

Supercongruences arising from hypergeometric series identities

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Abstract. By using some hypergeometric series identities, we prove two supercongruences on truncated hypergeometric series, one of which is related to a modular Calabi–Yau threefold, and the other is regarded as p -adic analogue of an identity due to Ramanujan.

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1 Introduction

Let

$$f(z) := \eta^4(2z)\eta^4(4z) = \sum_{n=1}^{\infty} a(n)q^n,$$

where $q = e^{2\pi iz}$ and the Dedekind eta function is given by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

For odd primes p , let $N(p)$ denote the number of solutions to the modular Calabi–Yau threefold:

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0$$

over the finite field with p elements. Ahlgren and Ono [1], van Geemen and Nygaard [12], and Verrill [14] showed by different methods that

$$a(p) = p^3 - 2p^2 - 7 - N(p).$$

In 2006, Kilbourn [6] proved that for any odd prime p ,

$$a(p) \equiv {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1 \end{matrix}; 1 \right]_{\frac{p-1}{2}} \pmod{p^3}. \quad (1.1)$$

Here the truncated hypergeometric series are given by

$${}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z \right]_n = \sum_{k=0}^n \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \cdot \frac{z^k}{k!},$$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

The first aim of this paper is to prove another supercongruence for $a(p)$.

Theorem 1.1 *For any prime $p \geq 5$, we have*

$$a(p) \equiv p \cdot {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & \frac{3}{4}, & \frac{5}{4} \end{matrix}; 1 \right]_{\frac{p-1}{2}} \pmod{p^3}. \quad (1.2)$$

In 1997, Van Hamme [13, (A.2)] proposed the following supercongruence conjecture.

Conjecture 1.2 *(Van Hamme, 1997) For any odd prime p , we have*

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1 \end{matrix}; -1 \right]_{\frac{p-1}{2}} \\ & \equiv \begin{cases} -p\Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^3} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \end{aligned} \quad (1.3)$$

where $\Gamma_p(\cdot)$ denotes the p -adic Gamma function.

The above supercongruence was regarded as p -adic analogue of the following identity due to Ramanujan (announced in his second letter to Hardy on February 27):

$${}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1 \end{matrix}; -1 \right] = \frac{2}{\Gamma\left(\frac{3}{4}\right)^4},$$

which was later proved by Hardy [5] and Watson [15]. The supercongruence (1.3) was first confirmed by McCarthy and Osburn [9].

In 2015, Swisher [11, Theorem 1.5] also showed that (1.3) holds modulo p^5 for primes $p \equiv 1 \pmod{4}$. Recently, Guo and Schlosser [4, Theorem 2.2] established an interesting q -analogue of a supercongruence closely related to (1.3). By using the software package **Sigma** due to Schneider [10], the author [7, Theorem 1.3] extended the case $p \equiv 3 \pmod{4}$ in (1.3) as follows.

Theorem 1.3 *Let $p \geq 5$ be a prime. For $p \equiv 3 \pmod{4}$, we have*

$${}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1 \end{matrix}; -1 \right]_{\frac{p-1}{2}} \equiv -\frac{p^3}{16} \Gamma_p\left(\frac{1}{4}\right)^4 \pmod{p^4}. \quad (1.4)$$

However, the proof of (1.4) in [7] is based on software package and seems unnatural. The second aim of this paper is to provide a human proof of (1.4) by hypergeometric series identities, which seems to be more natural.

The rest of this paper is organized as follows. Section 2 is devoted to recalling some properties of Gamma function and p -adic Gamma function. We prove Theorems 1.1 and 1.3 in Sections 3 and 4, respectively.

2 Preliminary results

We first recall some properties of Gamma function. The Gamma function $\Gamma(z)$ is an extension of the factorial function, which satisfies the functional equation:

$$\Gamma(z+1) = z\Gamma(z). \quad (2.1)$$

From the above equation, we immediately deduce that for complex numbers z and positive integers n ,

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \quad (2.2)$$

It also satisfies the following reflection formula and duplication formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad (2.3)$$

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \quad (2.4)$$

We next recall the definition and some basic properties of p -adic Gamma function. For more details, we refer to [3, Section 11.6]. Let p be an odd prime and \mathbb{Z}_p denote the set of all p -adic integers. For $x \in \mathbb{Z}_p$, the p -adic Gamma function is defined as

$$\Gamma_p(x) = \lim_{m \rightarrow x} (-1)^m \prod_{\substack{0 < k < m \\ (k,p)=1}} k,$$

where the limit is for m tending to x p -adically in $\mathbb{Z}_{>0}$.

We require several properties of p -adic Gamma function.

Lemma 2.1 (See [3, Section 11.6].) *For any odd prime p and $x, y \in \mathbb{Z}_p$, we have*

$$\Gamma_p(1) = -1, \quad (2.5)$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{s_p(x)}, \quad (2.6)$$

$$\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p} \quad \text{for } x \equiv y \pmod{p}, \quad (2.7)$$

where $s_p(x) \in \{1, 2, \dots, p\}$ with $s_p(x) \equiv x \pmod{p}$.

Lemma 2.2 (See [8, Lemma 17, (4)].) *Let p be an odd prime. If $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1, \dots, a+n-1$ in $p\mathbb{Z}_p$, then*

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \quad (2.8)$$

3 Proof of Theorem 1.1

Let ω be any primitive 3th root of unity. Letting $a = \frac{1}{2}$, $b = \frac{1-\omega p}{2}$, $c = \frac{1-\omega^2 p}{2}$, $k = \frac{3}{2}$, $m = \frac{p-1}{2}$ in [2, (1), page 32], we obtain

$$\begin{aligned} & {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1-\omega p}{2}, & \frac{1-\omega^2 p}{2}, & \frac{1-p}{2} \\ & 1 + \frac{\omega p}{2}, & 1 + \frac{\omega^2 p}{2}, & 1 + \frac{p}{2} \end{matrix}; 1 \right] \\ &= \frac{p \left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}}} {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1-\omega}{2}, & \frac{1-\omega^2 p}{2}, & \frac{1-p}{2} \\ & 1, & \frac{3}{4}, & \frac{5}{4} \end{matrix}; 1 \right]. \end{aligned} \quad (3.1)$$

By the fact that

$$(u + vp)(u + vp\omega)(u + vp\omega^2) = u^3 + v^3 p^3,$$

we have

$$(u + vp)_k (u + vp\omega)_k (u + vp\omega^2)_k \equiv (u)_k^3 \pmod{p^3}. \quad (3.2)$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} & {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & 1, & 1 \end{matrix}; 1 \right]_{\frac{p-1}{2}} \\ &\equiv \frac{p \left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}}} {}_4F_3 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & \frac{3}{4}, & \frac{5}{4} \end{matrix}; 1 \right]_{\frac{p-1}{2}} \pmod{p^3}. \end{aligned} \quad (3.3)$$

In order to prove (1.2), by (1.1) and (3.3) it suffices to show that

$$\frac{\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}}} \equiv 1 \pmod{p^3}. \quad (3.4)$$

By (3.2), we have

$$\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}} \equiv (1)_{\frac{p-1}{2}}^3 \pmod{p^3},$$

and so

$$\frac{\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}}} \equiv \frac{\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}}^3} \pmod{p^3}.$$

Furthermore, we have

$$\left(\frac{1-p}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} (1)_{\frac{p-1}{2}},$$

and

$$\left(\frac{1}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} \left(1 - \frac{p}{2}\right)_{\frac{p-1}{2}}.$$

Thus,

$$\frac{\left(\frac{1}{2}\right)_{\frac{p-1}{2}} \left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1 + \frac{\omega p}{2}\right)_{\frac{p-1}{2}} \left(1 + \frac{\omega^2 p}{2}\right)_{\frac{p-1}{2}}} \equiv \frac{\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(1 - \frac{p}{2}\right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}}^2} \pmod{p^3}. \quad (3.5)$$

We next evaluate the product on the right-hand side of (3.5) modulo p^4 :

$$\frac{\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(1 - \frac{p}{2}\right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}}^2} = \prod_{j=1}^{\frac{p-1}{2}} \left(1 - \frac{p^2}{4j^2}\right).$$

From the following Taylor expansion:

$$\prod_{j=1}^{\frac{p-1}{2}} (a_j + b_j x^2) = \prod_{j=1}^{\frac{p-1}{2}} a_j \cdot \left(1 + x^2 \sum_{j=1}^{\frac{p-1}{2}} \frac{b_j}{a_j}\right) + \mathcal{O}(x^4),$$

we deduce that

$$\frac{\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(1 - \frac{p}{2}\right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}}^2} \equiv 1 - \frac{p^2}{4} \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \pmod{p^4}.$$

By Wolstenholme's theorem, we have

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv \frac{1}{2} \left(\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} + \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{(p-j)^2} \right) = \frac{1}{2} \sum_{j=1}^{p-1} \frac{1}{j^2} \equiv 0 \pmod{p}.$$

It follows that

$$\frac{\left(1 + \frac{p}{2}\right)_{\frac{p-1}{2}} \left(1 - \frac{p}{2}\right)_{\frac{p-1}{2}}}{(1)_{\frac{p-1}{2}}^2} \equiv 1 \pmod{p^3}. \quad (3.6)$$

Combining (3.5) and (3.6), we complete the proof of (3.4).

4 A human proof of Theorem 1.3

Letting $a = \frac{1}{2}$, $x = 2n + \frac{3}{2}$ in [16, (14.1)], we obtain

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & -2n-1, & 2n+2, & \frac{1}{2}+y, & \frac{1}{2}-y \\ & \frac{1}{4}, & 2n+\frac{5}{2}, & -2n-\frac{1}{2}, & 1-y, & 1+y \end{matrix}; -1 \right] \\ &= \frac{\pi \Gamma(-2n - \frac{1}{2}) \Gamma(2n + \frac{5}{2}) \Gamma(1+y) \Gamma(1-y)}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) \Gamma(n + \frac{y}{2} + \frac{3}{2}) \Gamma(n - \frac{y}{2} + \frac{3}{2}) \Gamma(-n + \frac{y}{2}) \Gamma(-n - \frac{y}{2})}. \end{aligned} \quad (4.1)$$

Note that

$$\begin{aligned} & \Gamma\left(-2n - \frac{1}{2}\right) \Gamma\left(2n + \frac{5}{2}\right) \stackrel{(2.1)}{=} \left(2n + \frac{3}{2}\right) \Gamma\left(-2n - \frac{1}{2}\right) \Gamma\left(2n + \frac{3}{2}\right) \\ & \stackrel{(2.3)}{=} \frac{(4n+3)\pi}{2 \sin\left(\left(2n + \frac{3}{2}\right)\pi\right)} \\ & = -\frac{(4n+3)\pi}{2}, \end{aligned} \quad (4.2)$$

and

$$\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2}. \quad (4.3)$$

Also,

$$\Gamma(1+y)\Gamma(1-y) \stackrel{(2.4)}{=} \frac{1}{\pi} \Gamma\left(\frac{1+y}{2}\right) \Gamma\left(\frac{1-y}{2}\right) \Gamma\left(1 + \frac{y}{2}\right) \Gamma\left(1 - \frac{y}{2}\right). \quad (4.4)$$

Substituting (4.2)–(4.4) into the right-hand side of (4.1) gives

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & -2n-1, & 2n+2, & \frac{1}{2}+y, & \frac{1}{2}-y \\ & \frac{1}{4}, & 2n+\frac{5}{2}, & -2n-\frac{1}{2}, & 1-y, & 1+y \end{matrix}; -1 \right] \\ &= -(4n+3) \frac{\Gamma\left(\frac{1+y}{2}\right) \Gamma\left(\frac{1-y}{2}\right) \Gamma\left(1 + \frac{y}{2}\right) \Gamma\left(1 - \frac{y}{2}\right)}{\Gamma\left(n + \frac{y}{2} + \frac{3}{2}\right) \Gamma\left(n - \frac{y}{2} + \frac{3}{2}\right) \Gamma\left(-n + \frac{y}{2}\right) \Gamma\left(-n - \frac{y}{2}\right)} \\ & \stackrel{(2.2)}{=} -(4n+3) \frac{\left(-n + \frac{y}{2}\right)_{n+1} \left(-n - \frac{y}{2}\right)_{n+1}}{\left(\frac{1+y}{2}\right)_{n+1} \left(\frac{1-y}{2}\right)_{n+1}} \\ &= -(4n+3) \frac{\left(\frac{y}{2}\right)_{n+1} \left(-n + \frac{y}{2}\right)_{n+1}}{\left(-n + \frac{y-1}{2}\right)_{2n+2}}. \end{aligned} \quad (4.5)$$

Let i be any primitive 4th root of unity. Setting $n = \frac{p-3}{4}$ and $y = -\frac{ip}{2}$ in (4.5) yields

$$\begin{aligned} & {}_6F_5 \left[\begin{matrix} \frac{5}{4}, & \frac{1}{2}, & \frac{1-p}{2}, & \frac{1+p}{2}, & \frac{1-ip}{2}, & \frac{1+ip}{2} \\ & \frac{1}{4}, & 1 - \frac{p}{2}, & 1 + \frac{p}{2}, & 1 - \frac{ip}{2}, & 1 + \frac{ip}{2} \end{matrix}; -1 \right] \\ &= -\frac{p \left(-\frac{ip}{4}\right)_{\frac{p+1}{4}} \left(\frac{3-(i+1)p}{4}\right)_{\frac{p+1}{4}}}{\left(\frac{1-(i+1)p}{4}\right)_{\frac{p+1}{2}}}. \end{aligned} \quad (4.6)$$

By the fact that

$$(u+vp)(u-vp)(u+vp i)(u-vp i) = u^4 - v^4 p^4,$$

we have

$$(u+vp)_k(u-vp)_k(u+vp i)_k(u-vp i)_k \equiv (u)_k^4 \pmod{p^4}. \quad (4.7)$$

In order to prove (1.4), by (4.6) and (4.7) it suffices to show that

$$-\frac{p \left(-\frac{ip}{4}\right)_{\frac{p+1}{4}} \left(\frac{3-(i+1)p}{4}\right)_{\frac{p+1}{4}}}{\left(\frac{1-(i+1)p}{4}\right)_{\frac{p+1}{2}}} \equiv -\frac{p^3}{16} \Gamma_p \left(\frac{1}{4}\right)^4 \pmod{p^4}. \quad (4.8)$$

Note that

$$\left(-\frac{ip}{4}\right)_{\frac{p+1}{4}} \left(\frac{3-(i+1)p}{4}\right)_{\frac{p+1}{4}} = -\frac{p^2}{16} \prod_{j=1}^{\frac{p-3}{4}} \left(-\frac{p^2}{16} - j^2\right),$$

and

$$\left(\frac{1-(i+1)p}{4}\right)_{\frac{p+1}{2}} = \prod_{j=1}^{\frac{p+1}{4}} \left(-\frac{p^2}{16} - \left(-\frac{1}{2} + j\right)^2\right).$$

It follows that

$$\begin{aligned} -\frac{p \left(-\frac{ip}{4}\right)_{\frac{p+1}{4}} \left(\frac{3-(i+1)p}{4}\right)_{\frac{p+1}{4}}}{\left(\frac{1-(i+1)p}{4}\right)_{\frac{p+1}{2}}} &= \frac{p^3}{16} \cdot \frac{\prod_{j=1}^{\frac{p-3}{4}} \left(-\frac{p^2}{16} - j^2\right)}{\prod_{j=1}^{\frac{p+1}{4}} \left(-\frac{p^2}{16} - \left(-\frac{1}{2} + j\right)^2\right)} \\ &\equiv -\frac{p^3}{16} \cdot \frac{(1)_{\frac{p-3}{4}}^2}{\left(\frac{1}{2}\right)_{\frac{p+1}{4}}^2} \pmod{p^4} \\ &\stackrel{(2.8)}{=} -\frac{p^3}{16} \cdot \frac{\Gamma_p \left(\frac{p+1}{4}\right)^2 \Gamma_p \left(\frac{1}{2}\right)^2}{\Gamma_p(1)^2 \Gamma_p \left(\frac{p+3}{4}\right)^2}. \end{aligned}$$

Furthermore, by (2.5)–(2.7), we have

$$\begin{aligned} -\frac{p \left(-\frac{ip}{4}\right)_{\frac{p+1}{4}} \left(\frac{3-(i+1)p}{4}\right)_{\frac{p+1}{4}}}{\left(\frac{1-(i+1)p}{4}\right)_{\frac{p+1}{2}}} &\equiv -\frac{p^3}{16} \cdot \frac{\Gamma_p\left(\frac{1}{4}\right)^2}{\Gamma_p\left(\frac{3}{4}\right)^2} \pmod{p^4} \\ &= -\frac{p^3}{16} \Gamma_p\left(\frac{1}{4}\right)^4. \end{aligned}$$

This completes the proof of (4.8).

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