

AROUND THE q -BINOMIAL-EULERIAN POLYNOMIALS

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ABSTRACT. We find a combinatorial interpretation of Shareshian and Wachs' q -binomial-Eulerian polynomials, which leads to an alternative proof of their q - γ -positivity using group actions. Motivated by the sign-balance identity of Désarménien–Foata–Loday for the (des, inv) -Eulerian polynomials, we further investigate the sign-balance of the q -binomial-Eulerian polynomials. We show the unimodality of the resulting signed binomial-Eulerian polynomials by exploiting their continued fraction expansion and making use of a new quadratic recursion for the q -binomial-Eulerian polynomials. We finally use the method of continued fractions to derive a new (p, q) -extension of the γ -positivity of binomial-Eulerian polynomials which involves crossings and nestings of permutations.

1. INTRODUCTION

Let \mathfrak{S}_n be the set of all permutations of $[n] := \{1, 2, \dots, n\}$. For any permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the number of *descents*, the number of *excedances*, the *inversion number* and the *major index* of π are defined, respectively, by

$$\begin{aligned}\text{des}(\pi) &:= |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|, \\ \text{exc}(\pi) &:= |\{i \in [n-1] : \pi_i > i\}|, \\ \text{inv}(\pi) &:= |\{(i, j) \in [n] \times [n] : i < j \text{ and } \pi_i > \pi_j\}|, \\ \text{maj}(\pi) &:= \sum_{\pi_i > \pi_{i+1}} i.\end{aligned}$$

The first two statistics are *Eulerian statistics* whose enumerative polynomials give the n th *Eulerian polynomial* (cf. [28, Sec. 1.3])

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)},$$

while the other two statistics are *Mahonian statistics* with common generating function

$$[n]_q! := \prod_{i=1}^n (1 + q + \cdots + q^{i-1}).$$

The joint distributions of Eulerian and Mahonian statistics on permutations have been widely studied; see [2, 4, 9, 14, 17, 23, 24, 26, 27].

The (maj, exc) -*Eulerian polynomials* $A_n(t, q)$, which arise in Shareshian and Wachs' study of poset topology [23], are defined as

$$A_n(t, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{exc}(\pi)} q^{\text{maj}(\pi) - \text{exc}(\pi)}.$$

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Their exponential generating function has a nice q -analog of Euler's formula (see [9, 24]),

$$(1.1) \quad \sum_{n \geq 0} A_n(t, q) \frac{z^n}{(q; q)_n} = \frac{(1-t)e(z; q)}{e(tz; q) - te(z; q)},$$

where

$$(q; q)_n := \prod_{i=1}^n (1 - q^i) \quad \text{and} \quad e(z; q) := \sum_{n \geq 0} \frac{z^n}{(q; q)_n}.$$

An *admissible inversion* of a permutation $\pi \in \mathfrak{S}_n$ is an inversion pair (π_i, π_j) satisfying either of the following conditions:

- $1 < i$ and $\pi_{i-1} < \pi_i$ or
- there is some k such that $i < k < j$ and $\pi_i < \pi_k$.

Let $\text{ai}(\pi)$ be the number of admissible inversions of π . For example, the admissible inversions of 3142 are (3, 2) and (4, 2). So $\text{ai}(\pi) = 2$. The statistic of admissible inversions was first introduced by Shareshian and Wachs [23], who gave the interpretation

$$(1.2) \quad A_n(t, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{ai}(\pi)}.$$

The detailed proof of this interpretation was given by Linusson, Shareshian and Wachs [17] using Rees products of posets; see [4, 14] for alternative approaches and a generalization.

It is known (cf. [20]) that the Eulerian polynomials are the h -polynomials of dual permutohedra. Postnikov, Reiner, and Williams [21, Section 10.4] proved that the h -polynomials of dual stellohedra equal the binomial transformations

$$\tilde{A}_n(t) = 1 + t \sum_{m=1}^n \binom{n}{m} A_m(t)$$

of the Eulerian polynomials, and provided the combinatorial interpretation

$$(1.3) \quad \tilde{A}_n(t) = \sum_{\pi \in \text{PRW}_{n+1}} t^{\text{des}(\pi)},$$

where PRW_n is the set of permutations $\pi \in \mathfrak{S}_n$ such that the first ascent of π appears at the letter 1 if π has an ascent. For example,

$$\text{PRW}_1 = \{1\}, \quad \text{PRW}_2 = \{12, 21\}, \quad \text{and} \quad \text{PRW}_3 = \{123, 132, 213, 312, 321\}.$$

Shareshian and Wachs [25] called $\tilde{A}_n(t)$ *binomial-Eulerian polynomials* and introduced the *q -binomial-Eulerian polynomials*

$$\tilde{A}_n(t, q) = 1 + t \sum_{m=1}^n \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(t, q),$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}$$

are the q -binomial coefficients.

Even though an algebro-geometric interpretation of $\tilde{A}_n(t, q)$ has already been found in [25], no combinatorial interpretation of $\tilde{A}_n(t, q)$ is known similar to classical Eulerian

polynomials. Our first aim is to give such an interpretation, which is a q -analog of (1.3) and is similar to the interpretation (1.2) for $A_n(t, q)$.

Theorem 1.1. *For $n \geq 1$, the q -binomial-Eulerian polynomial $\tilde{A}_n(t, q)$ has the interpretation*

$$\tilde{A}_n(t, q) = \sum_{\pi \in \text{PRW}_{n+1}} t^{\text{des}(\pi)} q^{\text{ai}(\pi)}.$$

Recall that a polynomial $\sum_{i=0}^n h_i t^i$ in t with real coefficients is said to be *palindromic* if $h_i = h_{n-i}$ for all $0 \leq i \leq \lfloor n/2 \rfloor$. It is *unimodal* if

$$h_0 \leq h_1 \leq \dots \leq h_c \geq h_{c+1} \geq \dots \geq h_n \quad \text{for some } c.$$

A stronger property implying both the palindromicity and the unimodality is the γ -positivity. A polynomial of degree n in t with real coefficients is said to be γ -positive if it can be written in the basis

$$\{t^k (t+1)^{n-2k}\}_{0 \leq k \leq n/2}$$

with non-negative coefficients. Many interesting polynomials arising in enumerative and geometric combinatorics are palindromic and unimodal, some of which are even γ -positive; see [2, 3, 20].

For a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, we call σ_i ($1 \leq i \leq n$) a *double descent* (resp. *double ascent*, *peak*, *valley*) of σ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ (resp. $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$), where we use the convention $\sigma_0 = \sigma_{n+1} = +\infty$. In particular, σ_1 is a double descent if $\sigma_1 > \sigma_2$, and in this case we will call σ_1 the *initial double descent*. Denote by $\text{dd}(\sigma)$ (resp. $\text{da}(\sigma)$, $\text{peak}(\sigma)$, $\text{valley}(\sigma)$) the number of non-initial double descents (resp. double ascents, peaks, valleys) of σ . Foata and Schützenberger [10, Theorem 5.6] proved the following elegant γ -positivity expansion of the Eulerian polynomials

$$(1.4) \quad A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k}$ is the cardinality of the set

$$\Gamma_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{dd}(\sigma) = 0, \sigma_1 < \sigma_2 \text{ and } \text{des}(\sigma) = k\}.$$

The γ -positivity formula of Postnikov, Reiner, and Williams [21, Theorem 11.6] in the case of stellohedron asserts that

$$(1.5) \quad \tilde{A}_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \tilde{\gamma}_{n,k} t^k (1+t)^{n-1-2k},$$

where $\tilde{\gamma}_{n,k}$ counts permutations $\sigma \in \text{PRW}_{n+1}$ such that σ has no double ascents and $\text{asc}(\sigma) = k$, where $\text{asc}(\sigma) := n - 1 - \text{des}(\sigma)$.

The following q -analog of (1.4) was proved by various methods in [16, 17, 24, 25]:

$$A_n(t, q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k}(q) = \sum_{\sigma \in \Gamma_{n,k}} q^{\text{inv}(\sigma)}$, and a similar q - γ -positivity expansion for $\tilde{A}_n(t, q)$ was recently established by Shareshian and Wachs [25, Theorem 4.5].

Theorem 1.2 (Shareshian and Wachs). *Let*

$$\tilde{\Gamma}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{dd}(\sigma) = 0, \text{des}(\sigma) = k\}.$$

The q -binomial-Eulerian polynomials have the q - γ -positivity expansion

$$(1.6) \quad \tilde{A}_n(t, q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}(q) t^k (1+t)^{n-2k},$$

where

$$\tilde{\gamma}_{n,k}(q) = \sum_{\pi \in \tilde{\Gamma}_{n,k}} q^{\text{inv}(\pi)}.$$

Note that the combinatorial meanings of $\tilde{\gamma}_{n,k}(1)$ in (1.6) and $\tilde{\gamma}_{n,k}$ in (1.5) are apparently different. As observed in [25], the existence of expansion (1.6) with $\tilde{\gamma}_{n,k}(q) \in \mathbb{Z}[q]$ for $\tilde{A}_n(t, q)$ is equivalent to a symmetric q -Eulerian identity due independently to Chung–Graham [6] and Han–Lin–Zeng [13]. Theorem 1.2 was obtained from the principle specialization of an analogous symmetric function identity in [25]. Theorem 1.1 together with the so-called *Modified Foata–Strehl group action* on permutations enables us to give a combinatorial proof to Theorem 1.2. Our alternative approach has the advantage that makes the interpretation of $\tilde{\gamma}_{n,k}$ in (1.5) transparent; see Remark 3.1.

In 1992, Désarménien and Foata [8] showed the following sign-balance identity, which was conjectured by Loday [19],

$$(1.7) \quad \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} (-1)^{\text{inv}(\pi)} = \begin{cases} (1-t)^m A_m(t), & \text{if } n = 2m; \\ (1-t)^m A_{m+1}(t), & \text{if } n = 2m + 1. \end{cases}$$

This paper stems from the observation that identity (1.7) follows from a simple quadratic recursion (2.1) for the (inv, des) - q -Eulerian polynomials. This idea enables us to prove similar sign-balance identities for $A_n(t, q)$ and $\tilde{A}_n(t, q)$. It appears that the signed binomial-Eulerian polynomials $\tilde{A}_n(t, -1)$ have interesting properties which are observable from their first terms:

$$\begin{aligned} \tilde{A}_1(t, -1) &= 1 + t, \\ \tilde{A}_2(t, -1) &= 1 + t + t^2, \\ \tilde{A}_3(t, -1) &= 1 + 3t + 3t^2 + t^3, \\ \tilde{A}_4(t, -1) &= 1 + 3t + 5t^2 + 3t^3 + t^4, \\ \tilde{A}_5(t, -1) &= 1 + 7t + 15t^2 + 15t^3 + 7t^4 + t^5, \\ \tilde{A}_6(t, -1) &= 1 + 7t + 19t^2 + 25t^3 + 19t^4 + 7t^5 + t^6. \end{aligned}$$

Here is the central result of this paper.

Theorem 1.3. *For any $n \geq 1$, the signed binomial-Eulerian polynomial $\tilde{A}_n(t, -1)$ is palindromic and unimodal.*

Although the palindromicity of $\tilde{A}_n(t, -1)$ follows directly from the q - γ -positivity expansion (1.6) of $\tilde{A}_n(t, q)$, it is not clear how to derive the unimodality in Theorem 1.3 from Theorem 1.2. In showing the unimodality of $\tilde{A}_n(t, -1)$, we find a new quadratic recursion for $\tilde{A}_n(t, q)$.

Theorem 1.4. *The q -binomial-Eulerian polynomials satisfy the recurrence relation*

$$\tilde{A}_{n+1}(t, q) = (1 + t)\tilde{A}_n(t, q) + t \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k(t, q) \tilde{A}_{n-k}(t, q)$$

for $n \geq 0$ with initial value $\tilde{A}_0(t, q) = 1$.

As will be seen, two specializations of this recursion together with a continued fraction expansion conclude the desired unimodality of $\tilde{A}_n(t, -1)$ in Theorem 1.3. Via the machinery of continued fraction, we will also prove a new (p, q) -extension of the γ -positivity of binomial-Eulerian polynomials.

The rest of this paper is organized as follows. In Section 2, we show how to derive (1.7) and the sign-balance identity of the binomial-Eulerian polynomials using appropriate quadratic recursions and prove Theorem 1.4. In Section 3, we show Theorem 1.1 and present the Modified Foata–Strehl group action proof of Theorem 1.2. In Section 4, via the machinery of continued fraction, we prove the unimodality of $\tilde{A}_n(t, -1)$ and show a (p, q) -extension of the γ -positivity of binomial-Eulerian polynomials. We end this paper with two log-concavity conjectures.

2. QUADRATIC RECURSIONS AND SIGN-BALANCE OF q -BINOMIAL-EULERIAN POLYNOMIALS

In this section, we investigate the sign-balance of q -binomial-Eulerian polynomials. We begin with a new simple approach to identity (1.7). The following lemma is useful.

Lemma 2.1 (cf. [8]). *For any integers $m \geq i \geq 0$,*

$$\lim_{q \rightarrow -1} \begin{bmatrix} 2m \\ 2i \end{bmatrix}_q = \lim_{q \rightarrow -1} \begin{bmatrix} 2m + 1 \\ 2i \end{bmatrix}_q = \lim_{q \rightarrow -1} \begin{bmatrix} 2m + 1 \\ 2i + 1 \end{bmatrix}_q = \binom{m}{i} \quad \text{and} \quad \lim_{q \rightarrow -1} \begin{bmatrix} 2m \\ 2i + 1 \end{bmatrix}_q = 0.$$

Let us define the (inv, des)-Eulerian polynomials by

$$A_n^{\text{des,inv}}(t, q) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)} q^{\text{inv}(\pi)}.$$

Chow [5] gave a combinatorial proof of the quadratic recursion

$$(2.1) \quad A_{n+1}^{\text{des,inv}}(t, q) = (1 + tq^n)A_n^{\text{des,inv}}(t, q) + t \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_{n-k}^{\text{des,inv}}(t, q) A_k^{\text{des,inv}}(t, q).$$

Taking $q = 1$, we obtain

$$(2.2) \quad A_{n+1}(t) = (1 + t)A_n(t) + t \sum_{k=1}^{n-1} \binom{n}{k} A_{n-k}(t) A_k(t).$$

A new simple proof of (1.7). We proceed by induction on n . Assume that (1.7) holds for n up to $2m - 1$. It then follows from recursion (2.1) and Lemma 2.1 that

$$\begin{aligned} A_{2m}^{\text{des,inv}}(t, -1) &= (1-t)A_{2m-1}^{\text{des,inv}}(t, -1) + t \sum_{k=1}^{m-1} \binom{m-1}{k} A_{2(m-k)-1}^{\text{des,inv}}(t, -1) A_{2k}^{\text{des,inv}}(t, -1) \\ &\quad - t \sum_{k=0}^{m-2} \binom{m-1}{k} A_{2(m-k-1)}^{\text{des,inv}}(t, -1) A_{2k+1}^{\text{des,inv}}(t, -1) \\ &= (1-t)A_{2m-1}^{\text{des,inv}}(t, -1) = (1-t)^m A_m(t) \end{aligned}$$

and

$$\begin{aligned} A_{2m+1}^{\text{des,inv}}(t, -1) &= (1+t)A_{2m}^{\text{des,inv}}(t, -1) + t \sum_{k=1}^{m-1} \binom{m}{k} A_{2(m-k)}^{\text{des,inv}}(t, -1) A_{2k}^{\text{des,inv}}(t, -1) \\ &= (1-t)^m A_{m+1}(t), \end{aligned}$$

where the last equality follows from the recurrence relation (2.2). This completes the proof of (1.7) by induction. \square

The first author [14, Theorem 2] showed that one can derive the following quadratic recursion for $A_n(t, q)$, which is a q -analog of recursion (2.2):

$$(2.3) \quad A_{n+1}(t, q) = (1+t)A_n(t, q) + t \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_k(t, q) A_{n-k}(t, q).$$

By applying this recursion, the following major-balance identity can be proved through the same approach as (1.7), the details of which are omitted due to the similarity.

Theorem 2.2. *For $n \geq 1$, we have*

$$A_n(t, -1) = \begin{cases} (1+t)^m A_m(t), & \text{if } n = 2m; \\ (1+t)^m A_{m+1}(t), & \text{if } n = 2m + 1. \end{cases}$$

The above identity for even n appeared in [22, Corollary 6.2]. An immediate consequence of Theorem 2.2 and Lemma 2.1 is the following signed identity for $\tilde{A}_n(t, q)$.

Corollary 2.3. *For $n \geq 1$, we have*

$$\tilde{A}_n(t, -1) = \begin{cases} 1 + t \sum_{k=1}^m \binom{m}{k} (1+t)^k A_k(t), & \text{if } n = 2m; \\ 1 + t \sum_{k=0}^m \binom{m}{k} (1+t)^k (A_k(t) + A_{k+1}(t)), & \text{if } n = 2m + 1. \end{cases}$$

Here we use the convention $A_0(t) = 0$.

In the rest of this section, we give the proof of Theorem 1.4. The *Eulerian differential operator* δ_z used below is defined by

$$\delta_z(f(z)) := \frac{f(z) - f(qz)}{z}$$

for any formal power series $f(z)$ over the ring of real polynomials in q . It is not difficult to show for any variable α , that

$$(2.4) \quad \delta_z(e(\alpha z; q)) = \alpha e(\alpha z; q).$$

Proof of Theorem 1.4. We begin with the calculation of the exponential generating function of $\tilde{A}_n(t, q)$. By using (1.1), we can deduce that

$$\begin{aligned} \sum_{n \geq 0} \tilde{A}_n(t, q) \frac{z^n}{(q; q)_n} &= \sum_{n \geq 0} \left(1 + t \sum_{m=1}^n \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(t, q) \right) \frac{z^n}{(q; q)_n} \\ &= \sum_{n \geq 0} (1-t) \frac{z^n}{(q; q)_n} + \sum_{n \geq 0} \left(t \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(t, q) \right) \frac{z^n}{(q; q)_n} \\ &= (1-t)e(z; q) + t \left(\sum_{n \geq 0} \frac{z^n}{(q; q)_n} \right) \left(\sum_{n \geq 0} A_n(t, q) \frac{z^n}{(q; q)_n} \right) \\ &= (1-t)e(z; q) + te(z; q) \frac{(1-t)e(z; q)}{e(tz; q) - te(z; q)}, \end{aligned}$$

which is simplified to

$$(2.5) \quad \sum_{n \geq 0} \tilde{A}_n(t, q) \frac{z^n}{(q; q)_n} = \frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)}.$$

Applying the operator δ_z to both sides of (2.5) and using property (2.4) and the product rule of the Eulerian differential operator (see [14, Lemma 7]) yields

$$\begin{aligned} &\sum_{n \geq 0} \tilde{A}_{n+1}(t, r, q) \frac{z^n}{(q; q)_n} \\ &= \delta_z \left(\frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)} \right) \\ &= \frac{\delta_z((1-t)e(z; q)e(tz; q))}{e(tz; q) - te(z; q)} + \delta_z((e(tz; q) - te(z; q))^{-1}) (1-t)e(zq; q)e(tzq; q) \\ &= \frac{(1-t)e(tz; q)(te(zq; q) + e(z; q))}{e(tz; q) - te(z; q)} + \frac{(1-t)e(zq; q)e(tzq; q)(te(z; q) - te(tz; q))}{(e(tzq; q) - te(zq; q))(e(tz; q) - te(z; q))} \\ &= \frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)} + t \frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)} \frac{(1-t)e(zq; q)}{e(tzq; q) - te(zq; q)} \\ &\quad + \frac{t(1-t)e(zq; q)\Delta(t, q)}{(e(tz; q) - te(z; q))(e(tzq; q) - te(zq; q))}, \end{aligned}$$

where

$$\begin{aligned} \Delta(t, q) &:= te(tz; q)[e(z; q) - e(zq; q)] - [e(tz; q) - e(tzq; q)]e(z; q) \\ &= tze(tz; q)\delta_z(e(z; q)) - ze(z; q)\delta_z(e(tz; q)). \end{aligned}$$

Invoking (2.4) we see immediately that $\Delta(t, q) = 0$, and so

$$\sum_{n \geq 0} \tilde{A}_{n+1}(t, r, q) \frac{z^n}{(q; q)_n} = \frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)} + t \frac{(1-t)e(z; q)e(tz; q)}{e(tz; q) - te(z; q)} \frac{(1-t)e(zq; q)}{e(tzq; q) - te(zq; q)}.$$

Extracting the coefficient of $z^n/(q; q)_n$ from both sides, we obtain Theorem 1.4. \square

A direct consequence of Theorem 1.4 and Lemma 2.1 is the following recurrence relations for $\tilde{A}_n(t, -1)$, involving the signed Eulerian polynomials $A_n(t, -1)$.

Corollary 2.4. *For $n \geq 0$, we have*

$$(2.6) \quad \tilde{A}_{2n+1}(t, -1) = (1+t)\tilde{A}_{2n}(t, -1) + t \sum_{k=1}^n \binom{n}{k} A_{2k}(t, -1) \tilde{A}_{2n-2k}(t, -1)$$

and

$$\begin{aligned} \tilde{A}_{2n+2}(t, -1) &= (1+t)\tilde{A}_{2n+1}(t, -1) + t \sum_{k=1}^n \binom{n}{k} A_{2k}(t, -1) \tilde{A}_{2n+1-2k}(t, -1) \\ &\quad - t \sum_{k=0}^n \binom{n}{k} A_{2k+1}(t, -1) \tilde{A}_{2n-2k}(t, -1). \end{aligned}$$

3. PROOF OF THEOREMS 1.1 AND 1.2

We shall prove Theorems 1.1 and 1.2 in Subsections 3.1 and 3.2 respectively.

3.1. A combinatorial interpretation of $\tilde{A}_n(t, q)$. We need the following classical interpretation of the q -binomial coefficients (cf. [28, Prop. 1.3.17]):

$$(3.1) \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_q = \sum_{(\mathcal{A}, \mathcal{B})} q^{\text{inv}(\mathcal{A}, \mathcal{B})},$$

where the sum is over all ordered set partitions $(\mathcal{A}, \mathcal{B})$ of $[n]$ such that $|\mathcal{A}| = k$ and

$$\text{inv}(\mathcal{A}, \mathcal{B}) := \{(i, j) \in \mathcal{A} \times \mathcal{B} : i > j\}.$$

Proof of Theorem 1.1. We will show that the bivariate polynomial

$$\tilde{B}_n(t, q) := \sum_{\pi \in \text{PRW}_{n+1}} t^{\text{des}(\pi)} q^{\text{ai}(\pi)}$$

satisfies the same recurrence relation as $\tilde{A}_n(t, q)$ in Theorem 1.4. For each $0 \leq k \leq n$, let

$$\mathcal{B}_{n+1, k} := \{\pi \in \text{PRW}_{n+1} : \pi_{n+1-k} = n+1\}$$

and introduce the refinement $\tilde{B}_{n, k}(t, q)$ of $\tilde{B}_n(t, q)$ by

$$\tilde{B}_{n, k}(t, q) := \sum_{\pi \in \mathcal{B}_{n+1, k}} t^{\text{des}(\pi)} q^{\text{ai}(\pi)}.$$

It is clear that $\tilde{B}_n(t, q) = \sum_{k=0}^n \tilde{B}_{n,k}(t, q)$, $\tilde{B}_{n,n}(t, q) = t\tilde{B}_{n-1}(t, q)$ and $\tilde{B}_{n,0}(t, q) = \tilde{B}_{n-1}(t, q)$. The desired result then follows from the claim that

$$(3.2) \quad \tilde{B}_{n,k}(t, q) = t \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^k A_k(t, q) \tilde{B}_{n-1-k}(t, q) \quad \text{for any } 1 \leq k \leq n-1.$$

It remains to show the above claim. For a set X of distinct positive integers, we denote by $\binom{X}{m}$ the m -element subsets of X , by \mathfrak{S}_X the set of permutations of X and by PRW_X the set of all permutations in \mathfrak{S}_X whose first ascent entry is $\min(X)$. Let $\mathcal{W}(n, k)$ be the set of all triples (W, π_L, π_R) such that $W \in \binom{[n] \setminus \{1\}}{k}$, $\pi_L \in \text{PRW}_{[n] \setminus W}$ and $\pi_R \in \mathfrak{S}_W$. Note that for every permutation in $\mathcal{B}_{n+1,k}$ ($1 \leq k \leq n-1$), the entry $n+1$ appears to the right of the entry 1. Therefore, one can check easily that the mapping $\pi \mapsto (W, \pi_L, \pi_R)$ defined by

- $W = \{\pi_i : n+2-k \leq i \leq n+1\}$,
- $\pi_L = \pi_1 \pi_2 \cdots \pi_{n-k}$ and $\pi_R = \pi_{n+2-k} \pi_{n+3-k} \cdots \pi_{n+1}$,

is a bijection between $\mathcal{B}_{n+1,k}$ and $\mathcal{W}(n, k)$ satisfying

$$\text{des}(\pi) = \text{des}(\pi_L) + \text{des}(\pi_R) + 1$$

and

$$\text{ai}(\pi) = \text{ai}(\pi_L) + \text{ai}(\pi_R) + \text{inv}([n] \setminus W, W) + k.$$

It follows from this bijection and the interpretations (1.2) and (3.1) that claim (3.2) holds, which completes the proof. \square

As an example of Theorem 1.1, the permutations in PRW_4 with two descents are 1432, 3142, 4132, 2143, 4312, 4213 and 3214, which contribute the monomial $(2q^2 + 2q + 3)t^2$ to $\tilde{A}_3(t, q)$.

3.2. A group-action proof of the q - γ -positivity of $\tilde{A}_n(t, q)$. Let us review briefly the Modified Foata–Strehl group action originally inspired by work of Foata and Strehl [11]. Let $\sigma \in \mathfrak{S}_n$, for any $x \in [n]$, the x -factorization of σ reads $\sigma = w_1 w_2 x w_3 w_4$, where w_2 (resp. w_3) is the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all smaller than x . Following [11] we define $\varphi_x(\sigma) = w_1 w_3 x w_2 w_4$. For instance, if $x = 5$ and $\sigma = 63157248 \in \mathfrak{S}_8$, then $w_1 = 6, w_2 = 31, w_3 = \emptyset$ and $w_4 = 7248$. Thus $\varphi_x(\sigma) = 65317248$. Introduce the modified action φ'_x on σ by

$$\varphi'_x(\sigma) := \begin{cases} \varphi_x(\sigma), & \text{if } x \text{ is a double ascent or double descent of } \sigma; \\ \sigma, & \text{if } x \text{ is a valley or a peak of } \sigma. \end{cases}$$

It is clear that the φ'_x 's are involutions and commute. Therefore, for any subset $S \subseteq [n]$ we can define the function $\varphi'_S: \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by

$$\varphi'_S(\sigma) = \prod_{x \in S} \varphi'_x(\sigma),$$

where the multiplication is the composition of functions. Hence the group \mathbb{Z}_2^n acts on \mathfrak{S}_n via the functions φ'_S , where $S \subseteq [n]$. This action is called the *Modified Foata–Strehl action* (*MFS-action* for short) and has a nice visualization as depicted in Fig. 1. Note that this MFS-action is exactly the same as the version used in [16].

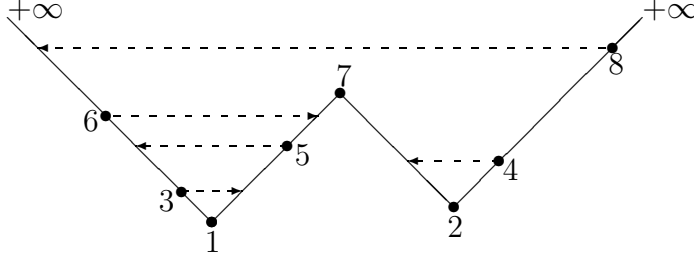


FIGURE 1. MFS-actions on 63157248

Proof of Theorem 1.2. For any permutation $\sigma \in \text{PRW}_{n+1}$ and $x \in [n+1]$, it is not hard to see that the permutation $\varphi_x(\sigma)$ still has the property that the entry 1 is the first ascent. Thus, the set PRW_{n+1} is invariant under the MFS-action. The MFS-action divides the set PRW_{n+1} into disjoint orbits. Moreover, if x is a double descent (resp. peak or valley) of σ , then x is a double ascent (resp. peak or valley) of the permutation $\varphi'_x(\sigma)$. In the orbit containing σ , we can choose the unique permutation with least descents (also coincident with the one without double descents), denoted $\bar{\sigma}$, as a representative element. Then, we have $\text{da}(\bar{\sigma}) = n - \text{peak}(\bar{\sigma}) - \text{valley}(\bar{\sigma})$ and $\text{des}(\bar{\sigma}) = \text{peak}(\bar{\sigma}) = \text{valley}(\bar{\sigma}) - 1$.

By [16, Lemma 7], the statistic “ai” is constant inside each orbit. Thus, by Theorem 1.1 and the above discussion, one may deduce that

$$\begin{aligned}
 \tilde{A}_n(t, q) &= \sum_{\sigma \in \text{PRW}_{n+1}} t^{\text{des}(\sigma)} q^{\text{ai}(\sigma)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{\bar{\sigma} \in \text{PRW}_{n+1} \cap \Gamma_{n+1, k}} q^{\text{ai}(\bar{\sigma})} \right) t^k (1+t)^{n-2k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{\bar{\sigma} \in \text{PRW}_{n+1} \cap \Gamma_{n+1, k}} q^{\text{inv}(\bar{\sigma})} \right) t^k (1+t)^{n-2k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{\pi \in \tilde{\Gamma}_{n, k}} q^{\text{inv}(\pi)} \right) t^k (1+t)^{n-2k},
 \end{aligned}$$

where the second last equality is a consequence of [16, Lemma 8], while the last equality follows from the simple one-to-one correspondence

$$\bar{\sigma}_1 \bar{\sigma}_2 \cdots \bar{\sigma}_n \mapsto (\bar{\sigma}_2 - 1) \cdots (\bar{\sigma}_n - 1)$$

between $\text{PRW}_{n+1} \cap \Gamma_{n+1, k}$ and $\tilde{\Gamma}_{n, k}$. Note that the first letter of each $\bar{\sigma} \in \text{PRW}_{n+1} \cap \Gamma_{n+1, k}$ must be 1. It is easy to check that the above correspondence is a bijection preserving both the number of descents and the number of inversions. This establishes (1.6). \square

Remark 3.1. In each orbit of the MFS-action on PRW_{n+1} , there is a unique permutation with least ascents, which is exactly the one with no double ascents. Thus, the interpretation of $\tilde{\gamma}_{n, k}$ in (1.5) due to Postnikov, Reiner and Williams is clear.

Define the γ -polynomial of $A_n(t, q)$ and $\tilde{A}_n(t, q)$ by

$$\Gamma_n(y, q) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) y^k \quad \text{and} \quad \tilde{\Gamma}_n(y, q) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}(q) y^k,$$

respectively. The following recurrence relation for $\tilde{\Gamma}_n(y, q)$ follows directly from Theorem 1.4 and the relationships

$$\tilde{A}_n(t, q) = (1+t)^n \tilde{\Gamma}_n(t/(1+t)^2, q) \quad \text{and} \quad A_n(t, q) = (1+t)^{n-1} \Gamma_n(t/(1+t)^2, q).$$

Corollary 3.2. *We have the following recursion for $\tilde{\Gamma}_n(y, q)$:*

$$(3.3) \quad \tilde{\Gamma}_{n+1}(y, q) = \tilde{\Gamma}_n(y, q) + y \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^k \Gamma_k(y, q) \tilde{\Gamma}_{n-k}(y, q).$$

Remark 3.3. One may also prove Theorem 1.2 by showing that the polynomials

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} y^k \left(\sum_{\pi \in \tilde{\Gamma}_{n,k}} q^{\text{inv}(\pi)} \right)$$

satisfy the same recurrence relation as $\tilde{\Gamma}_n(y, q)$ in (3.3).

4. CONTINUED FRACTIONS AND THE UNIMODALITY OF $\tilde{A}_n(t, -1)$

In this section, we present a proof of the unimodality of $\tilde{A}_n(t, -1)$ and give a new (p, q) -extension of the γ -positivity of $\tilde{A}_n(t)$, via the machine of continued fraction.

4.1. The unimodality of $\tilde{A}_n(t, -1)$. Since the product of two palindromic and unimodal polynomials is again palindromic and unimodal (cf. [29]), recursion (2.6) implies that Theorem 1.3 needs to be shown for even integers n only, that is, to show that the palindromic polynomial

$$(4.1) \quad A_m^*(t) := 1 + t \sum_{k=1}^m \binom{m}{k} (1+t)^k A_k(t)$$

is unimodal for any integer $m \geq 1$.

The polynomials $A_n^*(t)$ can be named the *binomial-Eulerian polynomials of type B*, since $(1+t)^n A_n(t)$ are the flag descent polynomials [1] over the Coxeter group of type B, namely,

$$(1+t)^n A_n(t) = \sum_{\sigma \in \mathfrak{B}_n} t^{\text{fdes}(\sigma)},$$

where \mathfrak{B}_n is the set of signed permutations of $[n]$ and $\text{fdes}(\sigma)$ is the number of flag descents of σ . In order to prove the unimodality of $A_n^*(t)$, we need some preparation.

Definition 4.1. For any permutation $\sigma \in \mathfrak{S}_n$, the numbers of *cycle peaks*, *cycle valley*, *cycle double rises*, *cycle double descents*, *fixed points* of σ are defined, respectively, by

$$\begin{aligned} \text{cpeak}(\sigma) &:= |\{i \in [n]: \sigma^{-1}(i) < i > \sigma(i)\}|, \\ \text{cval}(\sigma) &:= |\{i \in [n]: \sigma^{-1}(i) > i < \sigma(i)\}|, \\ \text{cdrise}(\sigma) &:= |\{i \in [n]: \sigma^{-1}(i) < i < \sigma(i)\}|, \\ \text{cdfall}(\sigma) &:= |\{i \in [n]: \sigma^{-1}(i) > i > \sigma(i)\}|, \\ \text{fix}(\sigma) &:= |\{i \in [n]: \sigma(i) = i\}|. \end{aligned}$$

For instance, if the cycle form of $\sigma \in \mathfrak{S}_7$ is (1462)(3)(57), then $\text{cpeak}(\sigma) = 2$, $\text{cval}(\sigma) = 2$, $\text{cdrise}(\sigma) = 1$, $\text{cdfall}(\sigma) = 1$ and $\text{fix}(\sigma) = 1$. Define

$$Q_n(a, b, c, d, \alpha) = \sum_{\sigma \in \mathfrak{S}_n} a^{\text{cval}(\sigma)} b^{\text{cpeak}(\sigma)} c^{\text{cdfall}(\sigma)} d^{\text{cdrise}(\sigma)} \alpha^{\text{fix}(\sigma)}.$$

We recall the following result from Zeng [32].

Lemma 4.2 (Zeng). *We have*

$$(4.2) \quad \sum_{n \geq 0} Q_n(a, b, c, d, \alpha) \frac{x^n}{n!} = e^{\alpha x} \frac{\alpha_1 - \alpha_2}{\alpha_1 e^{\alpha_2 x} - \alpha_2 e^{\alpha_1 x}},$$

where $\alpha_1 \alpha_2 = ab$ and $\alpha_1 + \alpha_2 = c + d$. Moreover,

$$(4.3) \quad \sum_{n=0}^{\infty} Q_n(a, b, c, d, \alpha) x^n = \frac{1}{1 - \gamma_0 x - \frac{\beta_1 x^2}{1 - \gamma_1 x - \frac{\beta_2 x^2}{1 - \gamma_2 x - \dots}}}$$

where $\gamma_n = n(c + d) + \alpha$ and $\beta_n = n^2 ab$.

Since $\text{exc}(\sigma) = \text{cval}(\sigma) + \text{cdrise}(\sigma)$, we have $A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} = Q_n(t, 1, 1, t, 1)$ and the well-known formula

$$(4.4) \quad \sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{x(t-1)}}$$

is a special case of (4.2).

Next we compute the exponential generating function of $A_n^*(t)$.

Lemma 4.3. *We have*

$$(4.5) \quad \sum_{n \geq 0} A_n^*(t) \frac{x^n}{n!} = \frac{(t-1)e^{t^2 x}}{t - e^{(t^2-1)x}}.$$

Proof. It follows from (4.1) and (4.4) that

$$\begin{aligned} \sum_{n \geq 0} A_n^*(t) \frac{x^n}{n!} &= (1-t)e^x + t \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} (1+t)^k A_k(t) \frac{x^n}{n!} \\ &= (1-t)e^x + te^x \sum_{n \geq 0} A_n(t) \frac{(1+t)^n x^n}{n!} \\ &= \frac{(t-1)e^{t^2 x}}{t - e^{(t^2-1)x}}, \end{aligned}$$

as desired. \square

We also need the following result [12, p. 306].

Lemma 4.4 (Jacobi–Rogers formula). *Let J_n be the sequence of coefficients in the expansion*

$$\sum_{n \geq 0} J_n x^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \dots}}}.$$

Then for $n \geq 1$, we have

$$J_n = b_0^n + \sum_{h \geq 0} \sum_{\substack{n_0, \dots, n_h \geq 1 \\ m_0, \dots, m_{h+1} \geq 0}} (b_0^{m_0} \cdots b_{h+1}^{m_{h+1}}) (\lambda_1^{n_0} \cdots \lambda_{h+1}^{n_h}) \cdot \rho(\mathbf{n}, \mathbf{m}),$$

where $2(n_0 + \cdots + n_h) + (m_0 + \cdots + m_{h+1}) = n$ and

$$\rho(\mathbf{n}, \mathbf{m}) := \prod_{j=0}^{h+1} \binom{n_j + n_{j-1} - 1}{n_{j-1} - 1} \binom{m_j + n_j + n_{j-1} - 1}{m_j}$$

with the convention $n_{-1} = 1$ and $n_{h+1} = 0$.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. By the discussions at the beginning of Section 4.1, we only need to show that $A_n^*(t)$ is unimodal for each $n \geq 1$.

Comparing the generating functions (4.5) and (4.2) we see that

$$A_n^*(t) = (1+t)^n Q_n \left(t, 1, 1, t, \frac{t^2 + t + 1}{1+t} \right).$$

It follows from (4.3) that

$$(4.6) \quad \sum_{n \geq 0} A_n^*(t) x^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \dots}}},$$

where $b_n = n(t+1)^2 + 1 + t + t^2$ and $\lambda_n = n^2t(1+t)^2$. In view of the Jacobi–Rogers formula (Lemma 4.4), we have

$$(4.7) \quad A_n^*(t) = b_0^n + \sum_{h \geq 0} \sum_{\substack{n_0, \dots, n_h \geq 1 \\ m_0, \dots, m_{h+1} \geq 0}} (b_0^{m_0} \cdots b_{h+1}^{m_{h+1}}) (\lambda_1^{n_0} \cdots \lambda_{h+1}^{n_h}) \cdot \rho(\mathbf{n}, \mathbf{m}),$$

where

$$(4.8) \quad 2(n_0 + \cdots + n_h) + (m_0 + \cdots + m_{h+1}) = n.$$

Note that both the polynomials $b_n = (n+1) + (2n+1)t + (n+1)t^2$ and $\lambda_n = n^2t(1+t)^2$ are palindromic and unimodal. In view of (4.8), each product in the summation (4.7) of $A_n^*(t)$ is also palindromic and unimodal with center of symmetry n . Hence $A_n^*(t)$ is palindromic and unimodal with center of symmetry n . \square

4.2. The log-convexity of $\tilde{A}_n(t)$ and $A_n^*(t)$. There has been recent interest in the log-convexity of combinatorial sequences or polynomials (cf. [18, 33]). Let \mathcal{L} be the operator which maps a sequence $\{f_n(q)\}_{n \geq 0}$ of polynomials with real coefficients to the polynomial sequence $\{g_n(q)\}_{n \geq 0}$ defined by

$$g_i(q) := f_{i+1}(q)f_{i-1}(q) - f_i(q)^2.$$

Then the sequence $\{f_n(q)\}_{n \geq 0}$ is called *k-q-log-convex* if $\mathcal{L}^k \{f_n(q)\}_{n \geq 0}$ is a sequence of polynomials with non-negative coefficients.

Before we proceed to show the log-convexity of $\tilde{A}_n(t)$ and $A_n^*(t)$, we need the following continued fraction expansion for the ordinary generating function of $\tilde{A}_n(t)$.

Lemma 4.5. *We have*

$$(4.9) \quad \sum_{n \geq 0} \tilde{A}_n(t)x^n = \frac{1}{1 - \gamma_0 x - \frac{\beta_1 x^2}{1 - \gamma_1 x - \frac{\beta_2 x^2}{1 - \gamma_2 x - \cdots}}},$$

where $\gamma_n = (n+1)(t+1)$ and $\beta_n = n^2t$.

Proof. By (2.5) we have

$$\sum_{n \geq 0} \tilde{A}_n(t) \frac{x^n}{n!} = \frac{(t-1)e^{tx}}{t - e^{(t-1)x}}.$$

Comparing with (4.2), we deduce that $\tilde{A}_n(t) = Q_n(t, 1, 1, t, t+1)$. The continued fraction expansion (4.9) then follows from (4.3). \square

Theorem 4.6. *The polynomial sequences $\{\tilde{A}_n(q)\}_{n \geq 1}$ and $\{A_n^*(q)\}_{n \geq 1}$ are 3-q-log-convex.*

Proof. By a criterion of Zhu [33, Theorem 2.2], it is routine to check (for instance, by Maple) that the continued fraction expansion (4.6) implies the 3-q-log-convexity of $\{A_n^*(q)\}_{n \geq 1}$. The 3-q-log-convexity of the sequence $\{\tilde{A}_n(q)\}_{n \geq 1}$ follows in the same fashion from the continued fraction expansion (4.9) for $\sum_{n \geq 0} \tilde{A}_n(t)x^n$. \square

4.3. **A new (p, q) -extension of the γ -positivity of $\tilde{A}_n(t)$ via continued fraction.** Let us introduce the polynomials $\hat{A}_n(t, p, q)$ by

$$(4.10) \quad \sum_{n \geq 0} \hat{A}_n(t, p, q) x^n = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - \dots}}},$$

where $b_n = (1+t)[n+1]_{p,q}$ and $\lambda_n = tq[n]_{p,q}^2$ with the usual notation $[n]_{p,q} = p^{n-1} + p^{n-2}q + \dots + pq^{n-2} + q^{n-1}$. In view of (4.9), we have

$$\hat{A}_n(t, 1, 1) = \tilde{A}_n(t),$$

so $\hat{A}_n(t, p, q)$ is a (p, q) -analog of the binomial-Eulerian polynomials. The first few values of $\hat{A}_n(t, p, q)$ are

$$\begin{aligned} \hat{A}_1(t, p, q) &= 1 + t, \\ \hat{A}_2(t, p, q) &= 1 + (2 + q)t + t^2, \\ \hat{A}_3(t, p, q) &= 1 + (3 + 2q + q^2 + pq)t + (3 + 2q + q^2 + pq)t^2 + t^3. \end{aligned}$$

The rest of this section is devoted to proving a (p, q) - γ -positivity decomposition of $\hat{A}_n(t, p, q)$, involving crossings, nestings and generalized patterns of permutations.

Definition 4.7. For any permutation $\sigma \in \mathfrak{S}_n$, the numbers of *crossings*, *nestings*, *drops*, *2-31 patterns*, *31-2 patterns* and *foremaxima* are defined, respectively, by

$$\begin{aligned} \text{cros}(\sigma) &:= |\{(i, j) \in [n] \times [n] : i < j \leq \sigma_i < \sigma_j \text{ or } i > j > \sigma_i > \sigma_j\}|, \\ \text{nest}(\sigma) &:= |\{(i, j) \in [n] \times [n] : i < j \leq \sigma_j < \sigma_i \text{ or } i > j > \sigma_j > \sigma_i\}|, \\ \text{drop}(\sigma) &:= |\{2 \leq i \leq n : \sigma_i < i\}|, \\ (2-31)\sigma &:= |\{(i, j) : 1 \leq i < j \leq n-1 \text{ and } \sigma_{j+1} < \sigma_i < \sigma_j\}|, \\ (31-2)\sigma &:= |\{(i, j) : 2 \leq i < j \leq n \text{ and } \sigma_i < \sigma_j < \sigma_{i-1}\}|, \\ \text{fmax}(\sigma) &:= |\{i \in [n] : \sigma_j < \sigma_i \text{ for all } 1 \leq j < i \text{ and } \sigma_i < \sigma_{i+1}\}|. \end{aligned}$$

For instance, if $\sigma = 42513 \in \mathfrak{S}_5$, then $\text{cros}(\sigma) = 1$, $\text{nest}(\sigma) = 1$, $\text{drop}(\sigma) = 2$, $(2-31)\sigma = 2$, $(31-2)\sigma = 2$ and $\text{fmax}(\sigma) = 0$.

The q -binomial-Eulerian polynomial $\hat{A}_n(t, 1, q)$ arose in Williams' enumeration of totally positive Grassmann cells (see [31, Lemma 5]), while the (p, q) -analog $\hat{A}_n(t, p, q)$ first appeared in the work of Corteel [7, Proposition 7], where she showed that

$$(4.11) \quad \hat{A}_n(t, p, q) = \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest}(\sigma)} q^{\text{cros}(\sigma) + \text{drop}(\sigma)} (1+t)^{\text{fix}(\sigma)} t^{\text{exc}(\sigma)}.$$

Consider the common enumerative polynomial (see [26, Theorem 5])

$$(4.12) \quad \begin{aligned} B_n(p, q, t, u, v, w, y) &= \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest}(\sigma)} q^{\text{cros}(\sigma)} t^{\text{drop}(\sigma)} u^{\text{cdrise}(\sigma)} v^{\text{cdfall}(\sigma)} w^{\text{cval}(\sigma)} y^{\text{fix}(\sigma)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} p^{(2-31)\sigma} q^{(31-2)\sigma} t^{\text{des}(\sigma)} u^{\text{da}^*(\sigma) - \text{fmax}(\sigma)} v^{\text{dd}(\sigma)} w^{\text{valley}^*(\sigma)} y^{\text{fmax}(\sigma)}, \end{aligned}$$

where $\text{da}^*(\sigma) = \text{da}(\sigma) + \chi(\sigma_1 < \sigma_2)$ and $\text{valley}^*(\sigma) = \text{valley}(\sigma) - \chi(\sigma_1 < \sigma_2)$. Since $\text{cdrise}(\sigma) + \text{cval}(\sigma) = \text{exc}(\sigma)$, it follows from (4.11) that

$$\hat{A}_n(t, p, q) = B_n(p, q, q, t, 1, t, 1+t).$$

This relationship together with interpretation (4.12) of $B_n(p, q, t, u, v, w, y)$ gives another interpretation for $\hat{A}_n(t, p, q)$:

$$(4.13) \quad \hat{A}_n(t, p, q) = \sum_{\sigma \in \mathfrak{S}_n} p^{(2-31)\sigma} q^{(31-2)\sigma + \text{des}(\sigma)} (1+t)^{\text{fmax}(\sigma)} t^{\text{des}(\sigma)},$$

in view of the symmetry $\hat{A}_n(t, p, q) = t^n \hat{A}_n(t^{-1}, p, q)$ proved below.

Theorem 4.8. *We have*

$$\hat{A}_n(t, p, q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \hat{\gamma}_k(p, q) t^k (1+t)^{n-2k},$$

where

$$(4.14) \quad \hat{\gamma}_k(p, q) = \sum_{\sigma \in \hat{\Gamma}_{n,k}} p^{\text{nest}(\sigma)} q^{\text{cros}(\sigma) + k} = \sum_{\sigma \in \tilde{\Gamma}_{n,k}} p^{(2-31)\sigma} q^{(31-2)\sigma + k}$$

with $\hat{\Gamma}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{cdfall}(\sigma) = 0, \text{drop}(\sigma) = k\}$.

Proof. Shin and the third author proved in [26, Eq. (34)] the following continued fraction expansion:

$$(4.15) \quad \sum_{n \geq 0} B_n(p, q, t, u, v, w, y) x^n = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - \dots}}}$$

with $b_n = yp^n + (qu + tv)[n]_{p,q}$ and $\lambda_n = tw[n]_{p,q}^2$. Let

$$P_n(p, q, y) := B_n(p, q, q, 1, 0, y, 1).$$

It follows from (4.15) that

$$(4.16) \quad \sum_{n \geq 0} P_n(p, q, y) x^n = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - \dots}}}$$

with $b_n = [n + 1]_{p,q}$ and $\lambda_n = yq[n]_{p,q}^2$. Comparing (4.16) with (4.10) yields

$$\hat{A}_n(t, p, q) = (1 + t)^n P_n(p, q, y),$$

where $y = \frac{t}{(1+t)^2}$. This is equivalent to

$$(4.17) \quad B_n(p, q, q, 1, 0, y, 1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \hat{\gamma}_{n,k}(p, q) y^k.$$

Since $\text{cval}(\sigma) = \text{drop}(\sigma)$ (resp. $\text{valley}^*(\sigma) = \text{des}(\sigma)$) whenever $\text{cdfall}(\sigma) = 0$ (resp. $\text{dd}(\sigma) = 0$), the interpretations of $\hat{\gamma}_{n,k}(p, q)$ in (4.14) then follow from (4.17) and the definition of $B_n(p, q, t, u, v, w, y)$. \square

Remark 4.9. Foata's first fundamental transformation (cf. [28, Prop. 1.3.1]) establishes a one-to-one correspondence between $\tilde{\Gamma}_{n,k}$ and $\hat{\Gamma}_{n,k}$.

5. CLOSING REMARKS

The elementary approach via quadratic recursion in Section 2 could be applied to prove other known or new sign-balance identities for the Eulerian distributions on restricted permutations, including the descent polynomials of André or Simsun permutations and the excedance polynomials of 321-avoiding permutations. The interested reader is referred to an extended version [15] of this paper for details.

Note that Wachs [30] used a combinatorial involution to prove (1.7). It would be interesting to find analogous combinatorial proof for Theorem 2.2 and Corollary 2.3. A combinatorial polynomial $h(t) = \sum_{k=0}^n a_k(q)t^k \in \mathbb{N}[q][t]$ is q -log-concave if $a_k(q)^2 - a_{k-1}(q)a_{k+1}(q) \in \mathbb{N}[q]$. We propose the following conjectures.

Conjecture 5.1. *The q -binomial-Eulerian polynomial $\tilde{A}_n(t, q)$ is q -log-concave for $n \geq 1$.*

Conjecture 5.2. *The signed binomial-Eulerian polynomial $\tilde{A}_n(t, -1)$ is log-concave for $n \geq 1$.*

The validation of Conjecture 5.2 would imply Theorem 1.3.

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