpha_s(m)$ be the degree of $g_m^{(0)}(\boldsymbol{x})$ with respect to x_s , and $\beta_s(m)$ be the degree of $g_m^{(1)}(\boldsymbol{x})$. Then we have the following estimation:

$$\operatorname{Ord}(x_m) \le 2 \sum_{s=-p-q}^{-1} |\alpha_s(m) + \alpha_s(m-p-q) - k\alpha_s(m-p) - k\alpha_s(m-q)| + \sum_{s=-p-q}^{-1} |c_s(m) - kc_s(m-p) - kc_s(m-q) + c_s(m-p-q)| + \sum_{s=-p-q}^{-1} k \left(c_s(m-p) + c_s(m-q) - \alpha_s(m-p) - \alpha_s(m-q) \right).$$
(9)

Proof Recall that

$$c_s(m) := \operatorname{Ord}_{x_s}(g_m) = \alpha_s(m) + \beta_s(m)$$

Since $g_m(\boldsymbol{x}), g_{m-p}(\boldsymbol{x}), g_{m-q}(\boldsymbol{x}), g_{m-p-q}(\boldsymbol{x})$ are pairwise coprime irreducible Laurent polynomials, we have

$$\operatorname{Ord}(x_m) = \operatorname{Ord}\left(\frac{g_m^{(0)}g_{m-p-q}^{(0)}}{(g_{m-p}^{(0)}g_{m-q}^{(0)})^k} \frac{g_m^{(1)}g_{m-p-q}^{(1)}}{(g_{m-p}^{(1)}g_{m-q}^{(1)})^k}\right)$$
$$\leq \operatorname{Ord}\left(\frac{g_m^{(0)}g_{m-p-q}^{(0)}}{(g_{m-p}^{(0)}g_{m-q}^{(0)})^k}\right) + \operatorname{Ord}\left(\frac{g_m^{(1)}g_{m-p-q}^{(1)}}{(g_{m-p}^{(1)}g_{m-q}^{(1)})^k}\right)$$

Next, since

$$\operatorname{Ord}_{x_s}\left(\frac{g_m^{(0)}g_{m-p-q}^{(0)}}{(g_{m-p}^{(0)}g_{m-q}^{(0)})^k}\right) = \alpha_s(m) + \alpha_s(m-p-q) - k\alpha_s(m-p) - k\alpha_s(m-q),$$

we have

$$\operatorname{Ord}\left(\frac{g_m^{(0)}g_{m-p-q}^{(0)}}{(g_{m-p}^{(0)}g_{m-q}^{(0)})^k}\right) \le \sum_s |\alpha_s(m) + \alpha_s(m-p-q) - k\alpha_s(m-p) - k\alpha_s(m-q)|,$$

where the summation moves from s = -p - q to s = -1. Therefore,

$$\operatorname{Ord}\left(\frac{g_m^{(1)}g_{m-p-q}^{(1)}}{(g_{m-p}^{(1)}g_{m-q}^{(1)})^k}\right) = \max\left[\sum_s c_s(m) + c_s(m-p-q) - \alpha_s(m) - \alpha_s(m-p-q), \sum_s k\left(c_s(m-p) + c_s(m-q) - \alpha_s(m-p) - \alpha_s(m-q)\right)\right]$$

$$= \sum_{s} k \left(c_s(m-p) + c_s(m-q) - \alpha_s(m-p) - \alpha_s(m-q) \right) \\ + \max \left[\sum_{s} c_s(m) - k c_s(m-p) - k c_s(m-q) + c_s(m-p-q) \\ - \left(\alpha_s(m) - k \alpha_s(m-p) - k \alpha_s(m-q) + \alpha_s(m-p-q) \right), 0 \right].$$

Therefore we obtain (9).

Following Proposition 2.6 plays the key role in our estimation of the upper bound:

Proposition 2.6

For arbitrary s $(-p - q \le s \le -1)$, there exist positive constants C_s , A_s such that

 $|c_s(m)| \le C_s \Lambda_{p,q}^m, \qquad |\alpha_s(m)| \le A_s \Lambda_{p,q}^m.$

Before its proof, let us complete the proof of Theorem 2.1. From Proposition 2.6 and (9) we have the upper bound for the algebraic entropy:

Corollary 2.7

We have $E_{p,q} \leq \log \Lambda_{p,q}$.

From Proposition 2.4 and Corollary 2.7 we obtain our main Theorem 2.1.

2.1 Proof of Proposition 2.6

The rest of this section is devoted to proving Proposition 2.6. Let us prepare an elementary lemma on a recurrence relation.

Lemma 2.8

For a sequence $(a_m)_{m=-p-q}^{\infty}$, let us define $A_m := a_m - k(a_{m-p} + a_{m-q}) + a_{m-p-q}$ (m = 0, 1, ...). Suppose that there exists an integer $m_0 \in \mathbb{Z}_{\geq p+q}$ such that $A_m = A_{m-p-q}$ for every $m \geq m_0$. Then there exists a positive constant C such that $|a_m| \leq C \Lambda_{p,q}^m$ for every $m \geq m_0$.

Proof $A_m = A_{m-p-q}$ is equivalent to

$$a_m - k(a_{m-p} + a_{m-q}) + k(a_{m-2p-q} + a_{m-p-2q}) - a_{m-2p-2q} = 0.$$

The characteristic polynomial of this linear recurrence is

$$\begin{split} \lambda^{2p+2q} &- k(\lambda^{p+2q} + \lambda^{2p+q}) + k(\lambda^q + \lambda^p) - 1 \\ &= (\lambda^{p+q} - 1) \left\{ (\lambda^{p+q} - k(\lambda^q + \lambda^p) + 1 \right\}, \end{split}$$

whose largest root with respect to the absolute value is $\Lambda_{p,q}$ from Note A.12. Therefore there exists a constant C > 0 such that $|a_m| \leq C \Lambda_{p,q}^m$.

2.1.1 The case of p = 1, q = 2:

First let us prove the former inequality on $c_s(m)$. We have

$$\begin{split} g_{-6} &= g_{-5} = g_{-4} = 1, \\ g_{-3} &= x_{-3}, \ g_{-2} = x_{-3}^k x_{-2}, \ g_{-1} = x_{-3}^{k^2 + k} x_{-2}^k x_{-1}. \end{split}$$

Let us denote by y_m the following degree: $y_m := c_{-3}(m) = \operatorname{Ord}_{x_{-3}}(g_m)$. By calculating $\boldsymbol{y} = (y_{-6}, y_{-5}, y_{-4}, \dots)$ we have

$$y_{-6} = y_{-5} = y_{-4} = 0, \quad y_{-3} = 1, \ y_{-2} = k, \ y_{-1} = k^2 + k$$

For $m \ge 0$ we have

$$g_m = \frac{-g_{m-1}^k g_{m-2}^k g_{m-6}^k + a g_{m-1}^k g_{m-3}^{k^2-1} g_{m-4}^{k^2+k} + b g_{m-2}^{k^2+k} g_{m-3}^{k^2-1} g_{m-5}^k}{g_{m-4}^k g_{m-5}^k}.$$

By comparing the degrees of the three terms on the right hand side

$$y_m^{(1)} := k(y_{m-1} + y_{m-2}) - k(y_{m-4} + y_{m-5}) + y_{m-6},$$

$$y_m^{(2)} := ky_{m-1} + (k^2 - 1)y_{m-3} + k^2y_{m-4} - ky_{m-5},$$

$$y_m^{(3)} := (k^2 + k)y_{m-2} + (k^2 - 1)y_{m-3} - ky_{m-4},$$

we obtain

$$y_m = \max[y_m^{(1)}, y_m^{(2)}, y_m^{(3)}], \tag{10}$$

unless an unexpected cancellation occurs. Precisely speaking, if the two terms, for example $g_{m-1}^k g_{m-2}^k g_{m-6}^k$ and $ag_{m-1}^k g_{m-3}^{k^2-1} g_{m-4}^{k^2+k}$ has the same degree with respect to all of x_s (s = -1, -2, -3) and the degree is greater than that of the third termit is possible that the degree satisfies $y_m < \max[y_m^{(1)}, y_m^{(2)}, y_m^{(3)}]$. This type of cancellation is inductively proved to be impossible later in this proof.

Let us define a sequence $Y_m := y_m - k(y_{m-1} + y_{m-2}) + y_{m-3}$. Then

$$Y_{-3}=1, \quad Y_{-2}=0, \quad Y_{-1}=0.$$

We shall prove that $Y_m = Y_{m-3}$ for every m. Let us define

$$Y_m^{(i)} := y_m^{(i)} - k(y_{m-1} + y_{m-2}) + y_{m-3} \qquad (i = 1, 2, 3).$$

Then we have

$$Y_m^{(1)} = Y_{m-3}, \qquad Y_m^{(2)} = -kY_{m-2}, \qquad Y_m^{(3)} = -kY_{m-1},$$

and

$$Y_m = \max[Y_{m-3}, -kY_{m-2}, -kY_{m-1}].$$
(11)

We shall use the notation (applicable only in this section) $Y_m = a_I$ to denote that $Y_m = a$ and the maximum/maxima in Equation (11) is attained on the *i*th term(s) for all $i \in I$. For example, $Y_0 = \max[1, 0, 0] = 1$ where maximum is attained on the first term 1, and thus we write $Y_0 = 1_1$. The successive iterations give

$$Y_0 = 1_1$$
, $Y_1 = \max[0, 0, -k] = 0_{1,2}$, $Y_2 = 0_{1,3}$, $Y_3 = 1_1$, $Y_4 = 0_{1,2}$, $Y_5 = 0_{1,3}$, \cdots ,

on condition that no unexpected cancellation occurs in (11).

Next we study $c_{-2}(m)$ and $c_{-1}(m)$. Note that $\operatorname{Ord}_{x-2}g_m = y_{m-1}$, $\operatorname{Ord}_{x-1}g_m = y_{m-2}$. We redefine $y_m := c_{-2}(m), y_m^{(i)}(i = 1, 2, 3)$ and $Y_m = y_m - k(y_{m-1} + y_{m-2}) + y_{m-3}$ (we use the same symbols y_m and Y_m as s = -3 to ease notation). Then Y_m satisfies Equation (11), with the initial condition

$$Y_{-3} = 0, Y_{-2} = 1, Y_{-1} = 0 \quad (s = -2).$$

If a cancellation does not occur, we have

$$Y_0 = 0_{1,3}, \quad Y_1 = 1_1, \quad Y_2 = 0_{1,2}, \quad Y_3 = 0_{1,3}, \quad Y_4 = 1_1, \quad Y_5 = 0_{1,2}, \cdots$$

For s = -1 let us redefine $y_m := c_{-1}(m), y_m^{(i)}(i = 1, 2, 3)$ and $Y_m = y_m - k(y_{m-1} + y_{m-2}) + y_{m-3}$. Then we have

$$Y_{-3} = 0, \ Y_{-2} = 0, \ Y_{-1} = 1,$$

 $Y_0 = 0_{1,2}, \quad Y_1 = 0_{1,3}, \quad Y_2 = 1_1, \quad Y_3 = 0_{1,2}, \quad Y_4 = 0_{1,3}, \quad Y_5 = 1_1, \cdots$

From these results, for any m, there exists at least one $s \in \{-1, -2, -3\}$ such that the right hand side of (11) attains its maximum only for one term (i.e., only one subscript in our notation). Thus the degrees of $y_m^{(i)}$ (i = 1, 2, 3) are all distinct from each other as rational functions of $\{x_{-1}, x_{-2}, x_{-3}\}$. Therefore it is proved inductively that no unexpected cancellation occurs while iterating (11) (and thus (10)) and that we have $Y_m - Y_{m-3} = 0$ for all $m \ge 0$. Thus, from Lemma 2.8, there exists a constant $C_s > 0$ such that $|y_m| \le C_s \Lambda_{1,2}^m$ for all s = -1, -2, -3. Now $|c_s(m)| \le C_s \Lambda_{1,2}^m$ (s = -1, -2, -3) is proved. Next let us prove the latter inequality on $\alpha_s(m)$. From Lemma 2.5, g_m has the factorization $g_m =$

Next let us prove the latter inequality on $\alpha_s(m)$. From Lemma 2.5, g_m has the factorization $g_m = g_m^{(0)}g_m^{(1)}$. Let z_m be the degree of $g_m^{(0)}$ with respect to x_{-3} : i.e., $z_m := \alpha_{-3}(m)$. It is clear that $z_m = y_m$ $(-6 \le m \le -1)$. For $m \ge 0$, z_m is iteratively defined as

$$z_m = \min[z_m^{(1)}, z_m^{(2)}, z_m^{(3)}], \tag{12}$$

where we use three auxiliary variables as

$$z_m^{(1)} := k(z_{m-1} + z_{m-2}) - k(z_{m-4} + z_{m-5}) + z_{m-6};$$

$$z_m^{(2)} := kz_{m-1} + (k^2 - 1)z_{m-3} + k^2 z_{m-4} - kz_{m-5};$$

$$z_m^{(3)} := (k^2 + k)z_{m-2} + (k^2 - 1)z_{m-3} - kz_{m-4}.$$

Equation (12) is true unless we encounter a non-trivial cancellation of terms just like in y_m . In order to avoid unexpected cancellations, it is sufficient that at least one of "degrees of monomial parts of g_m are distinct (discussion on Z_m below)" or "degrees of g_m are distinct (discussion on Y_m)" is satisfied. Let $Z_m := z_m - k(z_{m-1} + z_{m-2}) + z_{m-3}$. We have $Z_{-3} = 1$, $Z_{-2} = Z_{-1} = 0$. Let us define

$$Z_m^{(i)} := z_m^{(i)} - k(z_{m-1} + z_{m-2}) + z_{m-3} \qquad (i = 1, 2, 3)$$

Then

$$Z_m^{(1)} = Z_{m-3}, \qquad Z_m^{(2)} = -kZ_{m-2}, \qquad Z_m^{(3)} = -kZ_{m-1}$$

and thus

$$Z_m = \min_i [Z_m^{(i)}] = \min[Z_{m-3}, -kZ_{m-2}, -kZ_{m-1}].$$
(13)

It is easy to check that

 $Z_0 = 0_{2,3}, \quad Z_i = 0_{1,2,3} \ (1 \le i).$

Let us study the case of s = -2. Let us abuse the notation and redefine $z_m = \alpha_{-2}(m)$ and so on. Then Equation (13) is satisfied with the initial condition $Z_{-3} = 0$, $Z_{-2} = 1$, $Z_{-1} = 0$. Thus Z_m is periodic with period three for $m \ge 3$ as

$$Z_0 = -k_2, \quad Z_1 = 0_2, \quad Z_2 = 0_{1,3}, \quad Z_3 = -k_1, \quad Z_4 = 0_{1,2}, \quad Z_5 = 0_{1,3}, \cdots$$

The same discussion applies to $\alpha_{-1}(m)$. The redefined Z_m satisfies Equation (13) with the initial condition $Z_{-3} = 0$, $Z_{-2} = 0$, $Z_{-1} = 1$. Thus Z_m is again periodic with period three for $m \ge 3$ as

$$Z_0 = -k_3, \quad Z_1 = -k_2, \quad Z_2 = 1_1, \quad Z_3 = -k_{1,3}, \quad Z_4 = -k_{1,2}, \quad Z_5 = 1_1, \cdots$$

In the case of $m \equiv 0, 2 \mod 3, m \leq 4$, there exists at least one $s \in \{-1, -2\}$ such that the right hand side of (13) attains its minimum only for one term. In the case of $m \equiv 1 \mod 3, m \leq 4$, the degrees of the first two terms of the right hand side of g_m are zero with respect to x_{-2} and x_{-3} , they must have the following form:

$$x_{-1}^K G_1 + a x_{-1}^K G_2.$$

Here $K = z_m^{(1)} = z_m^{(2)}$ and G_1, G_2 are irreducible. On the other hand, from the discussion of Y_m , their degrees satisfy $\operatorname{Ord}(G_1) \neq \operatorname{Ord}(G_2)$, and therefore their highest order terms cannot be cancelled out. Therefore from Lemma 2.8, there exists a constant $A_s > 0$ such that $|z_m| \leq A_s \Lambda_{1,2}^m$ for any s = -1, -2, -3. Thus $|\alpha_s(m)| \leq A_s \Lambda_{1,2}^m$ (s = -1, -2, -3) is proved.

2.1.2 The case of $p = 1, q \ge 3$:

By successive iterations we have

$$g_{s} = 1 \quad (-2q - 2 \le s \le -q - 2), \quad g_{-q-1} = x_{-q-1}, \quad g_{-q} = x_{-q-1}^{k} x_{-q}, \cdots,$$
$$g_{-2} = x_{-q-1}^{k^{q-1}} x_{-q}^{k^{q-2}} \cdots x_{-3}^{k} x_{-2}, \quad g_{-1} = x_{-q-1}^{k^{q}+k} x_{-q}^{k^{q-1}} \cdots x_{-2}^{k} x_{-1}.$$

Let $y_m^{(s)} := c_s(m) = \operatorname{Ord}_{x_s}(g_m)$ for $s = -q - 1, \dots, -1$. For example,

$$y_s^{(-q-1)} = 0 \quad (-2q - 2 \le s \le -q - 2), \ y_{-q-1}^{(-q-1)} = 1, \ y_{-q}^{(-q-1)} = k, \cdots, y_{-2}^{(-q-1)} = k^{q-1}, \ y_{-1}^{(-q-1)} = k^q + k = 0$$

Just like the case of p = 1, q = 2, let us define

$$\begin{split} y_{m,1}^{(s)} &:= k(y_{m-1}^{(s)} + y_{m-q}^{(s)}) - k(y_{m-2-q}^{(s)} + y_{m-1-2q}^{(s)}) + y_{m-2-2q}^{(s)}, \\ y_{m,2}^{(s)} &:= ky_{m-1}^{(s)} + (k^2 - 1)y_{m-1-q}^{(s)} + k^2y_{m-2q}^{(s)} + ky_{m-2-q}^{(s)} - k(y_{m-2-q}^{(s)} + y_{m-1-2q}^{(s)}), \\ y_{m,3}^{(s)} &:= ky_{m-q}^{(s)} + (k^2 - 1)y_{m-q-1}^{(s)} + k^2y_{m-2}^{(s)} + ky_{m-1-2q}^{(s)} - k(y_{m-2-q}^{(s)} + y_{m-1-2q}^{(s)}). \end{split}$$

Let $Y_m^{(s)} := y_m^{(s)} - k(y_{m-1}^{(s)} + y_{m-q}^{(s)}) + y_{m-q-1}^{(s)}$ and $Y_{m,i}^{(s)} := y_{m,i}^{(s)} - k(y_{m-1}^{(s)} + y_{m-q}^{(s)}) + y_{m-q-1}^{(s)}$ for i = 1, 2, 3. Then

$$Y_{m,1}^{(s)} = Y_{m-1-q}^{(s)}, \qquad Y_{m,2}^{(s)} = -kY_{m-q}^{(s)}, \qquad Y_{m,3}^{(s)} = -kY_{m-1}^{(s)},$$

Therefore

$$Y_m^{(s)} = \max\left[Y_{m-1-q}^{(s)}, -kY_{m-q}^{(s)}, -kY_{m-1}^{(s)}\right].$$
(14)

Let us study the case of s = -q - 1 first. We have

$$(Y_{-q-1}^{(-q-1)}, Y_{-q}^{(-q-1)}, Y_{-q+1}^{(-q-1)}, \cdots, Y_{-1}^{(-q-1)}) = (1, 0, 0, ..., 0).$$

Therefore if there is no unexpected cancellation of terms, which shall be proved inductively in the course of the proof, we have

$$\begin{split} Y_0^{(-q-1)} &= \max[1, -k \cdot 0, -k \cdot 0] = 1_1, \quad Y_1^{(-q-1)} = \max[0, -k \cdot 0, -k \cdot 1] = 0_{1,2}, \\ Y_j^{(-q-1)} &= 0_{1,2,3} \; (2 \leq j \leq q-1), \quad Y_q^{(-q-1)} = \max[Y_{-1}^{(-q-1)}, -kY_0^{(-q-1)}, -kY_{q-1}^{(-q-1)}] = 0_{1,3}, \\ Y_{q+1}^{(-q-1)} &= \max[Y_0^{(-q-1)}, -kY_1^{(-q-1)}, -kY_q^{(-q-1)}] = 1_1. \end{split}$$

Therefore $Y_m^{(-q-1)}$ has period q+1.

Next we study the case of s = -q. Initial values of (14) are

$$(Y_{-q-1}^{(-q)}, Y_{-q}^{(-q)}, Y_{-q+1}^{(-q)}, \cdots, Y_{-1}^{(-q)}) = (0, 1, 0, ..., 0).$$

The iterations give

$$\begin{split} Y_0^{(-q)} &= 0_{1,3}, \quad Y_1^{(-q)} = 1_1, \quad Y_2^{(-q)} = 0_{1,2}, \quad Y_j^{(-q)} = 0_{1,2,3} \ (3 \le j \le q), \\ Y_{q+1}^{(-q)} &= \max[Y_0^{(-q)}, -kY_1^{(-q)}, -kY_q^{(-q)}] = 0_{1,3}. \end{split}$$

Therefore $Y_m^{(-q)}$ is also periodic with the period q + 1. The same discussion shows that $Y_m^{(s)}$ has period q+1 for all $-q+1 \leq s \leq -1$. When we fix s, the right hand side of Equation (14) attains its maximum only for one term if $m = s \pmod{q+1}$. Therefore no irregular cancellation is proved to be impossible. From Lemma 2.8, for each s there exists a constant $C_s > 0$ such that $|c_s(m)| \leq C_s \Lambda_{1,q}^n$.

From Lemma 2.8, for each s there exists a constant $C_s > 0$ such that $|c_s(m)| \leq C_s \Lambda_{1,q}^m$. Next let us study $\alpha_s(m)$ $(-q-1 \leq s \leq -1)$. Let $z_m^{(s)} = \alpha_s(m)$ and $Z_m^{(s)} := z_m^{(s)} - k(z_{m-1}^{(s)} + z_{m-q}^{(s)}) + z_{m-q-1}^{(s)}$. If no cancellation occurs we have

$$Z_m^{(s)} = \min[Z_{m-q-1}^{(s)}, -kZ_{m-q}^{(s)}, -kZ_{m-1}^{(s)}],$$
(15)

where the initial values are $Z_s^{(s)} = 1$, $Z_j^{(s)} = 0$ $(j \neq s)$. Let us prove that no cancellation occurs using a procedure similar to the (p,q) = (1,2) case. In the case of s = -1 we have:

$$\begin{split} &Z_{0}^{(-1)} = \min[0, 0, -k] = -k_{3}, \quad Z_{1}^{(-1)} = \min[0, 0, k^{2}] = 0_{1,2}, \quad Z_{j}^{(-1)} = 0_{1,2,3} \ (2 \leq j \leq q-2), \\ &Z_{q-1}^{(-1)} = \min[0, -k, 0] = -k_{2}, \quad Z_{q}^{(-1)} = \min[1, k^{2}, k^{2}] = 1_{1}, \quad Z_{q+1}^{(-1)} = \min[-k, 0, -k] = -k_{1,3}, \\ &Z_{q+2}^{(-1)} = \min[0, 0, k^{2}] = 0_{1,2}, \quad Z_{j}^{(-1)} = 0_{1,2,3} \ (q+3 \leq j \leq 2q-2), \quad Z_{2q-1}^{(-1)} = \min[0, k^{2}, 0] = 0_{1,3}, \\ &Z_{2q}^{(-1)} = \min[-k, -k, 0] = -k_{1,2}, \quad Z_{2q+1}^{(-1)} = \min[1, k^{2}, k^{2}] = 1_{1}, \cdots . \end{split}$$

Therefore $Z_m^{(-1)}$ has period q+1.

For $-2 \le s \le -q$, we have $Z_m^{(s)} = -k$ for $m \equiv q + s \pmod{q+1}$, which is the unique minimum in the right hand side of Equation (15). Otherwise $Z_m^{(s)} = 0$. Therefore $Z_m^{(s)} = Z_{m-q-1}^{(s)}$. For s = -q-1 we have $Z_{-q-1}^{(-q-1)} = 1, Z_m^{(-q-1)} = 0 \ (-q \le m \le -1)$ and

$$Z_0^{(-q-1)} = \min[1, 0, 0] = 0_{2,3}.$$

We have $Z_m^{(-q-1)} = 0$ for any $m \ge 0$. Thus $Z_m^{(-q-1)} = Z_{m-q-1}^{(-q-1)}$.

From the discussion above, for $m \equiv 0, 1, 2, ..., q - 2 \pmod{q+1}$, $Z_m^{(m-q)}$ has the unique minimum in the right hand side of Equation (15). For $m \equiv q$, $Z_m^{(-1)}$ has the unique minimum. For $m \equiv q-1$, monomials of $\boldsymbol{x} \setminus \{x_{-1}\}$ do not appear in the first and the second terms of g_m . Therefore the cancellation is impossible, taking into account the fact that these two terms have distinct degrees from the discussion of Y_m . Therefore, for each s, there exists a constant $A_s > 0$ such that $|\alpha_s(m)| \leq A_s \Lambda_{1,q}^m$.

2.1.3 The case of $q > p \ge 2$:

Since p and q are coprime, let us write lp < q < (l+1)p, r = q - lp where 0 < r < p. We shall use the same notations as previous parts. Let us define $y_m^{(s)} := c_s(m)$ for $s = -q - p, \dots, -1$. First let us derive $y_m^{(-q-p)}$. Values of $y_m^{(-q-p)}$ for $-q - p \le m \le -1$ are

$$\begin{aligned} y_{-q-p} &= 1, \ y_{-q-p+i} = 0 \ (i = 1, 2, \cdots, p-1), \\ y_{-q+tp} &= k^{t+1}, \ y_{-q+tp+i} = 0 \ (i = 1, 2, \cdots, p-1, t = 0, 1, \cdots, l-1, (i, t) \neq (r, l-1)), \\ y_{-p}(&= y_{-q+(l-1)p+r}) = k, \\ y_{-r} &= k^{l+1}, \ y_{j} = 0 \ (j = -r+1, \cdots, -1), \end{aligned}$$
(16)

where we have omitted the superscripts (-p-q) for simplicity. For example, when (q, p) = (17, 5), k = 2, we have

$$(y_{-22}^{(-22)}, \cdots, y_{-1}^{(-22)}) = (1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 4, 0, 0, 0, 0, 8, 0, 2, 0, 0, 16, 0).$$

In the case of s = -p - q + 1, we have

$$y_{-q-p}^{(-q-p+1)} = 0, \ y_{m+1}^{(-q-p+1)} = y_m^{(-q-p)} \ (m = -q-p, \cdots, -2),$$

which is derived by shifting the sequence (16) to the right and adding 0 to the left. In a similar manner we have

$$y_m^{(s)} = 0 (-q - p \le m \le s - 1), \ y_{m+s+q+p}^{(s)} = y_m^{(-q-p)} (-q - p \le m \le -q - p - s - 1),$$

for $-q - p + 2 \le s \le -1$. In particular, $(y_{-q-p}^{(-1)}, \cdots, y_{-1}^{(-1)}) = (0, 0, ..., 0, 1)$. Note that $y_m^{(s)} = 0$ $(m \le -p - q - 1)$ for any s. Let

$$Y_m^{(s)} = y_m^{(s)} - k(y_{m-p}^{(s)} + y_{m-q}^{(s)}) + y_{m-p-q}^{(s)}.$$

Then, for a fixed s, we have $Y_s^{(s)} = 1Y_j^{(s)} = 0 \ (-p - q \le j \le -1, \ j \ne s)$. Since we have

$$Y_m^{(s)} = \max[Y_{m-p-q}^{(s)}, -kY_{m-q}^{(s)}, -kY_{m-p}^{(s)}],$$
(17)

it is inductively proved that $Y_m^{(-p-q+j)} = 1$ $(m \equiv j \mod p+q)$, $Y_m^{(-p-q+j)} = 0$ $(m \not\equiv j \mod p+q)$ for j = 0, 1, ..., p+q-1. In the case of $Y_m^{(s)} = 1$, only the first term of the right hand side of (17) gives the maximum, and therefore there is no unexpected cancellation of the terms. Thus we have $Y_m^{(s)} = Y_{m-p-q}^{(s)}$ for all s.

For all s. Next let us investigate $z_m^{(s)} = \alpha_s(m)$ and $Z_m^{(s)} := z_m^{(s)} - k(z_{m-p}^{(s)} + z_{m-q}^{(s)}) + z_{m-p-q}^{(s)}$ for $-p - q \le s$. First $Z_m^{(s)} = Y_m^{(s)}$ for $-p - q \le s \le -1$. We readily obtain

$$Z_m^{(s)} = \min[Z_{m-p-q}^{(s)}, -kZ_{m-q}^{(s)}, -kZ_{m-p}^{(s)}].$$
(18)

It is proved in a similar manner to the previous case that $Z_m^{(s)}$ is periodic with respect to m with period p+q. The sketch is as follows: for a fixed $-p-q+1 \le s \le -1$ we have

$$Z_{2p+q+s}^{(s)} = -k, \ Z_{p+2q+s}^{(s)} = -k, \ Z_{2p+2q+s}^{(s)} = 1, \ \text{otherwise} \ Z_m^{(s)} = 0,$$

for $p + q \le m \le 2p + 2q - 1$, and this sequence continues periodically with period p + q. Moreover $Z_{2p+q+s}^{(s)}$ has the unique minimum 1 on the right hand side of (18). Therefore no cancellation is possible. For s = -p - q, we have $Z_m^{(-p-q)} = 0$ for all m, and this case neither denies a cancellation. The proof of Proposition 2.6 is now complete.

3 Further generalizations

Let us give a further generalization to Equation (4) via a reduction from the higher-dimensional lattice equation. Here is a higher dimensional analogue of (2):

$$x_{t+1,n} + x_{t-1,n} = \sum_{i=1}^{d} \left(\frac{a_i}{x_{t,n+e_i}^{k_i}} + \frac{b_i}{x_{t,n-e_i}^{l_i}} \right) \qquad (k_i, l_i \in 2\mathbb{Z}_+),$$
(19)

where each $\mathbf{e}_i \in \mathbb{Z}^d$ $(i = 1, 2, \dots, d)$ is the unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ whose *i*th component is 1, and $\mathbf{n} = \sum_{i=1}^d n_i \mathbf{e}_i$ denotes a point on the lattice. The set of initial variables are taken from those on t = 0 and t = 1 hyperplanes and evolve the equation towards $t \ge 2$. Equation (19) is proved to satisfy the coprimeness property [15] (the exact statement is that "two iterates $x_{t,\mathbf{n}}$ and $x_{t',\mathbf{n}'}$ are coprime in $\mathbb{Q}\left(\{x_{0,\mathbf{n}}, x_{1,\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}, \{a_i, b_i\}_{i=1}^d\right)$ on condition that |t - t'| > 2 or $|\mathbf{n} - \mathbf{n}'| > 2$ ") under the following condition:

$$\min_{1 \le i \le d} [k_i m_i - 1] > \max_{1 \le i \le d} [k_i m_i].$$
(20)

Let us give one of the reductions of (19) to one-dimensional lattice systems. Let $d = 2, \mathbf{n} = (n, m)$ and p > q > r be positive integers. Suppose that $x_N := x_{t,n,m}$ is constant if we fix one N = pt + qn + rm. Then we have the following recurrence relation:

$$x_{N+p} + x_{N-p} = \frac{a_1}{x_{N+q}^{k_1}} + \frac{b_1}{x_{N-q}^{m_1}} + \frac{a_2}{x_{N+r}^{k_2}} + \frac{b_2}{x_{N-r}^{m_2}}.$$
(21)

It is conjectured from several examples that, when the condition (20) is satisfied, the dynamical degree of (21) is equal to the largest real root of

$$\lambda^{2p} - k_1 \lambda^{p+q} - k_2 \lambda^{p+r} - m_2 \lambda^{p-r} - m_1 \lambda^{p-q} + 1 = 0,$$

which is the "characteristic" polynomial of its singularity structure. As for the second order systems (three-term recurrences), the relation between the degree growth and the singularity structures are well investigated. See [16, 17, 18] for details. By taking p = 5/2, q = 3/2, r = 1/2, $k_1 = m_2 = 4$, $k_2 = m_1 = 2$ and shifting N = n - 5/2, we have

$$x_n + x_{n-5} = \frac{1}{x_{n-4}^2} + \frac{1}{x_{n-3}^4} + \frac{1}{x_{n-2}^2} + \frac{1}{x_{n-1}^4}.$$
(22)

From a numerical experiment the dynamical degree of Equation (22) is estimated to be in (4.63551, 4.63552), while the largest real root of

$$\lambda^5 - 4\lambda^4 - 2\lambda^3 - 4\lambda^2 - 2\lambda + 1 = 0$$

is $4.6355149\cdots$. Here we have used the Diophantine calculation [19] for our estimation: the height of an iterate as a rational number is calculated instead of the its degree. The height H(r) of a non-zero rational number $r = \frac{p}{q}$, where p, q are pairwise coprime integers, is defined as $H(r) = \max(|p|, |q|)$ and serves as the arithmetic complexity of rationals. When we take arbitrary rational numbers as the initial variables, then every iterate $x_n \in \mathbb{Q}$. The speed of the growth of $\log H(x_n)$ is conjectured to be equal to that of deg x_n . Precisely speaking, the following limit

$$\lim_{n \to \infty} \frac{\log H(x_{n+1})}{\log H(x_n)}$$

is conjectured to converge to the dynamical degree of the mapping. Another example is

$$x_n + x_{n-6} = \frac{1}{x_{n-5}^2} + \frac{1}{x_{n-4}^2} + \frac{1}{x_{n-2}^2} + \frac{1}{x_{n-1}^2},$$

whose dynamical degree is estimated to be in (2.82320, 2.82322). This quantity is close to $2.8232019\cdots$, which is the largest real root of

$$\lambda^6 - 2\lambda^5 - 2\lambda^4 - 2\lambda^2 - 2\lambda + 1 = 0.$$

On the other hand, if we study

$$x_n + x_{n-6} = \frac{1}{x_{n-5}^2} + \frac{1}{x_{n-3}^4} + \frac{1}{x_{n-1}^2},$$

which does not satisfy (20), the estimation of its dynamical degree is in (2.61832, 2.61835), while the root of

$$\lambda^6 - 2\lambda^5 - 4\lambda^3 - 2\lambda + 1 = 0$$

is $\lambda = 2.6180339\cdots$. The discrepancy between these values seems to be beyond a numerical error and serves as a counter-example for the conjecture without (20). These are only *conjectural* topics, however we wish to give rigorous results in future correspondences.

4 Conclusion

In this article we have introduced a recurrence relation (4) through a reduction from the coprimenesspreserving extension to the discrete KdV equation (2). Equation (4) also satisfies the irreducibility and the coprimeness property and is considered as one generalisation of the Hietarinta-Viallet equation to a multi-term recurrence. As the main Theorem 2.1 we have derived that the algebraic entropy of (4) is given by the largest real root of the polynomial related to the singularity pattern of the equation. Although the proof is slightly complicated when obtaining the upper bound of the entropy, only elementary tools have been used. Finally we have introduced a higher-dimensional lattice equation (19). We have given several numerical simulations of the algebraic entropies of reduced mappings of (19) and have conjectured a property similar to Theorem 2.1.

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A Supplementary materials

A.1 Review on the coprimeness of Equation (3) and (5)

Let us review the results on the coprimeness property of the tau-function form of the coprimenesspreserving discrete KdV equation (3) and its reduction (5).

Theorem A.1 ([13, 20])

Let R be a unique factorization domain (UFD) and let $a, b \in R$ be nonzero. Then, Equation (3) has the Laurent property on any good domain, i.e. every iterate is a Laurent polynomial of the initial variables on any good domain. Moreover, every iterate is irreducible as a Laurent polynomial.

Here a nonempty subset $H \subset \mathbb{Z}^2$ is a good domain (with respect to Equation (3)) if it satisfies the following two conditions [21]:

- If $(t, n) \in H$, then $(t + 1, n), (t, n + 1) \in H$.
- For any $h \in H$, $\#\{h' \in H \mid h' \leq h\} < \infty$, where we denote by " \leq " the product order on the lattice \mathbb{Z}^2 : i.e., $h \leq h' \Leftrightarrow t \leq t'$ and $n \leq n'$ for $h = (t, n), h' = (t', n') \in \mathbb{Z}^2$.

Note that the first quadrant is a good domain with the initial variables on the L-shaped area $\{(t, n) | t = 0, 1, n \ge 0 \text{ or } n = 0, 1, t \ge 0\}$. It is proved that the reduction (5) also satisfies the Laurent, the irreducibility and the coprimeness properties.

Theorem A.2 ([20])

Let us denote by **f** the set of initial variables of (5). Then, for every iterate f_m we have

$$f_m \in \mathcal{R} := \mathbb{Z} \left[f^{\pm}, a, b \right].$$

Moreover each iterate is irreducible and arbitrary two iterates are pairwise coprime in \mathcal{R} .

Proof of Theorem A.2 is explained in [20] (Japanese article). From the discussion in [22], if a multidimensional lattice equation has the Laurent property on any good domain, then its reductions to lower-dimensional lattices preserve the Laurent property. Therefore, the Laurentness of (5) follows from the Theorem A.1 on the two-dimensional lattice equation (3). However, the irreducibility and the coprimeness do not trivially follow by the reduction and we need to prove them inductively with respect to m. In the case of (p,q) = (1,2) the induction process is not very different from that of the extended Hietarinta-Viallet equation [9], however when $p \ge 2$ or $q \ge 3$ the calculation is a bit complicated. First we show Lemma on the factorization of the Laurent polynomials, whose proof is given in [11]:

Lemma A.3 ([11], Lemma 2)

Let R be a UFD and $\{p_1, p_2, \dots, p_m\}$ and $\{q_1, q_2, \dots, q_m\}$ be two sets of independent variables satisfying for $j = 1, 2, \dots, m$ the following properties:

$$p_j \in R\left[q_1^{\pm}, q_2^{\pm}, \cdots, q_m^{\pm}\right], \qquad q_j \in R\left[p_1^{\pm}, p_2^{\pm}, \cdots, p_m^{\pm}\right],$$

and that q_j is irreducible as an element of $R\left[p_1^{\pm}, p_2^{\pm}, \cdots, p_m^{\pm}\right]$. Let f be an irreducible Laurent polynomial

$$f(p_1,\cdots,p_m) \in R\left[p_1^{\pm}, p_2^{\pm},\cdots,p_m^{\pm}\right],$$

and let g be another Laurent polynomial

$$g(q_1,\cdots,q_m) \in R\left[q_1^{\pm},q_2^{\pm},\cdots,q_m^{\pm}\right],$$

where $f(p_1, \dots, p_m) = g(q_1, \dots, q_m)$. Then the function g is factorized as

$$g(q_1, \cdots, q_m) = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} \tilde{g}(q_1, \cdots, q_m),$$

where $r_1, r_2, \cdots, r_m \in \mathbb{Z}$ and $\tilde{g}(q_1, \cdots, q_m)$ is irreducible in $R\left[q_1^{\pm}, q_2^{\pm}, \cdots, q_m^{\pm}\right]$.

A.1.1 The case of (p,q) = (1,2)

When (p,q) = (1,2) the equation (5) is

$$f_m = \frac{-f_{m-1}^k f_{m-2}^k f_{m-6} + a f_{m-1}^k f_{m-3}^{k^2 - 1} f_{m-4}^{k^2 + k} + b f_{m-2}^{k^2 + k} f_{m-3}^{k^2 - 1} f_{m-5}^k}{f_{m-4}^k f_{m-5}^k},$$
(23)

whose initial variables are $f = \{f_j\}_{j=-6}^{-1}$. Here are two Lemmata for the proof:

Lemma A.4

Each f_m $(m \ge 0)$ is a polynomial of f_{-6} , whose constant term is nonzero.

Lemma A.5

Let us substitute the following values in Equation (23):

$$a = b = 0, \quad f_{-6} = t, \quad f_{-5} = \dots = f_{-1} = 1.$$

Then f_m has a form $f_m = \pm t^{\alpha_m}$, where α_m is given by

$$\alpha_m = k(\alpha_{m-1} + \alpha_{m-2} - \alpha_{m-4} + \alpha_{m-5}) + \alpha_{m-6} \quad (m \ge 0),$$

$$\alpha_{-6} = 1, \quad \alpha_{-5} = \dots = \alpha_{-1} = 0.$$

In particular we have $\alpha_m > (k-1)\alpha_{m-1}$ for $m \ge 6$.

Now let us show Theorem A.2 for (p,q) = (1,2), using an induction with respect to m.

Step 1: irreducibility of f_m (m = 0, 1, 2) The iterate f_0 is linear with respect to f_{-6} , and thus is irreducible (from here on the irreducibility is considered in $\mathcal{R} = R[\{f_j^{\pm}\}_{j=-6}^{-1}]$ unless otherwise stated) and not invertible. Using Lemma A.3 for $R = \mathbb{Z}[a, b], \{p_j\}_{j=1}^6 = \{f_j\}_{j=-5}^0, \{q_j\}_{j=1}^6 = \{f_j\}_{j=-6}^{-1}, f_1$ is expressed as

$$f_1 = F_1 \prod_{j=-5}^{0} f_j^{r_j},$$

where F_1 is irreducible and each r_j is an integer. Since f_1 is a Laurent polynomial and f_0 is irreducible and non-invertible, we must have $r_0 \ge 0$. On the other hand we have

$$f_1 \equiv \frac{bf_{-1}^{k^2+k}f_{-2}^k}{f_{-3}^k} \mod f_0.$$

Since f_0 is linear with respect to f_{-6} , the iterate f_0 cannot divide f_1 , and therefore $r_0 = 0$. Moreover, r_j $(-5 \le j \le -1)$ are units in \mathcal{R} , thus f_1 is irreducible and is coprime with f_0 . From the irreducibility of f_1 in \mathcal{R} , the iterate f_2 is trivially irreducible in $R[\{f_j^{\pm}\}_{j=-5}^0]$ by shifting all the subscripts. Thus using Lemma A.3, we have $f_2 = f_0^{r_2} F_2$, where F_2 is irreducible in \mathcal{R} and r_2 is a non-negative integer. We can show from a simple computation that $f_2 \not\equiv 0 \mod f_0$. Thus $r_2 = 0$ and f_2 is irreducible.

Step 2: irreducibility of f_m (m = 3, 4, 5, 6) By a similar discussion to the previous step, we can inductively show the irreducibility of f_j and express $f_j = f_0^{r_j} F_j$ for j = 3, 4, 5, 6, where F_j is irreducible and r_j is a non-negative integer. The case of j = 3 is easy since it is readily obtained that $f_3 \not\equiv 0 \mod f_0$. Let us prove the irreducibility of f_4 . By a direct calculation we have

$$f_2 = \frac{\left(-f_0^k f_{-4} + a f_{-2}^{k^2 + k} f_{-1}^{k^2 - 1}\right) f_1^k + \mathcal{O}(f_0^{k^2 + k})}{f_{-2}^k f_{-3}^k}, \quad f_3 = \frac{-f_{-3} f_2^k f_1^k + \mathcal{O}(f_0^{k^2 + k})}{f_{-1}^k f_{-2}^k},$$

and

$$-f_3^k f_2^k f_{-2} + b f_2^{k^2 + k} f_1^{k^2 - 1} f_{-1}^k \equiv \frac{f_1^{k^2 - 1} f_2^{k^2 + k}}{f_{-1}^{k^2} f_{-2}^{k^2 - 1}} \left(-f_1 f_{-3}^k + b f_{-1}^{k^2 + k} f_{-2}^{k^2 - 1} \right) \mod f_0^{k+1}$$
$$\equiv \frac{f_1^{k^2 - 1} f_2^{k^2 + k}}{f_{-1}^{k^2} f_{-2}^{k^2 - 1}} \left(\frac{f_2^k f_{-2} - a f_0^{k^2 + k} f_{-2}^{k^2 - 1}}{f_{-4}^k} f_0^k \right) \mod f_0^{k+1}$$

Thus,

$$f_{4} = \frac{-f_{3}^{k} f_{2}^{k} f_{-2} + b f_{2}^{k^{2}+k} f_{1}^{k^{2}-1} f_{-1}^{k} + \mathcal{O}(f_{0}^{k^{2}+k})}{f_{0}^{k} f_{-1}^{k}}$$
$$\equiv \frac{f_{1}^{k^{2}-1} f_{2}^{k^{2}+k}}{f_{-1}^{k^{2}+k} f_{-2}^{k^{2}-1} f_{-4}^{k}} \left(f_{-1}^{k} f_{-5} - a f_{-3}^{k^{2}+k} f_{-2}^{k^{2}-1}\right) \neq 0 \mod f_{0}$$

Therefore f_4 is irreducible.

The case of f_5 is done as follows: by a direct calculation we have

$$-f_4^k f_3^k f_{-1} + a f_4^k f_2^{k^2 - 1} f_1^{k^2 + k} \equiv \frac{f_4^k f_2^{k^2 - 1} f_1^{k^2 + k} f_0^k f_{-4}}{f_{-1}^{k^2 - 1} f_{-2}^{k^2 + k}} \mod f_0^{k+1},$$

and thus

$$f_5 \equiv \frac{f_4^k f_2^{k^2 - 1} f_1^{k^2} f_{-4}}{f_{-1}^{k^2 - 1} f_{-2}^{k^2 + k}} + b \frac{f_3^{k^2 + k} f_2^{k^2 - 1}}{f_1^k} \mod f_0$$

$$= \frac{f_2^{k^2 - 1}}{f_{-1}^{k^2 - 1} f_{-2}^{k^2 + k} f_1^k} \left(f_4^k f_1^{k^2 + k} f_{-4} + b f_3^{k^2 + k} f_{-1}^{k^2 - 1} f_{-2}^{k^2 + k} \right)$$

It is sufficient to prove that $F := f_4^k f_1^{k^2+1} f_{-4} + b f_3^{k^2+k} f_{-1}^{k^2-1} f_{-2}^{k^2+k} \neq 0 \mod f_0$. By choosing the initial values $f_{-6} = a + b, f_{-5} = \cdots = f_{-1} = 1$, and by taking the parameters as a > 1, b > 0, we can show that F > 0. Thus F is not divisible by f_0 . Therefore f_5 is irreducible.

The irreducibility of f_6 is proved in a similar manner, since we can prove that

$$f_6 \equiv \frac{f_3^{k^2 - 1}}{f_2^k f_1^k} \left(a f_5^k f_2^{k^2 + k} + b f_4^{k^2 + k} f_1^k \right) \not\equiv 0 \mod f_0$$

Step 3: coprimeness of f_m $(0 \le m \le 6)$ Suppose that f_i and f_j (i > j) are not pairwise coprime, then we can express $f_i = uf_j$ where u is an invertible element in \mathcal{R} . On the other hand, from Lemma A.5, u must include the factor $t^{\alpha_i - \alpha_j}$. Therefore the constant term of f_j as a polynomial of t must be zero, which contradicts Lemma A.4.

Step 4: irreducibility and coprimeness of f_m $(m \ge 7)$ From Lemma A.3, f_7 is factorized in two ways as

$$f_7 = f_0^{r_0} F = f_1^{r_1} \cdots f_6^{r_6} F',$$

where F, F' are irreducible in \mathcal{R} and r_j are non-negative integers. Suppose that f_7 is not irreducible, then the factorization is limited to the form $f_7 = uf_0f_j$ where u is invertible and $j \in \{1, \dots, 6\}$. Thus we have

$$\alpha_7 > (k-1)\alpha_6 + 1 \ge \alpha_j + \alpha_3,$$

which contradicts Lemma A.4. The case of $m \ge 8$ can be done in the same manner. The pairwise coprimeness of f_i and f_j (i > j) is proved in exactly the same manner as in Step 3. The proof of Theorem A.2 for (p,q) = (1,2) is now complete.

A.1.2 The case of p = 1 and $q \ge 3$

Next let us prove the case of p = 1 and $q \ge 3$. It is worth noting that p, q, 2p, p + q, 2q, 2p + q, p + 2q are all distinct when $(p,q) \ne (1,2)$. Let us present four Lemmata, which are applicable for every (p,q) with $p \ge 1$ and $q \ge 2$.

Lemma A.6

For every $s \ge 0$, f_s is a polynomial of f_{-2p-2q} , $f_{-2p-2q+1}$, ..., $f_{-p-2q-1}$ whose constant term is nonzero. The statement is true even when we substitute a = 0 or b = 0.

Lemma A.7

Let $c_s^{(j)}$ be the degree of $f_s|_{a=b=0}$ with respect to $f_{-2p-2q+j}$ $(j=0,\ldots,p-1)$ and let $c_s = \left(c_s^{(0)},\ldots,c_s^{(p-1)}\right)$. Then we have the following properties:

- (i) $c_s^{(j)} = c_{s-j}^{(0)}$ for every $s \ge j$.
- (ii) If we have $f_s = u \prod_{i \in J} f_i$ where u is invertible, then $c_s = \sum_{i \in J} c_i$.
- (iii) Let f_s and f_r be irreducible Laurent polynomials. If $c_s \neq c_r$, f_s and f_r are pairwise coprime.

Proof (i) is trivial since we have

$$f_s|_{a=b=0} = -\frac{f_{s-p}^k f_{s-q}^k f_{s-2p-2q}^k}{f_{s-2p-q}^k f_{s-p-2q}^k}.$$

(ii) Since u is invertible, u does not depend on a, b. From Lemma A.6, u does not depend on $f_{-2p-2q}, \ldots, f_{-p-2q-1}$. By substituting a = b = 0 into $f_s = u \prod_{j \in J} f_j$, we obtain $c_s = \sum_{j \in J} c_j$.

(iii) Suppose that f_s and f_r are both irreducible but not coprime with each other. Then there exists an invertible element u such that $f_s = uf_r$. Thus we have $c_s = c_r$ from (ii).

Lemma A.8

The iterate f_s is irreducible if either $s \notin \{ip + jq \mid i, j \in \mathbb{Z}_{\geq 0}\}$ or $0 \leq s \leq p + q$ is satisfied.

Proof

Step 1 In the case of s = 0, ..., p - 1, f_s is linear with respect to $f_{-2p-2q+s}$ whose constant term is nonzero. Thus f_s is irreducible and is not a unit. Note that f_s does not depend on the initial variables $f_{-2p-2q+i}$ $(0 \le i \le p - 1, i \ne s)$.

Step 2 In the case of $s \neq ip + jq$ $(i, j \in \mathbb{Z}_{\geq 0})$, $f_m(m \geq 0)$ depends on f_{-2p-2q} if and only if m can be written as m = ip + jq where i, j are nonnegative integers. Thus f_s does not depend on f_{-2p-2q} . From Lemma A.3, by assuming the irreducibility of f_m for every $m \leq s - 1$, f_s can be factorized as $f_s = f_0^r F$, where r is a nonnegative integer and F is irreducible. Since f_s is independent of f_{-2p-2q} we must have r = 0. Thus f_s is irreducible.

Step 3 In the case of $1 \le s \le p+q$, let us define g_s as the value of f_s where we substitute the following values

$$f_{-2p-2q} = \frac{f_{-p-q}^{k^2-1}}{f_{-p}^k f_{-q}^k} \left(a f_{-p}^k f_{-2q}^{k^2} f_{-2p-q}^k + b f_{-q}^k f_{-2p}^{k^2} f_{-p-2q}^k \right),$$

$$f_m < 0 \quad (-2p - 2q \le m \le -p - q),$$

$$f_m > 0 \quad (-p - q + 1 \le m \le -1),$$

into the initial variables. Suppose that a, b > 0. It is clear that $g_0 = 0$ and g_s satisfies

$$g_{s} = \frac{-g_{s-2p-2q}g_{s-p}^{k}g_{s-q}^{k} + \left(ag_{s-p}^{k}g_{s-2q}^{k^{2}}g_{s-2p-q}^{k} + bg_{s-q}^{k}g_{s-2p}^{k^{2}}g_{s-p-2q}^{k}\right)g_{s-p-q}^{k^{2}-1}}{g_{s-p-2q}^{k}g_{s-2p-q}^{k}}.$$

Therefore $g_s > 0$. By the same argument as in Step 2, we conclude that f_s is irreducible.

Lemma A.9

Let us prepare two functions $C_{p,q}, \widetilde{C}_{p,q}$ by

$$C_{p,q} = \frac{f_p^{k^2 - 1} f_q^{k^2} f_{2p}^k}{f_{-q}^{k^2 - 1} f_{-p}^{k^2} f_{-2q}^k f_{p-q}^k}, \quad \widetilde{C}_{p,q} = \frac{f_p^{k^2} f_q^{k^2 - 1} f_{2q}^k}{f_{-p}^{k^2 - 1} f_{-q}^{k^2} f_{-2p}^k f_{-p+q}^k}$$

Then we have

$$f_{2p+q} \equiv C_{p,q} \left(f_{p-q}^k f_{-p-2q} - a f_{p-2q}^{k^2} f_{-p-q}^k f_{-q}^{k^2-1} \right) + \frac{a f_{p+q}^k f_{2p-q}^{k^2-1}}{f_{p-q}^k} \mod f_0,$$

$$f_{p+2q} \equiv \widetilde{C}_{p,q} \left(f_{-p+q}^k f_{-2p-q} - b f_{-2p+q}^{k^2} f_{-p-q}^k f_{-p}^{k^2-1} \right) + \frac{b f_{p+q}^k f_{-p+2q}^{k^2-1}}{f_{-p+q}^k} \mod f_0.$$

Now let us begin the proof of Theorem A.2 for $p = 1, q \ge 3$. Equation (5) is

$$f_{s} = \frac{f_{s-1}^{k} f_{s-q}^{k} \left(-f_{s-2-2q}\right) + \left(a f_{s-1}^{k} f_{s-2q}^{k^{2}} f_{s-2-q}^{k} + b f_{s-q}^{k} f_{s-2}^{k^{2}} f_{s-1-2q}^{k}\right) f_{s-1-q}^{k^{2}-1}}{f_{s-1-2q}^{k} f_{s-2-q}^{k}}$$

From Lemma A.8, f_s is irreducible for $0 \le s \le q+1$. Let g_s be the values of the iterates f_s when we substitute

$$f_m = \begin{cases} 2 & (m = -2q - 2) \\ -1 & (m = -2q) \\ 1 & (-2q - 1 \le m \le -1, m \ne -2q) \end{cases}$$

into the initial variables of f_s and take a = b = 1. Clearly $g_0 = 0$.

Step 1 It is readily obtained that if $g_s \neq 0$, then f_s is irreducible: From Lemma A.3, there exist a nonnegative integer r and an irreducible element F such that $f_s = f_0^r F$. If we assume that r > 0, then $g_s = 0$. From here on we shall prove $g_s \neq 0$ by a direct calculation. The first few terms are

$$g_s = \begin{cases} g_{s-2}^{k^2} & (3 \le s \le q-1) \\ g_{s-1}^k & (s=q) \\ -g_{s-1}^k & (s=q+1) \end{cases}$$

Therefore

$$g_s = \begin{cases} 1 & (3 \le s \le q - 1, s \text{ is odd}) \\ 2^{k^{s-2}} & (3 \le s \le q - 1, s \text{ is even}) \\ g_{q-1}^k & (s = q) \\ -g_{q-1}^{k^2} & (s = q + 1) \end{cases}.$$

When q is odd we have

$$g_{q-1} = 1, \quad g_q = 1, \quad g_{q+1} = -1,$$

and when q is even we have

$$g_{q-1} = 2^{k^{q-3}}, \quad g_q = 2^{k^{q-2}}, \quad g_{q+1} = -2^{k^{q-1}}.$$

Step 2: irreducibility of f_s : In the case of s = q + 2, from Lemma A.9 we have

$$g_{q+2} = g_{q-1}^{k^3} \neq 0.$$

In the case of $q + 3 \le s \le 2q - 1$, we have

$$g_{q+r+2} = \frac{-g_{s-1}^k g_{s-q}^k + \left(g_{s-1}^k g_{s-q-2}^k + g_{s-q}^k g_{s-2}^k\right) g_{s-1-q}^{k^2-1}}{g_{s-2-q}^k},$$

for $1 \leq r \leq q - 3$. Thus

$$g_{q+r+2} = \begin{cases} (N_r - 1) g_{q+r+1}^k + N_r g_{q+r}^{k^2} & (r \text{ is odd}) \\ (1 - N_r) g_{q+r+1}^k + N_r g_{q+r}^{k^2} & (r \text{ is even}) \end{cases}$$

where

$$N_r = 2^{k^{r-1}(k^2 - 1)}.$$

Therefore we have $g_{q+r+2} > 0$ if r is odd, and $g_{q+r+2} < 0$ if r is even. These inequalities are clear when r is odd. When r is even, we can prove this by

$$g_{q+r+2} = -(N_r - 1) \left((N_{r-1} - 1)g_{q+r}^k + N_{r-1}g_{q+r-1}^{k^2} \right)^k + N_r g_{q+r}^{k^2}$$

$$< - \left((N_r - 1)(N_{r-1} - 1)^k - N_r \right) g_{q+r}^{k^2}.$$

Therefore we have $g_s \neq 0$ for $q+3 \leq s \leq 2q-1$. In the case of s = 2q, since

$$f_{2q} \equiv \frac{-f_{2q-1}^k f_q^k f_{-2} + b f_q^k f_{2q-2}^{k} f_{-1}^k f_{q-1}^{k^2 - 1}}{f_{-1}^k f_{q-2}^k} \mod f_0.$$

we have

$$g_{2q} = \frac{g_q^k}{g_{q-2}^k} \left(-g_{2q-1}^k + g_{q-1}^{k^2-1} g_{2q-2}^{k^2} \right).$$

It is sufficient to prove that $G = -g_{2q-1}^k + g_{q-1}^{k^2-1}g_{2q-2}^{k^2} \neq 0$. If q is even, since $q \ge 4$ and $g_{q-1} = 1$, we must have $g_{2q-1} = \pm g_{2q-2}^k$ in order to achieve G = 0. This is not possible because

$$g_{2q-1} = (N_{q-3} - 1) g_{2q-2}^k + N_{q-3} g_{2q-3}^{k^2} > g_{2q-2}^k.$$

If q is odd, since $g_{q-1} = 2^{k^{q-3}}$ and thus $g_{q-1}^{k^2-1} = N_{q-2}$, we have

$$G = -g_{2q-1}^k + N_{q-2}g_{2q-2}^{k^2}.$$

In the case of q = 3, $G \neq 0$ is obtained by a direct calculation. In the case of $q \geq 5$, we must have $g_{2q-1} = \pm N_{q-3}g_{2q-2}^k$ when G = 0. However, since $g_{2q-1} = (1 - N_{q-3})g_{2q-2}^k + N_{q-3}g_{2q-3}^{k^2}$ whose right hand side is negative, we must have $g_{2q-1} = -N_{q-3}g_{2q-2}^k$, and therefore $g_{2q-2}^k = -N_{q-3}g_{2q-3}^{k^2}$, which is not possible. Thus $G \neq 0$.

In the case of s = 2q + 1, from Lemma A.9 we have

$$g_{2q+1} = g_q^{k^2 - 1} g_{2q}^k \left(g_{q-1}^k - g_{q-2}^{k^2} \right) + \frac{g_{q+1}^k g_{2q-1}^{k^2} g_q^{k^2 - 1}}{g_{q-1}^k}$$

If q is odd, $g_{2q+1} > 0$ is readily obtained since $g_{q-2} = 1$, $g_{q-1} = 2^{k^{q-3}}$. If q is even, we have

$$g_{q+r+2} \equiv 1 \mod 3$$
,

for $2 \leq r \leq q-3$. Since we have $g_{q-1}^k - g_{q-2}^{k^2} \equiv 0 \mod 3$ and that none of the four iterates $g_{q+1}, g_{2q-1}, g_q, g_{q-1}$ is divisible by 3, we have

$$g_{2q+1} \not\equiv 0 \mod 3$$

Thus $g_{2q+1} \neq 0$.

In the case of s = 2q + 2, $g_{2q+2} \neq 0$ is obtained from

$$g_{2q+2} = \frac{g_{q+1}^{k^2-1}}{g_1^k g_q^k} \left(g_{2q+1}^k g_2^{k^2} g_q^k + g_{q+2}^k g_{2q}^{k^2} g_1^k \right).$$

The proof of the irreducibility of f_s $(s \ge 2q + 3)$ is omitted.

Step 3: coprimeness of f_s : Let us prove that f_s and f_r are pairwise coprime if $s > r \ge 0$. The degrees $c_i^{(0)}$ in Lemma A.7 satisfy

$$\begin{split} c_0^{(0)} &= c_{-2q-2}^{(0)} = 1, \quad c_{-1}^{(0)} = \dots = c_{-2q-1}^{(0)} = 0, \\ c_j^{(0)} &= k \left(c_{j-1}^{(0)} + c_{j-q}^{(0)} - c_{j-2-q}^{(0)} - c_{j-1-2q}^{(0)} \right) + c_{j-2-2q}^{(0)} \quad (j \ge 1). \end{split}$$

Thus we have $c_j^{(0)} \ge k c_{j-1}^{(0)}$ for every $j \ge -2q - 1$ inductively. Therefore $c_s^{(0)} > c_r^{(0)}$. From Lemma A.7 (iii), f_s and f_r are coprime with each other.

A.1.3 The case of p = 2

When p = 2, q must be odd. Let g_s be the values of the iterates f_s when we substitute

$$f_m = \begin{cases} 2 & (m = -2p - 2q) \\ -1 & (m = -2q) \\ 1 & (-2p - 2q + 1 \le m \le -1, m \ne -2q) \end{cases}$$
(24)

into the initial variables of f_s and take a = b = 1. Clearly $g_0 = 0$. Our goal is to prove that $g_s \neq 0$ for every $s \ge 1$. The discussion goes in a similar manner to the case of $p = 1, q \ge 3$.

Step 1: irreducibility of f_s $(1 \le s \le 2q + 4)$: A direct calculation shows that

$$g_s = \begin{cases} N_i & (s = 4i) \\ 1 & (s \neq 4i) \end{cases}$$

for every $1 \le s \le q-1$, where we have defined $N_i = 2^{k^{2(i-1)}}$. Calculating further we have

$$g_q = g_{q-2}^k = 1, \quad g_{q+1} = g_{q-3}^{k^2}, \quad g_{q+2} = -g_q^k = -1, \quad g_{q+3} = g_{q-1}^{k^2},$$

where we have used the fact that q is odd. From Lemma A.9 we have

$$g_{q+4} = g_{q+2}^k g_{-q+4}^{k^2} g_2^{k^2 - 1} g_{2-q}^{-k} = 1 \neq 0$$

Thus we have $g_s \neq 0$ for $1 \leq s \leq q+4$. Next we show $g_s \neq 0$ for $q+5 \leq s \leq 2q-1$. By using r=s-q-4, we have

$$g_{q+r+4} = \left\{ \left(-g_{r+4}^k + g_{r+4-q}^{k^2} g_r^k g_{r+2}^{k^2-1} \right) g_{q+r+2}^k + g_{r+4}^k g_{r+2}^{k^2-1} g_{q+r}^{k^2} \right\} g_r^{-k},$$

and therefore,

$$g_{q+5} = g_{q+1}^{k^2}, \quad g_{q+6} = -1 + 2^{k^2 - 1} + 2^{k^2 - 1} = 2^{k^2} - 1 = N_2 - 1, \quad g_{q+7} = g_{q+3}^{k^2}.$$

Inductively we have the following expression for $3 \le r \le q-5$:

$$g_{q+r+4} = \begin{cases} \left(-\frac{N_{i+1}^k}{N_i^k} + 1\right) g_{q+r+2}^k + \frac{N_{i+1}^k}{N_i^k} g_{q+r}^{k^2} & (r = 4i) \\ g_{q+r}^{k^2} & (r = 4i+1, 4i+3) \\ \left(N_{i+1}^{k^2-1} - 1\right) g_{q+r+2}^k + N_{i+1}^{k^2-1} g_{q+r}^{k^2} & (r = 4i+2) \end{cases}$$

If r is odd, it is clear that $g_{q+r+4} \neq 0$. If r is even, $g_{q+r+4} \equiv g_{q+r+2} \mod 2$ and thus inductively $g_{q+r+4} \equiv 1 \mod 2 \neq 0$. Lastly we need to prove that $g_s \neq 0$ for $2q \leq s \leq 2q+4$ one by one. The case of $g_{q+r+4} = 1 \mod 2$) of Eastly we need to prove $g_{g} = -g_{2q-2}^k + g_{2q-4}^{k^2}$ and the fact that q is odd. We also have $g_{2q+1} \equiv 1 \mod 2$ by a direct calculation. In the case of s = 2q + 2, using the second equation in Lemma A.9, we have $g_{q+2} = -1$, $g_q = g_{q-2} = g_{q-4} = 1$ and $g_{2q+2} = g_{q+2}^k g_{2q-2}^{k^2-1} g_{q-2}^{-k} = g_{2q-2}^{k^2} \neq 0$. The cases of s = 2q + 3, 2q + 4 are omitted since they are proved by a direct calculation.

Step 2: coprimeness of f_s $(0 \le s \le 2p + 4)$: The variables $c_j^{(0)}$ in Lemma A.7 satisfy

$$\begin{aligned} c_0^{(0)} &= c_{-2q-4}^{(0)} = 1, \quad c_{-1}^{(0)} = \dots = c_{2q-3}^{(0)} = 0, \\ c_j^{(0)} &= k \left(c_{j-2}^{(0)} + c_{j-q}^{(0)} - c_{j-4-q}^{(0)} - c_{j-2-2q}^{(0)} \right) + c_{j-4-2q}^{(0)} \quad (j \ge 1). \end{aligned}$$

Thus we have $c_{2i}^{(0)} = k^i \ (0 \le i \le q - 1)$ and

$$\begin{aligned} c_{2i+1}^{(0)} &= 0 \left(0 \le i \le \frac{q-3}{2} \right), \ c_q^{(0)} = k, \ c_{q+2i}^{(0)} = (i+1)k^{i+1} - (i-1)k^{i-1} \ (1 \le i \le q-1), \\ c_{2q}^{(0)} &= k^q + k^2, \quad c_{2q+2}^{(0)} = k^{q+1} + 3k^3 - k, \quad c_{2q+4}^{(0)} = k^{q+2} + 6k^4 - 4k^2 + 1. \end{aligned}$$

Therefore $c_s \neq c_r$ for $0 \leq s < r \leq 2p + 4$. From (iii) of Lemma A.7, f_s and f_r must be pairwise coprime.

Step 3: irreducibility and coprimeness of f_s $(2q+5 \le s)$: Let us first prove the irreducibility of f_s for $s \ge 2q + 5$. From the previous steps, if we suppose that f_s is not irreducible then there exist a reversible element u and $1 \le r \le 2q + 4$ such that $f_s = uf_0 f_r$. Therefore from (ii) of Lemma A.7 we must have $c_s = c_0 + c_r$, and thus $c_s^{(0)} = c_r^{(0)} + 1$, $c_{s-1}^{(0)} = c_{r-1}^{(0)}$. However, this leads us to a contradiction since we can prove that $c_s \neq c_0 + c_r$ as follows: for $0 \le i \le q + 2$ we have $c_{2i}^{(0)} > c_{2i-1}^{(0)}$, $c_{i+2}^{(0)} \ge kc_i^{(0)}$ and $c_{i+q}^{(0)} \ge kc_i^{(0)}$. If $s \ge 2q + 5$ and s is even, we have $c_s^{(0)} \ge kc_{2q+4}^{(0)}$. Thus $c_s^{(0)} \ne c_r^{(0)} + 1$. If s is odd we have $c_{s-1}^{i+q-i} \ge c_{2q+4}^{(0)}$ and thus $c_{s-1}^{(0)} \ne c_{r-1}^{(0)}$. The coprimeness of f_s and f_r $(0 \le s \le r)$ is proved in a similar manner.

A.1.4 The case of p > 3

When $p \ge 3$, every pair from $\{ip, q+ip, 2q+ip \mid (i=0,1,2,...)\}$ is distinct from each other. Since p and q are pairwise coprime, we have the following Lemma A.10.

Lemma A.10

For $m, n \in \mathbb{Z}_{\geq 0}$ let us define s = mp + nq. Then s is uniquely expressed as

$$s = rpq + ip + jq$$
 $(r \in \mathbb{Z}_{\geq 0}, 0 \le i \le q - 1, 0 \le j \le p - 1).$

Moreover, this expression maps s to (r, i, j) bijectively.

From Lemma A.8, f_s is irreducible if $0 \le s \le p+q$ or $s \ne ip+jq$ $(i, j \in \mathbb{Z}_{>0})$. Let g_s be the same value as in (24) in the case of p = 2. Our goal is to prove the irreducibility of f_s for every $s \ge 1$. Basically we have only to prove that $g_s \neq 0$, however, for s = 2q we have $g_{2q} = 0$ and therefore another approach is necessary.

Step 1: irreducibility of f_s $(1 \le s \le 2p + 2q)$: Let us express s = rpq + ip + jq in the sense of Lemma A.10. First we study the case where $r = 0, 0 \le i \le q - 1$ and j = 0, 1. If j = 0, we have $g_{2p} \ne 0$ and

$$g_{ip} = -g_{(i-1)p}^{k} + g_{(i-1)p}^{k} + g_{(i-2)p}^{k^{2}} = g_{(i-2)p}^{k^{2}}.$$

Thus

$$g_{ip} = \begin{cases} 1 & (i \text{ :odd}) \\ 2^{k^{i-2}} & (i \text{ :even}) \end{cases}$$

Therefore $g_{ip} \neq 0$. Let us study the case of j = 1. For i = 0, 1, 2 we have

$$g_q = 1, \quad g_{q+p} = -1, \quad g_{q+2p} = 1.$$

Here we have used Lemma A.9 to obtain the value of g_{q+2p} . For $i \geq 3$,

$$g_{q+ip} = \frac{-g_{q+(i-1)p}^{k}g_{ip}^{k} + \left(g_{q+(i-1)p}^{k}g_{(i-2)p}^{k} + g_{ip}^{k}g_{q+(i-2)p}^{k^{2}}\right)g_{(i-1)p}^{k^{2}-1}}{g_{(i-2)p}^{k}}$$
$$= \begin{cases} -g_{q+(i-1)p}^{k} + N_{i-2}\left(g_{q+(i-1)p}^{k} + g_{q+(i-2)p}^{k^{2}}\right) & (i:\text{odd})\\ -(N_{i}/N_{i-2})g_{q+(i-1)p}^{k} + \left(g_{q+(i-1)p}^{k} + (N_{i}/N_{i-2})g_{q+(i-2)p}^{k^{2}}\right) & (i:\text{even}) \end{cases}$$

where $N_i = 2^{k^{i-2}}$. Since $g_{q+ip} \equiv 1 \mod 2$ we have $g_{q+ip} \neq 0$.

Next let us prove the irreducibility of f_{2q} . Since $g_{2q} = 0$, we cannot use the same argument as before using g_{2q} . Let h_s be the values of f_s when we substitute

$$f_m = \begin{cases} 2 & (m = -2p - 2q) \\ -1 & (m = -2p) \\ 1 & (-2p - 2q \le m \le -1, m \ne -2p) \end{cases}$$

in the initial variables of f_s and take a = b = 1. Then we have

$$h_0 = 0, \quad h_q = 1, \quad h_{2q} = 2 \neq 0$$

Thus f_{2q} is irreducible.

The proof for $g_{2q+p}, g_{2q+2p}, g_{3q} \neq 0$ is straightforward. From Lemma A.9 we have $g_{2q+p} = 1 \neq 0$. Using $g_{2q} = 0$, we have $g_{2q+2p} = 2^{k^2} \neq 0$. If $p \ge 4$, $g_{2q} = 0$ leads us to $g_{3q} = 1 \neq 0$. If p = 3, since 3q = pq, we can use the discussion in the next step.

Next let us prove the case of r = 1: i.e., s = rpq + ip + jq = pq + ip + jq. Note that from the condition $s \leq 2p+2q$, only the case s = pq with (p,q) = (3,4), (3,5) is possible. Therefore $g_{(p-1)q} = g_{2q} = 0$ and thus $g_{pq} = g_{(q-1)p}^k g_q^{k^2} \neq 0$. We have proved that f_s is irreducible for $0 \le s \le 2p + 2q$.

Step 2: coprimeness of f_s $(0 \le s \le 2p + 2q)$: First let us present several properties concerning the variable $c_s^{(0)}$ in Lemma A.7.

Lemma A.11 We have $c_{jp+i}^{(0)} < c_{jp}^{(0)}$ for any $j \in \mathbb{Z}_{\geq 0}$ and $0 < |i| \le p-1$. Moreover it holds that $c_{mp}^{(0)} \ge c_{(m-1)p}^{(0)}$ for any

Proof

For the former inequality, it is sufficient to show that, for a fixed i, $y_{m,n} := c_{mp+nq}^{(0)} - c_{mp+nq+i}^{(0)}$ satisfies $y_{m,0} > 0$. Note that $y_{m,n}$ satisfies $y_{m+q,n} = y_{m,n+p}$ with the initial values

$$y_{-2,-2} = 1, \quad y_{m,-2} = y_{m,-1} = y_{-2,n} = y_{-1,n} = 0 \quad (-1 \le m \le q - 1, -1 \le n \le p - 1),$$

and the recurrence relation for $y_{m,n}$ is

$$y_{m,n} = k \left(y_{m-1,n} + y_{m,n-1} - y_{m-2,n-1} - y_{m-1,n-2} \right) + y_{m-2,n-2}.$$

If we define $d_{m,n} = y_{m,n} - y_{m-1,n-1}$ we have $d_{m,n} = k(d_{m-1,n} + d_{m,n-1}) - d_{m-1,n-1}$, $d_{m+q,n} = d_{m,n+p}$, with the initial values $d_{-1,-1} = -1$, $d_{m,-1} = d_{-1,n} = 0$ ($0 \le m \le q - 1, 0 \le n \le p - 1$). Therefore it is readily obtained that $y_{m,n} > 0$ for $m, n \ge 0$. If m, n satisfy $-p \le m-n \le q$, we have $y_{m,n} = \sum_{\ell} d_{m-\ell,n-\ell}$, where the summation runs over $0 \le \ell \le \min(m, n)$. Thus $y_{m,n} > 0$. For a fixed $m \ge 0$, let us take $m_0 \in \mathbb{Z}_{\ge 0}$ such that $-q \le m_0(p+q) - m \le p$. Then $y_{m,0} = y_{m-m_0q,m_0p}$ and $-p \le m - m_0q - m_0p \le q$ indicate that $y_{m,0} > 0$. The latter inequality in Lemma A.11 is proved in a similar manner to the former one by defining $e_{m,n} := c_{mp+nq}^{(0)} - c_{(m-1)p+(n-1)q}^{(0)}$, which satisfies the same recurrence as that for $d_{m,n}$, and by proving $d_{m,0} \ge kd_{m-1,0}$.

Now, let us prove that f_s and f_t are pairwise coprime for $0 \le s < t \le 2p + 2q$. From (iii) of Lemma A.7, it is sufficient to prove that $c_s \ne c_t$. When we define

$$m_s = \max\left(c_s^{(0)}, \dots, c_{s-p+1}^{(0)}\right),$$

it is shown from Lemma A.11 that there exists $i \leq j$ such that $m_s = c_{ip}^{(0)}$, $m_t = c_{jp}^{(0)}$. If i < j then we have $m_s < m_t$ and thus $c_s \neq c_t$. If i = j then again from Lemma A.11 we have $c_s \neq c_t$ since they attain their maxima in the distinct elements.

Step 3: irreducibility and coprimeness of f_s $(2p+2q+1 \le s)$: For $s \ge 2p+2q+1$, let us assume that f_s is not irreducible. Then f_s must be factorized as $f_s = uf_r f_0$ using an invertible element u and a subscript r with $1 \le r \le 2p+2q$. From (ii) of Lemma A.7, we have $c_s = c_r + c_0$, which contradicts the calculations in the previous steps. It is readily obtained that f_s and f_t are pairwise coprime for every $0 \le s < t$ in the same manner as in the previous steps.

A.2 Comments on Proposition 2.4

Let us show two Lemmas on $\Lambda_{p,q}$.

Lemma A.12

We have $1 < \Lambda_{p,q} < k$. Moreover, $\Lambda_{p,q}$ is the largest absolute value among all the roots (including the imaginary ones) of (7).

Proof Let $f(\lambda) := \lambda^{p+q} - k(\lambda^p + \lambda^q) + 1$, then f(1) < 0 and f(k) > 0 and thus $1 < \Lambda_{p,q}$. For $x \ge \Lambda_{p,q}$ we have

$$\begin{aligned} f'(x) &= x^{-1} \left\{ p(x^q - k) x^p + q(x^p - k) x^q \right\} \\ &\geq x^{-1} \left\{ p(\Lambda_{p,q}^q - k) \Lambda_{p,q}^p + q(\Lambda_{p,q}^p - k) \Lambda_{p,q}^q \right\} \\ &= x^{-1} \left\{ q(k\Lambda_{p,q}^p - 1) + p(k\Lambda_{p,q}^q - 1) \right\} > 0. \end{aligned}$$

Thus for $x > \Lambda_{p,q}$ we have f(x) > 0 and thus $\Lambda_{p,q} < k$. Assume that there exists a root λ such that $|\lambda| > \Lambda_{p,q}$, then f(x) > 0 for $x := |\lambda|$. On the other hand, if we take $\lambda = xe^{-\sqrt{-1}\gamma}$, $f(\lambda) = 0$ is equivalent to

$$x^{p+q} - k(x^{p}e^{\sqrt{-1}q\gamma} + x^{q}e^{\sqrt{-1}p\gamma}) + e^{\sqrt{-1}(p+q)\gamma} = 0$$

Let us prove that no x, γ satisfy the above equality. It is sufficient to show that, for fixed 1 < u, y, zand u + y < 1 + z, we cannot find θ , ϕ that satisfy (25):

$$z - (ue^{\sqrt{-1}\theta} + ye^{\sqrt{-1}\phi}) + e^{\sqrt{-1}(\theta + \phi)} = 0.$$
(25)

By taking the real part of (25) we have

$$u + y - 1 + \cos(\theta + \phi) < z + \cos(\theta + \phi) = u\cos\theta + y\cos\phi$$

Thus

$$(1 - \cos \theta) + (1 - \cos \phi) \le u(1 - \cos \theta) + y(1 - \cos \phi) < 1 - \cos(\theta + \phi),$$

from which $0 \le (1 - \cos \theta)(1 - \cos \phi) < \sin \theta \sin \phi$ immediately follows. On the other hand, by taking the imaginary part of (25) we have

$$u\sin\theta + y\sin\phi = \sin(\theta + \phi),$$

and thus

$$(u - \cos \phi) \sin \theta + (y - \cos \theta) \sin \phi = 0$$

which is impossible since $\sin \theta \sin \phi > 0$.

Lemma A.13

There exists a constant c > 0 such that $d_m^* \ge c \Lambda_{p,q}^m$.

Proof Since the degree d_m^* satisfies $(d_{-2q-2p}^*, \cdots, d_{-1}^*) = (1, 0, 0, ..., 0)$ and

$$d_m^* - k(d_{m-p}^* + d_{m-q}^*) + k(d_{m-2p-q}^* + d_{m-p-2q}^*) - d_{m-2p-2q}^* = 0,$$

there exist suitable constants $c_i \in \mathbb{C}$ such that $d_m^* = \sum_{i=1}^{2p+2q} c_i \lambda_i^{m+2p+2q}$, where $\{\lambda_i\}$ consists of p+q roots of (7) in addition to the p+q-th root of unity. (Note that we omitted the case of multiple roots, however the discussion proceeds similarly to the simple roots.) Let $\lambda_{2p+2q} = \Lambda_{p,q}$ and we prove that $c_{2p+2q} \neq 0$. Let $\mathbf{c} := {}^t(c_1, c_2, ..., c_{2p+2q})\mathbf{e}_1 := {}^t(1, 0, 0, ..., 0)$ and \mathbf{A} be the square Vandermonde matrix generated by $\lambda_1, \lambda_2, ..., \lambda_{2p+2q}$. Then we have $\mathbf{Ac} = \mathbf{e}_1$. From Cramer's rule we have

$$c_{2p+2q} = -\frac{|A_{1.2p+2q}|}{|A|},$$

where $|\mathbf{A}_{1,2p+2q}|$ is the (1, 2p + 2q)-first minor of \mathbf{A} . The determinant of the Vandermonde matrix is nonzero, and $|\mathbf{A}_{1,2p+2q}| \neq 0$ is also satisfied, since $|\mathbf{A}_{1,2p+2q}| = \prod_{i=1}^{2p+2q-1} \lambda_i \times |B|$, where B is the square Vandermonde matrix generated by $\lambda_1, \lambda_2, ..., \lambda_{2p+2q-1}$. Thus $c_{2p+2q} \neq 0$. Therefore we can choose $0 < c \ll |c_{2p+2q}|$ so that $d_m^* \geq 0$ and $d_m^* \geq c\Lambda_{p,q}^m$.

A.3 Comment on Lemma 2.8

For a generic initial values there exists a constant c > 0 in addition to C > 0 such that $c\Lambda_{p,q}^m \leq |a_m| \leq C\Lambda_{p,q}^m$. In fact, the iterate a_m is expressed as

$$a_m = \sum_{i=1}^{2p+2q} c_i \lambda_i^m \qquad (c_i \in \mathbb{C}),$$

where $\lambda_1, \lambda_2, ..., \lambda_{2p+2q} = \Lambda_{p,q}$ are the roots of the characteristic polynomial. Since $\Lambda_{p,q}$ has the largest absolute value among the roots, we can find c > 0 such that $c\Lambda_{p,q}^m \leq |a_m|$ as long as $c_{-2p-2q} \neq 0$.

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