On stability of Euler flows on closed surfaces of positive genus

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Abstract

Incompressible flows of an ideal two-dimensional fluid on a closed orientable surface of positive genus are considered. Linear stability of harmonic, i.e. irrotational and incompressible, solutions to the Euler equations is shown using the Hodge-Helmholtz decomposition. We also demonstrate that any surface Euler flow is stable with respect to harmonic velocity perturbations.

Keywords: Surface Euler equations; Hodge-Helmholtz decomposition; Stability; Exterior calculus.
2000 MSC: 76E09, 58J90

1. Introduction

Recently, there has been a growing interest in the modeling of thin fluid flows on curved surfaces. Apart from its practical relevance, mathematical modeling of surface flows reveals many intriguing theoretical issues. In this paper we investigate how the absence of a boundary and the surface genus affect the stability of solutions to the Euler equations on a two-dimensional manifold.

Surfaces of positive genus have “handles” and admit divergence-free and curl-free (i.e. harmonic) vector fields. Harmonic velocity fields generate incompressible and irrotational flows, which are essential in theoretical fluid mechanics. Since the boundary is empty, all steady harmonic vector fields are solutions to Euler equation which may potentially drive the surface Euler flows to be unstable. It is worth noting that a harmonic vector field on a closed surface cannot be represented either by the gradient of a potential function or by the skew-gradient of a stream function.

2. Hodge-Helmholtz formulation of Euler equation

Let $(\mathcal{M}^2,\ g)$ be a compact connected orientable two-dimensional Riemannian manifold without boundary. Then there exist [1] a smooth *velocity* vector field v and a smooth *static pressure* function p_s on $\mathcal{M}$ such that, for sufficiently smooth vector fields F and v_0 on $\mathcal{M}$, the following Euler equations are satisfied:

$$v_t + \overline{\nabla}_v v + \overline{\nabla} p_s = F\,, \quad \operatorname{div} v = 0\,, \quad v\big|_{t=0} = v_0\,, \tag{1}$$

where $\overline{\nabla}$ is the Levi-Civita connection. Having chosen a volume form μ associated with g, one may define the Hodge $\star$ operator, the Hodge inner product $\langle .,.\rangle$, and codifferential δ.

Proposition 2.1. *Let f be a function, α, β be covector fields. Consider η, a skew-symmetric bilinear form defined on $\mathcal{M}^2$ equipped with volume form μ. Then the Hodge $\star$ operator maps vectors to vectors, functions to bilinear forms and vice versa, and the following properties hold true [2]:*

$$\begin{aligned} \star\star f &= f & \star\mu &= 1 \\ \star\star\alpha &= -\alpha & \alpha \wedge \star\beta &= (\alpha,\beta)\wedge\mu \\ \delta &= -\star d \star & \langle \delta\alpha, \beta\rangle &= \langle \alpha, d\beta\rangle \\ \langle \alpha,\beta\rangle &= \int_{\mathcal{M}} (\alpha,\beta)\wedge\mu & \langle f, \star\eta\rangle &= \int_{\mathcal{M}} f\wedge\eta \end{aligned}$$

The *rotational* form of the dual Euler equations (1) on a two-dimensional manifold can be obtained in terms of the covector field $v^\flat$ dual to v as follows:

$$v^\flat_t + \star d v^\flat \wedge \star v^\flat + dp = F^\flat\,, \qquad -\delta v^\flat = 0\,, \qquad v^\flat\big|_{t=0} = v^\flat_0\,, \tag{2}$$

where vorticity function $\omega = \star d v^\flat$ is the curl of vector field v, and $p = p_s + \frac{1}{2} v^\flat(v)$ is the Bernoulli pressure.

Hodge theory implies that an incompressible covector field on $\mathcal{M}$ can be uniquely decomposed into L^2-orthogonal terms:

$$v^\flat = \star d\psi + \gamma\,, \tag{3}$$

where ψ is the stream function and γ is a harmonic covector field. By substituting (3) into (2), one derives the Hodge-Helmholtz formulation of the Euler equations (initial conditions omitted):

$$\star d\psi_t + \gamma_t + \omega \wedge (-d\psi + \star\gamma) + dp = F^\flat\,, \qquad -\delta d\psi = \omega\,. \tag{4}$$

Remark 2.1. *The Hodge-Helmholtz decomposition of the first equation of* (4) *produces three independent relations thus matching with the number of unknowns* (ψ, ω, γ, p)*.*

3. Stability of harmonic solutions

Harmonic covector fields, i.e $v^\flat = \gamma$, have zero vorticity and thus represent all incompressible and irrotational velocity fields. The space of all harmonic vector fields is finite-dimensional, and its dimension is equal to the second Betti number, which is the doubled number of "handles" for a closed oriented surface with only one connected component; therefore, only a positive genus surface admits harmonic vector fields. Note that the image of a harmonic vector under the Hodge star operator is again harmonic.

We investigate stability of any harmonic solution (ψ, ω, γ, p)=(0, 0, γ_0, p_0) of the Euler equation (4) by the standard linearization:

$$\psi = 0 + \varepsilon\widetilde{\psi} + O(\varepsilon^2)\,, \quad \gamma = \gamma_0 + \varepsilon\widetilde{\gamma} + O(\varepsilon^2)\,, \quad p = p_0 + \varepsilon\widetilde{p} + O(\varepsilon^2)\,. \qquad (5)$$

Theorem 1. *A harmonic Euler flow on a closed surface is at most polynomially (linearly) unstable in the L^2 norm.*

Proof. To prove the statement, one estimates both parts of the velocity perturbation $\widetilde{v}^\flat = \star d\widetilde{\psi} + \widetilde{\gamma}$. To begin one substitutes (5) into (4) and derives the system that governs the evolution of small perturbations:

$$\star d\widetilde{\psi}_t + \widetilde{\gamma}_t + \widetilde{\omega} \wedge \star\gamma_0 + d\widetilde{p} = 0\,, \qquad -\delta d\widetilde{\psi} = \widetilde{\omega} \qquad (6)$$

We first estimate the stream function perturbation by taking $\star d$ of the first equation in (6) and obtain the transport equation for the vorticity perturbation along the main flow γ_0:

$$\widetilde{\omega}_t + \star(d\widetilde{\omega} \wedge \star\gamma_0) = 0 \qquad (7)$$

Note that L^2-norm of $\widetilde{\omega}$ remains constant over time:

$$\begin{aligned} -\left\langle \widetilde{\omega}, \widetilde{\omega} \right\rangle_t &= -2\left\langle \widetilde{\omega}_t, \widetilde{\omega} \right\rangle = 2\left\langle \star(d\widetilde{\omega} \wedge \star\gamma_0), \widetilde{\omega} \right\rangle = 2\int_{\mathcal{M}} \widetilde{\omega} \wedge d\widetilde{\omega} \wedge \star\gamma_0 = \\ &= \int_{\mathcal{M}} d(\widetilde{\omega} \wedge \widetilde{\omega}) \wedge \star\gamma_0 = \left\langle d(\widetilde{\omega} \wedge \widetilde{\omega}), \gamma_0 \right\rangle = \left\langle \widetilde{\omega} \wedge \widetilde{\omega}, \delta\gamma_0 \right\rangle = 0 \end{aligned} \qquad (8)$$

Using the first smallest positive eigenvalue λ_{min} of Hodge-Laplacian δd, one bounds the L^2-norm of $\widetilde{\psi}$ by the L^2-norm of $\widetilde{\omega}$:

$$\lambda_{min} \left\langle \widetilde{\psi}, \widetilde{\psi} \right\rangle \leq \left\langle \widetilde{\psi}, \delta d \widetilde{\psi} \right\rangle = \left\langle \widetilde{\psi}, -\widetilde{\omega} \right\rangle \leq \left\langle \widetilde{\psi}, \widetilde{\psi} \right\rangle^{\frac{1}{2}} \langle \widetilde{\omega}, \widetilde{\omega} \rangle^{\frac{1}{2}} ,$$
$$\left\langle \widetilde{\psi}, \widetilde{\psi} \right\rangle \leq \frac{1}{\lambda_{min}^2} \langle \widetilde{\omega}, \widetilde{\omega} \rangle$$

The evolution of the harmonic part of perturbation $\widetilde{\gamma}$ is studied by means of a Hodge-orthonormal basis $\{h^i\}$ of the finite-dimensional space of harmonic covector fields (summation over i is assumed):

$$\widetilde{\gamma} = c_i(t) h^i \,. \tag{9}$$

Taking the Hodge inner product of (6) with a basis covector field h^k results in the following equation for c_k:

$$\dot{c}_k(t) + \int_{\mathcal{M}} \widetilde{\omega} \wedge (\star\gamma_0, h^k)\mu = 0$$

Note that $(\star\gamma_0, h^k)$ are constant functions. We now show that $\widetilde{\gamma}$ grows at most linearly in time using Cauchy-Schwarz inequality:

$$|\dot{c}_k(t)| \leq \left(\int_{\mathcal{M}} \widetilde{\omega}^2 \mu \right)^{1/2} \left(\int_{\mathcal{M}} (\star\gamma_0, h^k)^2 \mu \right)^{1/2} = \langle \widetilde{\omega}, \widetilde{\omega} \rangle^{\frac{1}{2}} \left(\int_{\mathcal{M}} (\star\gamma_0, h^k)^2 \mu \right)^{1/2} ,$$

and since the right-hand side is constant over time due to (8), this concludes the proof. □

4. Stability of Euler flows with respect to harmonic perturbations

Theorem 1 shows that harmonic solutions to the Euler equations on a closed surface are linearly unstable. In this section, we would like to address the following question: is it possible for harmonic perturbations to destabilize an Euler flow on a closed surface with positive genus?

As in Section 3, let us perturb a solution (ψ_0, ω_0, γ_0, p_0) of (4) by a harmonic perturbation $\widetilde{v}^\flat = \widetilde{\gamma}$:

$$\widetilde{\gamma}_t + \omega_0 \wedge \star\widetilde{\gamma} + d\widetilde{p} = 0 \,. \tag{10}$$

Theorem 2. *Any Euler flow on a closed surface is L^2-stable with respect to harmonic perturbations.*

Proof. We employ decomposition (9). Taking the Hodge inner product of (10) with a basis vector h^i results in the following finite-dimensional system:

$$\dot{c}_i = c_j \int_{\mathcal{M}} \omega_0 \wedge h^i \wedge h^j \,. \tag{11}$$

The matrix $A_{ij} = \int_{\mathcal{M}} \omega_0 \wedge h^i \wedge h^j$ is skew-symmetric and even-dimensional. Hence the finite-dimensional system (11) preserves the norm of the vector c: $|c|^2_t = c \cdot c_t = c \cdot Ac = A^T c \cdot c = 0$, and the harmonic perturbation $\widetilde{\gamma}$ remains bounded. □

5. Conclusions

We have investigated the influence of a surface genus on the stability of solutions to surface Euler equations. It was shown that harmonic solutions are at most linearly unstable with a factor proportional to the L^2-norm of a perturbation vorticity. We also proved that Euler flows on a closed surface are not destabilized by harmonic perturbations.

6. Acknowledgments

The author gratefully thanks Boris Khesin who provided expertise that greatly assisted the research and also Maxim Olshanskii for constant interest in the work and helpful suggestions.

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