

A Hopf algebra on subgraphs of a graph

Xiaomeng Wang[†], Shou-Jun Xu, Xing Gao

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou, Gansu 730000, China

Abstract

In this paper, we construct a bialgebraic and further a Hopf algebraic structure on top of subgraphs of a given graph. Further, we give the dual structure of this Hopf algebraic structure. We study the algebra morphisms induced by graph homomorphisms, and obtain a covariant functor from a graph category to an algebra category.

Keywords: Hopf algebra, covariant functor, graph category, algebra category

1 Introduction

The Hopf algebraic theory has a long history and broad applications in mathematics and physics [2, 8, 14, 22, 24, 27, 28, 30, 33]. The Hopf algebra of rooted forests originated from the work of A. Dür [5] and had a very rich theory. For example, A. Connes and D. Kreimer introduced the Connes-Kreimer Hopf algebra of rooted forests in [2, 23] to study the renormalization of perturbative quantum field theory. In [2, 3, 4], they introduced commutative Hopf algebras of graphs to study the combinatorial structure of renormalization. It is also related to the Loday-Ronco Hopf algebra [26, 32] and Grossman-Larson Hopf algebra [13] of rooted trees. L. Foissy has done a lot of work on the Hopf algebra of rooted forests [8, 9, 10, 11]. As related results, various infinitesimal bialgebras on decorated rooted forests have been established in [16, 17, 31], via different Hochschild 1-cocycle conditions.

A totally assigned graph is a graph with a total order on its edges. In [6], G. E. Duchamp et al. built a Hopf algebra of totally assigned graphs. L. Foissy [12] constructed a Hopf algebra on all graphs to insert chromatic polynomial into the theory of combinatorial Hopf algebra, and gave a new proof of some classical results.

The concept of graph (incidence) Hopf algebras was introduced by Schmitt [29]. Namely, a graph algebra is a commutative, cocommutative, graded, connected Hopf algebra, whose basis elements correspond to finite graphs. Humpert and Martin [20] obtained a new nonrecursive formula for the antipode of a graph (incidence) Hopf algebra. In [25], the author introduced the 4-bialgebra of graphs that satisfies some relations.

The study of Hopf algebras of graphs is extensive, but the research on Hopf algebras of subgraphs has no result yet. Subgraph plays an important role in the study of graph theory. For example, we can study the perfect graph by the independence number and clique number of the induced subgraph. We can determine if a graph is an interval map by the induced subgraph and many more.

In the present paper, we construct a Hopf algebraic structure on subgraphs of a given graph. In particular, a combinatorial description of the coproduct is given. Narrowing ourself to the algebraic part of

E-mail address: wangxm2015@lzu.edu.cn

the aforementioned Hopf algebra on subgraphs of a given graph, we obtain an algebra morphism from a graph homomorphism. Using the language of categories, we obtain a covariant functor from the category of graphs to the category of algebras.

Here is the structure of the paper. In Section 2, we first review the definitions of free monoid and free module. Then we proceed to give an algebraic structure on subgraphs of a graph. Next, we construct a coalgebraic structure on subgraphs of a graph (Lemma 2.13), thereby we construct a bialgebra on the subgraph (Theorem 2.17). Finally, to make the coproduct Δ_G more explicit, we describe it in a combinatorial way (Eq. (5)).

In Section 3, continuing the line in Section 2, a Hopf algebra structure on subgraphs of a graph is given (Theorem 3.5). Further, we study the dual Hopf algebra structure. We conclude with a description of the dual Hopf algebra (Theorem 3.7). This section is also devoted to an algebra morphism induced by a graph homomorphism (Theorem 3.12). As an application, we obtain a functor from graph category to algebra category (Theorem 3.15).

Notation. Graphs considered in this paper are connected and undirected graphs without multiple edges and loops. We will be working over a unitary commutative base ring \mathbf{k} . Linear maps and tensor products are taken over \mathbf{k} . For an algebra A , we view $A \otimes A$ as an A -bimodule via

$$a \cdot (b \otimes c) := ab \otimes c \text{ and } (b \otimes c) \cdot a := b \otimes ca. \quad (1)$$

2 A bialgebra on subgraphs of a given graph

In this section, we build a bialgebraic structure on top of subgraphs of a given graph.

2.1 An algebraic structure on subgraphs of a given graph

Let us first review some notations on graphs and algebras, which are used throughout the remainder of the paper. For a graph G , denote by $V(G)$ and $E(G)$ the vertex set and edge set of G , respectively.

Definition 2.1. [7] Let G be a graph.

- (a) If its n vertices are distinguished from one another by labels such as $\{1, 2, \dots, n\}$, then G is said to be **labeled**.
- (b) If there is a path between any two vertices of G , then G is **connected**.
- (c) A **subgraph** of a graph G is simply a graph, all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$.
- (d) For $S \subseteq V(G)$, the **induced subgraph** $G[S]$ in G by S is the graph whose edge set consists of all of the edges in G that have both endpoints in S .

The concepts of free monoid and free module are needed later.

Definition 2.2. [18]

- (a) A **semigroup** is a nonempty set S together with a binary operation $\cdot : S \times S \rightarrow S$ which is associative:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \text{ for all } x, y, z \in S.$$

A semigroup S is called a **monoid** if it contains an element 1 with the property that

$$x \cdot 1 = 1 \cdot x = x \text{ for all } x \in S.$$

(b) Let $(S, \cdot_S, 1_S)$ and $(T, \cdot_T, 1_T)$ be monoids. A map $\phi : S \rightarrow T$ is called a **monoid morphism** if

$$\phi(x \cdot_S y) = \phi(x) \cdot_T \phi(y) \text{ and } \phi(1_S) = 1_T \text{ for all } x, y \in S.$$

(c) The **free monoid** on a set X is a monoid S together with a map $i : X \rightarrow S$ with the property that, for any monoid T and map $\phi : X \rightarrow T$, there exists a unique monoid morphism $\psi : S \rightarrow T$ such that $\psi \circ i = \phi$.

Definition 2.3. [21]

(a) Let \mathbf{k} be a ring. A (left) **k-module** is an additive abelian group M together with a function $\mathbf{k} \times M \rightarrow M$, $(k, x) \mapsto kx$ such that for all $k, l \in \mathbf{k}$ and $x, y \in M$:

- (i) $k(x + y) = kx + ky$;
- (ii) $(k + l)x = kx + lx$;
- (iii) $k(lx) = (kl)x$.

(b) Let M and N be modules over ring \mathbf{k} . A map $f : M \rightarrow N$ is a **k-module morphism** if

$$f(x + y) = f(x) + f(y) \text{ and } f(kx) = kf(x) \text{ for all } x, y \in M, k \in \mathbf{k}.$$

(c) The **free k-module** on a set X is a \mathbf{k} -module F together with a set map $i : X \rightarrow F$ with the property that, for any \mathbf{k} -module M and map $f : X \rightarrow M$, there exists a unique \mathbf{k} -module morphism $\bar{f} : F \rightarrow M$ such that $\bar{f} \circ i = f$.

A \mathbf{k} -module M is free if and only if it has a \mathbf{k} -linear basis [21].

In the rest of this paper, let G be a graph with vertices labelled by natural numbers and let \mathcal{G} be the set of nonempty connected subgraphs of G . Let $M(\mathcal{G}) = \langle \mathcal{G} \rangle$ be the free monoid on \mathcal{G} in which the multiplication is the concatenation, denoted by m_G and usually suppressed. The unit in $M(\mathcal{G})$ is the empty graph, denote by $\mathbb{1}$. An element F in $M(\mathcal{G})$ is a noncommutative product of connected subgraphs in \mathcal{G} :

$$F = \Gamma_1 \cdots \Gamma_n, \text{ where } n \geq 0 \text{ and } \Gamma_1, \cdots, \Gamma_n \in \mathcal{G}.$$

Here we employ the convention that $F = \mathbb{1}$ whenever $n = 0$. We define $\text{bre}(F) := n$ to be the **breadth** of F . For example,

$$\text{bre}(\mathbb{1}) = 0, \text{ bre}(\bullet_1) = 1, \text{ bre}(\mathbf{1}_1^2 \mathbf{1}_1^2 \bullet_1 \bullet_2) = 4.$$

Denote by

$$H(\mathcal{G}) := \mathbf{k}M(\mathcal{G})$$

the free \mathbf{k} -module spanned by $M(\mathcal{G})$. By linearity, the multiplication m_G can be extended to $H(\mathcal{G})$ and then the triple $(H(\mathcal{G}), m_G, \mathbb{1})$ is an algebra. Indeed, it is the free algebra on the set \mathcal{G} .

Example 2.4. Here are some examples of $H(\mathcal{G})$.

(a) If $G = \bullet_1$, then $\mathcal{G} = \{\bullet_1\}$, $M(\mathcal{G}) = \langle \bullet_1 \rangle$ and $H(\mathcal{G}) = \mathbf{k}\langle \bullet_1 \rangle$.

(b) If $G = \mathbf{1}_1^2$, then

$$\mathcal{G} = \{\bullet_1, \bullet_2, \mathbf{1}_1^2\}, M(\mathcal{G}) = \langle \bullet_1, \bullet_2, \mathbf{1}_1^2 \rangle \text{ and } H(\mathcal{G}) = \mathbf{k}\langle \bullet_1, \bullet_2, \mathbf{1}_1^2 \rangle.$$

(c) If $G = \mathbf{2}_2 \square_3^4$, then

$$\begin{aligned} \mathcal{G} &= \{\bullet_1, \bullet_2, \bullet_3, \bullet_4, \mathbf{1}_2^1, \mathbf{1}_3^4, \mathbf{1}_{1 \rightarrow 4}, \mathbf{2}_{\rightarrow 3}, \mathbf{2}_{\leftarrow 3}, \mathbf{2}_{\leftarrow 3}^4, \mathbf{2}_{\leftarrow 3}, \mathbf{1}_{\Gamma}^4, \mathbf{1}_{\Upsilon}^4, \mathbf{2}_{\mathbf{U}}^4, \mathbf{2}_{\mathbf{C}}^4, \mathbf{2}_{\mathbf{N}}^4, \mathbf{2}_{\mathbf{D}}^4, \mathbf{2}_{\mathbf{D}}^4\}, \\ M(\mathcal{G}) &= \langle \bullet_1, \bullet_2, \bullet_3, \bullet_4, \mathbf{1}_2^1, \mathbf{1}_3^4, \mathbf{1}_{1 \rightarrow 4}, \mathbf{2}_{\rightarrow 3}, \mathbf{2}_{\leftarrow 3}, \mathbf{2}_{\leftarrow 3}^4, \mathbf{2}_{\leftarrow 3}, \mathbf{1}_{\Gamma}^4, \mathbf{1}_{\Upsilon}^4, \mathbf{2}_{\mathbf{U}}^4, \mathbf{2}_{\mathbf{C}}^4, \mathbf{2}_{\mathbf{N}}^4, \mathbf{2}_{\mathbf{D}}^4, \mathbf{2}_{\mathbf{D}}^4 \rangle, \\ H(\mathcal{G}) &= \mathbf{k}\langle \bullet_1, \bullet_2, \bullet_3, \bullet_4, \mathbf{1}_2^1, \mathbf{1}_3^4, \mathbf{1}_{1 \rightarrow 4}, \mathbf{2}_{\rightarrow 3}, \mathbf{2}_{\leftarrow 3}, \mathbf{2}_{\leftarrow 3}^4, \mathbf{2}_{\leftarrow 3}, \mathbf{1}_{\Gamma}^4, \mathbf{1}_{\Upsilon}^4, \mathbf{2}_{\mathbf{U}}^4, \mathbf{2}_{\mathbf{C}}^4, \mathbf{2}_{\mathbf{N}}^4, \mathbf{2}_{\mathbf{D}}^4, \mathbf{2}_{\mathbf{D}}^4 \rangle. \end{aligned}$$

2.2 A coproduct on subgraphs of a given graph

In this subsection, we construct a coproduct on subgraphs of a graph. Let us first recall

Definition 2.5. [27, Definition 2.1] A **coalgebra** (C, Δ, ε) over \mathbf{k} is a \mathbf{k} -module C together with morphisms of \mathbf{k} -modules $\Delta := C \rightarrow C \otimes C$, called the **coproduct**, and $\varepsilon := C \rightarrow \mathbf{k}$, called the **counit**, such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \text{id}_C \otimes \Delta \\ C \otimes C & \xrightarrow{\Delta \otimes \text{id}_C} & C \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccccc} \mathbf{k} \otimes C & \xleftarrow{\varepsilon \otimes \text{id}_C} & C \otimes C & \xrightarrow{\text{id}_C \otimes \varepsilon} & C \otimes \mathbf{k} \\ & \searrow \beta_l & \uparrow \Delta & \swarrow \beta_r & \\ & & C & & \end{array}$$

are commutative. Here β_l and β_r are the isomorphisms defined by

$$\begin{aligned} \beta_l : C &\rightarrow \mathbf{k} \otimes C, \quad x \mapsto 1_{\mathbf{k}} \otimes x, \\ \beta_r : C &\rightarrow C \otimes \mathbf{k}, \quad x \mapsto x \otimes 1_{\mathbf{k}} \quad \text{for } x \in C. \end{aligned}$$

If a coalgebra (C, Δ, ε) satisfies $\tau \circ \Delta = \Delta$, where τ is a \mathbf{k} -linear map

$$\tau : C \otimes C \rightarrow C \otimes C, \quad x \otimes y \rightarrow y \otimes x \quad \text{for } x, y \in C,$$

then the coalgebra (C, Δ, ε) is said to be a **cocommutative coalgebra**.

Definition 2.6. [1, p. 54] Let (C, Δ, ε) be a coalgebra. A submodule D of a coalgebra C is called a **\mathbf{k} -subcoalgebra** of C if D is a coalgebra with the restriction of Δ and ε . Further, D is called a **simple \mathbf{k} -subcoalgebra** if it does not have any \mathbf{k} -subcoalgebras other than $\{0\}$ and D .

Now we define a coproduct Δ_G on $H(\mathcal{G})$. By linearity, we only need to define $\Delta_G(F)$ for basis elements $F \in M(\mathcal{G})$. If $\text{bre}(F) = 0$, then $F = \mathbb{1}$ and define

$$\Delta_G(F) := \Delta_G(\mathbb{1}) := \mathbb{1} \otimes \mathbb{1}. \quad (2)$$

If $\text{bre}(F) = 1$, then $F \in \mathcal{G}$ and we define

$$\Delta_G(F) := \sum_{V_1 \sqcup V_2 = V(F)} F[V_1] \otimes F[V_2], \quad (3)$$

where $F[V_i]$ are induced subgraphs of F by vertex sets V_i with $i = 1, 2$. Note

$$F[V_1], F[V_2] \in M(\mathcal{G}) \quad \text{and} \quad \Delta_G(F) \in H(\mathcal{G}) \otimes H(\mathcal{G}).$$

If $\text{bre}(F) > 1$, then $F = \Gamma_1 \cdots \Gamma_n$ for some $n \geq 2$ and $\Gamma_1, \dots, \Gamma_n \in \mathcal{G}$, and define

$$\Delta_G(F) := \Delta_G(\Gamma_1 \cdots \Gamma_n) := \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n). \quad (4)$$

Again note that

$$\Delta_G(F) \in H(\mathcal{G}) \otimes H(\mathcal{G}) \quad \text{by} \quad \Delta_G(\Gamma_i) \in H(\mathcal{G}) \otimes H(\mathcal{G}) \quad \text{for each } i = 1, \dots, n.$$

Let us give an example for better insight into Δ_G .

Example 2.7. Let $G = \mathbf{1}_1^2$. Then $\mathbf{1}_1^2$ and $\mathbf{1}_1^2 \mathbf{1}_1^2$ are in $M(\mathcal{G})$, and

$$\begin{aligned}\Delta_G(\mathbf{1}_1^2) &= \mathbb{1} \otimes \mathbf{1}_1^2 + \bullet_1 \otimes \bullet_2 + \bullet_2 \otimes \bullet_1 + \mathbf{1}_1^2 \otimes \mathbb{1}, \\ \Delta_G(\mathbf{1}_1^2 \mathbf{1}_1^2) &= \Delta_G(\mathbf{1}_1^2) \Delta_G(\mathbf{1}_1^2) \\ &= \mathbb{1} \otimes \mathbf{1}_1^2 \mathbf{1}_1^2 + \bullet_1 \otimes \mathbf{1}_1^2 \bullet_2 + \bullet_2 \otimes \mathbf{1}_1^2 \bullet_1 + \mathbf{1}_1^2 \otimes \mathbf{1}_1^2 \\ &\quad + \bullet_1 \otimes \bullet_2 \mathbf{1}_1^2 + \bullet_1 \bullet_1 \otimes \bullet_2 \bullet_2 + \bullet_1 \bullet_2 \otimes \bullet_2 \bullet_1 + \bullet_1 \mathbf{1}_1^2 \otimes \bullet_2 \\ &\quad + \bullet_2 \otimes \bullet_1 \mathbf{1}_1^2 + \bullet_2 \bullet_1 \otimes \bullet_1 \bullet_2 + \bullet_2 \bullet_2 \otimes \bullet_1 \bullet_1 + \bullet_2 \mathbf{1}_1^2 \otimes \bullet_1 \\ &\quad + \mathbf{1}_1^2 \otimes \mathbf{1}_1^2 + \mathbf{1}_1^2 \bullet_1 \otimes \bullet_2 + \mathbf{1}_1^2 \bullet_2 \otimes \bullet_1 + \mathbf{1}_1^2 \mathbf{1}_1^2 \otimes \mathbb{1}.\end{aligned}$$

2.3 A combinatorial description of the coproduct

This subsection is devoted to a combinatorial description of the coproduct Δ_G given in Subsection 2.2.

Let G be a graph and $F = \Gamma_1 \cdots \Gamma_n \in M(\mathcal{G})$ with $\Gamma_1, \dots, \Gamma_n \in \mathcal{G}$. Note that $V(F)$ is a multiset. For example, let $G = \frac{1}{2}\square_3^4$, $\Gamma = \mathbf{1}_1^2 \in \mathcal{G}$. Take $F := \Gamma\Gamma = \mathbf{1}_1^2 \mathbf{1}_1^2 \in M(\mathcal{G})$. Then $V(F) = \{1, 2, 1, 2\}$. Now we define

$$\Delta_G(F) := \sum_{U \uplus V = V(F)} F[U] \otimes F[V], \quad (5)$$

where $U \uplus V$ means that U and V can be expressed as

$$U = U_1 \cup \cdots \cup U_n \quad \text{and} \quad V = V_1 \cup \cdots \cup V_n,$$

with

$$U_i \sqcup V_i = V(\Gamma_i) \quad \text{for } i = 1, \dots, n.$$

Further, the coproduct Δ_G can be extended to $H(\mathcal{G}) = \mathbf{k}M(\mathcal{G})$ by linearity. Let us compute some examples for better understanding of Eq. (5).

Example 2.8. Let $G = \mathbf{1}_{\rightarrow 2} \mathbf{1}_3^4$ and $\Gamma_2 \Gamma_1 = \mathbf{1}_{3 \rightarrow 2}^4 \in M(\mathcal{G})$ with $\Gamma_1 = \mathbf{1}_{\rightarrow 2}$, $\Gamma_2 = \mathbf{1}_3^4$. Then

$$V(\Gamma_1) = \{1, 2\} \uplus \emptyset = \{1\} \uplus \{2\} = \{2\} \uplus \{1\} = \emptyset \uplus \{1, 2\},$$

$$V(\Gamma_2) = \{3, 4\} \uplus \emptyset = \{3\} \uplus \{4\} = \{4\} \uplus \{3\} = \emptyset \uplus \{3, 4\},$$

and so

$$\Delta_G(\Gamma_1) = \Delta_G(\mathbf{1}_{\rightarrow 2}) = \mathbf{1}_{\rightarrow 2} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{1}_{\rightarrow 2} + \bullet_1 \otimes \bullet_2 + \bullet_2 \otimes \bullet_1,$$

$$\Delta_G(\Gamma_2) = \Delta_G(\mathbf{1}_3^4) = \mathbf{1}_3^4 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{1}_3^4 + \bullet_3 \otimes \bullet_4 + \bullet_4 \otimes \bullet_3.$$

Further

$$\begin{aligned}\Delta_G(\mathbf{1}_3^4 \mathbf{1}_{\rightarrow 2}) &= \mathbf{1}_3^4 \mathbf{1}_{\rightarrow 2} \otimes \mathbb{1} + \mathbf{1}_3^4 \bullet_1 \otimes \bullet_2 + \mathbf{1}_3^4 \bullet_2 \otimes \bullet_1 + \mathbf{1}_3^4 \otimes \mathbf{1}_{\rightarrow 2} \\ &\quad + \bullet_3 \mathbf{1}_{\rightarrow 2} \otimes \bullet_4 + \bullet_3 \bullet_1 \otimes \bullet_4 \bullet_2 + \bullet_3 \bullet_2 \otimes \bullet_4 \bullet_1 + \bullet_3 \otimes \bullet_4 \mathbf{1}_{\rightarrow 2} \\ &\quad + \bullet_4 \mathbf{1}_{\rightarrow 2} \otimes \bullet_3 + \bullet_4 \bullet_1 \otimes \bullet_3 \bullet_2 + \bullet_4 \bullet_2 \otimes \bullet_3 \bullet_1 + \bullet_4 \otimes \bullet_3 \mathbf{1}_{\rightarrow 2} \\ &\quad + \mathbf{1}_{\rightarrow 2} \otimes \mathbf{1}_3^4 + \bullet_1 \otimes \mathbf{1}_3^4 \bullet_2 + \bullet_2 \otimes \mathbf{1}_3^4 \bullet_1 + \mathbb{1} \otimes \mathbf{1}_3^4 \mathbf{1}_{\rightarrow 2} \\ &= \Delta_G(\mathbf{1}_3^4) \Delta_G(\mathbf{1}_{\rightarrow 2}).\end{aligned}$$

Example 2.9. Consider $G = \mathbf{1}_3^4 \mathbf{1}_1^5$ and $F = \mathbf{1}_1^2 \mathbf{1}_1^5 \in M(\mathcal{G})$. Then

$$\begin{aligned}\Delta_G(\mathbf{1}_1^2) &= \mathbf{1}_1^2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{1}_1^2 + \cdot_1 \otimes \cdot_2 + \cdot_2 \otimes \cdot_1, \\ \Delta_G\left(\mathbf{1}_1^5\right) &= \mathbf{1}_1^5 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{1}_1^5 + \mathbf{1}_1^2 \otimes \cdot_5 + \mathbf{1}_2^5 \otimes \cdot_1 + \cdot_1 \otimes \mathbf{1}_2^5 + \cdot_5 \otimes \mathbf{1}_1^2 + \cdot_2 \otimes \cdot_1 \cdot_5 + \cdot_1 \cdot_5 \otimes \cdot_2,\end{aligned}$$

and

$$\begin{aligned}\Delta_G\left(\mathbf{1}_1^2 \mathbf{1}_1^5\right) &= \mathbf{1}_1^2 \mathbf{1}_1^5 \otimes \mathbb{1} + \mathbf{1}_1^5 \otimes \mathbf{1}_1^2 + \cdot_1 \mathbf{1}_1^5 \otimes \cdot_2 + \cdot_2 \mathbf{1}_1^5 \otimes \cdot_1 \\ &\quad + \mathbf{1}_1^2 \otimes \mathbf{1}_1^5 + \mathbb{1} \otimes \mathbf{1}_1^2 \mathbf{1}_1^5 + \cdot_1 \otimes \cdot_2 \mathbf{1}_1^5 + \cdot_2 \otimes \cdot_1 \mathbf{1}_1^5 \\ &\quad + \mathbf{1}_1^2 \mathbf{1}_1^2 \otimes \cdot_5 + \mathbf{1}_1^2 \otimes \mathbf{1}_1^2 \cdot_5 + \cdot_1 \mathbf{1}_1^2 \otimes \cdot_2 \cdot_5 + \cdot_2 \mathbf{1}_1^2 \otimes \cdot_1 \cdot_5 \\ &\quad + \mathbf{1}_1^2 \mathbf{1}_2^5 \otimes \cdot_1 + \mathbf{1}_2^5 \otimes \mathbf{1}_1^2 \cdot_1 + \cdot_1 \mathbf{1}_2^5 \otimes \cdot_2 \cdot_1 + \cdot_2 \mathbf{1}_2^5 \otimes \cdot_1 \cdot_1 \\ &\quad + \mathbf{1}_1^2 \cdot_1 \otimes \mathbf{1}_2^5 + \cdot_1 \otimes \mathbf{1}_1^2 \mathbf{1}_2^5 + \cdot_1 \cdot_1 \otimes \cdot_2 \mathbf{1}_2^5 + \cdot_2 \cdot_1 \otimes \cdot_1 \mathbf{1}_2^5 \\ &\quad + \mathbf{1}_1^2 \cdot_2 \otimes \cdot_1 \cdot_5 + \cdot_2 \otimes \mathbf{1}_1^2 \cdot_1 \cdot_5 + \cdot_1 \cdot_2 \otimes \cdot_2 \cdot_1 \cdot_5 + \cdot_2 \cdot_2 \otimes \cdot_1 \cdot_1 \cdot_5 \\ &\quad + \mathbf{1}_1^2 \cdot_5 \otimes \mathbf{1}_1^2 + \cdot_5 \otimes \mathbf{1}_1^2 \mathbf{1}_1^2 + \cdot_1 \cdot_5 \otimes \cdot_2 \mathbf{1}_1^2 + \cdot_2 \cdot_5 \otimes \cdot_1 \mathbf{1}_1^2 \\ &\quad + \mathbf{1}_1^2 \cdot_1 \cdot_5 \otimes \cdot_2 + \cdot_1 \cdot_5 \otimes \mathbf{1}_1^2 \cdot_2 + \cdot_1 \cdot_1 \cdot_5 \otimes \cdot_2 \cdot_2 + \cdot_2 \cdot_1 \cdot_5 \otimes \cdot_1 \cdot_2 \\ &= \Delta_G(\mathbf{1}_1^2) \Delta_G\left(\mathbf{1}_1^5\right).\end{aligned}$$

Now we prove that the combinatorial definition of Δ_G in Eq. (5) is the same as the one given in Subsection 2.2. For this, it suffices to show that Δ_G in Eq. (5) satisfies Eqs. (2)–(4). For the cases of Eqs. (2) and (3), it follows directly from the definition of Δ_G in Eq. (5). For the case of Eq. (4), we have

Lemma 2.10. Let Δ_G be given in Eq. (5) and $\Gamma_1, \dots, \Gamma_n \in \mathcal{G}$ with $n \geq 2$. Then

$$\Delta_G(\Gamma_1 \cdots \Gamma_n) = \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n).$$

Proof. We have

$$\begin{aligned}\Delta_G(\Gamma_1 \cdots \Gamma_n) &= \sum_{U \uplus V = V(\Gamma_1 \cdots \Gamma_n)} (\Gamma_1 \cdots \Gamma_n)[U] \otimes (\Gamma_1 \cdots \Gamma_n)[V] \quad (\text{by Eq. (5)}) \\ &= \sum_{\substack{U_i \sqcup V_i = V(\Gamma_i) \\ i=1, \dots, n}} (\Gamma_1 \cdots \Gamma_n)[U_1 \cup \cdots \cup U_n] \otimes (\Gamma_1 \cdots \Gamma_n)[V_1 \cup \cdots \cup V_n] \\ &= \sum_{\substack{U_i \sqcup V_i = V(\Gamma_i) \\ i=1, \dots, n}} \Gamma_1[U_1] \cdots \Gamma_n[U_n] \otimes \Gamma_1[V_1] \cdots \Gamma_n[V_n] \\ &= \left(\sum_{U_1 \sqcup V_1 = V(\Gamma_1)} \Gamma_1[U_1] \otimes \Gamma_1[V_1] \right) \cdots \left(\sum_{U_n \sqcup V_n = V(\Gamma_n)} \Gamma_n[U_n] \otimes \Gamma_n[V_n] \right) \\ &= \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n) \quad (\text{by Eq. (5)}).\end{aligned}$$

This completes the proof. □

So we conclude

Proposition 2.11. The Δ_G given in Eq. (5) coincides with the Δ_G given in Subsection 2.2.

Remark 2.12. The coproduct Δ_G in Eq. (5) is cocommutative.

2.4 A coalgebraic structure on subgraphs of a given graph

In this subsection, we obtain a coalgebra structure on subgraphs of a given graph G . Firstly, we define a linear map $\varepsilon_G : H(\mathcal{G}) \rightarrow \mathbf{k}$ by taking

$$\varepsilon_G(F) := \begin{cases} 1_{\mathbf{k}}, & \text{if } F = \mathbb{1}, \\ 0, & \text{if } F \neq \mathbb{1}. \end{cases} \quad (6)$$

Lemma 2.13. Let G be a graph. Then the triple $(H(\mathcal{G}), \Delta_G, \varepsilon_G)$ is a coalgebra.

Proof. We first show the coassociativity

$$(\text{id} \otimes \Delta_G)\Delta_G(F) = (\Delta_G \otimes \text{id})\Delta_G(F) \text{ for } F \in H(\mathcal{G}).$$

By linearity, it suffices to consider basis elements $F \in M(\mathcal{G})$. We have

$$\begin{aligned} (\text{id} \otimes \Delta_G)\Delta_G(F) &= (\text{id} \otimes \Delta_G) \left(\sum_{U \uplus V = V(F)} F[U] \otimes F[V] \right) \quad (\text{by Eq. (5)}) \\ &= \sum_{U \uplus V = V(F)} F[U] \otimes \Delta_G(F[V]) \\ &= \sum_{U \uplus V = V(F)} F[U] \otimes \left(\sum_{V' \uplus V'' = V(F[V])} (F[V][V'] \otimes (F[V][V''])) \right) \quad (\text{by Eq. (5)}) \\ &= \sum_{U \uplus V = V(F)} F[U] \otimes \left(\sum_{V' \uplus V'' = V(F[V])} F[V'] \otimes F[V''] \right) \\ &= \sum_{U \uplus V' \uplus V'' = V(F)} F[U] \otimes F[V'] \otimes F[V''] \quad (\text{by } V(F[V]) = V) \\ &= \sum_{U \uplus V' \uplus V'' = V(F)} (F[U \uplus V'])[U] \otimes (F[U \uplus V'])[V'] \otimes F[V''] \\ &= \sum_{W \uplus V'' = V(F)} \left(\sum_{U \uplus V' = W} F[W][U] \otimes F[W][V'] \right) \otimes F[V''] \\ &= \sum_{W \uplus V'' = V(F)} \left(\sum_{U \uplus V' = V(F[W])} F[W][U] \otimes F[W][V'] \right) \otimes F[V''] \quad (\text{by } V(F[W]) = W) \\ &= \sum_{W \uplus V'' = V(F)} \Delta_G(F[W]) \otimes F[V''] \quad (\text{by } U \uplus V' = W) \\ &= (\Delta_G \otimes \text{id}) \left(\sum_{W \uplus V'' = V(F)} F[W] \otimes F[V''] \right) \\ &= (\Delta_G \otimes \text{id})\Delta_G(F). \end{aligned}$$

Thus Δ_G is coassociative. Next we prove the counicity of ε_G . For $F \in M(\mathcal{G})$,

$$\begin{aligned}
(\varepsilon_G \otimes \text{id})\Delta_G(F) &= (\varepsilon_G \otimes \text{id}) \left(\sum_{U \uplus V = V(F)} F[U] \otimes F[V] \right) \quad (\text{by Eq. (5)}) \\
&= (\varepsilon_G \otimes \text{id}) \left(\mathbb{1} \otimes F + F \otimes \mathbb{1} + \sum_{\substack{U \uplus V = V(F) \\ U, V \neq V(F)}} F[U] \otimes F[V] \right) \\
&= \varepsilon_G(\mathbb{1}) \otimes F + \varepsilon_G(F) \otimes \mathbb{1} + \sum_{\substack{U \uplus V = V(F) \\ U, V \neq V(F)}} \varepsilon_G(F[U]) \otimes F[V] \\
&= \mathbf{1}_{\mathbf{k}} \otimes F = \beta_l(F) \quad (\text{by Eq. (6)}).
\end{aligned}$$

With the same argument, we can show $(\text{id} \otimes \varepsilon_G)\Delta_G = \beta_r$. Here β_l and β_r are the isomorphisms defined in Definition 2.5. This completes the proof. \square

Corollary 2.14. Let G be a graph and G' be a subgraph of graph G . Then the triple $(H(\mathcal{G}'), \Delta_G, \varepsilon_G)$ is a subcoalgebra of the coalgebra $(H(\mathcal{G}), \Delta_G, \varepsilon_G)$.

Proof. It follows from the Lemma 2.13 and Eqs. (5) and (6). \square

Corollary 2.15. Let G be a graph. Then $(H(\bullet), \Delta_G, \varepsilon_G)$ is a simple subcoalgebra, for any $\bullet \in V(G)$.

Proof. It follows directly from the Corollary 2.14 and Eq. (5). \square

2.5 A bialgebraic structure on subgraphs of a given graph

Combing the algebraic structure obtained in Subsection 2.1 and the coalgebraic structure obtained in Subsection 2.4, we build a bialgebraic structure on top of subgraphs of a given graph. Let us review the concept of bialgebras.

Definition 2.16. (a) [15, p. 49] Let (H, m_H, u_H) and (L, m_L, u_L) be two algebras. A map $f : H \rightarrow L$ is called an **algebra morphism** if

$$m_L \circ (f(x) \otimes f(y)) = f \circ m_H(x \otimes y) \quad \text{and} \quad u_L(k) = f \circ u_H(k) \quad \text{for all } x, y \in H, k \in \mathbf{k}.$$

(b) [15, p. 51] A **bialgebra** is a quintuple $(H, m, u, \Delta, \varepsilon)$, where (H, m, u) is an algebra and (H, Δ, ε) is a coalgebra such that $\Delta : H \rightarrow H \otimes H$ and $\varepsilon : H \rightarrow \mathbf{k}$ are morphisms of algebras.

Now, we arrive at our main result in this subsection.

Theorem 2.17. Let G be a graph. Then the quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a bialgebra.

Proof. In Subsection 2.1, we know that the triple $(H(\mathcal{G}), m_G, \mathbb{1})$ is an algebra. The triple $(H(\mathcal{G}), \Delta_G, \varepsilon_G)$ is a coalgebra by Lemma 2.13. Further, the coproduct Δ_G is an algebra morphism by Eq. (4). We are left to prove that ε_G is an algebra morphism. For this, let $F_1, F_2 \in M(\mathcal{G})$. Then by Eq. (6)

$$\varepsilon_G(F_1 F_2) = \begin{cases} \mathbf{1}_{\mathbf{k}}, & \text{if } F_1 \text{ and } F_2 \text{ are empty graphs,} \\ 0, & \text{others,} \end{cases}$$

and

$$\varepsilon_G(F_1) \varepsilon_G(F_2) = \begin{cases} \mathbf{1}_{\mathbf{k}}, & \text{if } F_1 \text{ and } F_2 \text{ are empty graphs,} \\ 0, & \text{others.} \end{cases}$$

Thus, $\varepsilon_G(F_1 F_2) = \varepsilon_G(F_1) \varepsilon_G(F_2)$ and ε_G is an algebra morphism. This completes the proof. \square

3 A Hopf algebra on subgraphs of a given graph

In this section, we construct a Hopf algebraic structure on subgraphs of a given graph G .

3.1 A Hopf algebraic structure on subgraphs of a graph

Let us review some concepts needed later.

Definition 3.1. [1, p. 61] Let (C, Δ, ε) be a coalgebra and (A, m, u) be an algebra. For $f, g \in \text{Hom}(C, A)$,

$$f * g := m \circ (f \otimes g) \circ \Delta$$

is said to be the **convolution** of f and g .

Definition 3.2. [1, p. 61] Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. Then the triple $(\text{Hom}(H, H), *, u \circ \varepsilon)$ is a \mathbf{k} -algebra. A linear endomorphism S of H is called an **antipode** for H if it is the inverse of id_H under the convolution product:

$$S * \text{id}_H = \text{id}_H * S = u \circ \varepsilon.$$

A bialgebra with an antipode is called a **Hopf algebra**.

Definition 3.3. [15, Definition 2.3.1] A bialgebra $(H, m, u, \Delta, \varepsilon)$ is called a **graded bialgebra** if there are \mathbf{k} -submodules $H^{(n)}, n \geq 0$, of H such that

$$(a) \quad H = \bigoplus_{n=0}^{\infty} H^{(n)};$$

$$(b) \quad H^{(p)}H^{(q)} \subseteq H^{(p+q)}, \quad p, q \geq 0;$$

$$(c) \quad \Delta(H^{(n)}) \subseteq \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}, \quad n \geq 0, \quad 0 \leq p, q \leq n.$$

Elements of $H^{(n)}$ are said to have degree n . The H is called **connected** if $H^{(0)} = \mathbf{k}$ and $\ker \varepsilon = \bigoplus_{n \geq 1} H^{(n)}$.

Lemma 3.4. [2, 11, 15] Any connected graded bialgebra H is a Hopf algebra. The antipode S is given by:

$$S(x) = \sum_{k \geq 0} (e - \text{id}_H)^{*k}(x).$$

It is also defined by $S(1_H) = 1_H$ and recursively by any of the two formulas:

$$S(x) = -x - \sum_{(x)} S(x')x'',$$

$$S(x) = -x - \sum_{(x)} x'S(x'') \quad \text{for } x \in \ker \varepsilon.$$

Here x' and x'' are from

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x''.$$

We proceed to prove that $H(\mathcal{G})$ is a connected graded bialgebra. For this, denoted by

$$H(\mathcal{G})^{(n)} := \mathbf{k} \{F \in M(\mathcal{G}) \mid |V(F)| = n\} \quad \text{for } n \geq 0,$$

where $V(F)$ is the multiset of vertices of F . Now we arrive at one of our main results in this section.

Theorem 3.5. *Let G be a graph. Then*

- (a) *The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a connected graded bialgebra.*
- (b) *The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a Hopf algebra with antipode $S = \sum_{k \geq 0} (e - \text{id}_H)^{*k}$.*

Proof. (a). The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a bialgebra by Theorem 2.17. By the definition of $H(\mathcal{G})$, we obtain

$$H(\mathcal{G}) = \bigoplus_{n=0}^{\infty} H(\mathcal{G})^{(n)}.$$

Let $F_1 \in H(\mathcal{G})^{(p)}$ and $F_2 \in H(\mathcal{G})^{(q)}$ with $p, q \geq 0$. Then

$$|V(F_1)| = p, |V(F_2)| = q \quad \text{and} \quad |V(F_1 F_2)| = pq,$$

whence

$$H(\mathcal{G})^{(p)} H(\mathcal{G})^{(q)} \subseteq H(\mathcal{G})^{(p+q)}.$$

By Eq. (5),

$$\Delta_G(H(\mathcal{G})^{(n)}) \subseteq \bigoplus_{p+q=n} H(\mathcal{G})^{(p)} \otimes H(\mathcal{G})^{(q)}.$$

Thus, the quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a graded bialgebra. Further,

$$H(\mathcal{G})^{(0)} = \mathbf{k}\{\mathbb{1}\} = \mathbf{k} \quad \text{and} \quad \ker \varepsilon = \bigoplus_{n \geq 1} H(\mathcal{G})^{(n)} \quad \text{by Eq. (6).}$$

Therefore, the quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a connected graded bialgebra.

(b). It follows from Lemma 3.4 and Item (a). □

3.2 The dual Hopf algebra

In this subsection, we consider the dual Hopf algebra of $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$. The following elementary result is fundamental.

Lemma 3.6. [11] Let $H = \bigoplus_{n=0}^{\infty} H^{(n)}$ be a graded bialgebra and $H^{(n)}$ be finite dimensional. Then

- (a) The graded dual H^* is $\bigoplus_{n=0}^{\infty} (H^{(n)})^*$. Note that H^* is also a graded bialgebra, and $H^{**} \simeq H$.
- (b) $H \otimes H$ is also a graded bialgebra with $(H \otimes H)^{(n)} = \sum_{i=0}^n H^{(i)} \otimes H^{(n-i)}$ for all $n \in \mathbb{N}$. Moreover, $(H \otimes H)^* \simeq H^* \otimes H^*$.

The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a connected graded bialgebra by Theorem 3.5 and $H(\mathcal{G})^{(n)}$ is finite dimensional. Then its connected graded dual inherits also a graded Hopf algebra by Lemma 3.6, denoted by $(H(\mathcal{G})^*, \Delta_G^*, \varepsilon_G^*, m_G^*, \mathbb{1}^*)$.

For each $F \in M(\mathcal{G})$, we define

$$Z_F : H(\mathcal{G}) \longrightarrow \mathbf{k}, \quad F' \longrightarrow \delta_{F, F'} \quad \text{for } F' \in M(\mathcal{G}), \quad (7)$$

where $\delta_{F, F'}$ is Kronecker function. As $\{Z_F \mid F \in M(\mathcal{G})^{(n)}\}$ is a basis of $(H(\mathcal{G})^{(n)})^*$, $\{Z_F \mid F \in M(\mathcal{G})\}$ is a basis of $H(\mathcal{G})^*$.

The following result is a combinatorial description of the multiplication in $H(\mathcal{G})^*$.

Theorem 3.7. Let G be graph and $F_1, F_2 \in M(\mathcal{G})$. The product of Z_{F_1} and Z_{F_2} is given by

$$Z_{F_1}Z_{F_2} = \sum_{F \in M(\mathcal{G})} n(F_1, F_2; F)Z_F, \quad (8)$$

where $n(F_1, F_2; F)$ is the number of $F_1 \otimes F_2$ in $\Delta_G(F)$.

Proof. Let G be a graph. We consider the basis elements of $H(\mathcal{G})^*$. For any $F_1, F_2 \in M(\mathcal{G})$, we suppose

$$Z_{F_1}Z_{F_2} = \sum_{F' \in M(\mathcal{G})} c_{F_1, F_2}^{F'} Z_{F'}.$$

For any $F \in M(\mathcal{G})$, we have

$$\sum_{F' \in M(\mathcal{G})} c_{F_1, F_2}^{F'} Z_{F'}(F) = c_{F_1, F_2}^F Z_F(F) = c_{F_1, F_2}^F,$$

and

$$\begin{aligned} Z_{F_1}Z_{F_2}(F) &= \Delta_G^*(Z_{F_1} \otimes Z_{F_2})(F) = (Z_{F_1} \otimes Z_{F_2})(\Delta_G(F)) \\ &= (Z_{F_1} \otimes Z_{F_2}) \left(\sum_{U \uplus V = V(F)} F[U] \otimes F[V] \right) \\ &= \sum_{U \uplus V = V(F)} Z_{F_1}(F[U]) \otimes Z_{F_2}(F[V]) \\ &= \sum_{U \uplus V = V(F)} \delta_{F_1, F[U]} \otimes \delta_{F_2, F[V]} \\ &= n(F_1, F_2; F) \quad (\text{by Eq. (5)}). \end{aligned}$$

Thus $c_{F_1, F_2}^F = n(F_1, F_2; F)$. This completes the proof. \square

Example 3.8. Let G be a graph in Item (c) of Example 2.4. We consider the product of $H(\mathcal{G})^*$.

(a) Let $F_1 = \bullet_1$ and $F_2 = \mathbf{1}_2^1$. Then

$$Z_{\bullet_1} Z_{\mathbf{1}_2^1} = Z_{\bullet_1 \mathbf{1}_2^1} + Z_{\mathbf{1}_2^1 \bullet_1}.$$

(b) Let $F_1 = \bullet_3$ and $F_2 = \mathbf{1}_2^1$. We have

$$Z_{\bullet_3} Z_{\mathbf{1}_2^1} = Z_{\bullet_3 \mathbf{1}_2^1} + Z_{\mathbf{1}_2^1 \bullet_3} + Z_{\mathbf{1}_2^1 \mathbf{1}_3}.$$

The coproduct m_G^* of $H(\mathcal{G})^*$ is given by

Lemma 3.9. Let G be graph and $\Gamma_1 \cdots \Gamma_n \in M(\mathcal{G})$. The coproduct of $Z_{\Gamma_1 \cdots \Gamma_n} \in M(\mathcal{G})^*$ is given by

$$m_G^*(Z_{\Gamma_1 \cdots \Gamma_n}) = \sum_{0 \leq i \leq n} Z_{\Gamma_1 \cdots \Gamma_i} \otimes Z_{\Gamma_{i+1} \cdots \Gamma_n}, \quad (9)$$

with the convention that $Z_{\Gamma_1 \Gamma_0} = \mathbb{1}$ and $Z_{\Gamma_{n+1} \Gamma_n} = \mathbb{1}$.

Proof. We consider the basis elements of $H(\mathcal{G})^*$. Suppose

$$m^*(Z_{\Gamma_1 \cdots \Gamma_n}) = \sum_{F', F'' \in M(\mathcal{G})} c_{F', F''} Z_{F'} \otimes Z_{F''}.$$

For any $F_1, F_2 \in M(\mathcal{G})$, we have

$$m_G^*(Z_{\Gamma_1 \cdots \Gamma_n})(F_1 \otimes F_2) = Z_{\Gamma_1 \cdots \Gamma_n}(m_G(F_1 \otimes F_2)) = Z_{\Gamma_1 \cdots \Gamma_n}(F_1 F_2) = \delta_{\Gamma_1 \cdots \Gamma_n, F_1 F_2},$$

and

$$\begin{aligned} & \left(\sum_{F', F'' \in M(\mathcal{G})} c_{F', F''} (Z_{F'} \otimes Z_{F''}) \right) (F_1 \otimes F_2) \\ &= \sum_{F', F'' \in M(\mathcal{G})} c_{F', F''} Z_{F'}(F_1) \otimes Z_{F''}(F_2) \\ &= \sum_{F', F'' \in M(\mathcal{G})} c_{F', F''} \delta_{F', F_1} \otimes \delta_{F'', F_2} \\ &= c_{F_1, F_2}. \end{aligned}$$

Thus, $c_{F_1, F_2} = 1$ if $\Gamma_1 \cdots \Gamma_n = F_1 F_2$ and $c_{F_1, F_2} = 0$ otherwise. This completes the proof. \square

Example 3.10. Let G be a graph in Item (c) of Example 2.4. We consider the coproduct of $H(\mathcal{G})^*$. Let $\Gamma_1 = \cdot_1$ and $\Gamma_2 = \mathbf{1}_2^1$. Then

$$Z_{\cdot_1} \mathbf{1}_2^1 = Z_{\cdot_1} \mathbf{1}_2^1 \otimes \mathbb{1} + \mathbb{1} \otimes Z_{\cdot_1} \mathbf{1}_2^1 + Z_{\cdot_1} \otimes Z_{\mathbf{1}_2^1}.$$

3.3 The algebra morphisms induced from graph homomorphisms

In this subsection, we are going to consider algebra morphisms induced by graph homomorphisms. For this, let us recall the concept of a graph homomorphism.

Definition 3.11. [19, p. 4] Let G_1 and G_2 be two graphs. A **homomorphism** of G_1 to G_2 , written as $f : G_1 \rightarrow G_2$, is a mapping $f : V(G_1) \rightarrow V(G_2)$ such that $(f(u), f(v)) \in E(G_2)$ whenever $(u, v) \in E(G_1)$.

Next, we give the main result in this subsection. Let $f : G_1 \rightarrow G_2$ be a graph homomorphism. Then f can be restricted to any subgraphs of G_1 , still denoted by f . We define the linear map $H(f) : H(\mathcal{G}_1) \rightarrow H(\mathcal{G}_2)$ given by

$$H(f)(\mathbb{1}_{G_1}) := \mathbb{1}_{G_2}$$

and

$$H(f)(F) := H(f)(\Gamma_1 \cdots \Gamma_n) := f(\Gamma_1) \cdots f(\Gamma_n) \text{ for } F = \Gamma_1 \cdots \Gamma_n \in M(\mathcal{G}_1) \text{ with } n \geq 1. \quad (10)$$

Theorem 3.12. *The map $H(f) : H(\mathcal{G}_1) \rightarrow H(\mathcal{G}_2)$ is an algebra morphism.*

Proof. Let

$$F_1 = \Gamma_1 \cdots \Gamma_m \text{ and } F_2 = \Gamma'_1 \cdots \Gamma'_n \in M(\mathcal{G}_1).$$

Then by Eq. (10),

$$\begin{aligned} H(f)(F_1 F_2) &= H(f)(\Gamma_1 \cdots \Gamma_m \Gamma'_1 \cdots \Gamma'_n) = f(\Gamma_1) \cdots f(\Gamma_m) f(\Gamma'_1) \cdots f(\Gamma'_n) \\ &= H(f)(\Gamma_1 \cdots \Gamma_m) H(f)(\Gamma'_1 \cdots \Gamma'_n) \\ &= H(f)(F_1) H(f)(F_2). \end{aligned}$$

Thus $H(f)$ is an algebra morphism. This completes the proof. \square

As an application of Theorem 3.12, we obtain a functor from the graph category to the algebra category. Let us recall

Definition 3.13. [21, Definition 7.1] A **category** is a class \mathfrak{C} of objects (denoted A, B, C, \dots) together with

- (a) a class of disjoint sets, denoted $\text{hom}(A, B)$, one for each pair of objects in \mathfrak{C} ; (an element f of $\text{hom}(A, B)$ is called a **morphism** from A to B and is denoted $f : A \rightarrow B$);
- (b) for each triple (A, B, C) of objects of \mathfrak{C} a function:

$$\text{hom}(B, C) \times \text{hom}(A, B) \rightarrow \text{hom}(A, C);$$

(for morphisms $f : A \rightarrow B, g : B \rightarrow C$, this function is written $(g, f) \mapsto g \circ f$ and $g \circ f : A \rightarrow C$ is called the **composite** of f and g); all subject to the two axioms:

- (i) **Associativity.** If $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$ are morphisms of \mathfrak{C} , then $h \circ (g \circ f) = (h \circ g) \circ f$.
- (ii) **Identity.** For each object B of \mathfrak{C} there exists a morphism $1_B : B \rightarrow B$ such that for any $f : A \rightarrow B, g : B \rightarrow C$,

$$1_B \circ f = f \quad \text{and} \quad g \circ 1_B = g.$$

Definition 3.14. [21, Definition 1.1] Let \mathfrak{C} and \mathfrak{D} be categories. A **covariant functor** \mathbf{T} from \mathfrak{C} to \mathfrak{D} (denoted $T : \mathfrak{C} \rightarrow \mathfrak{D}$) is a pair of functions (both denoted by T), an object function that assigns to each object C of \mathfrak{C} an object $T(C)$ of \mathfrak{D} and a morphism function which assigns to each morphism $f : C \rightarrow D$ of \mathfrak{C} a morphism

$$T(f) : T(C) \rightarrow T(D)$$

of \mathfrak{D} , such that

- (a) $T(1_C) = 1_{T(C)}$ for every identity morphism 1_C of \mathfrak{C} ;
- (b) $T(g \circ f) = T(g) \circ T(f)$ for any two morphisms f, g of \mathfrak{C} whose composite $g \circ f$ is defined.

Let \mathfrak{G} be the class of all graphs, for $G_1, G_2 \in \mathfrak{G}$, $\text{hom}(G_1, G_2)$ is the set of all graph homomorphisms $f : G_1 \rightarrow G_2$. Then \mathfrak{G} is a category, called the **graph category**. Let \mathfrak{S} be the **k-algebra category** whose objects are all **k**-algebra; the $\text{hom}(A_1, A_2)$ is the set of all **k**-algebra morphisms $\varphi : A_1 \rightarrow A_2$.

Now we arrive at our main result of this section.

Theorem 3.15. *Let \mathfrak{G} be the graph category and \mathfrak{S} the **k**-algebra category. Define*

$$H(-) : \mathfrak{G} \rightarrow \mathfrak{S}, \quad G \mapsto H(G), \quad f \mapsto H(f),$$

where G is a graph in \mathfrak{G} , $f : G_1 \rightarrow G_2$ is a graph homomorphism and $H(f) : H(G_1) \rightarrow H(G_2)$ is the algebra morphism obtained in Theorem 3.12. Then $H(-)$ is a covariant functor.

Proof. By Theorem 3.12, it suffices to prove Items (a) and (b) in Definition 3.14. For Item (a), let $F = \Gamma_1 \cdots \Gamma_n \in M(\mathcal{G})$. Then by Eq. (10),

$$\begin{aligned} H(1_G)(F) &= H(1_G)(\Gamma_1 \cdots \Gamma_n) = 1_G(\Gamma_1) \cdots 1_G(\Gamma_n) = \Gamma_1 \cdots \Gamma_n \\ &= 1_{H(G)}(\Gamma_1) \cdots 1_{H(G)}(\Gamma_n) = 1_{H(G)}(\Gamma_1 \cdots \Gamma_n) = 1_{H(G)}(F). \end{aligned}$$

For Item (b), let

$$f : G_1 \rightarrow G_2 \text{ and } g : G_2 \rightarrow G_3,$$

be graph homomorphisms. For $F = \Gamma_1 \cdots \Gamma_n \in M(\mathcal{G}_1)$,

$$\begin{aligned} H(g) \circ H(f)(F) &= H(g) \circ H(f)(\Gamma_1 \cdots \Gamma_n) = H(g)(f(\Gamma_1) \cdots f(\Gamma_n)) \\ &= g(f(\Gamma_1)) \cdots g(f(\Gamma_n)) \\ &= (g \circ f)(\Gamma_1) \cdots (g \circ f)(\Gamma_n) \\ &= H(g \circ f)(\Gamma_1 \cdots \Gamma_n) \\ &= H(g \circ f)(F), \end{aligned}$$

where the third equation employs the fact that the image of a connected subgraph of a graph homomorphism is a connected subgraph. Thus $H(g) \circ H(f)(F) = H(g \circ f)(F)$. This completes the proof. \square

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References

- [1] E. Abe, Hopf algebras, Cambridge university press, 1977.
- [2] A. Connes and D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* **199(1)** (1998), 203-242.
- [3] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.* **210** (2000), 249-273.
- [4] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β -function, diffeomorphisms and the renormalization group, *Comm. Math. Phys.* **216** (2001), 215-241.
- [5] A. Dür, Möbius functions, incidence algebras and power series representations, Lecture Notes in Mathematics **1202**, Springer-Verlag, Berlin 1986.
- [6] G. E. Duchamp, L. Foissy, N. Hoang-Nghia, D. Manchon and A. Tanasa, A combinatorial non-commutative Hopf algebra of graphs, arXiv: 1307.3928v3.
- [7] R. Diestel, Graph Theory, Springer-Verlag, 2006.
- [8] L. Foissy, Finite-dimensional comodules over the Hopf algebra of rooted trees, *J. Algebra* **255(1)** (2002), 89-120.
- [9] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés I, *Bull. Sci. Math.* **126** (2002), 193-239.
- [10] L. Foissy, The infinitesimal Hopf algebra and the poset of planar forests, *J. Algebra. Comb.* **30** (2009), 277-309.

- [11] L. Foissy, Introduction to Hopf Algebra of Rooted Trees, available at <http://loic.foissy.free.fr/pageperso/preprint3.pdf>.
- [12] L. Foissy, Chromatic polynomials and bialgebras of graphs, arXiv: 1611.04303v2.
- [13] R. Grossman and R.G. Larson, Hopf-algebraic structure of families of trees, *J. Algebra* **126**(1) (1989), 184-210.
- [14] L. Guo, Algebraic Birkhoff decomposition and its applications, Automorphic Forms and the Langlands Program, International Press, 2008, 283-323.
- [15] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.
- [16] X. Gao and X. Wang, Infinitesimal unitary Hopf algebras and planar rooted forests, *J. Algebra. Comb.* **49** (2018), 437-460.
- [17] X. Gao and Y. Zhang, Planar rooted forests and weighted infinitesimal Hopf algebras: an operated algebra approach, arXiv:1810.10790.
- [18] J.M. Howie, Fundamentals of semigroup theory, oxford university press, 2003.
- [19] P. Hell and J. Nešetřil, Graphs and Homomorphisms, oxford university press, 2004.
- [20] B. Humpert and J. L. Martin, The incidence hopf algebra of graphs, *SIAM J. Discrete math.* (2012), 555-570.
- [21] T.W. Hungerford, Algebra, Springer science+business media, Inc, 1974.
- [22] C. Kassel, Quantum groups, Springer-Verlag, 1995.
- [23] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, *Adv. Theor. Math. Phys.* **2** (1998), 303-334.
- [24] D. Kreimer, On overlapping Divergences, *Comm. Math. Phys.* **204** (1999), 669-689.
- [25] S. K. Lando, On a Hopf algebra in graph theory, *J. Comb. Theory B* **80**, 104-121, (2000).
- [26] J.-L. Loday and M. O. Ronco, Hopf algebra of the planar binary trees, *Adv. Math.* **139**(2) (1998), 293-309.
- [27] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, *Annals of Mathematics* **2** (2012), 211-264.
- [28] I. Moerdijk, On the Connes-Kreimer construction of Hopf algebras, *Contemp. Math.* **271** (2001), 311-321.
- [29] W. R. Schmitt, Incidence Hopf algebra, *J. Pure Appl. Algebra* **96**, (1994), 299-330.
- [30] M. Sweedler, Hopf algebras, Benjamin, New York, 1969.
- [31] Y. Zhang, D. Chen, X. Gao and Y. F. Luo, Weighted infinitesimal unitary bialgebras on rooted forests and weighted cocycles, to appear in *Pacific J. Math.*, arXiv:1812.0145.
- [32] Y. Zhang and X. Gao, Hopf algebras of planar binary trees: an operated algebra approach, *J. Algebraic Combin.* (2019), <https://doi.org/10.1007/s10801-019-00885-8>, arXiv:1905.09430.

- [33] T. J. Zhang, X. Gao and L. Guo, Hopf algebras of rooted forests, cocycles and free Rota-Baxter algebras, *Journal. Math. Phys* **57** (2016), 101701.