

A NEW PROOF AND EXTENSION OF THE ODDS-THEOREM

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ABSTRACT. There are n independent Bernoulli random variables I_k with parameters p_k that are observed sequentially. We consider a generalization of the Last-Success-Problem considering w_k positive payments if the player successfully predicts that the last "1" occurs in the variable I_k . We establish the optimal strategy and the expected profit in similar terms to the Odds-Theorem. The proof provided here is an alternative proof to the one Bruss provides in his Odds-Theorem (case $w_i = 1$) that is even simpler and more elementary than his proof.

Keywords: Secretary problem; Last-Success-Problem; Odds-Theorem; Threshold strategy; Stopping problem

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1. INTRODUCTION

The Last-Success-Problem is the problem of maximizing the probability of stopping on the last success in a finite sequence of Bernoulli trials. The framework is as follows. There are n Bernoulli random variables which are observed sequentially. The problem is to find a stopping rule to maximize the probability of stopping on the last "1". We restrict ourselves here to the case in which the random variables are independent. This problem has been studied by Hill and Krengel [7] and Hsiao and Yang [8] for the case in which the random variables are independent and was simply and elegantly solved by T.F. Bruss in [2] with the following famous result.

Theorem 1. (*Odds-Theorem, F.T. Bruss 2000*). *Let I_1, I_2, \dots, I_n be n independent Bernoulli random variables with known n . We denote by $(i = 1, \dots, n)$ p_i , the parameter of I_i ; i.e. ($p_i = P(I_i = 1)$). Let $q_i = 1 - p_i$ and $r_i = p_i/q_i$. We define the index*

$$s = \begin{cases} \max\{1 \leq k \leq n : \sum_{j=k}^n r_j \geq 1\}, & \text{if } \sum_{i=1}^n r_i \geq 1 ; \\ 1, & \text{otherwise} \end{cases}$$

To maximize the probability of stopping on the last "1" of the sequence, it is optimal to stop on the first "1" we encounter among the variables I_s, I_{s+1}, \dots, I_n .

The optimal win probability is given by

$$\mathcal{V}(p_1, \dots, p_n) := \left(\prod_{j=s}^n q_j \right) \left(\sum_{i=s}^n r_i \right)$$

This theorem was extended by T. Ferguson (see [4]) in several ways. First, considering an infinite number of Bernoulli variables. Second, the payoff for not stopping is allowed to be different from the payoff for stopping on a success that is not the last success. Third, the Bernoulli variables are allowed to be dependent.

In this paper, we present a generalization of the Last-Success-Problem, considering a positive payoff, w_k , if the player stops on the last success and this occurs at the k -th event. We establish the optimal strategy and the expected profit in similar terms to the Odds-Theorem. The proof we provide constitutes an alternative proof to the one provided by Bruss that is even simpler and more elementary.

2. THRESHOLD STRATEGIES

In this section, we shall show that, under certain conditions, the optimal strategy is a threshold strategy. Dynamic programming provides the probability of winning and the optimal strategy in a simple way. In what follows, we shall take into account the following definitions.

Definition 1. Let us define the following functions.

- $\mathbb{E}_{\text{Stop}}(k)$ is the expected profit if we stop at the k -th event with $I_k = 1$

$$\mathbb{E}_{\text{Stop}}(k) := w_k \prod_{i=k+1}^n (1 - p_i)$$

- $\mathbb{E}_{\text{Keep}}(k)$ is the expected profit after observing the r -th event and continuing (not stopping) in order to adopt the optimal strategy later on. The dynamic program that defines it by recurrence is:

$$\mathbb{E}_{\text{Keep}}(n) = 0$$

$$\mathbb{E}_{\text{Keep}}(k) = p_{k+1} \cdot \max \{ \mathbb{E}_{\text{Stop}}(k+1), \mathbb{E}_{\text{Keep}}(k+1) \} + (1 - p_k) \cdot \mathbb{E}_{\text{Keep}}(k+1)$$

Proposition 1. *With the above definitions, it is obvious that the following strategy is optimal:*

- ◇ *Stop if $I_k = 1$ and $\mathbb{E}_{\text{Stop}}(k) > \mathbb{E}_{\text{Keep}}(k)$ and continue otherwise.*

In addition, using this strategy, the expected profit is $\mathbb{E}_{\text{Keep}}(0)$.

Definition 2. We denote by the stopping set the set of indices in which the decision to stop is optimal if the corresponding event is successful. That is:

$$\Upsilon_n := \{k : \mathbb{E}_{\text{Stop}}(k) > \mathbb{E}_{\text{Keep}}(k)\}$$

Example 1. *Let us consider 9 random Bernoulli variables with the following parameters, p_i , and payoffs, w_i :*

$$\{p_1 = \frac{1}{6}, p_2 = \frac{1}{10}, p_3 = \frac{1}{12}, p_4 = \frac{1}{3}, p_5 = \frac{1}{12}, p_6 = \frac{1}{10}, p_7 = \frac{1}{5}, p_8 = \frac{1}{10}, p_9 = \frac{1}{12}\}$$

$$\{w_1 = 7, w_2 = 4, w_3 = 9, w_4 = 10, w_5 = 6, w_6 = 3, w_7 = 9, w_8 = 9, w_9 = 1\}$$

The corresponding dynamic program returns:

$$\text{ExpectedProfit} = \mathbb{E}_{\text{Keep}}(0) = \frac{6721}{2000}$$

and the stopping set

$$\text{StoppingSet} = \{4, 5, 7, 8, 9\}$$

Definition 3. If the stopping set has a single stopping island, $\Upsilon_n = \{k : k \geq \mathbf{k}\}$, we shall say that the optimal strategy is a *threshold strategy* and, in this case, $\mathbf{k} := \min \Upsilon_n$ is the *optimal threshold*. In the terminology of Chow, Robbins and Siegmund (see [3]), we also state that the problem is a *monotone problem*, which is not the case in the aforementioned example.

Remark. Note that, for the optimal threshold, we have that

$$\mathbf{k} = \min\{k : \mathbb{E}_{\text{Stop}}(k) > \mathbb{E}_{\text{Keep}}(k)\} = 1 + \max\{k : \mathbb{E}_{\text{Stop}}(k) \leq \mathbb{E}_{\text{Keep}}(k)\}$$

The following two easy results characterize monotone problems.

Proposition 2. *The problem is monotone if and only if for all $0 < k < n$*

$$\mathbb{E}_{\text{Stop}}(k) > \mathbb{E}_{\text{Keep}}(k) \Rightarrow \mathbb{E}_{\text{Stop}}(k+1) > \mathbb{E}_{\text{Keep}}(k+1)$$

Proposition 3. *The problem is monotone if and only if for all $0 < k \leq n$*

$$\mathbb{E}_{\text{Stop}}(k) - \mathbb{E}_{\text{Keep}}(k)$$

change sign at the most once.

With the following result, we present a sufficient condition for the problem to be monotone. In particular, when the payment function, w_k , is non-decreasing, the problem is monotone.

Proposition 4. *If $w_{k+1} \geq (1 - p_{k+1})w_k$ for all $k \in \{1, \dots, n-1\}$, then the problem is monotone.*

Proof. If we take into account the associated dynamic program, we see that $\mathbb{E}_{\text{Keep}}(k)$ is non-increasing

$$\mathbb{E}_{\text{Keep}}(k) = p_{k+1} \max\{\mathbb{E}_{\text{Stop}}(k+1), \mathbb{E}_{\text{Keep}}(k+1)\} + (1-p_k)\mathbb{E}_{\text{Keep}}(k+1) \geq \mathbb{E}_{\text{Keep}}(k+1)$$

On the other hand, \mathbb{E}_{Stop} is non-decreasing since

$$\frac{\mathbb{E}_{\text{Stop}}(k+1)}{\mathbb{E}_{\text{Stop}}(k)} = \frac{w_{k+1} \prod_{i=k+2}^n (1-p_i)}{w_k \prod_{i=k+1}^n (1-p_i)} = \frac{w_{k+1}}{w_k(1-p_{k+1})} \geq 1$$

As a consequence, given that \mathbb{E}_{Stop} is non-decreasing and \mathbb{E}_{Keep} is non-increasing,

$$\mathbb{E}_{\text{Stop}}(k) \geq \mathbb{E}_{\text{Keep}}(k) \Rightarrow \mathbb{E}_{\text{Stop}}(k+1) \geq \mathbb{E}_{\text{Keep}}(k+1)$$

and we are able to use Proposition 2. \square

With Proposition 2, it became evident that for the problem to be monotone, it is sufficient for $\mathbb{E}_{\text{Stop}}(r)$ to be non-decreasing. However, this is not a necessary condition. Actually, the problem is monotone if and only if the difference $\mathbb{E}_{\text{Stop}}(k) - \mathbb{E}_{\text{Keep}}(k)$ presents one change of sign at the most. However, the verification of this statement presents difficulties as the dynamic program does not allow us to know an explicit expression of $\mathbb{E}_{\text{Keep}}(r)$. We shall see how to overcome this difficulty below.

Definition 4. Let us denote by $\bar{\mathbb{E}}_{\text{Keep}}(k)$ the expected profit after observing the k -th event and continuing in order to stop on the next success to be found.

$$\bar{\mathbb{E}}_{\text{Keep}}(k) := \sum_{i=k+1}^n \left(\prod_{j=k+1}^{i-1} (1-p_j) \right) \cdot p_i \cdot \mathbb{E}_{\text{Stop}}(i)$$

In other words, $\bar{\mathbb{E}}_{\text{Keep}}(k)$ is the expected profit using the strategy of stopping on the first success after the k -th event.

It is clear from the definition itself that $\overline{\mathbb{E}}_{\text{Keep}}(k) \leq \mathbb{E}_{\text{Keep}}(k)$.

Lemma 1. *Let r_0 be such that $\mathbb{E}_{\text{Stop}}(r) > \overline{\mathbb{E}}_{\text{Keep}}(r)$ for every $r > r_0$. Then, $\mathbb{E}_{\text{Stop}}(r) > \mathbb{E}_{\text{Keep}}(r)$ for every $r > r_0$.*

Proof. Given r_0 , let us consider the set $S = \{r > r_0 : \mathbb{E}_{\text{Stop}}(r) \leq \mathbb{E}_{\text{Keep}}(r)\}$. It is necessary to prove that $S = \emptyset$. Let us assume that S is nonempty and let r' be its maximum. This means that $\mathbb{E}_{\text{Stop}}(r') \leq \mathbb{E}_{\text{Keep}}(r')$ and $\mathbb{E}_{\text{Stop}}(r') > \overline{\mathbb{E}}_{\text{Keep}}(r')$, while $\mathbb{E}_{\text{Stop}}(\mathbf{r}') > \mathbb{E}_{\text{Keep}}(\mathbf{r}')$ for all $\mathbf{r}' > r'$; but this is a contradiction. This is because if $\mathbb{E}_{\text{Stop}}(\mathbf{r}') > \mathbb{E}_{\text{Keep}}(\mathbf{r}')$ for all $\mathbf{r}' > r'$, then $\mathbb{E}_{\text{Keep}}(r') = \overline{\mathbb{E}}_{\text{Keep}}(r')$. \square

Using this lemma, it is possible to reformulate Proposition 2 and Proposition 3 in terms of $\overline{\mathbb{E}}_{\text{Keep}}(r)$, which we can know explicitly.

Proposition 5. *If for all $0 < r < n$ the following is true*

$$\mathbb{E}_{\text{Stop}}(r) > \overline{\mathbb{E}}_{\text{Keep}}(r) \Rightarrow \mathbb{E}_{\text{Stop}}(r+1) > \overline{\mathbb{E}}_{\text{Keep}}(r+1)$$

then the problem is monotone.

Proof. Let r_0 be the minimum of the stopping set. $\mathbb{E}_{\text{Stop}}(r_0) > \overline{\mathbb{E}}_{\text{Keep}}(r_0)$ and using the hypothesis inductively, we have that $\mathbb{E}_{\text{Stop}}(r) > \overline{\mathbb{E}}_{\text{Keep}}(r)$ for all $r \geq r_0$. We thus find ourselves within the conditions of Lemma 1 and hence $\mathbb{E}_{\text{Stop}}(r) > \mathbb{E}_{\text{Keep}}(r)$ for all $r \geq r_0$.

Proposition 6. *The problem is monotone if and only if for all $0 < k \leq n$*

$$\mathbb{E}_{\text{Stop}}(k) - \overline{\mathbb{E}}_{\text{Keep}}(k)$$

change sign at the most once.

\square

Proposition 7. *If the problem is monotone and \mathbf{k} is the optimal threshold, then*

$$\begin{aligned} \mathbb{E}_{\text{Stop}}(r) > \mathbb{E}_{\text{Keep}}(r) &\iff \mathbb{E}_{\text{Stop}}(r) > \overline{\mathbb{E}}_{\text{Keep}}(r) \\ \mathbb{E}_{\text{Keep}}(r) &= \begin{cases} \overline{\mathbb{E}}_{\text{Keep}}(r), & \text{if } r \geq \mathbf{k} ; \\ \overline{\mathbb{E}}_{\text{Keep}}(\mathbf{k} - 1), & \text{if } r < \mathbf{k} \end{cases} \end{aligned}$$

3. THE EXTENDED ODDS-THEOREM

Theorem 2. *Let I_1, I_2, \dots, I_n be n independent Bernoulli random variables with parameter p_i . Let w_i be real positive numbers that represent the payments a player receives for indicating the last "1" in the variable I_i . We define the index (with auxiliary $w_0 := 0$)*

$$\mathbf{s} = \max \left\{ k : \sum_{j=k}^n \frac{w_j \cdot p_j}{1 - p_j} \geq w_{k-1} \right\}$$

If the problem is monotone, then \mathbf{s} is the optimal threshold. That is, to maximize the expected profit, it is optimal to stop on the first "1" we encounter among the variables $I_{\mathbf{s}}, \dots, I_n$. Furthermore, with this strategy, the expected profit is:

$$\mathbb{E} = \begin{cases} \left(\prod_{j=\mathbf{s}}^n (1 - p_j) \right) \sum_{i=\mathbf{s}}^n \frac{w_i \cdot p_i}{1 - p_i}, & \text{if } p_{\mathbf{s}} < 1 ; \\ w_{\mathbf{s}} \cdot \prod_{j=\mathbf{s}+1}^n (1 - p_j), & \text{if } p_{\mathbf{s}} = 1 \end{cases}$$

Proof. Recall that the optimal threshold is

$$\mathbf{k} = 1 + \max\{k : \mathbb{E}_{\text{Stop}}(k) \leq \mathbb{E}_{\text{Keep}}(k)\} = \max\{k : \mathbb{E}_{\text{Stop}}(k-1) \leq \mathbb{E}_{\text{Keep}}(k-1)\}.$$

We shall first assume that $p_{\mathbf{k}} < 1$ and hence $p_k < 1$ for all $k > \mathbf{k}$. Bear in mind that if $p_k = 1$ for some $k > \mathbf{k}$, then $\mathbb{E}_{\text{Stop}}(\mathbf{k}) = 0$, which would be a contradiction. We shall first prove that $\mathbf{s} = \mathbf{k}$.

$$\mathbf{k} = \max\left\{k : \mathbb{E}_{\text{Stop}}(k-1) \leq \sum_{i=k}^n \left(\prod_{j=k}^{i-1} (1-p_j) \right) \cdot p_i \cdot \mathbb{E}_{\text{Stop}}(i) \right\}$$

as

$$\mathbb{E}_{\text{Stop}}(i) = w_i \prod_{t=i+1}^n (1-p_t)$$

$$\mathbf{k} = \max \left\{ k : w_{k-1} \prod_{t=k}^n (1-p_t) \leq \sum_{i=k}^n \left(\prod_{j=k}^{i-1} (1-p_j) \right) \cdot p_i \cdot w_i \prod_{t=i+1}^n (1-p_t) \right\}$$

$$\mathbf{k} = \max \left\{ k : w_{k-1} \prod_{t=k}^n (1-p_t) \leq \sum_{i=k}^n \left(\prod_{j=k}^n (1-p_j) \right) \cdot \frac{p_i}{1-p_i} \cdot w_i \right\}$$

$$\mathbf{k} = \max \left\{ k : w_{k-1} \leq \sum_{i=k}^n \frac{p_i}{1-p_i} \cdot w_i \right\} = \mathbf{s}$$

As to the value of the expected profit, which is in fact $\overline{\mathbb{E}}_{\text{Keep}}(\mathbf{s}-1)$, we have

$$\overline{\mathbb{E}}_{\text{Keep}}(\mathbf{s}-1) = \sum_{i=\mathbf{s}}^n \left(\left(\prod_{j=\mathbf{s}}^{i-1} (1-p_j) \right) p_i \cdot \mathbb{E}_{\text{Stop}}(i) \right)$$

$$\overline{\mathbb{E}}_{\text{Keep}}(\mathbf{s}-1) = \sum_{i=\mathbf{s}}^n \left(\left(\prod_{j=\mathbf{s}}^{i-1} (1-p_j) \right) p_i \cdot w_i \prod_{t=i+1}^n (1-p_t) \right)$$

and, carrying out the same operations as before, we have that

$$\mathbb{E} = \overline{\mathbb{E}}_{\text{Keep}}(\mathbf{s}-1) = \left(\prod_{j=\mathbf{s}}^n (1-p_j) \right) \sum_{i=\mathbf{s}}^n \frac{w_i \cdot p_i}{1-p_i}$$

If $p_{\mathbf{k}} = 1$, the proof that $\mathbf{s} = \mathbf{k}$ is the same. As for the expected profit, bearing in mind that we shall stop at the \mathbf{s} -th variable with probability 1, then

$$\mathbb{E} = \overline{\mathbb{E}}_{\text{Keep}}(\mathbf{s}-1) = \mathbb{E}_{\text{Stop}}(\mathbf{s}) = w_{\mathbf{s}} \cdot \left(\prod_{j=\mathbf{k}+1}^n (1-p_j) \right).$$

□

The previous proposition has as its particular case the famous Odds-Theorem (Theorem 1) when considering $w_i = 1$. The proof provided here is even more elementary and simpler than that provided by Bruss. The preparatory results cannot be said to be absolutely original in substance, but they are so in terms of their elucidation and hence the paper may be said to be fully self-contained.

4. SOME APPLICATION EXAMPLES

4.1. The Best-choice Duration Problem. Let us consider the secretary problem with a payment $w_k = (n - k + 1)$ for selecting the best secretary in the k -th interview. Within the context of this paper, we have n independent Bernoulli random variables with parameters $p_k = 1/k$ and payoffs w_k . It is not difficult (though not straightforward) to see that the problem is monotone. In this case, its proof requires using Proposition 6.

$$\mathbf{s}_n = \max \left\{ k : \sum_{j=k}^n \frac{w_j \cdot p_j}{1 - p_j} \geq w_{k-1} \right\} = \max \left\{ k : \sum_{j=k}^n \frac{(n - j + 1) \cdot \frac{1}{j}}{1 - \frac{1}{j}} \geq n - j + 2 \right\}$$

$$\mathbf{s}_n = \max \left\{ k : \frac{2n - 2k + 3}{n} \leq \sum_{i=k-2}^{n-1} \frac{1}{i} \right\}$$

from which it is can easily be seen that \mathbf{s}_n/n tends to *rumour's constant*, which is the solution to the equation $2 - 2x + \log(x) = 0$

$$\vartheta := -\frac{1}{2}W(-2e^{-2}) = 0.203187869\dots \text{ (A106533 in OEIS)}$$

and the asymptotic expected profit is $\mathbb{E}_n \sim n \cdot \vartheta(1 - \vartheta) = n \cdot 0.161902\dots$

Remark. Ferguson et al. in [5], within the context of the Best-choice Duration Problem, consider a payoff of $(n - k + 1)/n$ and find the above asymptotic values erroneously approximated as 0.20388... and 0.1618....

Remark. If we consider $w_k := 1 - k/n$ and $p_k = 1/k$, the problem is equivalent to the secretary problem considering a cost of $1/n$ for each interview and a payment of 1 for success. The asymptotic values are the same as in the example and can be calculated in another way in [1].

4.2. The Best-choice and Minimal Duration Problem. To the best of our knowledge, there is no study in the literature of this problem, which consists in considering in the secretary problem a payment for success equal to the number of interviews carried out. In the terms of this paper, we shall have n independent Bernoulli random variables with parameters $p_k = 1/k$ and payoffs $w_k = k$. In this case, it is clear that the problem is monotone (optimal threshold strategy) as w_k is increasing.

$$\mathbf{s}_n = \max \left\{ k : \sum_{j=k}^n \frac{w_j \cdot p_j}{1 - p_j} \geq w_{k-1} \right\} = \max \left\{ k : \sum_{j=k}^n \frac{j \cdot \frac{1}{j}}{1 - \frac{1}{j}} \geq k - 1 \right\}$$

Denoting by $\mathbf{H}(k)$ the k -th harmonic number, we have

$$\sum_{i=k}^n \frac{i \frac{1}{i}}{1 - \frac{1}{i}} = 1 - k + n + (\mathbf{H}(n-1) - \mathbf{H}(k-2)) = 1 - k + n + \sum_{i=k-2}^{n-1} \frac{1}{i}$$

$$\mathbf{s}_n = \max \left\{ k : 1 - k + n + \sum_{i=k-2}^{n-1} \frac{1}{i} \geq k - 1 \right\} = \max \left\{ k : \sum_{i=k-2}^{n-1} \frac{1}{i} \geq 2k - n - 2 \right\}$$

from which it can easily be seen that s_n/n tends to $1/2$ and the asymptotic expected profit is

$$\mathbb{E}_n \sim \frac{n}{4}$$

4.3. n Bernoulli variables with the same parameter $p_k = 1/n$ and $w_k = k$. Let us consider n independent Bernoulli random variables with parameters $p_k = 1/k$ and payoffs $w_k = k$. The problem is monotone, as w_k is increasing.

$$s_n = \max \left\{ k : \sum_{j=k}^n \frac{(1-k+n)(k+n)}{2(-1+n)} \geq k-1 \right\}$$

$$s_n = \left\lfloor \frac{3-2n+\sqrt{1+8n^2}}{2} \right\rfloor \approx \frac{3}{2} + (-1+\sqrt{2})n$$

$$\mathbb{E}_n \sim n(-1+\sqrt{2})e^{-2+\sqrt{2}} = n \cdot 0.230579\dots$$

4.4. n Bernoulli variables with the same parameter $p_k = p$ and $w_k = n-k+1$. Let us consider n independent Bernoulli random variables with parameters $p_k = p$ and payoffs $w_k = n-k+1$. In this case, the problem is monotone as

$$\mathbb{E}_{\text{Stop}}(k) = (1-k+n)(1-p)^{-k+n}$$

$$\overline{\mathbb{E}}_{\text{Keep}}(k) = \frac{(-1+k-n)(k-n)(1-p)^{-1-k+n}p}{2}$$

and $\mathbb{E}_{\text{Stop}}(k) - \overline{\mathbb{E}}_{\text{Keep}}(k)$ change sign at the most once.

Making

$$\Omega := \left\{ k : \sum_{j=k}^n \frac{-((-2+k-n)(-1+k-n)p)}{2(-1+p)} \geq n-k+2 \right\}$$

$$s_n = \begin{cases} \max \Omega, & \text{if } \Omega \neq \emptyset ; \\ 1, & \text{if } \Omega = \emptyset \end{cases}$$

$$s_n = \begin{cases} \left\lfloor 3 + n - \frac{2}{p} \right\rfloor, & \text{if } n > \frac{2(1-p)}{p} ; \\ 1, & \text{if } n \leq \frac{2(1-p)}{p} \end{cases}$$

$$\mathbb{E}_n = \begin{cases} \frac{(1-p)^{-3+\lceil 2/p \rceil} p(-2+\lceil 2/p \rceil)(-1+\lceil 2/p \rceil)}{2}, & \text{if } n > \frac{2(1-p)}{p} ; \\ \frac{n(1+n)(1-p)^{-1+n}p}{2}, & \text{if } n \leq \frac{2(1-p)}{p} \end{cases}$$

As expected, the optimal threshold is less than $n - \lceil 1/p \rceil + 2$, which is the value we obtain when considering the Last-Success-Problem with parameters $p_i = p$ (see [6] and [9]).

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