

BESSEL FUNCTIONS AND THE WAVE EQUATION

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ABSTRACT. We solve the Cauchy problem for the n -dimensional wave equation using elementary properties of the Bessel functions.

With $\nabla^2 = D_{x_1 x_1}^2 + \cdots + D_{x_n x_n}^2$ the Laplacian in \mathbb{R}^n , where

$$D_{x_k x_k}^2 = \frac{\partial^2}{\partial x_k^2}, \quad 1 \leq k \leq n,$$

and D_t and D_{tt} indicating the first and second order derivatives with respect to the variable $t \in \mathbb{R}$, respectively, the wave equation in the upper half-space \mathbb{R}_+^{n+1} is given by

$$(1) \quad D_{tt}^2 u(x, t) = \nabla^2 u(x, t), \quad x \in \mathbb{R}^n, t > 0,$$

and the Cauchy problem for this equation consists of finding $u(x, t)$ that satisfies (1) subject to the initial conditions

$$u(x, 0) = \varphi(x) \quad \text{and} \quad D_t u(x, 0) = \psi(x), \quad x \in \mathbb{R}^n,$$

where for simplicity we shall take φ and ψ in $\mathcal{S}(\mathbb{R}^n)$.

Applying the Fourier transform to (1) in the space variables, considering t as a parameter, it readily follows that $\widehat{\nabla^2 u}(\xi, t) = -|\xi|^2 \widehat{u}(\xi, t)$, and so \widehat{u} satisfies

$$D_{tt}^2 \widehat{u}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t) = 0, \quad \xi \in \mathbb{R}^n, t > 0,$$

subject to

$$\widehat{u}(\xi, 0) = \widehat{\varphi}(\xi) \quad \text{and} \quad D_t \widehat{u}(\xi, 0) = \widehat{\psi}(\xi), \quad \xi \in \mathbb{R}^n.$$

For each fixed $\xi \in \mathbb{R}^n$ this resulting ordinary differential equation in t is the simple harmonic oscillator equation with constant angular frequency $|\xi|$, and so

$$\widehat{u}(\xi, t) = \widehat{\varphi}(\xi) \cos(t|\xi|) + \widehat{\psi}(\xi) \frac{\sin(t|\xi|)}{|\xi|}, \quad \xi \in \mathbb{R}^n, t > 0.$$

Hence, the Fourier inversion formula gives for $(x, t) \in \mathbb{R}_+^{n+1}$,

$$(2) \quad u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) \cos(t|\xi|) e^{i\xi \cdot x} d\xi \\ + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\psi}(\xi) \frac{\sin(t|\xi|)}{|\xi|} e^{i\xi \cdot x} d\xi.$$

Since the first integral in (2) can be obtained from the second by differentiating with respect to t , we will concentrate on the latter. The idea is to

interpret $\sin(|\xi|t)/|\xi|$ as the Fourier transform of a tempered distribution, and the key ingredient for this are the following representation formulas established in [1].

Representation Formulas. *Assume that n is an odd integer greater than or equal to 3. Then, with $d\sigma$ the element of surface area on $\partial B(0, R)$,*

$$(3) \quad \frac{\sin(R|\xi|)}{|\xi|} = c_n \left(\frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-3)/2} \left(\frac{1}{\omega_n R} \int_{\partial B(0, R)} e^{-ix \cdot \xi} d\sigma(x) \right),$$

where $R > 0$, ω_n is the surface measure of the unit ball in \mathbb{R}^n , and $c_n^{-1} = (n-2)(n-4) \cdots 1$.

On the other hand, if n is an even integer greater than or equal to 2,

$$(4) \quad \frac{\sin(R|\xi|)}{|\xi|} = d_n \left(\frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-2)/2} \left(\frac{1}{v_n} \int_{B(0, R)} \frac{1}{\sqrt{R^2 - |x|^2}} e^{-ix \cdot \xi} dx \right),$$

where $R > 0$, $d_n^{-1} = n(n-2)(n-4) \cdots 2$, and v_n is the volume of the unit ball in \mathbb{R}^n .

The purpose of this note is to establish (3) and (4) using elementary properties of Bessel functions. $J_\nu(x)$, the Bessel function of order ν , is defined as the solution of the second order linear equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0.$$

Several basic properties of the Bessel functions follow readily from their power series expression [2]. They include the recurrence formula

$$(5) \quad \frac{d}{dx} (x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x),$$

the integral representation of Poisson type

$$(6) \quad J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^1 (1-s^2)^{\nu-1/2} e^{ixs} ds,$$

and the identity

$$(7) \quad J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{x^{1/2}} \sin(x),$$

for $x > 0$.

We will consider the odd dimensional case first. The dimensional constant c_n may vary from appearance to appearance until it is finally determined at the end of the proof. To begin recall that for $n \geq 3$, as established in (18) in [1],

$$\frac{1}{\omega_{n-1} R} \int_{\partial B(0, R)} e^{-ix \cdot \xi} d\sigma(x) = R^{n-2} \int_{-1}^1 e^{iR|\xi|s} (1-s^2)^{(n-3)/2} ds,$$

which combined with (6) above with $\nu - 1/2 = (n - 3)/2$ there, i.e., $\nu = (n - 2)/2$, gives

$$\begin{aligned} \frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) &= c_n R^{n-2} \frac{J_{(n-2)/2}(R|\xi|)}{(R|\xi|)^{(n-2)/2}} \\ &= c_n \frac{1}{|\xi|^{n-2}} (R|\xi|)^{(n-2)/2} J_{(n-2)/2}(R|\xi|). \end{aligned}$$

Now, by (5) we obtain that

$$\frac{\partial}{\partial R} \left(\frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right) = c_n \frac{1}{|\xi|^{n-2}} |\xi| (R|\xi|)^{(n-2)/2} J_{(n-4)/2}(R|\xi|),$$

or

$$\frac{1}{R} \frac{\partial}{\partial R} \left(\frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right) = c_n \frac{1}{|\xi|^{n-4}} (R|\xi|)^{(n-4)/2} J_{(n-4)/2}(R|\xi|).$$

Thus, applying the above reasoning $(n - 3)/2$ times, (7) gives

$$\begin{aligned} \left(\frac{1}{R} \frac{\partial}{\partial R} \right)^{(n-3)/2} \left(\frac{1}{\omega_n R} \int_{\partial B(0,R)} e^{-ix \cdot \xi} d\sigma(x) \right) \\ &= c_n \frac{1}{|\xi|} (R|\xi|)^{1/2} J_{1/2}(R|\xi|) \\ &= c_n \frac{1}{|\xi|} (R|\xi|)^{1/2} \frac{\sin(R|\xi|)}{(R|\xi|)^{1/2}} \\ &= c_n \frac{\sin(R|\xi|)}{|\xi|}. \end{aligned}$$

The value of c_n is readily obtained as in [1], and (3) has been established.

To consider the case n even, one generally proceeds at this point by a reasoning akin to Hadamard's method of descent, i.e., the desired result for the wave equation in even dimension n is derived from the result in odd dimension $n + 1$, as is done for instance in [1] for the representation formulas. On the other hand, Bessel functions provide the desired result for the wave equation in even dimensions directly, by a method akin to ascent: the result for the wave equation for dimension $n = 2$ is obtained explicitly, and for even dimension $n + 2$ is obtained from the result in even dimension n .

We will first prove a preliminary result. The dimensional constant d_n may vary from appearance to appearance until it is finally determined at the end of the proof.

Lemma. *The following three statements hold.*

$$(8) \quad \int_0^\infty \sin(R\rho) J_0(t\rho) d\rho = \frac{1}{\sqrt{R^2 - t^2}} H(R - t), \quad R, t > 0,$$

where H denotes the Heavyside function.

Furthermore, for $\nu \geq 1$,

$$(9) \quad \left(\frac{1}{R} \frac{\partial}{\partial R} \right) \left(\int_0^\infty \sin(R\rho) \rho^{\nu-1} J_{\nu-1}(t\rho) d\rho \right) = \frac{1}{t} \int_0^\infty \sin(R\rho) \rho^\nu J_\nu(t\rho) d\rho,$$

and, consequently, for $1 \leq j \leq \nu$,

$$(10) \quad \left(\frac{1}{R} \frac{\partial}{\partial R}\right)^j \left(\int_0^\infty \sin(R\rho) \rho^{\nu-j} J_{\nu-j}(t\rho) d\rho\right) = \frac{1}{t^j} \int_0^\infty \sin(R\rho) \rho^\nu J_\nu(t\rho) d\rho.$$

Proof. (8) is Formula (6) in [2], page 405.

Now,

$$\begin{aligned} \frac{\partial}{\partial R} \left(\int_0^\infty \sin(R\rho) \rho^{\nu-1} J_{\nu-1}(t\rho) d\rho\right) &= - \int_0^\infty \cos(R\rho) \rho^\nu J_{\nu-1}(t\rho) d\rho \\ &= -\frac{1}{t^\nu} \int_0^\infty \cos(R\rho) (t\rho)^\nu J_{\nu-1}(t\rho) d\rho, \end{aligned}$$

which, by (1), equals

$$\frac{-1}{t^{\nu+1}} \int_0^\infty \cos(R\rho) \frac{\partial}{\partial \rho} ((t\rho)^\nu J_\nu(t\rho)) d\rho = \frac{R}{t} \int_0^\infty \sin(R\rho) \rho^\nu J_\nu(t\rho) d\rho,$$

which proves (9).

(10) follows by repeated applications of (9), and we have finished. \square

Finally, recall that the Fourier transform of a radial function f on \mathbb{R}^n is given by the expression [2],

$$\widehat{f}(|\xi|) = d_n \frac{1}{|\xi|^{(n-2)/2}} \int_0^\infty \rho^{n/2} f(\rho) J_{(n-2)/2}(|\xi|\rho) d\rho.$$

In particular, we have

$$(11) \quad \int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} d\xi = d_n \frac{1}{|x|^{(n-2)/2}} \int_0^\infty \rho^{n/2} \frac{\sin(R\rho)}{\rho} J_{(n-2)/2}(|x|\rho) d\rho.$$

Let now $n = 2k$ be an even integer. Then by (11),

$$\int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} d\xi = d_n \frac{1}{|x|^{(k-1)}} \int_0^\infty \sin(R\rho) \rho^{k-1} J_{k-1}(|x|\rho) d\rho,$$

and, therefore, (10) with $\nu = j = k - 1$ there yields

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\sin(R|\xi|)}{|\xi|} e^{-ix \cdot \xi} d\xi &= d_n \frac{1}{|x|^{(k-1)}} \int_0^\infty \sin(R\rho) \rho^{(k-1)} J_{k-1}(|x|\rho) d\rho \\ &= d_n \left(\frac{1}{R} \frac{\partial}{\partial R}\right)^{(k-1)} \left(\int_0^\infty \sin(R\rho) J_0(|x|\rho) d\rho\right) \\ &= d_n \left(\frac{1}{R} \frac{\partial}{\partial R}\right)^{(k-1)} \left(\frac{1}{\sqrt{R^2 - |x|^2}} H(R - |x|)\right). \end{aligned}$$

Thus by the Fourier inversion formula,

$$\begin{aligned} \frac{\sin(R|\xi|)}{|\xi|} &= d_n \left(\frac{1}{R} \frac{\partial}{\partial R}\right)^{(n-2)/2} \left(\int_{\mathbb{R}^n} \frac{1}{\sqrt{R^2 - |x|^2}} H(R - |x|) e^{-ix \cdot \xi} dx\right) \\ &= d_n \left(\frac{1}{R} \frac{\partial}{\partial R}\right)^{(n-2)/2} \left(\frac{1}{v_n} \int_{B(0,R)} \frac{1}{\sqrt{R^2 - |x|^2}} e^{-ix \cdot \xi} dx\right), \end{aligned}$$

The constant d_n is readily determined as in [1], and we have finished.

REFERENCES

- [1] A. Torchinsky, *The Fourier transform and the wave equation*, Amer. Math. Monthly 118 (2011) no.7, 599-609.
- [2] G. N. Watson, *A treatise on the theory of Bessel functions*. *Cambridge Mathematical Library*. Cambridge University Press, Cambridge, 1995.