Further results on the least Q-eigenvalue of a graph with fixed domination number*

Guanglong Yu[a†] Yarong Wu[b] Mingqing Zhai[c]

[a]Department of Mathematics, Lingnan normal nniversity, Zhanjiang, 524048, Guangdong, China
[b]SMU college of art and science, Shanghai maritime university, Shanghai, 200135, China
[c] School of mathematics and finance, Chuzhou university, Chuzhou, 239000, Anhui, China

Abstract

In this paper, we proceed on determining the minimum q_{min} among the connected nonbipartite graphs on $n \geq 5$ vertices and with domination number $\frac{n+1}{3} < \gamma \leq \frac{n-1}{2}$. Further results obtained are as follows:

(i) among all nonbipartite connected graph of order $n \geq 5$ and with domination number $\frac{n-1}{2}$, the minimum q_{min} is completely determined;

(ii) among all nonbipartite graphs of order $n \geq 5$, with odd-girth $g_o \leq 5$ and domination number at least $\frac{n+1}{3} < \gamma \leq \frac{n-2}{2}$, the minimum q_{min} is completely determined.

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1 Introduction

All graphs considered in this paper are connected, undirected and simple, i.e., no loops or multiple edges are allowed. We denote by $\parallel S \parallel$ the *cardinality* of a set S, and denote by $G = G[V(G), E(G)]$ a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$ where $\parallel V(G) \parallel = n$ is the *order* and $\parallel E(G) \parallel = m$ is the *size*.

In a graph, if vertices v_i and v_j are adjacent (denoted by $v_i \sim v_j$), we say that they *dominate* each other. A vertex set D of a graph G is said to be a *dominating set* if every vertex of $V(G) \setminus D$ is adjacent to (dominated by) at least one vertex in D. The *domination number* $\gamma(G)$ (γ, for short) is the minimum cardinality of all dominating sets of G. For a graph G, a dominating set is called a *minimal dominating set* if its cardinality is $\gamma(G)$. A well known result about $\gamma(G)$ is that for a graph G of order n containing no isolated vertex, $\gamma \leq \frac{n}{2}$ [12]. A comprehensive study of issues relevant to dominating set of a graph has been undertaken because of its good applications [8], [19].

Recall that $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (or *Q-matrix*) of G, where $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_i = d_{eg}(v_i)$ being the degree of vertex v_i ($1 \leq i \leq n$), and $A(G)$ is the adjacency matrix of G. The signless Laplacian has attracted the attention of many researchers and it is being promoted by many researchers [1], [2]-[6], [15].

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†E-mail addresses: yglong01@163.com.

The least eigenvalue of $Q(G)$, denote by $q_{min}(G)$ or q_{min}, is called the *least Q-eigenvalue* of G. Because $Q(G)$ is positive semi-definite, we have $q_{min}(G) \geq 0$. From [2], we know that, for a connected graph G, $q_{min}(G) = 0$ if and only if G is bipartite. Consequently, in [7], q_{min} was studied as a measure of nonbipartiteness of a graph. One can notice that there are quite a few results about q_{min}. In [1], D.M. Cardoso et al. determined the graphs with the the minimum q_{min} among all the connected nonbipartite graphs with a prescribed number of vertices. In [6], L. de Lima et al. surveyed some known results about q_{min} and also presented some new results. In [9], S. Fallat, Y. Fan investigated the relations between q_{min} and some parameters reflecting the graph bipartiteness. In [15], Y. Wang, Y. Fan investigated q_{min} of a graph under some perturbations, and minimized q_{min} among the connected graphs with fixed order which contains a given nonbipartite graph as an induced subgraph. Recently, in [14], the authors determined all non-bipartite hamiltonian graphs whose q_{min} attains the minimum.

Recall that a *lollipop graph* $L_{g,l}$ is a graph composed of a cycle $\mathbb{C} = v_1v_2\cdots v_gv_1$ and a path $\mathbb{P} = v_gv_{g+1}\cdots v_{g+l}$ with $l \geq 1$. For given g and l, a graph of order n is called a $F_{g,l}$*-graph* if it is obtained by attaching $n-g-l$ pendant vertices to some nonpendant vertices of a $L_{g,l}$. If $l = 1$, a $F_{g,l}$-graph is also called a *sunlike* graph. In a graph, a vertex is called a *p-dominator* (or *support vertex*) if it dominates a pendant vertex. In a $F_{g,l}$-graph if each p-dominator other than v_{g+l-1} is attached with exactly one pendant vertex, then this graph is called a $\mathcal{F}_{g,l}$*-graph*. A $\mathcal{F}_{g,l}$-graph is called a $\mathcal{F}^{\circ}_{g,l}$*-graph* if v_g is a p-dominator. In the following paper, for unity, for a $\mathcal{F}_{g,l}$-graph, $\mathbb{C}$ and $\mathbb{P}$ are expressed as above.

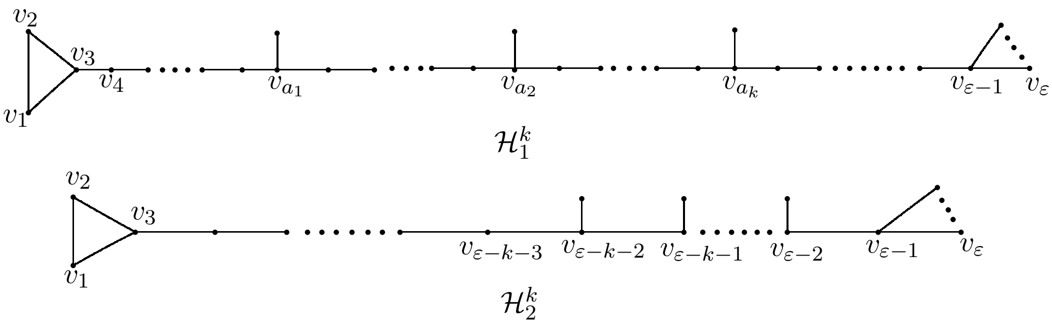

Fig. 1.1. $\mathcal{H}_1^k$, $\mathcal{H}_2^k$

Let $\mathcal{H}_1^k$ be a $\mathcal{F}_{3,\varepsilon-3}$-graph of order $n \geq 4$ where there are $k \geq 0$ p-dominators among v_1, v_2, $\ldots$, $\varepsilon - 2$ ($\varepsilon \geq 3$. see Fig. 1.1). If $k \geq 1$, in $\mathcal{H}_1^k$, suppose $v_{a_j}s$ are p-dominators where $1 \leq j \leq k$, $1 \leq a_1 < a_2 < \cdots < a_k \leq \varepsilon - 2$, and suppose v_{τ_j} is the pendant vertex attached to v_{a_j}. Let $\mathcal{H}_2^k = \mathcal{H}_1^k - \sum\limits_{j=1}^{k} v_{\tau_j}v_{a_j} + \sum\limits_{j=1}^{k} v_{\tau_j}v_{\varepsilon-2-k+j}$ (see Fig. 1.1). If $k = 0$, then $\mathcal{H}_1^0 = \mathcal{H}_2^0$. If $\alpha \geq 1$, we denoted by $\mathscr{H}_{3,\alpha}$ the graph $\mathcal{H}_2^{\alpha-1}$ of order n in which there are α p-dominators and $v_{\varepsilon-1}$ has only one pendant vertex (where $\varepsilon = n - \alpha + 1$); if $\alpha = 0$, we let $\mathscr{H}_{3,0} = C_3 = v_1v_2v_3v_1$.

In [10] and [17], the authors first considered the relation between q_{min} of a graph and its domination number. Among all the nonbipartite graphs with both order $n \geq 4$ and domination number $\gamma \leq \frac{n+1}{3}$, they characterized the graphs with the minimum q_{min}. A remaining open problem is that how about the q_{min} of the connected nonbipartite graph on n vertices with domination number $\frac{n+1}{3} < \gamma \leq \frac{n}{2}$. In [18], the authors proceeded on considering this problem. Among the nonbipartite graphs of order $n = 4$, the minimum q_{min} is completely determined; among the nonbipartite graphs of order n and with given domination number $\frac{n}{2}$, the minimum q_{min} is completely determined; further results about the domination number, the q_{min} of a graph as well as their relation are represented. An open problem still left is that how to determine the minimum q_{min} of the connected nonbipartite graph on $n \geq 5$ vertices with domination number $\frac{n+1}{3} < \gamma \leq \frac{n-1}{2}$. Let $\mathbb{S} = \mathscr{H}_{3,\alpha}$ be of order $n \geq 4$ where α is the least integer such that $\lceil \frac{n-2\alpha-2}{3} \rceil + \alpha = \gamma$. In [18], the authors represented some structural characterizations about the minimum q_{min} for this problem, and conjectured that such $\mathbb{S}$ has the smallest q_{min}. However, the problem seems really difficult to solve. Motivated by proceeding on solving this problem, we go on with our research and get some further results as follows.

Theorem 1.1 *Let G be a nonbipartite connected graph of order $n \geq 5$ and with domination number $\frac{n-1}{2}$. Then $q_{min}(G) \geq q_{min}(\mathscr{H}_{3,\frac{n-1}{2}})$ with equality if and only if $G \cong \mathscr{H}_{3,\frac{n-1}{2}}$.*

Theorem 1.2 *Among all nonbipartite graphs of order $n \geq 5$, with odd-girth $g_o \leq 5$ (length of the shortest odd cycle in this graph) and domination number $\frac{n+1}{3} < \gamma \leq \frac{n-2}{2}$, then the least q_{min} attains the minimum uniquely at a $\mathscr{H}_{3,\alpha}$ where $\alpha \leq \frac{n-3}{2}$ is the least integer such that $\lceil \frac{n-2\alpha-2}{3} \rceil + \alpha = \gamma$.*

2 Preliminary

In this section, we introduce some notations and some working lemmas.

Denote by P_n, C_n, K_n, a *path*, a n-cycle (of length n), a *complete* graph of order n respectively. If k is odd, we say C_k an *odd cycle*. The *girth* of a graph G, denoted by g, is the length of the shortest cycle in G. The *odd-girth* for a nonbipartite graph G, denoted by $g_o(G)$ or g_o, is the length of the shortest odd cycle in this graph. $G - v_iv_j$ denotes the graph obtained from G by deleting the edge $v_iv_j \in E(G)$, and let $G - v_i$ denote the graph obtained from G by deleting the vertex v_i and the edges incident with v_i. Similarly, $G + v_iv_j$ is the graph obtained from G by adding an edge v_iv_j between its two nonadjacent vertices v_i and v_j. Given an vertex set S, $G - S$ denotes the graph obtained by deleting all the vertices in S from G and the edges incident with any vertex in S.

A connected graph G of order n is called a *unicyclic* graph if $\|E(G)\| = n$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. Denoted by $d_{istG}(v_i, v_j)$ the *distance* between two vertices v_i and v_j in a graph G.

For a graph G of order n, let $X = (x_1, x_2, \ldots, x_n)^T \in R^n$ be defined on $V(G)$, i.e., each vertex v_i is mapped to the entry x_i; let $|x_i|$ denote the *absolute value* of x_i. One can find that $X^TQ(G)X = \sum_{v_iv_j \in E(G)}(x_i + x_j)^2$. In addition, for an arbitrary unit vector $X \in R^n$, $q_{min}(G) \leq X^TQ(G)X$, with equality if and only if X is an eigenvector corresponding to $q_{min}(G)$.

Lemma 2.1 **[3]** *Let G be a graph on n vertices and m edges, and let e be an edge of G. Let $q_1 \geq q_2 \geq \cdots \geq q_n$ and $s_1 \geq s_2 \geq \cdots \geq s_n$ be the Q-eigenvalues of G and $G - e$ respectively. Then $0 \leq s_n \leq q_n \leq \cdots \leq s_2 \leq q_2 \leq s_1 \leq q_1$.*

Let G_1 and G_2 be two disjoint graphs, and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. The *coalescence* of G_1 and G_2, denoted by $G_1(v_1) \diamond G_2(v_2)$ or $G_1(u) \diamond G_2(u)$, is obtained from G_1, G_2 by identifying

v_1 with v_2 and forming a new vertex u where for $i = 1, 2$, G_i can be trivial (that is, G_i is only one vertex). For a connected graph $G = G_1(u) \diamond G_2(u)$, $i = 1, 2$, G_i is called a *branch* of G with root u. For a vector $X = (x_1, x_2, \ldots, x_n)^T \in R^n$ defined on $V(G)$, a branch H of G is called a *zero branch* with respect to X if $x_i = 0$ for all $v_i \in V(H)$; otherwise, it is called a *nonzero branch* with respect to X.

Lemma 2.2 **[15]** *Let G be a connected graph which contains a bipartite branch H with root v_s, and let X be an eigenvector of G corresponding to $q_{min}(G)$.*

(i) *If $x_s = 0$, then H is a zero branch of G with respect to X;*

(ii) *If $x_s \neq 0$, then $x_p \neq 0$ for every vertex $v_p \in V(H)$. Furthermore, for every vertex $v_p \in V(H)$, $x_p x_s$ is either positive or negative depending on whether v_p is or is not in the same part of the bipartite graph H as v_s; consequently, $x_p x_t < 0$ for each edge $v_p v_t \in E(H)$.*

Lemma 2.3 **[15]** *Let G be a connected nonbipartite graph of order n, and let X be an eigenvector of G corresponding to $q_{min}(G)$. T is a tree which is a nonzero branch of G with respect to X and with root v_s. Then $|x_t| < |x_p|$ whenever v_p, v_t are vertices of T such that v_t lies on the unique path from v_s to v_p.*

Lemma 2.4 **[16]** *Let $G = G_1(v_2) \diamond T(u)$ and $G^* = G_1(v_1) \diamond T(u)$, where G_1 is a connected nonbipartite graph containing two distinct vertices v_1, v_2, and T is a nontrivial tree. If there exists an eigenvector $X = (x_1, x_2, \ldots, x_k, \ldots)^T$ of G corresponding to $q_{min}(G)$ such that $|x_1| > |x_2|$ or $|x_1| = |x_2| > 0$, then $q_{min}(G^*) < q_{min}(G)$.*

Lemma 2.5 **[16]** *Let $G = C(v_0) \diamond B(v_0)$ be a graph of order n, where $C = v_0 v_1 v_2 \cdots v_{2k}$ is a cycle of length $2k+1$, and B is a bipartite graph of order $n-2k$. Then there exists an eigenvector $X = (x_0, x_1, x_2, \ldots, x_{2k})^T$ corresponding to $q_{min}(G)$ satisfying the following:*

(i) $|x_0| = \max\{|x_i| \mid v_i \in V(C)\} > 0$;

(ii) $x_i = x_{2k-i+1}$ *for* $i = 1, 2, \ldots, k$;

(iii) $x_i x_{i-1} \leq 0$ *for* $i = 1, 2, \ldots, k$, $x_{2k} x_0 \leq 0$ *and* $x_{2k-i+1} x_{2k-i+2} \leq 0$ *for* $i = 2, \ldots, k$.

Moreover, if $2k+1 < n$, then the multiplicity of $q_{min}(G)$ is one, and then any eigenvector corresponding to $q_{min}(G)$ satisfies (i), (ii), (iii).

Lemma 2.6 **[5]** *Let G be a connected graph of order n. Then $q_{min} < \delta$, where δ is the minimal vertex degree of G.*

Lemma 2.7 **[17]** *Let G be a nonbipartite graph with domination number $\gamma(G)$. Then G contains a nonbipartite unicyclic spanning subgraph H with both $g_o(H) = g_o(G)$ and $\gamma(H) = \gamma(G)$.*

Lemma 2.8 **[17]** *Suppose a graph G contains pendant vertices. Then*

(i) *there must be a minimal dominating set of G containing all of its p-dominators but no any pendant vertex;*

(ii) *if v is a p-dominator of G and at least two pendant vertices are adjacent to v, then any minimal dominating set of G contains v but no any pendant vertex adjacent to v.*

Lemma 2.9 **[11]** *(i) For a path P_n, we have $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

(ii) For a cycle C_n, we have $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.

We define the corona G of graphs G_1 and G_2 as follows. The *corona* $G = G_1 \circ G_2$ is the graph formed from one copy of G_1 and $\parallel V(G_1) \parallel$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

Lemma 2.10 **[13]** *Let G be a graph of order n. $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Denote by $C^*_{3,\,k}$ the graph obtained by attaching a C_3 to an end vertex of a path of length k and attaching $n-3-k$ pendant vertices to the other end vertex of this path.

Lemma 2.11 **[17]** *Among all the nonbipartite graphs with both order $n \geq 4$ and domination number $\gamma \leq \frac{n+1}{3}$, we have*

(i) if $n = 3\gamma - 1,\ 3\gamma,\ 3\gamma + 1$, then the graph with the minimal least Q-eigenvalue attains uniquely at $C^*_{3,\,n-4}$;

(ii) if $n \geq 3\gamma + 2$, then the graph with the minimal least Q-eigenvalue attains uniquely at $C^*_{3,\,3\gamma-3}$.

Lemma 2.12 **[18]** *Among all nonbipartite unicyclic graphs of order n, and with both domination number γ and girth g ($g \leq n-1$), the minimum q_{min} attains at a $\mathcal{F}_{g,l}$-graph G for some l. Moreover, for this graph G, suppose that $X = (x_1,\ x_2,\ x_3,\ \ldots,\ x_n)^T$ is a unit eigenvector corresponding to $q_{min}(G)$. Then we have that $|x_g| > 0$, and $|x_{g+l-1}| = \max\{|x_i| \mid v_i$ is a p-dominator$\}$.*

In $\mathcal{H}^k_2$, for $j = 1,\ 2,\ \ldots,\ k$, suppose $v_{\tau_{\varepsilon-2-k+j}}$ is the pendant vertex attached to vertex $v_{\varepsilon-2-k+j}$. Suppose $v_{\omega_1},\ v_{\omega_2},\ \ldots,\ v_{\omega_s}$ are the pendant vertices attached to vertex $v_{\varepsilon-1}$. If $s \geq 2$, let $\mathcal{H}^k_3 = \mathcal{H}^k_2 - v_{\varepsilon-1-k}v_{\tau_{\varepsilon-1-k}} + v_{\varepsilon-1}v_{\tau_{\varepsilon-1-k}} - \sum\limits_{j=2}^{s} v_{\varepsilon-1}v_{\omega_j} + \sum\limits_{j=2}^{s} v_{\omega_1}v_{\omega_j}$. Let $\mathcal{H}^{k-1}_4 = \mathcal{H}^k_2 - v_{\varepsilon-1-k}v_{\tau_{\varepsilon-1-k}} + v_{\varepsilon-1}v_{\tau_{\varepsilon-1-k}}$, $\mathcal{H}^{k-2}_5 = \mathcal{H}^{k-1}_4 - v_{\varepsilon-k}v_{\tau_{\varepsilon-k}} + v_{\varepsilon-1}v_{\tau_{\varepsilon-k}}$.

Lemma 2.13 **[18]**

(i) $\gamma(\mathcal{H}^k_1) \leq \gamma(\mathcal{H}^k_2)$.

(ii) *If $\varepsilon - k - 1 \leq 2$, then $\gamma(\mathcal{H}^k_2) = k + 1$ and $\gamma(\mathcal{H}^{k-1}_4) = \gamma(\mathcal{H}^k_2) - 1$;*

(iii) *If $\varepsilon - k - 1 \geq 3$, then $\gamma(\mathcal{H}^k_2) = \lceil \frac{\varepsilon-k-4}{3} \rceil + k + 1$;*

(iv) *$\gamma(\mathcal{H}^k_2) \leq \gamma(\mathcal{H}^k_3)$;*

(v) *If $\varepsilon - k - 1 \geq 3$, $\frac{\varepsilon-k-4}{3} \neq t$ where t is a nonnegative integral number, then $\gamma(\mathcal{H}^{k-1}_4) = \gamma(\mathcal{H}^k_2) - 1$;*

(vi) *If $\varepsilon - k - 1 \geq 3$, $\frac{\varepsilon-k-4}{3} = t$ where t is a nonnegative integral number, $\gamma(\mathcal{H}^{k-1}_4) = \gamma(\mathcal{H}^k_2)$, $\gamma(\mathcal{H}^{k-2}_5) = \gamma(\mathcal{H}^k_2) - 1$.*

Lemma 2.14 **[18]**

(i) *$\gamma(\mathscr{H}_{3,0}) = 1$;*

(ii) *If $\alpha \geq 1$ and $n - 2\alpha \leq 2$, then $\gamma(\mathscr{H}_{3,\alpha}) = \alpha$;*

(iii) *If $\alpha \geq 1$ and $n - 2\alpha \geq 3$, then $\gamma(\mathscr{H}_{3,\alpha}) = \lceil \frac{n-2\alpha-2}{3} \rceil + \alpha$.*

3 Domination number and the structure of a graph

Let G^* be a sunlike graph of order n and with both girth g and k p-dominators $v_1, v_2, \ldots, v_k$ on $\mathbb{C}$.

Lemma 3.1 *Let G be a sunlike graph of order n and with both girth g and k p-dominators on $\mathbb{C}$. Then $\gamma(G) \le \gamma(G^*)$, where $\gamma(G^*) = k + \lceil \frac{g-k-2}{3} \rceil$.*

Proof. Suppose $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are the k p-dominators on $\mathbb{C}$ in G, where $1 \le i_1 < i_2 < \cdots < i_k \le g$. Suppose that there exists some $1 \le z \le k$ such that $i_{z+1} - i_z \ge 2$, where if $z = k$, we let $i_{k+1} = i_1$ and $i_{k+1} - i_k = i_1 + g - i_k$. Let $H = G - \sum_{s=i_z+1}^{i_{z+1}-1} v_s$.

Assertion 1 If $i_{z+1} - i_z \le 3$, then $\gamma(H) = \gamma(G)$. By Lemma 2.8, there is a minimal dominating set D of G which contains all the k p-dominators but no any pendant vertex. Thus both $v_{i_{z+1}}$ and v_{i_z} are in D. Note the minimality of D and $2 \le i_{z+1} - i_z \le 3$. Then $D \cap \{v_{i_z+1}\} = \emptyset$ if $i_{z+1} - i_z = 2$; $D \cap \{v_{i_z+1}, v_{i_{z+1}-1}\} = \emptyset$ if $i_{z+1} - i_z = 3$. Thus D is also a dominating set of H. This implies that $\gamma(H) \le \gamma(G)$. Note that for H, by Lemma 2.8, there is a minimal dominating set D' which contains all the k p-dominators but no any pendant vertex. Thus both $v_{i_{z+1}}$ and v_{i_z} are in D'. Then v_{i_z+1} is dominated by D' if $i_{z+1} - i_z = 2$; v_{i_z+1}, $v_{i_{z+1}-1}$ are is dominated by D' if $i_{z+1} - i_z = 3$. Consequently, D' is also a dominating set of G. This implies that $\gamma(G) \le \gamma(H)$. As a result, it follows that $\gamma(H) = \gamma(G)$. And then our assertion holds.

Assertion 2 If $i_{z+1} - i_z \ge 4$, then $\gamma(G) = \gamma(H) + \gamma(P_{i_z, i_{z+1}})$ where $P_{i_z, i_{z+1}} = v_{i_z+2} v_{i_z+3} \cdots v_{i_{z+1}-2}$. By Lemma 2.8, there is a minimal dominating set D of G which contains all the k p-dominators but no any pendant vertex. Thus both $v_{i_{z+1}}$ and v_{i_z} are in D. We claim that at most one of v_{i_z+1}, v_{i_z+2} is in D. Otherwise, suppose that both v_{i_z+1} and v_{i_z+2} are in D. Then $D \setminus \{v_{i_z+1}\}$ is also a dominating set of G, which contradicts the minimality of D. Consequently, our claim holds. Similarly, we get that at most one of $v_{i_{z+1}-2}$, $v_{i_{z+1}-1}$ is in D. Thus we let $D^\circ = ((D \cup \{v_{i_z+2}, v_{i_{z+1}-2}\}) \setminus \{v_{i_z+1}, v_{i_{z+1}-1}\}) \cap V(P_{i_z, i_{z+1}})$ if $v_{i_z+1} \in D$, $v_{i_{z+1}-1} \in D$; let $D^\circ = ((D \cup \{v_{i_z+2}\}) \setminus \{v_{i_z+1}\}) \cap V(P_{i_z, i_{z+1}})$ if $v_{i_z+1} \in D$ and $v_{i_{z+1}-1} \notin D$; let $D^\circ = ((D \cup \{v_{i_{z+1}-2}\}) \setminus \{v_{i_{z+1}-1}\}) \cap V(P_{i_z, i_{z+1}})$ if $v_{i_z+1} \notin D$ and $v_{i_{z+1}-1} \in D$; let $D^\circ = (D \cap V(P_{i_z, i_{z+1}})$ if $v_{i_z+1} \notin D$ and $v_{i_{z+1}-1} \notin D$. Note that $D^* = D \setminus (V(P_{i_z, i_{z+1}}) \cup \{v_{i_z+1}, v_{i_{z+1}-1}\})$ is a dominating set of H, $D^\circ \cup D^*$ is a dominating set of G with cardinality $\gamma(G)$, and note that D° is a dominating set of $P_{i_z, i_{z+1}}$. Thus $\gamma(P_{i_z, i_{z+1}}) \le \| D^\circ \|$. Note that both $v_{i_{z+1}-1}$ and v_{i_z+1} are dominated by D^*. Consequently, for any minimal dominating set B of $P_{i_z, i_{z+1}}$, then $B \cup D^*$ is also a dominating set of G. Note that $\| B \| = \gamma(P_{i_z, i_{z+1}}) \le \| D^\circ \|$. As a result, $\| B \cup D^* \| \le \| D \| = \gamma(G)$. Note that the minimality of D. Then $\| D^\circ \| = \| B \| = \gamma(P_{i_z, i_{z+1}})$, and then it follows that $\gamma(G) = \gamma(H) + \gamma(P_{i_z, i_{z+1}})$.

Denote by $\tau_{i_j, i_{j+1}}$ the dominating index where we let $i_{k+1} = i_1$ if $i = k$. Let $\tau_{i_j, i_{j+1}} = 0$ if $i_{j+1} - i_j \le 3$; let $\tau_{i_j, i_{j+1}} = \gamma(P_{i_j, i_{j+1}})$ if $i_{j+1} - i_j \ge 4$. Thus from Assertion 1, Assertion 2 and Lemma 2.8, we get that $\gamma(G) = k + \sum_{i=1}^{k} \tau_{i_j, i_{j+1}}$. By Lemma 2.9, it follows that $\tau_{i_j, i_{j+1}} = \gamma(P_{i_j, i_{j+1}}) = \lceil \frac{i_{j+1} - i_j - 3}{3} \rceil$ if $i_{j+1} - i_j \ge 4$. Note that for any two nonnegative integers x and y, we have $\lceil \frac{x}{3} \rceil + \lceil \frac{y}{3} \rceil \le \lceil \frac{x+y}{3} \rceil$. Then

$$\sum_{i=1}^{k} \tau_{i_j, i_{j+1}} = \sum_{\tau_{i_s, i_{s+1}} \ne 0} \tau_{i_s, i_{s+1}} \le \left\lceil \frac{\sum_{\tau_{i_s, i_{s+1}} \ne 0} (i_{s+1} - i_s - 3)}{3} \right\rceil \le \left\lceil \frac{g-k-2}{3} \right\rceil.$$

Thus $\gamma(G) \leq k+\lceil \frac{g-k-2}{3} \rceil$. Noting that by Assertion1 and Assertion 2, we have $\gamma(G^*) = k+\lceil \frac{g-k-2}{3} \rceil$. Then the result follows as desired. This completes the proof. $\square$

Theorem 3.2 *Suppose that $\mathcal{G}$ is a nonbipartite $\mathcal{F}_{g,l}$-graph with $\gamma(\mathcal{G}) = \frac{n-1}{2}$, $g \geq 5$ and order $n \geq g+1$, and suppose there are exactly f vertices of the unique cycle $\mathbb{C}$ such that none of them is p-dominator. Then we get*

(i) *if $f = g$, then $g = 5$;*

(ii) *if $f \neq g$, then $f \leq 3$ and $f \neq 2$;*

(iii) *if $f = 3$, then the three vertices are consecutive on $\mathbb{C}$, i.e., they are v_{i-1}, v_i, v_{i+1} for some $1 \leq i < g$, and each in $(V(\mathbb{C}) \setminus \{v_{i-1}, v_i, v_{i+1}\}) \cup V(\mathbb{P} - v_{g+l})$ is a p-dominator (if $i = 1$, then $v_{i-1} = v_g$).*

Proof. Denote by A the set of vertices of $\mathbb{C}$ and the pendant vertices attached to $\mathbb{C}$. Let $\parallel A \parallel = z$, and let $A' = V(\mathcal{G}) \setminus A$. Then $\gamma(\mathcal{G}) \leq \gamma(\mathcal{G}[A]) + \gamma(\mathcal{G}[A'])$. Note that $A' = \emptyset$, or $\mathcal{G}[A']$ is connected with at least 2 vertices. Suppose $f \geq 4$.

(i) $f = g$. Then $z - f = 0$. This means that there is no p-dominator on $\mathbb{C}$. So, $\mathcal{G}[A']$ is connected with at least 2 vertices. Thus, if $f \geq 9$, by Lemma 2.9, then $\gamma(\mathcal{G}) \leq \lceil \frac{f}{3} \rceil + \gamma(\mathcal{G}[A']) \leq \frac{n-f}{2} + \frac{f+2}{3} < \frac{n-1}{2}$. Therefore $f \leq 7$.

Note that g is odd and $g = f$ now. Thus if $\gamma(\mathcal{G}[A']) < \frac{n-f}{2}$, then $\gamma(\mathcal{G}) \leq \lceil \frac{f}{3} \rceil + \gamma(\mathcal{G}[A']) < \frac{n-1}{2}$. Hence, it follows that $\gamma(\mathcal{G}[A']) = \frac{n-f}{2}$. Combined with Lemma 2.10, it follows that $\mathcal{G}[A'] = P_{\frac{n-f}{2}} \circ K_1$. Here, suppose $P_{\frac{n-f}{2}} = v_{a_1} v_{a_2} \cdots v_{a_t}$ with $t = \frac{n-f}{2}$, and suppose v_{τ_1} is the unique pendant vertex attached to v_{a_1}. By Lemma 2.8, $V(P_{\frac{n-f}{2}})$ is a minimal dominating set of $\mathcal{G}[A']$.

Assume that $f = 7$. Note that $\mathcal{G}$ is a $\mathcal{F}_{g,l}$-graph. If $\mathcal{G} = \mathbb{C} + v_g v_{a_1} + \mathcal{G}[A']$, then $V(P_{\frac{n-f}{2}}) \cup \{v_2, v_5\}$ is a dominating set of $\mathcal{G}$; if $\mathcal{G} = \mathbb{C} + v_g v_{\tau_1} + \mathcal{G}[A']$, then $(V(P_{\frac{n-f}{2}}) \setminus \{v_{a_1}\}) \cup \{v_2, v_5, v_{\tau_1}\}$ is a dominating set of $\mathcal{G}$. This implies that $\gamma(\mathcal{G}) \leq \frac{n-7}{2} + 2 < \frac{n-1}{2}$ which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Thus, it follows that $g = 5$.

(ii) $f \neq g$. Note that there is no the case that $z - f = 1$. Then $z - f \geq 2$. By Lemma 3.1, $\gamma(\mathcal{G}[A]) \leq \gamma(\mathcal{G}^*[A]) = g - f + \lceil \frac{f-2}{3} \rceil \leq \frac{z-f}{2} + \lceil \frac{f-2}{3} \rceil$, where $\mathcal{G}^*[A]$ is a sunlike graph with vertex set A, $\mathbb{C}$ contained in it and $g - f$ p-dominators $v_1, v_2, \ldots, v_{g-f}$ (defined as $\mathcal{G}^*$ in Lemma 3.1). Thus, if $f \geq 4$, then $\gamma(\mathcal{G}) \leq \frac{z-f}{2} + \lceil \frac{f-2}{3} \rceil + \gamma(\mathcal{G}[A']) \leq \frac{n-f}{2} + \lceil \frac{f-2}{3} \rceil \leq \frac{n-f}{2} + \frac{f}{3} < \frac{n-1}{2}$. This contradicts that $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Consequently, $f \leq 3$.

Suppose $f = 2$ and suppose that v_j, v_k of $\mathbb{C}$ are the exact 2 vertices such that neither of them is p-dominator. Note that by Lemma 2.8, there is a minimal dominating set D of $\mathcal{G} - v_j - v_k$ which contains all p-dominators but no any pendant vertex. Note that the vertices of $\mathbb{C}$ other than v_j, v_k are all p-dominators in both $\mathcal{G} - v_j - v_k$ and $\mathcal{G}$. Thus, each of v_j, v_k is adjacent to at least one p-dominator on $\mathbb{C}$. So, D is also a dominating set of $\mathcal{G}$. Note that there is no isolated vertex in $\mathcal{G} - v_j - v_k$. Then $\gamma(\mathcal{G} - v_j - v_k) \leq \frac{n-2}{2}$, and then $\gamma(\mathcal{G}) \leq \frac{n-2}{2}$, which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Then (ii) follows.

(iii) Suppose v_a, v_b, v_c are the exact 3 vertices of $\mathbb{C}$ such that none of them is p-dominator. If the 3 vertices v_a, v_b, v_c are not consecutive, then each of them can be dominated by its adjacent p-dominator. Note that by Lemma 2.8, there are a minimal dominating set D of $\mathcal{G} - v_a - v_b - v_c$ which contains all p-dominators but no any pendant vertex. Thus such D is also a dominating set of $\mathcal{G}$. Note that there is no isolated vertex in $\mathcal{G}-v_a-v_b-v_c$. So, $\gamma(\mathcal{G}) \leq \| D \| = \gamma(\mathcal{G}-v_a-v_b-v_c) \leq \frac{n-3}{2}$, which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Therefore, the 3 vertices v_a, v_b, v_c are consecutive.

Suppose that the 3 vertices are v_{i-1}, v_i, v_{i+1} for some $1 \leq i \leq g$ (here, if $i = g$, we let $v_{i+1} = v_1$; if $i = 1$, we let $v_{i-1} = v_g$). Let $H = \mathcal{G} - v_{i-1} - v_i - v_{i+1}$. Note that there is no isolated vertex in H. Thus, $\gamma(H) \leq \frac{n-3}{2}$. Next, we claim that $\gamma(H) = \frac{n-3}{2}$.

Claim 1 $\gamma(H) = \frac{n-3}{2}$. Otherwise, suppose $\gamma(H) < \frac{n-3}{2}$, and suppose D is a minimal dominating set of H. Then $D \cup \{v_i\}$ is a dominating set D of $\mathcal{G}$. Thus, $\mathcal{G} < 1 + \frac{n-3}{2} < \frac{n-1}{2}$, which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Then the claim holds.

By Lemma 2.10, $H = \mathcal{L} \circ K_1$ for some acyclic graph $\mathcal{L}$ of order $\frac{n-3}{2}$.

Claim 2 For any minimal dominating set D of H, in $\mathcal{G}$, at least one of v_{i-1}, v_i, v_{i+1} can not be dominated by D. Otherwise, D is a dominating set of $\mathcal{G}$ too. Hence, $\gamma(\mathcal{G}) \leq \frac{n-3}{2}$, which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. Then the claim holds.

If $i = g$, then let $H = H_1 \cup H_2$, where $H_1 = \mathcal{G}[A] - v_{g-1} - v_g - v_1$, $H_2 = \mathcal{G}[A'] = P_{\frac{n-z}{2}} \circ K_1$ (if $n = z$, then H_2 is empty). Here, suppose $P_{\frac{n-z}{2}} = v_{a_1} v_{a_2} \cdots v_{a_t}$ with $t = \frac{n-z}{2}$, and suppose v_{τ_1} is the unique pendant vertex attached to v_{a_1}. Thus there are two possible cases for G, i.e., $\mathcal{G} = \mathcal{G}[A] + v_g v_{a_1} + H_2$ or $\mathcal{G} = \mathcal{G}[A] + v_g v_{\tau_1} + H_2$. Let $\mathcal{Z} = (\mathbb{C} \setminus \{v_{g-1}, v_g, v_1\}) \cup V(P_{\frac{n-z}{2}})$. Note that the vertices in $\mathcal{Z}$ are all p-dominators in $\mathcal{G}$. If $\mathcal{G} = \mathcal{G}[A] + v_g v_{a_1} + H_2$, then $\mathcal{Z}$ is also a dominating set of $\mathcal{G}$; if $\mathcal{G} = \mathcal{G}[A] + v_g v_{\tau_1} + H_2$, then $(\mathcal{Z} \setminus \{v_{a_1}\}) \cup \{v_{\tau_1}\}$ is a dominating set of $\mathcal{G}$. Thus it follows that $\gamma(\mathcal{G}) \leq \frac{n-3}{2} < \frac{n-1}{2}$ which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. This implies $i \neq g$.

If $i \neq 1, g-1$, then H is connected. Let $\mathcal{Z} = (V(\mathbb{C}) \setminus \{v_{i-1},\ v_i,\ v_{i+1}\}) \cup V(\mathbb{P} - v_{g+l})$, where $\mathbb{P} = v_g v_{g+1} \cdots v_{g+l}$. Then each vertex in $\mathcal{Z}$ is a p-dominator in $\mathcal{G}$.

If $i = 1$, then let $H = H_1 \cup H_2$, where $H_1 = \mathcal{G}[A] - v_g - v_1 - v_2$, $H_2 = \mathcal{G}[A'] = P_{\frac{n-z}{2}} \circ K_1$ (if $n = z$, then H_2 is empty). Here, suppose $P_{\frac{n-z}{2}} = v_{a_1} v_{a_2} \cdots v_{a_t}$ with $t = \frac{n-z}{2}$, and suppose v_{τ_1} is the unique pendant vertex attached to v_{a_1}. Thus there are two possible cases for G, i.e., $\mathcal{G} = \mathcal{G}[A] + v_g v_{a_1} + H_2$ or $\mathcal{G} = \mathcal{G}[A] + v_g v_{\tau_1} + H_2$. We say that $\mathcal{G} \neq \mathcal{G}[A] + v_g v_{\tau_1} + H_2$. Otherwise, suppose $\mathcal{G} = \mathcal{G}[A] + v_g v_{\tau_1} + H_2$. Note that $n - z$ is even now and $\mathcal{G} - \{v_2, v_1, v_g, v_{a_1}, v_{\tau_1}\}$ has no isolated vertex. Then for $\mathcal{G} - \{v_2, v_1, v_g, v_{a_1}, v_{\tau_1}\}$, it has a dominating set $\mathbb{D}$ with $\| \mathbb{D} \| \leq \frac{n-5}{2}$. Then $\mathbb{D} \cup \{v_1, v_{\tau_1}\}$ is a dominating set of $\mathcal{G}$, which contradicts $\gamma(\mathcal{G}) = \frac{n-1}{2}$. This implies that $\mathcal{G} = \mathcal{G}[A] + v_g v_{a_1} + H_2$. It follows that each one in $(V(\mathbb{C}) \setminus \{v_g,\ v_1,\ v_2\}) \cup V(\mathbb{P} - v_{g+l})$ is a p-dominator. Similarly, for $i = g-1$, we get that each one in $(V(\mathbb{C}) \setminus \{v_{g-2},\ v_{g-1},\ v_g\}) \cup V(\mathbb{P} - v_{g+l})$ is a p-dominator. Then (iii) follows. $\square$

4 The q_{min} among uncyclic graphs

Lemma 4.1 [18] *Let G be a nonbipartite unicyclic graph of order n and with the odd cycle $\mathcal{C} = v_1 v_2 \cdots v_g v_1$ in it. There is a unit eigenvector $X = (\, x_1,\ x_2,\ \ldots,\ x_g,\ x_{g+1},\ x_{g+2},\ \ldots,\ x_{n-1},\ x_n\,)^T$ corresponding to $q_{min}(G)$, in which suppose $|x_1| = \min\{|x_1|,\ |x_2|,\ \ldots,\ |x_g|\}$, $|x_s| = \max\{|x_1|,\ |x_2|,\ \ldots,\ |x_g|\}$ where $s \geq 2$, satisfying that*

(i) $|x_1| < |x_s|$;

(ii) $|x_1| = 0$ *if and only if* $x_g = -x_2 \neq 0$*; if* $|x_1| = 0$ *and* $x_i x_{i+1} \neq 0$ *for some* $1 \leq i \leq g-1$*, then* $x_i x_{i+1} < 0$*; moreover, if* $x_j \neq 0$*, then* $sgn(x_j) = (-1)^{dist_H(v_1,v_j)}$ *where* $H = G - v_1 v_g$*.*

(iii) *if* $|x_1| > 0$*, then*

(1) *if* $3 \leq s \leq g-1$*, then* $|x_2| < \cdots < |x_{s-2}| < |x_{s-1}| \leq |x_s|$ *and* $|x_g| < |x_{g-1}| < \cdots < |x_{s+2}| < |x_{s+1}| \leq |x_s|$*;*

(2) *if* $|x_2| > |x_g|$*, then* $x_1 x_g > 0$*; for* $1 \leq i \leq g-1$*,* $x_i x_{i+1} < 0$*;* $|x_1| \leq |x_g|$*;*

(3) *if* $|x_2| < |x_g|$*, then* $x_1 x_2 > 0$*; for* $2 \leq i \leq g-1$*,* $x_i x_{i+1} < 0$*;* $x_g x_1 < 0$*;* $|x_1| \leq |x_2|$*;*

(4) *if* $|x_2| = |x_g|$*, then* $|x_1| \leq |x_2|$*, and exactly one of* $x_1 x_g > 0$ *and* $x_1 x_2 > 0$ *holds, where*

(4.1) *if* $x_1 x_g > 0$*, then for* $1 \leq i \leq g-1$*,* $x_i x_{i+1} < 0$*;*

(4.2) *if* $x_1 x_2 > 0$*, then* $x_i x_{i+1} < 0$ *for* $2 \leq i \leq g-1$ *and* $x_g x_1 < 0$*;*

(5) *at least one of* $|x_{s+1}|$ *and* $|x_{s-1}|$ *is less than* $|x_s|$*.*

Lemma 4.2 **[18]** *If* $\mathcal{G}$ *is a nonbipartite* $\mathcal{F}^{\circ}_{g,l}$*-graph with* $g \geq 5$*,* $n \geq g+1$*, then there is a graph* $\mathbb{H}$ *with girth* 3 *and order* n *such that* $\gamma(\mathcal{G}) \leq \gamma(\mathbb{H})$ *and* $q_{min}(\mathbb{H}) < q_{min}(\mathcal{G})$*.*

Lemma 4.3 **[18]** *Suppose that* G *is a nonbipartite* $\mathcal{F}_{3,l}$*-graph of order* n *where* $\mathbb{C} = v_1 v_2 v_3 v_1$*.* $X = (x_1, x_2, \ldots, x_n)^T$ *is a unit eigenvector corresponding to* $q_{min}(G)$*. Then* $|x_3| = \max\{|x_1|, |x_2|, |x_3|\}$*.*

Theorem 4.4 *Among all nonbipartite unicyclic graphs of order* $n \geq 5$ *with girth* 3 *and domination number at least* $\frac{n+1}{3} < \gamma \leq \frac{n}{2}$*, if* $\gamma = \frac{n-1}{2}$*, the* q_{min} *attains the minimum uniquely at* $\mathscr{H}_{3,\frac{n-3}{2}}$*.*

Proof. The result follows from Lemmas 2.4, 2.12, 2.13, 4.3 and Theorem 3.2 □

Let $\mathcal{K} = \{G|\ G$ be a nonbipartite $\mathcal{F}^{\circ}_{g,l}$-graph of order $n \geq 4$ and domination number at least $\frac{n+1}{3} < \gamma \leq \frac{n}{2}$, where g is any odd number at least 3 and l is any positive integral number$\}$ and $q_{\mathcal{K}} = \min\{q_{min}(G)|\ G \in \mathcal{K}\}$.

Lemma 4.5 **[18]**

(i) *If* $n = 4$*, the* $q_{\mathcal{K}}$ *attains uniquely at* $\mathscr{H}_{3,1}$*;*

(ii) *If* $n \geq 5$ *and* $n - 2\gamma \geq 2$*, then the least* $q_{\mathcal{K}} > q_{min}(\mathscr{H}_{3,\alpha})$ *where* $\alpha \leq \frac{n-3}{2}$ *is the least integer such that* $\lceil \frac{n-2\alpha-2}{3} \rceil + \alpha = \gamma$*.*

Lemma 4.6 *For a nonbipartite* $\mathcal{F}_{g,l}$*-graph graph* G *of order* $n \geq 5$ *and with* $g = 5$*, there exists a graph* $\mathbb{H}$ *such that* $g(\mathbb{H}) = 3$*,* $\gamma(G) \leq \gamma(\mathbb{H})$ *and* $q_{min}(\mathbb{H}) < q_{min}(G)$*.*

Proof. If $n = 5$, then $G = C_5$. And then the result follows from Lemma 2.11. Next we consider the case that $n \geq 6$. By Lemma 2.6, we get that $q_{min}(G) < 1$.

Case 1 There is no p-dominator on $\mathbb{C}$. Then G is like G_1 (see G_1 in Fig. 4.1). By Lemma 2.5, there is a unit eigenvector $X = (x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{n-1}, x_n)^T$ corresponding to $q_{min}(G)$ such that $|x_5| = \max\{|x_1|, |x_2|, |x_3|, |x_4|, |x_5|\} > 0$, and $x_1 = x_4$, $x_2 = x_3$. By Lemma 4.1, we get that $|x_2| > 0$, $|x_2| < |x_1|$ and $x_2 x_1 < 0$. Let $\mathbb{H} = G - v_3 v_4 + v_3 v_1$. By Lemma 2.4, we get that $q_{min}(\mathbb{H}) < q_{min}(G)$. Let $B_1 = \mathbb{H}[v_1, v_2, v_3]$, $B_2 = \mathbb{H} - \{v_1, v_2, v_3\}$. As Lemma 3.1, we can get a minimal dominating set D of $\mathbb{H}$, which contains all p-dominators but no any pendant vertex and no v_1, such that $D = \{v_2\} \cup D_2$, where $\{v_2\}$ is a dominating set of B_1, D_2 is a dominating set of B_2. Note that D is also a dominating set of G. So, $\gamma(G) \leq \gamma(\mathbb{H})$.

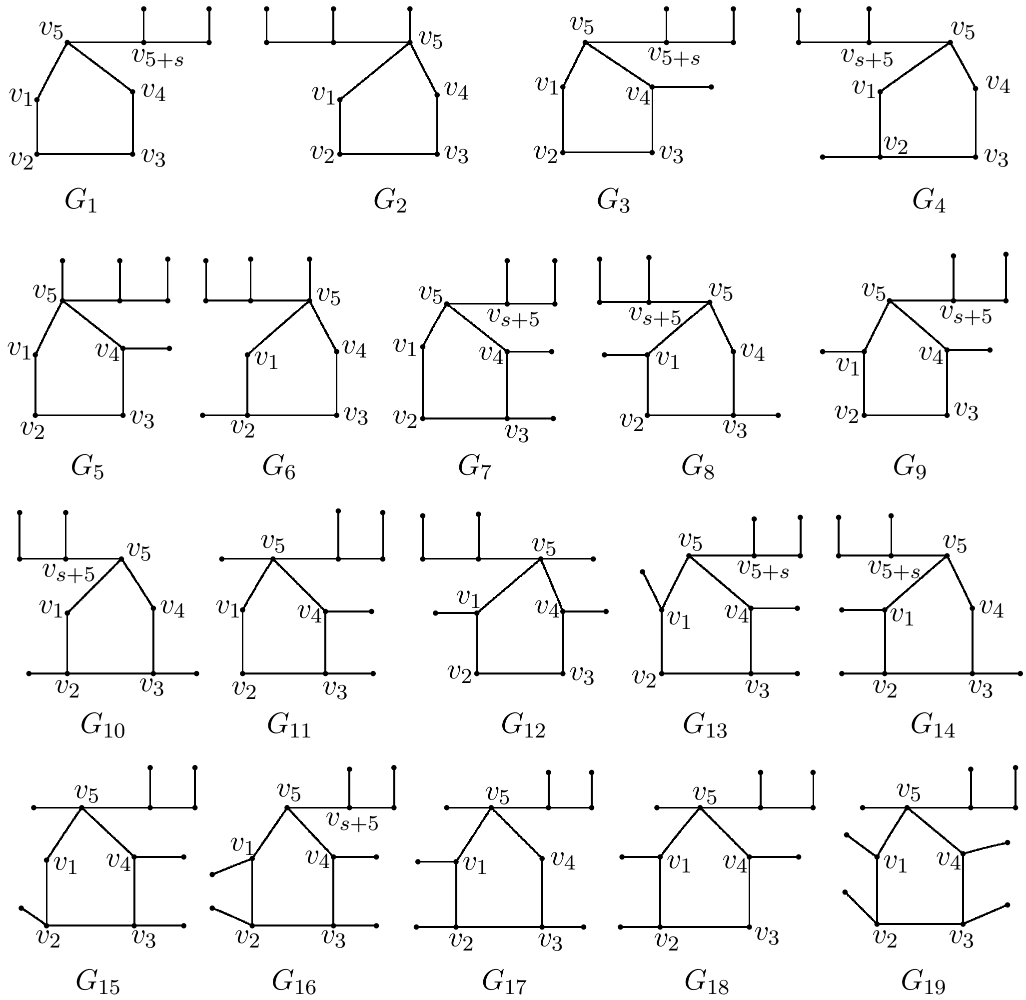

Fig. 4.1. $G_1 - G_{19}$

Case 2 There is only 1 p-dominator on $\mathbb{C}$ (see $G_2 - G_4$ in Fig. 4.1).

Subcase 2.1 For G_2, let $\mathbb{H} = G_2 - v_3v_4 + v_3v_1$. As Case 1, it is proved that $\gamma(G_2) \leq \gamma(\mathbb{H})$ and $q_{min}(\mathbb{H}) < q_{min}(G_2)$.

Subcase 2.2 For G_3, suppose $X = (\, x_1, x_2, \ldots, x_{n-1}, x_n\,)^T$ is a unit eigenvector corresponding to $q_{min}(G_3)$.

Claim $|x_4| > |x_1|$, $|x_5| > |x_3|$. Denote by v_k the pendant vertex attached to v_4. Suppose $0 < |x_4| \leq |x_1|$. Let $G_3^{'} = G_3 - v_4v_k + v_1v_k$. By Lemma 2.4, then $q_{min}(G_3^{'}) < q_{min}(G_3)$. This is a contradiction because $G_3^{'} \cong G_3$. Suppose $|x_4| = |x_1| = 0$. By Lemma 4.1, we get that $x_2 \neq 0$, $x_3 \neq 0$. By $q_{min}(G_3)x_2 = 2x_2 + x_3$, $q_{min}(G_3)x_3 = 2x_3 + x_2$, we get $x_2^2 = x_3^2$. Suppose $x_2 > 0$. Then we get $q_{min}(G_3)x_2 = 2x_2 + x_3 \geq x_2$. This means that $q_{min}(G_3) \geq 1$ which contradicts $q_{min}(G_3) < 1$. Thus, $|x_4| > |x_1|$. Similarly, we get $|x_5| > |x_3|$. Then the claim holds.

Suppose $|x_1| = \min\{|x_1|, |x_2|, |x_3|\}$ and $x_1 \geq 0$. If $|x_2| > |x_5|$, by Lemma 4.1, suppose $x_1x_5 \geq 0$. Let $H = G_3 - v_1v_5$. Also by Lemma 4.1, suppose for any $j \neq 1, 5$, $\text{sgn}x_j = (-1)^{d_{istH}(v_j, v_1)}$. Let $\mathbb{H} = G_3 - v_1v_5 + v_3v_1$. Because $|x_5| > |x_3|$, it follows that $q_{min}(\mathbb{H}) \leq X^TQ(\mathbb{H})X < X^TQ(G_3)X = q_{min}(G_3)$. Let $B_1 = \mathbb{H}[v_1, v_2]$, $B_2 = \mathbb{H} - \{v_1, v_2\}$. As Lemma 3.1, we can get a minimal dominating set D of $\mathbb{H}$, which contains all p-dominators but no any pendant vertex and no v_3, such that $D = \{v_1\} \cup D_2$, where D_2 is a dominating set of B_2. Note that D is also a dominating set of G_3. So, $\gamma(G_3) \le \gamma(\mathbb{H})$. If $|x_2| < |x_5|$, by Lemma 4.1, $x_1x_2 \ge 0$. Let $H = G_3 - v_1v_2$. Also by Lemma 4.1, suppose for any $j \ne 1, 2$, $\text{sgn}x_j = (-1)^{d_{istH}(v_j,v_1)}$. Let $\mathbb{H} = G_3 - v_1v_5 + v_3v_1$. Because $|x_5| > |x_3|$, it follows that $q_{min}(\mathbb{H}) < q_{min}(G_3)$ similarly. As the case that $|x_2| > |x_5|$, it is proved that $\gamma(G_3) \le \gamma(\mathbb{H})$. If $|x_2| = |x_5|$, by Lemma 4.1, without loss of generality, suppose $x_1x_5 \ge 0$. Let $\mathbb{H} = G_3 - v_1v_5 + v_3v_1$. As the case that $|x_2| > |x_5|$, it is proved that $q_{min}(\mathbb{H}) < q_{min}(G_3)$, $\gamma(G_3) \le \gamma(\mathbb{H})$.

For the both cases that $|x_2| = \min\{|x_1|, |x_2|, |x_3|\}$ and $|x_3| = \min\{|x_1|, |x_2|, |x_3|\}$. As the case that $|x_1| = \min\{|x_1|, |x_2|, |x_3|\}$, it is proved that there exists a graph $\mathbb{H}$ such that $g(\mathbb{H}) = 3$, $\gamma(G_3) \le \gamma(\mathbb{H})$ and $q_{min}(\mathbb{H}) < q_{min}(G_3)$.

In a same way, for G_4, it is proved that there exists a graph $\mathbb{H}$ such that $g(\mathbb{H}) = 3$, $\gamma(G_4) \le \gamma(\mathbb{H})$ and $q_{min}(\mathbb{H}) < q_{min}(G_4)$.

And in a same way, for the cases that **Case 3** there is exactly 2 p-dominators on $\mathbb{C}$ (see $G_5 - G_{10}$ in Fig. 4.1); **Case 4** there is exactly 3 p-dominators on $\mathbb{C}$ (see $G_{11} - G_{15}$ in Fig. 4.1); **Case 5** there is exactly 4 p-dominators on $\mathbb{C}$ (see $G_{16} - G_{18}$ in Fig. 4.1); **Case 6** there is exactly 5 p-dominators on $\mathbb{C}$ (see G_{19} in Fig. 4.1), it is proved that the exists a a graph $\mathbb{H}$ such that $g(\mathbb{H}) = 3$, $\gamma(G) \le \gamma(\mathbb{H})$ and $q_{min}(\mathbb{H}) \le q_{min}(G)$. Thus, the result follows as desired. □

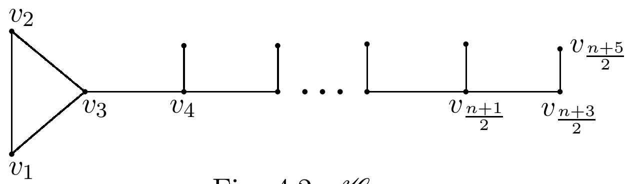

Fig. 4.2. $\mathscr{H}_{3,\frac{n-3}{2}}$

Lemma 4.7 *Let G be a nonbipartite $\mathcal{F}_{g,l}$-graph of order n for some l and with domination number $\frac{n-1}{2}$. Then $q_{min}(G) \ge q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$ with equality if and only if $G \cong \mathscr{H}_{3,\frac{n-3}{2}}$ (see Fig. 4.2).*

Proof. Because G is nonbipartite, g is odd. If G is a $\mathcal{F}^{\circ}_{g,l}$-graph, then the theorem follows from Lemma 4.5. If $g = 3$, then the theorem follows from Theorem 4.4. For $g = 5$, the theorem follows from Lemma 4.6. Next we consider the case that G is not a $\mathcal{F}^{\circ}_{g,l}$-graph and suppose $g \ge 7$.

Let $X = (x_1, x_2, \ldots, x_n)^T$ is a unit eigenvector corresponding to $q_{min}(G)$. Suppose $x_a = \min\{|x_1|, |x_2|, \ldots, |x_g|\}$. Note that by Theorem 3.2, in G, there are at most 3 consecutive vertices of $\mathbb{C}$ such that none of them is p-dominator, and there are 2 cases as follows to consider.

Case 1 In G, there is exactly one vertex of $\mathbb{C}$ which is not p-dominator. Note that G is not a $\mathcal{F}^{\circ}_{g,l}$-graph. Then $n \ge g+2$ and v_g is the only one vertex which is not p-dominator on $\mathbb{C}$. By a same discussion in the proof of Lemma 4.3 (see [18]), it is proved that $x_g = \max\{|x_1|, |x_2|, \ldots, |x_{g-1}|, |x_g|\}$. Then we suppose $a \le g-1$. By Lemma 4.1, if $a \le g-3$, without loss of generality, suppose $x_{a+1} \le x_{a-1}$, $x_{a+1}x_a \ge 0$, $|x_{a-1}| \ge |x_{a+2}|$. Let $G_1 = G - v_av_{a-1} + v_av_{a+2}$ (if $|x_{a-1}| \le |x_{a+2}|$ and $a \ge 2$, let $G_1 = G - v_{a+1}v_{a+2} + v_{a+1}v_{a-1}$; if $a = 1$, let $G_1 = G - v_1v_g + v_1v_3$). If $a = g-2$, suppose $|x_{g-1}| \le |x_{g-3}|$, $x_{g-1}x_{g-2} \ge 0$, and then let $G_1 = G - v_{g-1}v_g + v_{g-1}v_{g-3}$. If $a = g-1$, because $|x_g| \ge |x_{g-2}|$, then suppose $x_{g-1}x_{g-2} \ge 0$. Let $G_1 = G - v_{g-1}v_g + v_{g-1}v_{g-3}$. Note that

$\gamma(G_1) \leq \frac{n-1}{2}$. As the proof of Lemma 4.2, we get that $\gamma(G) \leq \gamma(G_1) = \frac{n-1}{2}$, $q_{min}(G_1) < q_{min}(G)$. Note that $g(G_1) = 3$. Then the theorem follows from Theorem 4.4.

Case 2 In G, there are exactly 3 consecutive vertices of $\mathbb{C}$ such that each of them is not p-dominator. Note that G is not a $\mathcal{F}^{\circ}_{g,l}$-graph. Combined with Theorem 3.2, the 3 vertices of $\mathbb{C}$ such that each of them is not p-dominator are v_{g-2}, v_{g-1}, v_g or v_g, v_1, v_2. Without loss of generality, we suppose the 3 vertices are v_{g-2}, v_{g-1}, v_g. By Lemma 2.12, $|x_g| > 0$. We say that $|x_g| > |x_{g-2}|$. Otherwise, suppose $|x_g| \leq |x_{g-2}|$. Let $G' = G - v_g v_{g+1} + v_{g+1} v_{g-2}$. Then by Lemma 2.4, $q_{min}(G') < q_{min}(G)$. This is a contradiction because $G' \cong G$. Hence $|x_g| > |x_{g-2}|$. And then $a \leq g - 1$.

Subcase 2.1 $a \leq g - 4$. By Lemma 4.1, without loss of generality, suppose $x_{a+1} \leq x_{a-1}$, $x_{a+1} x_a \geq 0$. As Case 1, it is proved that the theorem holds.

Subcase 2.2 $a = g - 3$. By Lemma 4.1, suppose $x_{g-2} \leq x_{g-4}$, $x_{g-2} x_{g-3} \geq 0$; suppose $|x_{g-4}| \geq |x_{g-1}|$. Denote by $v_{\tau_{g-3}}$ the pendant vertex attached to v_{g-3}. Let $G_1 = G - v_{g-3} v_{g-4} + v_{g-3} v_{g-1} - v_{g-3} v_{\tau_{g-3}} + v_g v_{\tau_{g-3}}$ (if $x_{g-4} \leq x_{g-1}$, let $G_1 = G - v_{g-2} v_{g-1} + x_{g-2} x_{g-4}$). As Case 1, it is proved that the theorem holds.

Subcase 2.3 $a = g - 2$. By Lemma 4.1, suppose $x_{g-1} \leq x_{g-3}$, $x_{g-1} x_{g-2} \geq 0$; suppose $|x_{g-3}| \geq |x_g|$. Denote by $v_{\tau_{g-3}}$ the pendant vertex attached to v_{g-3}. Let $G_1 = G - v_{g-2} v_{g-3} + v_{g-2} v_g$ (if $x_{g-3} \leq x_g$, let $G_1 = G - v_{g-1} v_g + x_{g-1} x_{g-3} - v_{g-3} v_{\tau_{g-3}} + v_g v_{\tau_{g-3}}$). As Case 1, it is proved that the theorem holds.

Subcase 2.4 $a = g - 1$. Note $|x_g| > |x_{g-2}|$. By Lemma 4.1, $x_{g-2} x_{g-1} \geq 0$. Without loss of generality, suppose $x_{g-3} \geq x_g$, let $G_1 = G - v_{g-2} v_{g-3} + v_{g-2} v_g$ (if $x_{g-3} \leq x_g$, let $G_1 = G - v_{g-1} v_g + x_{g-1} x_{g-3} - v_g v_{g+1} + v_{g-3} v_{g+1}$). As Case 1, it is proved that the theorem holds. This completes the proof. $\square$

By Lemmas 2.12, 4.7, we get the following Theorem 4.8.

Theorem 4.8 *Let G be a nonbipartite connected unicyclic graph of order $n \geq 3$ and with domination number $\frac{n-1}{2}$. Then $q_{min}(G) \geq q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$ with equality if and only if $G \cong \mathscr{H}_{3,\frac{n-3}{2}}$.*

5 Proof of main results

Proof of Theorem 1.1. By Lemmas 2.1, 2.7, then G contains a nonbipartite unicyclic spanning subgraph H with $g_o(H) = g_o(G)$, $\gamma(H) = \gamma(G)$ and $q_{min}(H) \leq q_{min}(G)$. By Theorem 4.8, it follows that $q_{min}(H) \geq q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$ with equality if and only if $H \cong \mathscr{H}_{3,\frac{n-3}{2}}$. Thus it follows that $q_{min}(G) \geq q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$.

Suppose that $q_{min}(G) = q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$. Then $q_{min}(H) = q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$ and $H \cong \mathscr{H}_{3,\frac{n-3}{2}}$. For convenience, we suppose that $H = \mathscr{H}_{3,\frac{n-3}{2}}$. Suppose that Y is a unit eigenvector corresponding to $q_{min}(G)$. Note that $q_{min}(\mathscr{H}_{3,\frac{n-3}{2}}) = q_{min}(H) \leq Y^T Q(H) Y \leq Y^T Q(G) Y = q_{min}(G)$. Because we suppose that $q_{min}(G) = q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$, it follows that $Y^T Q(H) Y = Y^T Q(G) Y$ and $Q(H)Y = q_{min}(H)Y$.

For $\mathscr{H}_{3,\frac{n-3}{2}}$ (see Fig. 4.2), we claim that $y_3 > y_1$, $y_3 > y_2$. Otherwise, suppose that $y_3 \leq y_1$. Let $H' = \mathscr{H}_{3,\frac{n-3}{2}} - v_3 v_4 + v_1 v_4$. By Lemma 2.4, it follows that $q_{min}(H') < q_{min}(\mathscr{H}_{3,\frac{n-3}{2}})$. This is a contradiction because $H' \cong H \cong \mathscr{H}_{3,\frac{n-3}{2}}$. Thus our claim holds.

If $G \neq H$, combined with Lemma 2.3, then for any edge $v_iv_j \notin E(H)$, it follows that $x_i + x_j \neq 0$, and then $Y^TQ(H)Y < Y^TQ(G)Y$, which contradicts $Y^TQ(H)Y = Y^TQ(G)Y$. Then it follows that $q_{min}(G) = q_{min}(\mathscr{H}_{3,\frac{n-1}{2}})$ if and only if $G \cong \mathscr{H}_{3,\frac{n-1}{2}}$. This completes the proof. □

In a same way, with Lemmas 2.13, 2.14 and 4.6, Theorem 1.2 is proved.

Remark It can be seen that the conjecture in [18] that $\mathbb{S}$ has the smallest q_{min} holds for the graphs with domination number $\gamma = \frac{n-1}{2}$ and the graphs with girth at most 5. With references [17] and [18], it can also be seen that the minimum q_{min} of the connected nonbipartite graph on $n \geq 5$ vertices, with domination number $\frac{n+1}{3} < \gamma \leq \frac{n-2}{2}$ and girth $g \geq 5$, is still open.

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