

# The reduced formula of the characteristic polynomial of hypergraphs and the spectrum of hyperpaths

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## Abstract

In this paper, we give a reduced formula of the characteristic polynomial of  $k$ -uniform hypergraphs with a pendant edge. And the explicit characteristic polynomial and all distinct eigenvalues of  $k$ -uniform hyperpath are given.

*Keywords:* Hypergraph, Tensor, Characteristic polynomial, Reduced formula, Poisson formula

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## 1. Introduction

For a positive integer  $n$ , let  $[n] = \{1, \dots, n\}$ . A  $k$ -order  $n$ -dimension complex tensor  $\mathcal{T} = (t_{i_1 \dots i_k})$  is a multidimensional array with  $n^k$  entries on complex number field  $\mathbb{C}$ , where  $i_j \in [n]$ ,  $j = 1, \dots, k$ . Denote the set of  $n$ -dimension complex vector and the set of  $k$ -order  $n$ -dimension complex tensor by  $\mathbb{C}^n$  and  $\mathbb{C}^{[k, n]}$ , respectively. For  $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n$ ,  $\mathcal{T}x^{k-1}$  is a vector in  $\mathbb{C}^n$  whose  $i$ -th component is defined as

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \dots, i_k=1}^n t_{ii_2 \dots i_k} x_{i_2} \cdots x_{i_k}.$$

If there exist  $\lambda \in \mathbb{C}$  and a nonzero vector  $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$  such that  $\mathcal{T}x^{k-1} = \lambda x^{[k-1]}$ , then  $\lambda$  is called an *eigenvalue* of  $\mathcal{T}$  and  $x$  is called an *eigenvector* of  $\mathcal{T}$  corresponding to  $\lambda$ , where  $x^{[k-1]} = (x_1^{k-1}, \dots, x_n^{k-1})^T$  (see [8, 11]). The *characteristic polynomial*  $\phi_{\mathcal{T}}(\lambda)$  of tensor  $\mathcal{T}$  is defined as the resultant  $\text{Res}(\lambda x^{[k-1]} - \mathcal{T}x^{k-1})$ . And  $\phi_{\mathcal{T}}(\lambda)$  is a monic polynomial in  $\lambda$  of degree  $n(k-1)^{n-1}$  (see [3]).

A hypergraph  $H = (V(H), E(H))$  is called *k-uniform* if each edge of  $H$  contains exactly  $k$  distinct vertices. When  $k = 2$ ,  $H$  is a graph. The tensor  $\mathcal{A}_H = (a_{i_1 i_2 \dots i_k}) \in$

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$\mathbb{C}^{[k,n]}$  is the *adjacency tensor* of a  $k$ -uniform hypergraph  $H$  with vertex set  $V(H) = \{1, 2, \dots, n\}$ , where

$$a_{i_1 i_2 \dots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \dots, i_k\} \in E(H), \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of tensor  $\mathcal{A}_H$  is called the characteristic polynomial of the hypergraph  $H$  (see [2]). For a vector  $y = (y_{i_1}, y_{i_2}, \dots, y_{i_t})^T \in \mathbb{C}^t$ , let  $y^S = \prod_{i \in S} y_i$  for  $S \subseteq \{i_1, i_2, \dots, i_t\}$ , where  $i_1, i_2, \dots, i_t$  are distinct nonnegative integers. For a vertex  $i \in V(H)$ ,  $E_i(H)$  denotes the set of edges incident with vertex  $i$ . Then

$$(\mathcal{A}_H x^{k-1})_i = \sum_{e \in E_i(H)} x^{e \setminus \{i\}}$$

for  $x = (x_1, \dots, x_n)^T \in \mathbb{C}^n$  and  $i \in [n]$ .

The characteristic polynomial of graph is an important research topic in spectral graph theory. In 1962, Harary gave the structural parameter representation of the determinant of the adjacency matrix of graphs [6]. In 1964, Sachs gave the coefficients of characteristic polynomial which is usually known as Sachs Coefficient Theorem using the result of Harary [12]. In 1971, Harary et al. gave a reduced formula of the characteristic polynomial of graphs with a pendant edge [5].

**Theorem 1.1.** [5] *Let  $G_v$  denote the graph obtained from  $G$  by adding a pendant edge at the vertex  $v$ . Let  $G - v$  denote the graph obtained from  $G$  by removing  $v$  together with all edges incident to  $v$ . Then*

$$\phi_{G_v}(\lambda) = \lambda \phi_G(\lambda) - \phi_{G-v}(\lambda).$$

In 1973, Lovász and Pelikán gave the characteristic polynomial of paths [9].

**Theorem 1.2.** [9] *The characteristic polynomial of a path  $P_m$  of length  $m$  is*

$$\phi_{P_m}(\lambda) = \sum_{q=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^q C_{m+1-q}^q \lambda^{m+1-2q},$$

where  $C_{m+1-q}^q$  is a combinatorial number.

In [10](Page 73 of [10]), the author gave all the distinct eigenvalues of  $P_m$ .

**Theorem 1.3.** [10] *The distinct eigenvalues of  $P_m$  are  $2 \cos \frac{\pi t}{m+2}$ ,  $t = 1, 2, \dots, m+1$ .*

In 2012, Cooper and Dutle gave the characteristic polynomial of a  $k$ -uniform hyperpath with one edge [2]. In 2015, Shao et al. gave some properties of the

characteristic polynomial of hypergraphs whose spectrum are  $k$ -symmetric [13]. In 2015, Cooper and Dutle gave the characteristic polynomial of 3-uniform hyperstars [3]. In 2019, Bao et al. gave the characteristic polynomial of  $k$ -uniform hyperstars and the characteristic polynomial of hypergraphs with a cut vertex under some assumptions [15].

In this paper, we give a reduced formula of the characteristic polynomials of  $k$ -uniform hypergraphs with pendent edges. And using this formula we give the explicit characteristic polynomial of hyperpaths. All distinct eigenvalues of  $k$ -uniform hyperpath are given.

## 2. Preliminary

In this section, we introduce the Poisson formula and some properties of the resultant which are used in the proof of our main results.

The  $k$ -uniform hyperpath  $P_m^{(k)}$  is the  $k$ -uniform hypergraph which obtained by adding  $k - 2$  vertices with degree one to each edge of the path  $P_m$ . In 2012, Cooper and Dutle gave the characteristic polynomial of  $P_1^{(k)}$  [2].

**Lemma 2.1.** [2] *The characteristic polynomial of the  $k$ -uniform hyperpath  $P_1^{(k)}$  is*

$$\phi_{P_1^{(k)}}(\lambda) = \lambda^{k(k-1)^{k-1}-k^{k-1}} (\lambda^k - 1)^{k^{k-2}}.$$

In this paper, the *Poisson formula* of resultants is important to compute the characteristic polynomials of hypergraphs.

**Lemma 2.2.** (Poisson formula)[1, 4, 7] *Let  $F_0, F_1, \dots, F_n$  be homogeneous polynomials of respective degrees  $d_0, \dots, d_n$  in  $K[x_0, \dots, x_n]$ , where  $K$  is an algebraically closed field. For  $0 \leq i \leq n$ , let  $\overline{F}_i = F_i|_{x_0=0}$  and  $f_i = F_i|_{x_0=1}$ . Let  $\mathcal{V}$  be the set of simultaneous zeros of the system of polynomials  $f_1, f_2, \dots, f_n$ , that is,  $\mathcal{V}$  is the affine*

*variety defined by the polynomials. If  $\text{Res} \begin{pmatrix} \overline{F}_1 \\ \vdots \\ \overline{F}_n \end{pmatrix} \neq 0$ , then  $\mathcal{V}$  is a zero-dimensional variety (a finite set of points), and*

$$\text{Res} \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_n \end{pmatrix} = \text{Res} \begin{pmatrix} \overline{F}_1 \\ \vdots \\ \overline{F}_n \end{pmatrix}^{d_0} \prod_{p \in \mathcal{V}} (f_0(p))^{m(p)}$$

$$\operatorname{Res} \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_n \end{pmatrix} = \operatorname{Res} \begin{pmatrix} \overline{F_1} \\ \vdots \\ \overline{F_n} \end{pmatrix}^{d_0} \prod_{p \in \mathcal{V}} (f_0(p))^{m(p)},$$

where  $m(p)$  is the multiplicity of a point  $p \in \mathcal{V}$ .

**Lemma 2.3.** (Page 97 and 102 of [1]) Let  $F_0, F_1, \dots, F_n$  be homogeneous polynomials of respective degrees  $d_0, \dots, d_n$  in  $K[x_0, \dots, x_n]$ , where  $K$  is an algebraically closed field. Then

$$(1) \operatorname{Res} \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ \lambda F_n \end{pmatrix} = \lambda^{d_0 d_1 \cdots d_{n-1}} \operatorname{Res} \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_n \end{pmatrix};$$

$$(2) \operatorname{Res} \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \\ x_n^d \end{pmatrix} = \operatorname{Res} \begin{pmatrix} F_0|_{x_n=0} \\ F_1|_{x_n=0} \\ \vdots \\ F_{n-1}|_{x_n=0} \end{pmatrix}^d.$$

For a  $k$ -uniform hypergraph  $H$  with vertices set  $V(H) = \{0, 1, \dots, n\}$ , let

$$F^H = \lambda x^{[k-1]} - \mathcal{A}_H x^{k-1} = (F_0^H, F_1^H, \dots, F_n^H)^\top,$$

$$\overline{F^H} = (F_1^H, F_2^H, \dots, F_n^H)^\top|_{x_0=0} = (\overline{F_1^H}, \overline{F_2^H}, \dots, \overline{F_n^H})^\top,$$

$$f_i^H = F_i^H|_{x_0=1}, i = 0, 1, 2, \dots, n.$$

where  $x = (x_0, x_1, \dots, x_n)^\top \in \mathbb{C}^{n+1}$ . Let  $\mathcal{V}^H$  denote the affine variety defined by the polynomials  $f_1^H, f_2^H, \dots, f_n^H$ . From Lemma 2.1,

$$\phi_H(\lambda) = \operatorname{Res}(F^H) = \operatorname{Res}(\overline{F^H})^{k-1} \prod_{p \in \mathcal{V}^H} (f_0^H(p))^{m_H(p)}, \quad (1)$$

where  $m_H(p)$  is the multiplicity of a point  $p \in \mathcal{V}^H$ .

### 3. Main results

In this section, a reduced formula of the characteristic polynomials of  $k$ -uniform hypergraphs with a pendant edge and the explicit characteristic polynomial of  $k$ -uniform hyperpath are given. These results generalize the results given by Harary et al. [5] and Lovász et al. [9].

Cooper and Dutle gave the characteristic polynomial of  $P_1^{(k)}$  via the trace of tensor [2]. Let  $V(P_1^{(k)}) = \{0, 1, \dots, k-1\}$ . And  $\mathcal{V}$  denotes the affine variety defined by the polynomials  $f_1^{P_1^{(k)}}, f_2^{P_1^{(k)}}, \dots, f_{k-1}^{P_1^{(k)}}$ . In order to give the reduced formula of the characteristic polynomials of  $k$ -uniform hypergraphs with a pendant edge, we first give  $m(p)$  for  $p \in \mathcal{V}$ .

**Theorem 3.1.** *Let  $V(P_1^{(k)}) = \{0, 1, \dots, k-1\}$ . And  $\mathcal{V}$  denotes the affine variety defined by the polynomials  $f_1^{P_1^{(k)}}, f_2^{P_1^{(k)}}, \dots, f_{k-1}^{P_1^{(k)}}$ . Then  $\sum_{0 \neq p \in \mathcal{V}} m(p) = k^{k-2}$  and  $m(0) = (k-1)^{k-1} - k^{k-2}$  for  $0 \in \mathcal{V}$ .*

*Proof.* For the hypergraph  $H = P_1^{(k)}$  with one edge  $e = \{0, 1, \dots, k-1\}$ ,  $F_i^H = \lambda x_i^{k-1} - x^{e \setminus \{i\}}$  for  $i = 0, 1, \dots, k-1$ , where  $x = (x_0, x_1, \dots, x_{k-1})^T \in \mathbb{C}^k$ . From Eq. (1),

$$\phi_H(\lambda) = \text{Res}(F^H) = \text{Res}(\overline{F^H})^{k-1} \prod_{p \in \mathcal{V}} (f_0^H(p))^{m(p)}. \quad (2)$$

Since  $\overline{F_i^H} = F_i^H|_{x_0=0} = (\lambda x_i^{k-1} - x^{e \setminus \{i\}})|_{x_0=0} = \lambda x_i^{k-1}$  for  $i \in [k-1]$ ,  $\text{Res}(\overline{F^H})$  is the characteristic polynomial of the  $k$ -order  $(k-1)$ -dimension null tensor. From the definition of tensor eigenvalues, we know that the eigenvalues of the null tensor are zero. And from  $\text{Res}(\overline{F^H})$  is monic polynomials of degree  $(k-1)^{k-1}$ , we get

$$\text{Res}(\overline{F^H}) = \lambda^{(k-1)^{k-1}}. \quad (3)$$

For  $p = (p_1, p_2, \dots, p_{k-1})^T \in \mathcal{V}$ . When  $p = 0$ , we have  $p^{e \setminus \{0\}} = p_1 p_2 \cdots p_{k-1} = 0$ . When  $p \neq 0$ , we get  $f_i^H(p) = \lambda p_i^{k-1} - p^{e \setminus \{0, i\}} = 0$  for all  $i \in [k-1]$ . Then

$$\lambda p_1^k = \lambda p_2^k = \cdots = \lambda p_{k-1}^k = p^{e \setminus \{0\}} \quad (4)$$

and  $\lambda^{k-1} p_1^k p_1^k \cdots p_{k-1}^k = \lambda^{k-1} (p^{e \setminus \{0\}})^k = (p^{e \setminus \{0\}})^{k-1}$ . Note that  $\lambda$  is an indeterminate of the characteristic polynomials  $\phi_H(\lambda)$ . From Eq.(4), we know that  $p^{e \setminus \{0\}} \neq 0$ . Then  $p^{e \setminus \{0\}} = \frac{1}{\lambda^{k-1}}$ . From the above discussion, we obtain

$$p^{e \setminus \{0\}} = \begin{cases} 0, & p = 0, \\ \frac{1}{\lambda^{k-1}}, & p \neq 0. \end{cases} \quad (5)$$

Hence,

$$f_0^H(p) = \lambda - p^{e \setminus \{0\}} = \begin{cases} \lambda, & p = 0, \\ \lambda - \frac{1}{\lambda^{k-1}}, & p \neq 0. \end{cases}$$

By Eq. (2) and Eq. (3), we have

$$\begin{aligned}
\phi_H(\lambda) &= \lambda^{(k-1)^k} \prod_{p \in \mathcal{V}} (f_0^H(p))^{m(p)} \\
&= \lambda^{(k-1)^k} \prod_{0=p \in \mathcal{V}} \lambda^{m(p)} \prod_{0 \neq p \in \mathcal{V}} \left( \lambda - \frac{1}{\lambda^{k-1}} \right)^{m(p)} \\
&= \lambda^{(k-1)^k} \lambda^{m(0)} \left( \lambda - \frac{1}{\lambda^{k-1}} \right)^{\sum_{0 \neq p \in \mathcal{V}} m(p)} \\
&= \lambda^{(k-1)^k + m(0) - (k-1) \sum_{0 \neq p \in \mathcal{V}} m(p)} (\lambda^k - 1)^{\sum_{0 \neq p \in \mathcal{V}} m(p)}. \tag{6}
\end{aligned}$$

Comparing Lemma 2.1 with Eq.(6), we obtain  $\sum_{0 \neq p \in \mathcal{V}} m(p) = k^{k-2}$  and  $m(0) = (k-1)^{k-1} - k^{k-2}$ .  $\square$

Let  $H$  be a  $k$ -uniform hypergraph with vertices set  $V(H) = \{0, 1, 2, \dots, n\}$ . From Eq.(1), we have

$$\phi_H(\lambda) = \text{Res} \left( \begin{array}{c} \overline{F_1^H} \\ \overline{F_2^H} \\ \vdots \\ \overline{F_n^H} \end{array} \right)^{k-1} \prod_{p \in \mathcal{V}^H} (f_0^H(p))^{m_H(p)}. \tag{7}$$

For  $v \in V(H)$ ,  $H - v$  denotes the  $k$ -uniform hypergraph obtained from  $H$  by removing vertex  $v$  and all edges incident to vertex  $v$ . Without loss of generality, let the vertex  $v = 0$ . Since

$$\begin{aligned}
\overline{F_i^H} &= \left( \lambda x_i^{k-1} - \sum_{e \in E_i(H)} x^{e \setminus \{i\}} \right) \Big|_{x_0=0} \\
&= \lambda x_i^{k-1} - \sum_{\substack{e \in E_i(H) \\ 0 \notin e}} x^{e \setminus \{i\}} \\
&= \lambda x_i^{k-1} - \sum_{e \in E_i(H-0)} x^{e \setminus \{i\}}
\end{aligned}$$

for  $x = (x_0, x_1, \dots, x_n)^T \in \mathbb{C}^{n+1}$  and  $i \in V(H-0)$ , we have  $\text{Res} \left( \begin{array}{c} \overline{F_1^H} \\ \overline{F_2^H} \\ \vdots \\ \overline{F_n^H} \end{array} \right) = \phi_{H-0}(\lambda)$ .

Then from Eq.(7), we know

$$\begin{aligned}\phi_H(\lambda) &= \phi_{H-0}^{k-1} \prod_{p \in \mathcal{V}^H} (f_0^H(p))^{m_H(p)} \\ &= \phi_{H-0}^{k-1}(\lambda) \prod_{p \in \mathcal{V}^H} \left( \lambda - \sum_{e \in E_0(H)} p^{e \setminus \{0\}} \right)^{m_H(p)}.\end{aligned}$$

So for  $v \in V(H)$ ,

$$\phi_H(\lambda) = \phi_{H-v}^{k-1}(\lambda) \prod_{p \in \mathcal{V}^H} \left( \lambda - \sum_{e \in E_v(H)} p^{e \setminus \{v\}} \right)^{m_H(p)}, \quad (8)$$

where  $\mathcal{V}^H$  is affine variety defined by polynomials  $F_i^H|_{x_v=1}$  for all  $i \in V(H-v)$ .

Let

$$M_H(\lambda, \frac{t}{\lambda^{k-1}}) = \prod_{p \in \mathcal{V}^H} \left( \lambda - \sum_{e \in E_v(H)} p^{e \setminus \{v\}} - \frac{t}{\lambda^{k-1}} \right)^{m_H(p)}, \quad (9)$$

where  $t$  is a nonnegative integer. And  $M_H(\lambda, \frac{t}{\lambda^{k-1}})$  is called the “ $\frac{t}{\lambda^{k-1}}$ -translational fraction” of  $M_H(\lambda, 0)$ . By Eq.(8) and Eq.(9), we know  $M_H(\lambda, 0) = \frac{\phi_H(\lambda)}{\phi_{H-v}^{k-1}(\lambda)}$ .

We give the reduced formula of the characteristic polynomial of  $k$ -uniform hypergraphs with pendent edges as follows. When  $k = 2$ , this result is the Theorem 1.1.

**Theorem 3.2.** *Let  $H$  be a  $k$ -uniform hypergraph with  $n$  vertices.  $H_v$  denotes the  $k$ -uniform hypergraph obtained from  $H$  by adding a pendent edge at the vertex  $v$ . Let  $H-v$  be the  $k$ -uniform hypergraph obtained from  $H$  by removing  $v$  and all edges incident to  $v$ . Then*

$$\phi_{H_v}(\lambda) = \lambda^{(k-1)^{n+k-1}} \phi_{H-v}(\lambda)^{(k-1)^k} M_H(\lambda, 0)^{(k-1)^{k-1} - k^{k-2}} M_H(\lambda, \frac{1}{\lambda^{k-1}})^{k^{k-2}},$$

where  $M_H(\lambda, 0) = \frac{\phi_H(\lambda)}{\phi_{H-v}(\lambda)^{k-1}}$ ,  $M_H(\lambda, \frac{1}{\lambda^{k-1}})$  is the  $\frac{1}{\lambda^{k-1}}$ -translational fraction of  $M_H(\lambda, 0)$ .

*Proof.* Without loss of generality, let the vertex  $v = 0$ . Let  $V(H) = \{0, 1, \dots, n-1\}$  and  $V(H_0) = \{0, 1, \dots, n, n+1, \dots, n+k-2\}$ . Then the pendent edge of  $H_0$  is  $e_0 = \{0, n, n+1, \dots, n+k-2\}$ . Let  $\phi_1 = \text{Res} \left( \frac{F^H}{F^{e_0}} \right)$  and

$\phi_2 = \prod_{p \in \mathcal{V}^{H_0}} (\lambda - \sum_{e \in E_0(H)} p^{e \setminus \{0\}} - p^{e_0 \setminus \{0\}})^{m_{H_0}(p)}$ . It follows from Eq.(1) that

$$\begin{aligned}
\phi_{H_0}(\lambda) &= \text{Res}(\overline{F^{H_0}})^{k-1} \prod_{p \in \mathcal{V}^{H_0}} (f_0^{H_0}(p))^{m_{H_0}(p)} \\
&= \text{Res}(\overline{F^{H_0}})^{k-1} \prod_{p \in \mathcal{V}^{H_0}} (\lambda - \sum_{e \in E_0(H_0)} p^{e \setminus \{0\}})^{m_{H_0}(p)} \\
&= \text{Res} \left( \frac{\overline{F^H}}{\overline{F^{e_0}}} \right)^{k-1} \prod_{p \in \mathcal{V}^{H_0}} (\lambda - \sum_{e \in E_0(H)} p^{e \setminus \{0\}} - p^{e_0 \setminus \{0\}})^{m_{H_0}(p)} \\
&= \phi_1^{k-1} \phi_2.
\end{aligned} \tag{10}$$

Since  $\overline{F_i^{e_0}} = F_i^{H_0}|_{x_0=0} = \lambda x_i^{k-1}$  for  $i \in e_0 \setminus \{0\}$ , from Lemma 2.3 (1), we have

$$\phi_1 = \text{Res} \left( \frac{\overline{F^H}}{\overline{F^{e_0}}} \right) = \text{Res} \begin{pmatrix} \overline{F^H} \\ \lambda x_n^{k-1} \\ \vdots \\ \lambda x_{n+k-3}^{k-1} \\ \lambda x_{n+k-2}^{k-1} \end{pmatrix} = \lambda^{(k-1)n+k-3} \text{Res} \begin{pmatrix} \overline{F^H} \\ \lambda x_n^{k-1} \\ \vdots \\ \lambda x_{n+k-3}^{k-1} \\ x_{n+k-2}^{k-1} \end{pmatrix}.$$

Since  $\overline{F^H}|_{x_{n+k-2}=0} = \overline{F^H}$ , it follows from that Lemma 2.3 (2) that  $\text{Res} \begin{pmatrix} \overline{F^H} \\ \lambda x_n^{k-1} \\ \vdots \\ \lambda x_{n+k-3}^{k-1} \\ x_{n+k-2}^{k-1} \end{pmatrix} =$

$\text{Res} \begin{pmatrix} \overline{F^H} \\ \lambda x_n^{k-1} \\ \vdots \\ \lambda x_{n+k-3}^{k-1} \end{pmatrix}^{k-1}$ . Then  $\phi_1 = \lambda^{(k-1)n+k-3} \text{Res} \begin{pmatrix} \overline{F^H} \\ \lambda x_n^{k-1} \\ \vdots \\ \lambda x_{n+k-3}^{k-1} \end{pmatrix}^{k-1}$ . By repeating

the above process, we obtain  $\phi_1 = \lambda^{(k-1)n+k-2} \text{Res}(\overline{F^H})^{(k-1)^{k-1}}$ .

From  $\overline{F_i^H} = F_i^{H_0}|_{x_0=0} = \lambda x_i^{k-1} - \sum_{\substack{e \in E_i(H) \\ 0 \notin e}} x^{e \setminus \{i\}} = \lambda x_i^{k-1} - \sum_{e \in E_i(H-0)} x^{e \setminus \{i\}}$  for

$i \in V(H-0)$ , it yields that  $\text{Res}(\overline{F^H}) = \phi_{H-0}(\lambda)$ . We obtain

$$\phi_1 = \text{Res} \left( \frac{\overline{F^H}}{\overline{F^{e_0}}} \right) = \lambda^{(k-1)n+k-2} \phi_{H-0}(\lambda)^{(k-1)^{k-1}}.$$

Note that  $\mathcal{V}^{H_0} = \mathcal{V}^H \times \mathcal{V}^{e_0}$ . For  $p \in \mathcal{V}^{H_0}$ , we have vector  $p = \begin{pmatrix} q \\ r \end{pmatrix}$ , where



$q \in \mathcal{V}^H$ ,  $r \in \mathcal{V}^{e_0}$ . Let  $r = (r_n, r_{n+1}, \dots, r_{n+k-2})^T \in \mathcal{V}^{e_0}$ . From Eq. (5), we know that

$$r^{e_0 \setminus \{0\}} = r_n \cdots r_{n+k-2} = \begin{cases} 0, & r = 0, \\ \frac{1}{\lambda^{k-1}}, & r \neq 0. \end{cases}$$

By Lemma 3.1, we have  $m_{e_0}(0) = (k-1)^{k-1} - k^{k-2}$  for  $0 \in \mathcal{V}^{e_0}$  and  $\sum_{0 \neq r \in \mathcal{V}^{e_0}} m_{e_0}(r) = k^{k-2}$ . Hence

$$\begin{aligned} \phi_2 &= \prod_{p \in \mathcal{V}^{H_0}} (\lambda - \sum_{e \in E_0(H)} p^{e \setminus \{0\}} - p^{e_0 \setminus \{0\}})^{m_{H_0}(p)} = \prod_{\substack{q \in \mathcal{V}^H \\ r \in \mathcal{V}^{e_0}}} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}} - r^{e_0 \setminus \{0\}})^{m_H(q)m_{e_0}(r)} \\ &= \prod_{\substack{q \in \mathcal{V}^H \\ 0 \neq r \in \mathcal{V}^{e_0}}} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}} - r^{e_0 \setminus \{0\}})^{m_H(q)m_{e_0}(r)} \prod_{\substack{q \in \mathcal{V}^H \\ 0 \neq r \in \mathcal{V}^{e_0}}} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}} - r^{e_0 \setminus \{0\}})^{m_H(q)m_{e_0}(r)} \\ &= \prod_{q \in \mathcal{V}^H} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}})^{m_H(q)((k-1)^{k-1} - k^{k-2})} \prod_{q \in \mathcal{V}^H} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}} - \frac{1}{\lambda^{k-1}})^{m_H(q)k^{k-2}}. \end{aligned}$$

By Eq. (9), we have

$$M_H(\lambda, 0) = \frac{\phi_H(\lambda)}{\phi_{H-0}(\lambda)^{k-1}} = \prod_{q \in \mathcal{V}^H} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}})^{m_H(q)}$$

and

$$M_H(\lambda, \frac{1}{\lambda^{k-1}}) = \prod_{q \in \mathcal{V}^H} (\lambda - \sum_{e \in E_0(H)} q^{e \setminus \{0\}} - \frac{1}{\lambda^{k-1}})^{m_H(q)}.$$

Then

$$\phi_2 = \prod_{p \in \mathcal{V}^{H_0}} (\lambda - \sum_{e \in E_0(H)} p^{e \setminus \{0\}} - p^{e_0 \setminus \{0\}})^{m_{H_0}(p)} = M_H(\lambda, 0)^{(k-1)^{k-1} - k^{k-2}} M_H(\lambda, \frac{1}{\lambda^{k-1}})^{k^{k-2}}.$$

Substituting  $\phi_1$  and  $\phi_2$  into Eq.(10), the proof is completed.  $\square$

In Theorem 3.2, when  $k = 2$ ,  $H$  is a graph with  $n$  vertices.  $\phi_H(\lambda)$  and  $\phi_{H-v}(\lambda)$  are polynomials of degree  $n$  and  $n - 1$ , respectively. It follows from Eq. (9) that

$$M_H(\lambda, 0) = \frac{\phi_H(\lambda)}{\phi_{H-v}(\lambda)} = \lambda + \frac{\phi_H(\lambda) - \lambda\phi_{H-v}(\lambda)}{\phi_{H-v}(\lambda)},$$

$$M_H(\lambda, \frac{1}{\lambda}) = \lambda + \frac{\phi_H(\lambda) - \lambda\phi_{H-v}(\lambda)}{\phi_{H-v}(\lambda)} - \frac{1}{\lambda} = \frac{\phi_H(\lambda)}{\phi_{H-v}(\lambda)} - \frac{1}{\lambda}.$$

Then

$$\begin{aligned}\phi_{H_v}(\lambda) &= \lambda\phi_{H-v}(\lambda) \left( \frac{\phi_H(\lambda)}{\phi_{H-v}(\lambda)} - \frac{1}{\lambda} \right) \\ &= \lambda\phi_H(\lambda) - \phi_{H-v}(\lambda).\end{aligned}$$

Hence, the Theorem 3.2 is the Theorem 1.1 when  $k = 2$ .

Let  $H_v^s$  denote the  $k$ -uniform hypergraph obtained from hypergraph  $H$  by adding  $s$  pendent edges at the vertex  $v$ . By Theorem 3.2, we give a reduced formula of the characteristic polynomial of  $H_v^s$ .

**Theorem 3.3.** *Let  $H$  be a  $k$ -uniform hypergraph with  $n$  vertices.  $H_v^s$  denotes the  $k$ -uniform hypergraph obtained from  $H$  by adding  $s$  pendent edges at the vertex  $v$ . Let  $H - v$  be the  $k$ -uniform hypergraph obtained from  $H$  by removing  $v$  together with all edges incident to  $v$ . Then*

$$\phi_{H_v^s}(\lambda) = \lambda^{s(k-1)n+s(k-1)} \phi_{H-v}(\lambda)^{(k-1)^{s(k-1)+1}} \prod_{t=0}^s \left( M_H(\lambda, \frac{t}{\lambda^{k-1}}) \right)^{C_s^t K_1^{s-t} K_2^t},$$

where  $M_H(\lambda, \frac{t}{\lambda^{k-1}})$  is  $\frac{t}{\lambda^{k-1}}$ -translational fraction of  $M_H(\lambda, 0) = \frac{\phi_H(\lambda)}{\phi_{H-v}(\lambda)^{k-1}}$ ,  $K_1 = (k-1)^{k-1} - k^{k-2}$ ,  $K_2 = k^{k-2}$  and  $C_s^t$  is a combinatorial number.

*Proof.* From Theorem 3.2, we have

$$\begin{aligned}\phi_{H_v}(\lambda) &= \lambda^{(k-1)n+k-1} \phi_{H-v}(\lambda)^{(k-1)^k} M_H(\lambda, 0)^{(k-1)^{k-1}-k^{k-2}} M_H(\lambda, \frac{1}{\lambda^{k-1}})^{k^{k-2}} \\ &= (\phi_{H-v}(\lambda))^{k-1} M_H(\lambda, 0)^{K_1} M_H(\lambda, \frac{1}{\lambda^{k-1}})^{K_2}.\end{aligned}$$

Then

$$M_{H_v}(\lambda, 0) = \frac{\phi_{H_v}(\lambda)}{(\phi_{H-v}(\lambda))^{k-1}} = M_H(\lambda, 0)^{K_1} M_H(\lambda, \frac{1}{\lambda^{k-1}})^{K_2}, \quad (11)$$

and  $M_{H_v}(\lambda, \frac{1}{\lambda^{k-1}}) = M_H(\lambda, \frac{1}{\lambda^{k-1}})^{K_1} M_H(\lambda, \frac{2}{\lambda^{k-1}})^{K_2}$ . From Eq.(11), we get

$$\begin{aligned}M_{H_v^2}(\lambda, 0) &= M_{H_v}(\lambda, 0)^{K_1} M_{H_v}(\lambda, \frac{1}{\lambda^{k-1}})^{K_2} \\ &= \prod_{t=0}^2 M_H(\lambda, \frac{t}{\lambda^{k-1}})^{C_2^t K_1^{2-t} K_2^t}.\end{aligned}$$

By induction, we obtain

$$M_{H_v^s}(\lambda) = \frac{\phi_{H_v^s}(\lambda)}{\phi_{H_v^s-v}(\lambda)^{k-1}} = \prod_{t=0}^s M_H(\lambda, \frac{t}{\lambda^{k-1}})^{C_s^t K_1^{s-t} K_2^t}$$

for  $s \geq 1$ . Then

$$\begin{aligned} \phi_{H_v^s}(\lambda) &= \phi_{H_v^s-v}(\lambda)^{k-1} \prod_{t=0}^s M_H(\lambda, \frac{t}{\lambda^{k-1}})^{C_s^t K_1^{s-t} K_2^t} \\ &= \lambda^{s(k-1)n+s(k-1)} \phi_{H-v}(\lambda)^{(k-1)^{s(k-1)+1}} \prod_{t=0}^s M_H(\lambda, \frac{t}{\lambda^{k-1}})^{C_s^t K_1^{s-t} K_2^t}. \end{aligned}$$

□

We use Theorem 3.2 to get the characteristic polynomial and all distinct eigenvalues of  $P_m^{(k)}$ . And we express the characteristic polynomial of  $P_m^{(k)}$  by the characteristic polynomial of path. For convenience, we prove it by induction.

**Theorem 3.4.** *The characteristic polynomial of the  $k$ -uniform hyperpath  $P_m^{(k)}$  of length  $m$  is*

$$\phi_{P_m^{(k)}}(\lambda) = \prod_{j=0}^m \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j,m)},$$

where  $\phi_{P_j}(\lambda) = \sum_{t=0}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^t C_{j+1-t}^t \lambda^{j+1-2t}$  is the characteristic polynomial of the path  $P_j$ ,

$$a(j,m) = \begin{cases} K_2^m, & j = m, \\ ((m-j+1)K_1 + 2K_2) K_1 K_2^j (k-1)^{(m-j-2)(k-1)}, & 1 \leq j \leq m-1, \\ \frac{2}{k} [m(k-1) + 1] (k-1)^{m(k-1)} - \sum_{r=1}^m (r+1)a(r,m), & j = 0, \end{cases}$$

$K_1 = (k-1)^{k-1} - k^{k-2}$  and  $K_2 = k^{k-2}$ .

*Proof.* By the reduced formula in Theorem 3.2 and induction, we give this proof. When  $m = 1$ , it follows from Lemma 2.1 that

$$\begin{aligned} \phi_{P_1^{(k)}}(\lambda) &= \lambda^{k(k-1)^{k-1} - k^{k-1}} (\lambda^k - 1)^{k^{k-2}} \\ &= (\lambda^k)^{K_1} (\lambda^k - 1)^{K_2} \\ &= \prod_{j=0}^1 \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j,1)}. \end{aligned}$$

Let  $V(P_1^{(k)}) = \{0, 1, \dots, k-1\}$ . Then

$$\phi_{\mathcal{P}_1^{(k)}-0}(\lambda) = \lambda^{(k-1)^{k-1}} = \lambda^{K_1+K_2}.$$

So

$$M_{\mathcal{P}_1^{(k)}}(\lambda, 0) = \frac{\phi_{\mathcal{P}_1^{(k)}}(\lambda)}{\phi_{\mathcal{P}_1^{(k)}-0}(\lambda)^{k-1}} = \lambda^{K_1} \left( \lambda - \frac{1}{\lambda^{k-1}} \right)^{K_2}$$

and

$$M_{\mathcal{P}_1^{(k)}}\left(\lambda, \frac{1}{\lambda^{k-1}}\right) = \left( \lambda - \frac{1}{\lambda^{k-1}} \right)^{K_1} \left( \lambda - \frac{2}{\lambda^{k-1}} \right)^{K_2}.$$

Then from the reduced formula in Theorem 3.2, we get

$$\begin{aligned} \phi_{\mathcal{P}_2^{(k)}}(\lambda) &= \lambda^{(2(k-1)+1)(K_1+K_2)^2-2K_1K_2k-K_2^2k} (\lambda^k - 1)^{2K_1K_2} (\lambda^k - 2)^{K_2^2} \\ &= \prod_{j=0}^2 \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j;2)}. \end{aligned}$$

Assume that for  $m_0 \geq 2$ ,

$$\phi_{\mathcal{P}_{m_0}^{(k)}}(\lambda) = \prod_{j=0}^{m_0} \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j,m_0)}$$

and

$$\phi_{\mathcal{P}_{m_0-1}^{(k)}}(\lambda) = \prod_{j=0}^{m_0-1} \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j,m_0-1)}.$$

Let  $v$  be the pendent vertex in  $P_{m_0}^{(k)}$ . Then

$$\begin{aligned} \phi_{\mathcal{P}_{m_0}^{(k)}-v}(\lambda) &= \lambda^{(k-2)(k-1)^{m_0(k-1)-1}} \left( \phi_{\mathcal{P}_{m_0-1}^{(k)}}(\lambda) \right)^{(k-1)^{k-2}} \\ &= \lambda^{(k-2)(k-1)^{m_0(k-1)-1}} \left( \prod_{j=0}^{m_0-1} \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j,m_0-1)} \right)^{(k-1)^{k-2}}. \end{aligned}$$

So

$$\begin{aligned}
M_{P_{m_0}^{(k)}}(\lambda, 0) &= \frac{\phi_{P_{m_0}^{(k)}}(\lambda)}{\phi_{P_{m_0}^{(k)}-v}(\lambda)^{k-1}}, \\
&= \lambda^{K_1(K_1+K_2)^{m_0-1}} \prod_{j=1}^{m_0-1} \left( \lambda^{\frac{2-k}{2}} \frac{\phi_{P_j}(\lambda^{\frac{k}{2}})}{\phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} \right)^{(K_1+K_2)^{m_0-1-j} K_2^j K_1} \left( \lambda^{\frac{2-k}{2}} \frac{\phi_{P_{m_0}}(\lambda^{\frac{k}{2}})}{\phi_{P_{m_0-1}}(\lambda^{\frac{k}{2}})} \right)^{K_2^{m_0}}.
\end{aligned} \tag{12}$$

By Theorem 1.2, we have  $\phi_{P_0}(\lambda) = \lambda$ ,  $\phi_{P_1}(\lambda) = \lambda^2 - 1$ . By Theorem 1.1, we have

$$\phi_{P_j}(\lambda) = \lambda \phi_{P_{j-1}}(\lambda) - \phi_{P_{j-2}}(\lambda). \tag{13}$$

It follows from Eq.(13) that  $\phi_{P_{-1}}(\lambda) = 1$  when  $j = 1$ . Replace  $\lambda$  with  $\lambda^{\frac{k}{2}}$  in Eq. (13), we get  $\phi_{P_j}(\lambda^{\frac{k}{2}}) = \lambda^{\frac{k}{2}} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}}) - \phi_{P_{j-2}}(\lambda^{\frac{k}{2}})$ . Then

$$\frac{\phi_{P_j}(\lambda^{\frac{k}{2}})}{\phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} = \lambda^{\frac{k}{2}} - \frac{\phi_{P_{j-2}}(\lambda^{\frac{k}{2}})}{\phi_{P_{j-1}}(\lambda^{\frac{k}{2}})}.$$

Therefore,

$$\lambda^{\frac{2-k}{2}} \frac{\phi_{P_j}(\lambda^{\frac{k}{2}})}{\phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} = \lambda - \frac{\phi_{P_{j-2}}(\lambda^{\frac{k}{2}})}{\lambda^{\frac{k-2}{2}} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})}. \tag{14}$$

From Eq. (12) and Eq. (14), it yields that

$$\begin{aligned}
M_{P_{m_0}^{(k)}}(\lambda, 0) &= \\
&\lambda^{K_1(K_1+K_2)^{m_0-1}} \prod_{j=1}^{m_0-1} \left( \lambda - \frac{\phi_{P_{j-2}}(\lambda^{\frac{k}{2}})}{\lambda^{\frac{k-2}{2}} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} \right)^{(K_1+K_2)^{m_0-1-j} K_1 K_2^j} \left( \lambda - \frac{\phi_{P_{m_0-2}}(\lambda^{\frac{k}{2}})}{\lambda^{\frac{k-2}{2}} \phi_{P_{m_0-1}}(\lambda^{\frac{k}{2}})} \right)^{K_2^{m_0}}.
\end{aligned}$$

Next, we give the representation of  $M_{P_{m_0}^{(k)}}\left(\lambda, \frac{1}{\lambda^{k-1}}\right)$ . Since

$$\begin{aligned} \lambda - \frac{\phi_{P_{j-2}}(\lambda^{\frac{k}{2}})}{\lambda^{\frac{k-2}{2}} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} - \frac{1}{\lambda^{k-1}} &= \lambda^{\frac{2-k}{2}} \frac{\phi_{P_j}(\lambda^{\frac{k}{2}})}{\phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} - \frac{1}{\lambda^{k-1}} \\ &= \frac{\lambda^{\frac{k}{2}} \phi_{P_j}(\lambda^{\frac{k}{2}}) - \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})}{\lambda^{k-1} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})} \\ &= \frac{\phi_{P_{j+1}}(\lambda^{\frac{k}{2}})}{\lambda^{k-1} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})}, \end{aligned}$$

we have

$$\begin{aligned} M_{P_{m_0}^{(k)}}\left(\lambda, \frac{1}{\lambda^{k-1}}\right) &= \\ \left(\frac{\lambda^k - 1}{\lambda^{k-1}}\right)^{K_1(K_1+K_2)^{m_0-1}} \prod_{j=1}^{m_0-1} \left(\frac{\phi_{P_{j+1}}(\lambda^{\frac{k}{2}})}{\lambda^{k-1} \phi_{P_{j-1}}(\lambda^{\frac{k}{2}})}\right)^{(K_1+K_2)^{m_0-1-j} K_1 K_2^j} &\left(\frac{\phi_{P_{m_0+1}}(\lambda^{\frac{k}{2}})}{\lambda^{k-1} \phi_{P_{m_0-1}}(\lambda^{\frac{k}{2}})}\right)^{K_2^{m_0}}. \end{aligned}$$

Then from Theorem 3.2, we obtain

$$\begin{aligned} \phi_{P_{m_0+1}^{(k)}}(\lambda) &= \lambda^{(k-1)(m_0+1)(k-1)+1} \left(\phi_{P_{m_0-v}^{(k)}}(\lambda)\right)^{(k-1)^k} M_{P_{m_0}^{(k)}}(\lambda)^{K_1} M_{P_{m_0}^{(k)}}\left(\lambda, \frac{1}{\lambda^{k-1}}\right)^{K_2} \\ &= \prod_{j=0}^{m_0+1} \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j, m_0+1)}. \end{aligned}$$

By induction, we get

$$\phi_{P_m^{(k)}}(\lambda) = \prod_{j=0}^m \phi_{P_j}(\lambda^{\frac{k}{2}})^{a(j, m)}.$$

□

From Theorem 1.3, we know that all the distinct eigenvalues of a path  $P_m$  are  $2 \cos \frac{\pi t}{m+2}$ ,  $t = 1, 2, \dots, m+1$  i.e.  $\phi_{P_m}(\lambda) = \prod_{t=1}^{m+1} \left(\lambda - 2 \cos \frac{\pi}{m+2} t\right)$ . Then  $\phi_{P_m}(\lambda^{\frac{k}{2}}) = \prod_{t=1}^{m+1} \left(\lambda^{\frac{k}{2}} - 2 \cos \frac{\pi}{m+2} t\right)$ . From Theorem 3.4, we directly get the following result.

**Theorem 3.5.** *The distinct eigenvalues of the  $k$ -uniform hyperpath  $P_m^{(k)}$  are the different numbers of  $\left(2 \cos \frac{\pi}{j+2} t\right)^{\frac{2}{k}} e^{i \frac{2\pi}{k} \theta}$  for all  $j \in [m]$ ,  $t \in [j+1]$  and  $\theta \in [k]$ , where  $i^2 = -1$ .*

Let  $\rho(P_m^{(k)})$  be the spectral radius of  $P_m^{(k)}$ . In 2016, Lu and Man proved that  $\lim_{m \rightarrow \infty} \rho(P_m^{(k)}) = \sqrt[k]{4}$  (see [14]). From Theorem 3.5, we know that  $\rho(P_m^{(k)}) = (2 \cos \frac{\pi}{m+2})^{\frac{2}{k}}$ .

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## References

## References

- [1] D.A. Cox, J. Little and D. O’Shea. Using Algebraic Geometry, 185 of Graduate Texts in Mathematics. Springer, New York, second edition, 2005.
- [2] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436(2012)3268-3292.
- [3] J. Cooper and A. Dutle. Computing hypermatrix spectra with the poisson product formula. *Linear Multilinear Algebra*, 63(2015)956-970.
- [4] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky. Discriminants, Resultants and Multidimensional Determinants. Birkhauser, Boston, 1994.
- [5] F. Harary, C. King, A. Mowshowitz and R. C. Read. Cospectral graphs and digraphs. *Bull. Lond. Math. Soc.*, 3(1971)321-328.
- [6] F. Harary. The determinant of the adjacency matrix of a graph. *SIAM Rev.*, 4(1962)202-210.
- [7] J.P. Jouanolou. Le formalisme du resultant. *Adv. Math.*, 90(1991)117-263.
- [8] L.H. Lim. Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, December 13-15, (2005)129-132.
- [9] L. Lovász and J. Pelikán. On the eigenvalues of trees. *Period. Math. Hungar.*, 3(1973)175-182.
- [10] D. Cvetković, M. Doob and H. Sachs. Spectral of Graphs. New York : Academic Press, 1980.
- [11] L. Qi. Eigenvalues of a real supersymmetric tensor. *J. Symbolic Comput.*, 40(2005)1302-1324.
- [12] H. Sachs. Beziehungen zwischen den in einem graphen enthaltenen kreisen und seinem charakteristischen polynom. *Publi. Math.*, 11(1964)119-134.
- [13] J.Y. Shao, L. Qi and S. Hu. Some new trace formulas of tensors with applications in spectral hypergraph theory. *Linear Multilinear Algebra*, 63(2015)971-992.
- [14] L. Lu and S. Man. Connected hypergraphs with small spectral radius. *Linear Algebra Appl.*, 509(2016)206-227.
- [15] Y. Bao, Y. Fan, Y. Wang and M. Zhu. A combinatorial method for computing characteristic polynomials of starlike hypergraphs. *J. Algebraic Combin.*, Doi: 10.1007/s10801-019-00886-7.