ERROR ESTIMATION OF THE BESSE RELAXATION SCHEME FOR A SEMILINEAR HEAT EQUATION

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Abstract. The solution to the initial and Dirichlet boundary value problem for a semilinear, one dimensional heat equation is approximated by a numerical method that combines the Besse relaxation scheme in time (C. R. Acad. Sci. Paris Sér. I, vol. 326 (1998)) with a central finite difference method in space. A new, composite stability argument is developed, leading to an optimal, second-order error estimate in the discrete $L^{\infty}_t(H^1_x)$–norm. It is the first time in the literature where an error estimate for fully discrete approximations based on the Besse relaxation scheme is provided.

1. Introduction

1.1. **Formulation of the problem.** Let $T>0$, $x_a,\ x_b\in\mathbb{R}$ with $x_b>x_a$, $\mathcal{I}:=[x_a,x_b]$ and $u:[0,T]\times\mathcal{I}\to\mathbb{R}$ be the solution of the following initial and boundary value problem:

$$u_t = u_{xx} + g(u)\,u + f \quad \text{on} \ \ [0,T]\times\mathcal{I}, \tag{1.1}$$

$$u(t,x_a) = u(t,x_b) = 0 \quad \forall\, t\in[0,T], \tag{1.2}$$

$$u(0,x) = u_0(x) \quad \forall\, x\in\mathcal{I}, \tag{1.3}$$

where $g:\mathbb{R}\to\mathbb{R}$, $f:[0,T]\times\mathcal{I}\to\mathbb{R}$ and $u_0:\mathcal{I}\to\mathbb{R}$ with

$$u_0(x_a) = u_0(x_b) = 0. \tag{1.4}$$

Furthermore, we assume that the data f, u_0 and g are smooth enough and compatible, in order to guarantee the existence and uniqueness of a solution u to the problem above that is sufficiently smooth for our purposes.

Two decades ago, for the discretization in time of the nonlinear Schrödinger equation, C. Besse [4] introduced a new linear-implicit time-stepping method (called *Relaxation Scheme*) as an attempt to avoid the numerical solution of the nonlinear systems of algebraic equations that the application of the implicit Crank-Nicolson method yields. The proposed time discretization technique, combined with a finite element or a finite difference space discretization, is computationally efficient (see, e.g., [3], [8], [6]) and performs as a second order method (see, e.g., [5], [8]). Later, C. Besse [5] analyzing the Relaxation Scheme as a semidiscrete in time method to approximate the solution of the Cauchy problem (i.e. without the presence of boundary conditions) shows, using that it is local well-posedness and convergent without concluding a convergent rate with respect to the time-step. Until today, in spite of the results in [5], there is no scientific work in the literature providing an error estimate for the Relaxation Scheme. Since the Relaxation Scheme can not be classified as a Runge-Kutta or a linear multistep method, a natural question arises: "is the Relaxation Scheme a special method or a representative member of a new family of linear implicit time-discretization methods?" One way moving toward to find an answer is first to understand its convergence and then to construct methods with similar characteristics.

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The aim of the work at hands is to contribute to the understanding of the convergence nature of the Besse relaxation scheme, by investigating its use, along with a finite difference space discretization, to obtain approximations of the solution to the parabolic problem (1.1)-(1.4). By building up a proper stability argument and using energy techniques, we are able to prove an optimal, second order error estimate in a discrete $L^{\infty}_t(H^1_x)$–norm. The result is new and opens the discussion on the applicability and the extension of the Relaxation Scheme to other non-linear evolution equations.

1.2. **Formulation of the numerical method.**

1.2.1. *Notation.* Let $\mathbb{N}$ be the set of all positive integers and $\mathsf{L} := x_b - x_a$. For given $N \in \mathbb{N}$, we define a uniform partition of the time interval $[0,T]$ with time-step $\tau := \frac{T}{N}$, nodes $t_n := n\,\tau$ for $n = 0,\dots,N$, and intermediate nodes $t^{n+\frac12} = t_n + \frac{\tau}{2}$ for $n = 0,\dots,N-1$. Also, for given $J \in \mathbb{N}$, we consider a uniform partition of $\mathcal{I}$ with mesh-width $h := \frac{\mathsf{L}}{J+1}$ and nodes $x_j := x_a + j\,h$ for $j = 0,\dots,J+1$. Then, we introduce the discrete spaces

$$\mathfrak{X}_h := \left\{ (v_j)_{j=0}^{J+1} :\ v_j \in \mathbb{R},\ j = 0,\dots,J+1 \right\} \quad \text{and} \quad \mathfrak{X}_h^{\circ} := \left\{ (v_j)_{j=0}^{J+1} \in \mathfrak{X}_h :\ v_0 = v_{J+1} = 0 \right\},$$

a discrete product operator $\cdot \otimes \cdot : \mathfrak{X}_h \times \mathfrak{X}_h \to \mathfrak{X}_h$ by

$$(v \otimes w)_j = v_j\, w_j, \quad j = 0,\dots,J+1, \quad \forall\, v, w \in \mathfrak{X}_h,$$

and a discrete Laplacian operator $\Delta_h : \mathfrak{X}_h^{\circ} \to \mathfrak{X}_h^{\circ}$ by

$$\Delta_h v_j := \tfrac{v_{j-1} - 2v_j + v_{j+1}}{h^2}, \quad j = 1,\dots,J, \quad \forall\, v \in \mathfrak{X}_h^{\circ}.$$

In addition, we introduce operators $\mathsf{I}_h : \mathsf{C}(\mathcal{I};\mathbb{R}) \to \mathfrak{X}_h$ and $\mathsf{I}_h^{\circ} : \mathsf{C}(\mathcal{I};\mathbb{R}) \to \mathfrak{X}_h^{\circ}$, which, for given $z \in \mathsf{C}(\mathcal{I};\mathbb{R})$, are defined by $(\mathsf{I}_h z)_j := z(x_j)$ for $j = 0,\dots,J+1$ and $z \in \mathsf{C}(\mathcal{I};\mathbb{R})$ and $(\mathsf{I}_h^{\circ} z)_j := z(x_j)$ for $j = 1,\dots,J$. Finally, for $\ell \in \mathbb{N}$ and for any function $q : \mathbb{R}^{\ell} \to \mathbb{R}$ and any $w = (w^1,\dots,w^{\ell}) \in (\mathfrak{X}_h)^{\ell}$, we define $q(w) \in \mathfrak{X}_h$ by $(q(w))_j := q\left(w_j^1,\dots,w_j^{\ell}\right)$ for $j = 0,\dots,J+1$.

1.2.2. *The Besse Relaxation Finite Difference method.* The Besse Relaxation Finite Difference (BRFD) method combines a standard finite difference discetization in space with the Besse relaxation scheme in time (cf. [4]). Its algorithm consists of the following steps:

`Step I`: Define $U^0 \in \mathfrak{X}_h^{\circ}$ by

$$U^0 := u^0 \tag{1.5}$$

and then find $U^{\frac12} \in \mathfrak{X}_h^{\circ}$ such that

$$\tfrac{U^{\frac12} - U^0}{(\tau/2)} = \Delta_h \left(\tfrac{U^{\frac12} + U^0}{2} \right) + g\big(u^0\big) \otimes \left(\tfrac{U^{\frac12} + U^0}{2} \right) + \mathsf{I}_h^{\circ} \left[\tfrac{f(t^{\frac12},\cdot) + f(t_0,\cdot)}{2} \right]. \tag{1.6}$$

`Step II`: Define $\Phi^{\frac12} \in \mathfrak{X}_h$ by

$$\Phi^{\frac12} := g(U^{\frac12}) \tag{1.7}$$

and then find $U^1 \in \mathfrak{X}_h^{\circ}$ such that

$$\tfrac{U^1 - U^0}{\tau} = \Delta_h \left(\tfrac{U^1 + U^0}{2} \right) + \Phi^{\frac12} \otimes \left(\tfrac{U^1 + U^0}{2} \right) + \mathsf{I}_h^{\circ} \left[\tfrac{f(t_1,\cdot) + f(t_0,\cdot)}{2} \right]. \tag{1.8}$$

`Step III`: For $n = 1,\dots,N-1$, first define $\Phi^{n+\frac12} \in \mathfrak{X}_h$ by

$$\Phi^{n+\frac12} := 2\,g(U^n) - \Phi^{n-\frac12} \tag{1.9}$$

and then find $U^{n+1} \in \mathfrak{X}_h^{\circ}$ such that

$$\tfrac{U^{n+1} - U^n}{\tau} = \Delta_h \left(\tfrac{U^{n+1} + U^n}{2} \right) + \Phi^{n+\frac12} \otimes \left(\tfrac{U^{n+1} + U^n}{2} \right) + \mathsf{I}_h^{\circ} \left[\tfrac{f(t_{n+1},\cdot) + f(t_n,\cdot)}{2} \right]. \tag{1.10}$$

Obviously, the numerical method above requires, at each time step, the solution of a tridiagonal linear system of algebraic equations.

1.3. **An overview of the paper.** In the error analysis of the (BRFD) method, we face the locally Lipschitz nonlinearity of the problem by introducing the (MBRFD) scheme (see Section 4.2), which follows from the (BRFD) method after molifying properly the terms with nonlinear structure (cf. [1], [9], [7]). The (MBRFD) approximations depend on a parameter $\delta > 0$ and have the following key property: when their discrete L^∞-norm is bounded by δ, then they are also (BRFD) approximations, because, in that case, the molifier (see (4.1)) acts as an indentity. Assuming that δ is large enough and τ is sufficiently small, for the non computable (BRFD) approximations, first we show that are well-defined (see Proposition 4.1), and then we establish an optimal, second order error estimate in the discrete H^1-norm (see Theorem 4.2). Letting h and τ be sufficiently small (see (4.58)) and applying a discrete Sobolev inequality (see (2.1)), the latter convergence result implies that the discrete L^∞–norm of the (MBRFD) approximations are lower than δ and thus they, also, are (BRFD) approximations. Finally, we are show that the (BRFD) approximations are unique and hence inherit the convergence properties of the (MBRFD) scheme (see Theorem 4.3), i.e. that there exist constants C_1 and C_2, independent of τ and h, such that

$$\left| U^{\frac{1}{2}} - \mathsf{I}_h^{\circ}[u(t^{\frac{1}{2}},\cdot)] \right|_{1,h} \le C_1\,(\tau^2 + \tau^{\frac{1}{2}}\,h^2)$$

and

$$\max_{0\le n\le N} \Big[\left| \Phi^{n+\frac{1}{2}} - \mathsf{I}_h[g(u(t^{n+\frac{1}{2}},\cdot))] \right|_{1,h} + \left| U^n - \mathsf{I}_h^{\circ}[u(t_n,\cdot)] \right|_{1,h} \Big] \le C_2\,(\tau^2 + h^2),$$

where $|\cdot|_{1,h}$ is a discrete H^1–norm which is stronger than the discrete L^∞–norm.

At every time-step, the (BRFD) method computes first an approximation of $g(u)$ at the midpoint of the current time interval (see (1.7) and (1.9)) and then an approximation of u at the next time node (see (1.8) and (1.10)). However, the computation of the approximations of $g(u)$ at the midpoints is a simple postprocessing procedure and has no obvious discrete dynamic structure. The stability argument we employ is based first on taking a discrete derivative of the error equation that corresponds to (1.9) (see (4.27)) and then on including the discrete L^2 and discrete H^1 norm of the time increment of the error in the stability norm (see (4.32) and (4.52)).

We close this section by giving a brief overview of the paper. In Section 2, we introduce additional notation and provide a series of auxiliary results. Section 3 is dedicated to the estimation of several type of consistency errors and of the approximation error of a discrete elliptic projection. In Section 4, we define a modified version of the (BRFD) method, and then analyze its convergence properties and arrive at a set of conditions that ensure the well-posedness and convergence of the (BRFD) method.

2. Preliminaries

Let us introduce another discrete space by $\mathfrak{S}_h := \left\{ (z_j)_{j=0}^J : \ z_j \in \mathbb{R}, \quad j = 0,\dots,J \right\}$ and the discrete space derivative operator $\delta_h : \mathfrak{X}_h \to \mathfrak{S}_h$ by

$$\delta_h v_j := \tfrac{v_{j+1}-v_j}{h}, \quad j = 0,\dots,J, \quad \forall\, v \in \mathfrak{X}_h.$$

We define on $\mathfrak{S}_h$ an inner product $(\!(\cdot,\cdot)\!)_{0,h}$ by $(\!(z,v)\!)_{0,h} := h\sum_{j=0}^J z_j\,v_j$ for $z, v \in \mathfrak{S}_h$, and we will denote by $\|\cdot\|_{0,h}$ the corresponding norm, i.e. $\|z\|_{0,h} := \left[(\!(z,z)\!)_{0,h}\right]^{1/2}$ for $z \in \mathfrak{S}_h$. Also, we define a discrete maximum norm $\|\cdot\|_{\infty,h}$ on $\mathfrak{S}_h$ by $\|v\|_{\infty,h} := \max_{0\le j\le J}|v_j|$ for $v \in \mathfrak{S}_h$.

We provide $\mathfrak{X}_h^{\circ}$ with the discrete inner product $(\cdot,\cdot)_{0,h}$ given by $(v,z)_{0,h} := h\sum_{j=1}^J v_j\,z_j$ for $v, z \in \mathfrak{X}_h^{\circ}$, and we shall denote by $\|\cdot\|_{0,h}$ its induced norm, i.e. $\|v\|_{0,h} := \left[(v,v)_{0,h}\right]^{1/2}$ for $v \in \mathfrak{X}_h^{\circ}$. Also, we equip $\mathfrak{X}_h$ with a discrete L^∞-norm $|\cdot|_{\infty,h}$ defined by $|w|_{\infty,h} := \max_{0\le j\le J+1}|w_j|$ for $w \in \mathfrak{X}_h$, and with a discrete H^1-seminorm $|\cdot|_{1,h}$ given by $|w|_{1,h} := \|\delta_h w\|_{0,h}$ for $w \in \mathfrak{X}_h$. It is easily seen that $|\cdot|_{1,h}$ becomes a norm when it is restricted on $\mathfrak{X}_h^{\circ}$ and satisfies the following useful inequalities:

$$|v|_{\infty,h} \le \mathsf{L}^{1/2}\,|v|_{1,h}, \tag{2.1}$$

$$\|v\|_{0,h} \le \mathsf{L}\,|v|_{1,h} \tag{2.2}$$

for $v \in \mathfrak{X}_h^\circ$. In the sequel, we present a series of auxiliary results that they will be in often use in the rest of the work.

Lemma 2.1. *For all $v, z \in \mathfrak{X}_h^\circ$ it holds that*

$$(\Delta_h v, z)_{0,h} = -(\!(\delta_h v, \delta_h z)\!)_{0,h} = (v, \Delta_h z)_{0,h}, \tag{2.3}$$

$$(\Delta_h v, v)_h = -|v|_{1,h}^2. \tag{2.4}$$

Proof. Let $v, z \in \mathfrak{X}_h^\circ$. First, we establish (2.3) proceeding as follows:

$$(\Delta_h v, z)_{0,h} = \sum_{j=1}^{J} \left[(\delta_h v)_j - (\delta_h v)_{j-1} \right] z_j = \sum_{j=0}^{J} (\delta_h v)_j \, z_j - \sum_{j=0}^{J} (\delta_h v)_j \, z_{j+1} = -(\!(\delta_h v, \delta_h z)\!)_{0,h}.$$

Then, we set $z = v$ in (2.3) to get (2.4). □

Lemma 2.2. *Let $\mathfrak{g} \in C_b^2(\mathbb{R};\mathbb{R})$. Then, for $v, w \in \mathfrak{X}_h^\circ$, it holds that*

$$|\mathfrak{g}(v) - \mathfrak{g}(w)|_{1,h} \le \mathfrak{g}'_\infty \, |v - w|_{1,h} + \mathfrak{g}''_\infty \, \|\delta_h w\|_{\infty,h} \, \|v - w\|_{0,h} \tag{2.5}$$

where $\mathfrak{g}'_\infty := \sup_{\mathbb{R}} |\mathfrak{g}'|$ and $\mathfrak{g}''_\infty := \sup_{\mathbb{R}} |\mathfrak{g}''|$.

Proof. Let $v, w \in \mathfrak{X}_h^\circ$. First, we define $\mathfrak{a}^s$, $\mathfrak{b}^s \in \mathfrak{S}_h$ by $\mathfrak{a}_j^s := s\, v_{j+1} + (1-s)\, v_j$ and $\mathfrak{b}_j^s := s\, w_{j+1} + (1-s)\, w_j$ for $j = 0, \dots, J$ and $s \in [0,1]$. Then, we use the mean value theorem, to conclude that

$$\delta_h(\mathfrak{g}(v) - \mathfrak{g}(w)) = \mathcal{L}^A + \mathcal{L}^B \tag{2.6}$$

where $\mathcal{L}^A, \mathcal{L}^B \in \mathfrak{S}_h$ given by $\mathcal{L}_j^A := (\delta_h(v-w))_j \int_0^1 \mathfrak{g}'(\mathfrak{a}_j^s)\, ds$ and $\mathcal{L}_j^B := \delta_h w_j \int_0^1 \left[\mathfrak{g}'(\mathfrak{a}_j^s) - \mathfrak{g}'(\mathfrak{b}_j^s)\right] ds$ for $j = 0, \dots, J$. Observing that

$$\left|\mathcal{L}_j^A\right| \le \sup_{\mathbb{R}} |\mathfrak{g}'| \, |(\delta_h(v-w))_j|, \quad j = 0, \dots, J,$$

and

$$\begin{aligned} \left|\mathcal{L}_j^B\right| &\le |(\delta_h w)_j| \sup_{\mathbb{R}} |\mathfrak{g}''| \left| \int_0^1 \left[s(v_{j+1} - w_{j+1}) + (1-s)(v_j - w_j) \right] ds \right| \\ &\le \tfrac{1}{2} |(\delta_h w)_j| \sup_{\mathbb{R}} |\mathfrak{g}''| \left(|v_{j+1} - w_{j+1}| + |v_j - w_j| \right), \quad j = 0, \dots, J, \end{aligned}$$

we, easily, arrive at

$$\|\mathcal{L}^A\|_{0,h} \le \sup_{\mathbb{R}} |\mathfrak{g}'| \, \|\delta_h(v-w)\|_{0,h}, \tag{2.7}$$

$$\|\mathcal{L}^B\|_{0,h} \le \|\delta_h w\|_{\infty,h} \sup_{\mathbb{R}} |\mathfrak{g}''| \, \|v - w\|_{0,h}. \tag{2.8}$$

Thus, (2.5) follows as a simple consequence of (2.6), (2.7) and (2.8). □

Lemma 2.3. *Let $\mathfrak{g} \in C_b^3(\mathbb{R};\mathbb{R})$. Then, for $v^a, v^b, z^a, z^b \in \mathfrak{X}_h^\circ$, it holds that*

$$\begin{aligned} \|\mathfrak{g}(v^a) - \mathfrak{g}(v^b) - \mathfrak{g}(z^a) + \mathfrak{g}(z^b)\|_{0,h} \le\, &\mathfrak{g}''_\infty \, |z^a - z^b|_{\infty,h} \, \|v^b - z^b\|_{0,h} \\ &+ \left(\mathfrak{g}'_\infty + \mathfrak{g}''_\infty \, |z^a - z^b|_{\infty,h} \right) \|v^a - v^b - z^a + z^b\|_{0,h} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} |\mathfrak{g}(v^a) - \mathfrak{g}(v^b) - \mathfrak{g}(z^a) + \mathfrak{g}(z^b)|_{1,h} \le\, &\mathcal{F}^A(v^a, v^b) \, |v^a - v^b - z^a + z^b|_{1,h} \\ &+ \mathcal{F}^B(z^a, z^b) \, \left(\|v^a - v^b - z^a + z^b\|_{0,h} + \|v^b - z^b\|_{0,h} \right) \\ &+ \mathcal{F}^C(z^a, z^b) \, \left(|v^a - v^b - z^a + z^b|_{1,h} + |v^b - z^b|_{1,h} \right), \end{aligned} \tag{2.10}$$

where $\mathfrak{g}'_\infty := \sup_{\mathbb{R}} |\mathfrak{g}'|$, $\mathfrak{g}''_\infty := \sup_{\mathbb{R}} |\mathfrak{g}''|$,

$$\begin{aligned} \mathcal{F}^A(v^a, v^b) &:= \mathfrak{g}'_\infty + \tfrac{\mathsf{L}^{1/2}}{2} \, \mathfrak{g}''_\infty \left(|v^a|_{1,h} + |v^b|_{1,h} \right), \\ \mathcal{F}^B(z^a, z^b) &:= \mathfrak{g}''_\infty \, \|\delta_h(z^a - z^b)\|_{\infty,h}, \\ \mathcal{F}^C(z^a, z^b) &:= |z^a - z^b|_{1,h} \left[\mathfrak{g}''_\infty + \mathsf{L}\, \mathfrak{g}'''_\infty \left(\|\delta_h z^a\|_{\infty,h} + \|\delta_h z^b\|_{\infty,h} \right) \right] \end{aligned}$$

and $\mathfrak{g}'''_\infty := \sup_{\mathbb{R}} |\mathfrak{g}'''|$.

Proof. Let $v^a, v^b, z^a, z^b \in \mathfrak{X}_h^\circ$. We simplify the notation, first, by defining $\mathfrak{a}^s$, $\mathfrak{b}^s \in \mathfrak{X}_h^\circ$ by $\mathfrak{a}^s := s\,v^a + (1-s)\,v^b$ and $\mathfrak{b}^s := s\,z^a + (1-s)\,z^b$ for $s \in [0,1]$, and then, by introducing $\mathfrak{f} \in \mathfrak{X}_h$ by $\mathfrak{f} := \int_0^1 \mathfrak{g}'(\mathfrak{a}^s)\,ds$ and $\mathfrak{t} \in \mathfrak{X}_h^\circ$ by $\mathfrak{t} := \int_0^1 [\mathfrak{g}'(\mathfrak{a}^s) - \mathfrak{g}'(\mathfrak{b}^s)]\,ds$. Also, we set $e^a := v^a - z^a$ and $e^b := v^b - z^b$.

Part I. First, we use the definition of $\mathfrak{f}$ and the mean value theorem, to get

$$|\mathfrak{f}|_{\infty,h} \le \mathfrak{g}'_\infty \tag{2.11}$$

and

$$\begin{aligned}|\delta_h \mathfrak{f}_j| &\le \tfrac{1}{h} \int_0^1 |\mathfrak{g}'(\mathfrak{a}^s_{j+1}) - \mathfrak{g}'(\mathfrak{a}^s_j)|\,ds\\ &\le \mathfrak{g}''_\infty \int_0^1 \left|s\,\delta_h v^a_j + (1-s)\,\delta_h v^b_j\right|\,ds\\ &\le \tfrac{1}{2}\,\mathfrak{g}''_\infty \left(|\delta_h v^a_j| + |\delta_h v^b_j|\right), \quad j = 0,\dots,J,\end{aligned}$$

which, obviously, yields

$$|\mathfrak{f}|_{1,h} \le \tfrac{1}{2}\,\mathfrak{g}''_\infty \left(|v^a|_{1,h} + |v^b|_{1,h}\right). \tag{2.12}$$

Next, we use the definition of $\mathfrak{t}$ and the mean value theorem, to obtain

$$\begin{aligned}|\mathfrak{t}_j| &\le \mathfrak{g}''_\infty \int_0^1 |\mathfrak{a}^s_j - \mathfrak{b}^s_j|\,ds\\ &\le \mathfrak{g}''_\infty \int_0^1 |s\,(v^a_j - v^b_j - z^a_j + z^b_j) + (v^b_j - z^b_j)|\,ds\\ &\le \mathfrak{g}''_\infty \left(|v^a_j - v^b_j - z^a_j + z^b_j| + |v^b_j - z^b_j|\right), \quad j = 1,\dots,J,\end{aligned}$$

which, leads to

$$\|\mathfrak{t}\|_{0,h} \le \mathfrak{g}''_\infty \left(\|e^a - e^b\|_{0,h} + \|e^b\|_{0,h}\right). \tag{2.13}$$

Finally, for $s \in [0,1]$, we apply (2.5) and (2.2), to arrive at

$$\begin{aligned}|\mathfrak{g}'(\mathfrak{a}^s) - \mathfrak{g}'(\mathfrak{b}^s)|_{1,h} &\le \mathfrak{g}''_\infty\,|\mathfrak{a}^s - \mathfrak{b}^s|_{1,h} + \mathfrak{g}'''_\infty\,|\!|\!|\delta_h \mathfrak{b}^s|\!|\!|_{\infty,h}\,\|\mathfrak{a}^s - \mathfrak{b}^s\|_{0,h}\\ &\le \left(\mathfrak{g}''_\infty + \mathsf{L}\,\mathfrak{g}'''_\infty\,|\!|\!|\delta_h \mathfrak{b}^s|\!|\!|_{\infty,h}\right)|\mathfrak{a}^s - \mathfrak{b}^s|_{1,h}\\ &\le \left(\mathfrak{g}''_\infty + \mathsf{L}\,\mathfrak{g}'''_\infty\,|\!|\!|\delta_h \mathfrak{b}^s|\!|\!|_{\infty,h}\right)\left(|e^a - e^b|_{1,h} + |e^b|_{1,h}\right).\end{aligned} \tag{2.14}$$

Observing that $\delta_h \mathfrak{t} = \int_0^1 \delta_h\,[\mathfrak{g}'(\mathfrak{a}^s) - \mathfrak{g}'(\mathfrak{b}^s)]\,ds$ and using (2.14) we have

$$\begin{aligned}|\mathfrak{t}|_{1,h} &\le \int_0^1 |\mathfrak{g}'(\mathfrak{a}^s) - \mathfrak{g}'(\mathfrak{b}^s)|_{1,h}\,ds\\ &\le \left[\mathfrak{g}''_\infty + \mathsf{L}\,\mathfrak{g}'''_\infty \left(|\!|\!|\delta_h z^a|\!|\!|_{\infty,h} + |\!|\!|\delta_h z^b|\!|\!|_{\infty,h}\right)\right]\left(|e^a - e^b|_{1,h} + |e^b|_{1,h}\right).\end{aligned} \tag{2.15}$$

Part II. Using the mean value theorem, we obtain

$$\mathfrak{g}(v^a) - \mathfrak{g}(v^b) - \mathfrak{g}(z^a) + \mathfrak{g}(z^b) = \mathfrak{L}^A + \mathfrak{L}^B, \tag{2.16}$$

where $\mathfrak{L}^A$, $\mathfrak{L}^B \in \mathfrak{X}_h^\circ$ are defined by $\mathfrak{L}^A := (v^a - v^b - z^a + z^b) \otimes \mathfrak{f}$ and $\mathfrak{L}^B := (z^a - z^b) \otimes \mathfrak{t}$. Thus, using (2.11) and (2.13), we have

$$\begin{aligned}\|\mathfrak{L}^A\|_{0,h} &\le \mathfrak{g}'_\infty\,\|e^a - e^b\|_{0,h},\\ \|\mathfrak{L}^B\|_{0,h} &\le \mathfrak{g}''_\infty\,|z^a - z^b|_{\infty,h}\left(\|e^a - e^b\|_{0,h} + \|e^b\|_{0,h}\right).\end{aligned} \tag{2.17}$$

The desired inequality (2.9) follows, easily, as a simple outcome of (2.16) and (2.17).

Part III. For the discrete derivative of $\mathfrak{L}^A$ and $\mathfrak{L}^B$, we, easily, obtain the following formulas:

$$(\delta_h \mathfrak{L}^A)_j = \delta_h (v^a - v^b - z^a + z^b)_j\, \mathfrak{f}_{j+1} + (v^a_j - v^b_j - z^a_j + z^b_j)\, (\delta_h \mathfrak{f})_j,$$
$$(\delta_h \mathfrak{L}^B)_j = \delta_h (z^a - z^b)_j\, \mathfrak{t}_{j+1} + (z^a - z^b)_j\, (\delta_h \mathfrak{t})_j$$

for $j = 0, \ldots, J$, which yield

$$\begin{aligned} |\mathfrak{L}^A|_{1,h} &\le |e^a - e^b|_{1,h}\, |\mathfrak{f}|_{\infty,h} + |e^a - e^b|_{\infty,h}\, |\mathfrak{f}|_{1,h}, \\ |\mathfrak{L}^B|_{1,h} &\le \|\delta_h (z^a - z^b)\|_{\infty,h}\, \|\mathfrak{t}\|_{0,h} + |z^a - z^b|_{\infty,h}\, |\mathfrak{t}|_{1,h}. \end{aligned} \tag{2.18}$$

Using (2.18), (2.1), (2.11) and (2.12), we have

$$|\mathfrak{L}^A|_{1,h} \le \left[\mathfrak{g}'_\infty + \tfrac{\mathsf{L}^{1/2}}{2}\, \mathfrak{g}''_\infty \left(|v^a|_{1,h} + |v^b|_{1,h} \right) \right] |e^a - e^b|_{1,h}. \tag{2.19}$$

Combining (2.18), (2.13), (2.15) and (2.1), we arrive at

$$\begin{aligned} |\mathfrak{L}^B|_{1,h} \le\, & \mathfrak{g}''_\infty\, \|\delta_h (z^a - z^b)\|_{\infty,h} \left(\|e^a - e^b\|_{0,h} + \|e^b\|_{0,h} \right) \\ & + |z^a - z^b|_{1,h} \left[\mathfrak{g}''_\infty + \mathsf{L}\, \mathfrak{g}'''_\infty \left(\|\delta_h z^a\|_{\infty,h} + \|\delta_h z^b\|_{\infty,h} \right) \right] \left(|e^a - e^b|_{1,h} + |e^b|_{1,h} \right). \end{aligned} \tag{2.20}$$

Finally, (2.10) follows, easily, in view of (2.16), (2.19) and (2.20). □

3. Consistency Errors

To simplify the notation, we set $t^{\frac14} := \frac{\tau}{4}$, $u^{\frac14} := \mathsf{I}_h[u(t^{\frac14}, \cdot)]$, $u^n := \mathsf{I}_h[u(t_n, \cdot)]$ for $n = 0, \ldots, N$, and $u^{n+\frac12} := \mathsf{I}_h[u(t^{n+\frac12}, \cdot)]$ for $n = 0, \ldots, N-1$. In view of the Dirichlet boundary conditions (1.2) and the compatibility conditions (1.4), it holds that $u^{\frac14} \in \mathfrak{X}^\circ_h$, $u^n \in \mathfrak{X}^\circ_h$ for $n = 0, \ldots, N$ and $u^{n+\frac12} \in \mathfrak{X}^\circ_h$ for $n = 0, \ldots, N-1$.

3.1. **Time consistency error at the nodes.**

Let $\mathsf{r}^{\frac14} \in \mathfrak{X}_h$ be defined by

$$\frac{u^{\frac12} - u^0}{(\tau/2)} = \mathsf{I}_h \left[\frac{u_{xx}(t^{\frac12}, \cdot) + u_{xx}(t_0, \cdot)}{2} \right] + g(u^0) \otimes \left(\frac{u^{\frac12} + u^0}{2} \right) + \mathsf{I}_h \left[\frac{f(t^{\frac12}, \cdot) + f(t_0, \cdot)}{2} \right] + \mathsf{r}^{\frac14} \tag{3.1}$$

and let $\mathsf{r}^{n+\frac12} \in \mathfrak{X}_h$ be specified by

$$\frac{u^{n+1} - u^n}{\tau} = \mathsf{I}_h \left[\frac{u_{xx}(t_{n+1}, \cdot) + u_{xx}(t_n, \cdot)}{2} \right] + g(u^{n+\frac12}) \otimes \left(\frac{u^{n+1} + u^n}{2} \right) + \mathsf{I}_h \left[\frac{f(t_{n+1}, \cdot) + f(t_n, \cdot)}{2} \right] + \mathsf{r}^{n+\frac12} \tag{3.2}$$

for $n = 0, \ldots, N-1$. Assuming that the solution u is smooth enough on $[0, T] \times \mathcal{I}$, and using (1.4) and the Dirichlet boundary conditions (1.2), we conclude that $u_{xx}(t, x) = -f(t, x)$ for $t \in [0, T]$ and $x \in \{x_a, x_b\}$. Thus, we have $\mathsf{r}^{\frac14} \in \mathfrak{X}^\circ_h$ and $\mathsf{r}^{n+\frac12} \in \mathfrak{X}^\circ_h$ for $n = 0, \ldots, N-1$.

Substracting (1.1) with $(t, x) = (t^{\frac14}, x_j)$ from (3.1), and (1.1) with $(t, x) = (t^{n+\frac12}, x_j)$ from (3.2), we get

$$\mathsf{r}^{\frac14} = \mathsf{r}^{\frac14}_A - \mathsf{r}^{\frac14}_B - \mathsf{r}^{\frac14}_C - \mathsf{r}^{\frac14}_D, \quad \mathsf{r}^{n+\frac12} = \mathsf{r}^{n+\frac12}_A - \mathsf{r}^{n+\frac12}_B - \mathsf{r}^{n+\frac12}_C - \mathsf{r}^{n+\frac12}_D, \quad n = 0, \ldots, N-1, \tag{3.3}$$

where $\mathsf{r}^{\frac14}_A, \mathsf{r}^{\frac14}_C, \mathsf{r}^{n+\frac12}_A, \mathsf{r}^{n+\frac12}_C \in \mathfrak{X}^\circ_h$ and $\mathsf{r}^{\frac14}_B, \mathsf{r}^{\frac14}_D, \mathsf{r}^{n+\frac12}_B, \mathsf{r}^{n+\frac12}_D \in \mathfrak{X}_h$ be defined by

$$\mathsf{r}^{n+\frac12}_A := \frac{u^{n+1} - u^n}{\tau} - \mathsf{I}_h\big[u_t(t^{n+\frac12}, \cdot)\big], \quad \mathsf{r}^{n+\frac12}_B := \mathsf{I}_h \left[\frac{u_{xx}(t_{n+1}, \cdot) + u_{xx}(t_n, \cdot)}{2} - u_{xx}(t^{n+\frac12}, \cdot) \right],$$
$$\mathsf{r}^{n+\frac12}_C := g(u^{n+\frac12}) \otimes \left[\frac{u^{n+1} + u^n}{2} - u^{n+\frac12} \right], \quad \mathsf{r}^{n+\frac12}_D := \mathsf{I}_h \left[\frac{f(t_{n+1}, \cdot) + f(t_n, \cdot)}{2} - f(t^{n+\frac12}, \cdot) \right]$$

and

$$\mathsf{r}_A^{\frac14} := \tfrac{u^{\frac12}-u^0}{(\tau/2)} - \mathsf{I}_h\big[u_t\big(t^{\frac14},\cdot\big)\big], \quad \mathsf{r}_B^{\frac14} := \mathsf{I}_h^{\circ}\left[\tfrac{u_{xx}(t^{\frac12},\cdot)+u_{xx}(t_0,\cdot)}{2} - u_{xx}(t^{\frac14},\cdot)\right],$$
$$\mathsf{r}_C^{\frac14} := -\Big[g(u^{\frac14}) - g(u^0)\Big] \otimes u^{\frac14} + g(u^0)\otimes\left[\tfrac{u^{\frac12}+u^0}{2} - u^{\frac14}\right],$$
$$\mathsf{r}_D^{\frac14} := \mathsf{I}_h\left[\tfrac{f(t^{\frac12},\cdot)+f(t_0,\cdot)}{2} - f(t^{\frac14},\cdot)\right].$$

Applying the Taylor formula we obtain

$$\begin{aligned}
(\mathsf{r}_A^{n+\frac12})_j &= \tfrac{\tau^2}{2}\int_0^{\frac12}\Big[\, s^2\,u_{ttt}(t_n+s\,\tau,x_j) + (\tfrac12 - s)^2\,u_{ttt}(t^{n+\frac12}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_C^{n+\frac12})_j &= \tfrac{g(u(t^{n+\frac12},x_j))}{2}\,\tau^2\int_0^{\frac12}\Big[\, s\,u_{tt}(t_n+s\,\tau,x_j) + (\tfrac12 - s)\,u_{tt}(t^{n+\frac12}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_B^{n+\frac12})_j &= \tfrac{\tau^2}{2}\int_0^{\frac12}\Big[\, s\,u_{xxtt}(t_n+s\,\tau,x_j) + (\tfrac12 - s)\,u_{xxtt}(t^{n+\frac12}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_D^{n+\frac12})_j &= \tfrac{\tau^2}{2}\int_0^{\frac12}\Big[\, s\,f_{tt}(t_n+\tau\,s,x_j) + (\tfrac12 - s)\,f_{tt}(t^{n+\frac12}+\tau\,s,x_j)\,\Big]\,ds
\end{aligned} \tag{3.4}$$

for $j=0,\dots,J+1$ and $n=0,\dots,N-1$, and

$$\begin{aligned}
(\mathsf{r}_A^{\frac14})_j &= \tfrac{\tau^2}{2}\int_0^{\frac14}\Big[\, s^2\,u_{ttt}(s\,\tau,x_j) + (\tfrac14 - s)^2\,u_{ttt}(t^{\frac14}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_C^{\frac14})_j &= -\,u(t^{\frac14},x_j)\,\tau\int_0^{\frac14} g'(u(s\,\tau,x_j))\,u_t(s\,\tau,x_j)\,ds\\
&\quad + \tfrac{g(u_0(x_j))}{2}\,\tau^2\int_0^{\frac14}\Big[\, s\,u_{tt}(s\,\tau,x_j) + (\tfrac14 - s)\,u_{tt}(t^{\frac14}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_B^{\frac14})_j &= \tfrac{\tau^2}{2}\int_0^{\frac14}\Big[\, s\,u_{xxtt}(s\,\tau,x_j) + (\tfrac14 - s)\,u_{xxtt}(t^{\frac14}+s\,\tau,x_j)\,\Big]\,ds,\\
(\mathsf{r}_D^{\frac14})_j &= \tfrac{\tau^2}{2}\int_0^{\frac14}\Big[\, s\,f_{tt}(t_n+\tau\,s,x_j) + (\tfrac14 - s)\,f_{tt}(t^{n+\frac12}+\tau\,s,x_j)\,\Big]\,ds
\end{aligned} \tag{3.5}$$

for $j=0,\dots,J+1$. Then, from (3.3), (3.4) and (3.5), we arrive at

$$\|\mathsf{r}_A^{\frac14}\|_{0,h} + \|\mathsf{r}_B^{\frac14}\|_{0,h} + \|\mathsf{r}_D^{\frac14}\|_{0,h} + \max_{0\le n\le N-1}\|\mathsf{r}^{n+\frac12}\|_{0,h} \le \widehat{\mathsf{C}}_{1,1}\,\tau^2, \tag{3.6}$$

$$\|\mathsf{r}_C^{\frac14}\|_{0,h} \le \widehat{\mathsf{C}}_{1,2}\,\tau \tag{3.7}$$

and

$$\max_{0\le n\le N-1}|\mathsf{r}^{n+\frac12}|_{1,h} \le \widehat{\mathsf{C}}_{1,3}\,\tau^2, \tag{3.8}$$

$$|\mathsf{r}_C^{\frac14}|_{1,h} \le \widehat{\mathsf{C}}_{1,4}\,\tau. \tag{3.9}$$

3.2. **Space consistency error.** Also, let $\mathsf{s}^{\frac14}\in\mathfrak{X}_h^{\circ}$ be defined by

$$\tfrac{u^{\frac12}-u^0}{(\tau/2)} = \Delta_h\left(\tfrac{u^{\frac12}+u^0}{2}\right) + g(u^0)\otimes\left(\tfrac{u^{\frac12}+u^0}{2}\right) + \mathsf{I}_h^{\circ}\left[\tfrac{f(t^{\frac12},\cdot)+f(t_0,\cdot)}{2}\right] + \mathsf{s}^{\frac14} \tag{3.10}$$

and, for $n=0,\dots,N-1$, let $\mathsf{s}^{n+\frac12}\in\mathfrak{X}_h^{\circ}$ be given by

$$\tfrac{u^{n+1}-u^n}{\tau} = \Delta_h\left(\tfrac{u^{n+1}+u^n}{2}\right) + g(u^{n+\frac12})\otimes\left(\tfrac{u^{n+1}+u^n}{2}\right) + \mathsf{I}_h^{\circ}\left[f\,\tfrac{f(t_{n+1},\cdot)+f(t_n,\cdot)}{2}\right] + \mathsf{s}^{n+\frac12}. \tag{3.11}$$

Subtracting (3.10) from (3.1) and (3.11) from (3.2), we obtain

$$
\begin{aligned}
\mathsf{r}^{\frac14}-\mathsf{s}^{\frac14}&=\mathsf{I}_h^{\circ}\left[\tfrac{u_{xx}(t^{\frac12},\cdot)+u_{xx}(t_0,\cdot)}{2}\right]-\Delta_h\left(\tfrac{u^{\frac12}+u^0}{2}\right),\\
\mathsf{r}^{n+\frac12}-\mathsf{s}^{n+\frac12}&=\mathsf{I}_h^{\circ}\left[\tfrac{u_{xx}(t_{n+1},\cdot)+u_{xx}(t_n,\cdot)}{2}\right]-\Delta_h\left(\tfrac{u^{n+1}+u^n}{2}\right),\quad n=0,\dots,N-1.
\end{aligned}
\tag{3.12}
$$

The use of the Taylor formula yields

$$
\begin{aligned}
\left(\mathsf{I}_h^{\circ}[u_{xx}(t,\cdot)]-\Delta_h\left(\mathsf{I}_h[u(t,\cdot)]\right)\right)_j=&\tfrac{h^2}{6}\int_0^1(1-y)^3\,u_{xxxx}(t,x_j+h\,y)\,dy\\
&+\tfrac{h^2}{6}\int_0^1 y^3\,u_{xxxx}(t,x_{j-1}+h\,y)\,dy,
\end{aligned}
$$

for $j=1,\dots,J$ and $t\in[0,T]$, which along with (3.12) yields

$$
\|\mathsf{s}^{\frac14}-\mathsf{r}^{\frac14}\|_{0,h}+\max_{0\le n\le N-1}\|\mathsf{s}^{n+\frac12}-\mathsf{r}^{n+\frac12}\|_{0,h}\le\widehat{C}_2\,h^2.\tag{3.13}
$$

3.3. **Time consistency error at the intermediate nodes.** For $n=1,\dots,N-1$, let $\mathsf{r}^n\in\mathfrak{X}_h^{\circ}$ be determined by

$$
\tfrac{g(u^{n+\frac12})+g(u^{n-\frac12})}{2}=g(u^n)+\mathsf{r}^n.\tag{3.14}
$$

Setting $w(t,x)=g(u(t,x))$ and using, again, the Taylor formula we have

$$
\mathsf{r}_j^n=\tfrac12\,\tau^2\int_0^{\frac12}\left[\,(\tfrac12-s)\,w_{tt}(t_n+s\,\tau,x_j)+s\,w_{tt}(t^{n-\frac12}+s\,\tau,x_j)\right]\,ds\tag{3.15}
$$

for $j=0,\dots,J+1$ and $n=1,\dots,N-1$, which, easily, yields

$$
\max_{1\le n\le N-1}\|\mathsf{r}^n\|_{0,h}+\max_{1\le n\le N-1}|\mathsf{r}^n|_{1,h}\le\widehat{\mathsf{C}}_{3,1}\,\tau^2,\tag{3.16}
$$

$$
\max_{2\le n\le N-1}\|\mathsf{r}^n-\mathsf{r}^{n-1}\|_{0,h}+\max_{2\le n\le N-1}|\mathsf{r}^n-\mathsf{r}^{n-1}|_{1,h}\le\widehat{\mathsf{C}}_{3,2}\,\tau^3.\tag{3.17}
$$

3.4. **A Discrete Ellliptic Projection.** Let $v\in\mathsf{C}^2(\mathcal{I};\mathbb{R})$. Then, we define $\mathsf{R}_h(v)\in\mathfrak{X}_h^{\circ}$ (cf. [2]) by requiring

$$
\Delta_h(\mathsf{R}_h v)=\mathsf{I}_h^{\circ}(v'').\tag{3.18}
$$

Using the Taylor formula, it follows that

$$
\Delta_h(\mathsf{I}_h^{\circ}v)-\mathsf{I}_h^{\circ}(v'')=\tfrac{h^2}{12}\,\mathsf{r}^{\mathrm{E}}(v)\tag{3.19}
$$

where $\mathsf{r}^{\mathrm{E}}(v)\in\mathfrak{X}_h^{\circ}$ is defined by

$$
(\mathsf{r}^{\mathrm{E}}(v))_j:=\int_0^1\left[\,(1-y)^3\,v''''(x_j+h\,y)+y^3\,v''''(x_{j-1}+h\,y)\,\right]\,dy,\quad j=1,\dots,J.\tag{3.20}
$$

First, subtract (3.18) from (3.19) to get

$$
\Delta_h(\mathsf{I}_h^{\circ}v-\mathsf{R}_h v)=\tfrac{h^2}{12}\,\mathsf{r}^{\mathrm{E}}(v).\tag{3.21}
$$

Then, take the $(\cdot,\cdot)_{0,h}$–inner product of both sides of (3.21) with $(\mathsf{I}_h^{\circ}v-\mathsf{R}_h v)$ and use (2.4), the Cauchy-Schwarz inequality and (2.2) to obtain

$$
|\mathsf{R}_h v-\mathsf{I}_h^{\circ}v|_{1,h}\le\tfrac{\mathsf{L}}{12}\,h^2\,\|\mathsf{r}^{\mathrm{E}}(v)\|_{0,h}.\tag{3.22}
$$

Finally, we use (3.22) to have

$$
\begin{aligned}
\left|\mathsf{R}_h\left[\tfrac{u(t_{n+1},\cdot)-u(t_n,\cdot)}{\tau}\right]-\left(\tfrac{u^{n+1}-u^n}{\tau}\right)\right|_{1,h}&\le\tfrac{\mathsf{L}}{12}\,h^2\left\|\mathsf{r}^{\mathrm{E}}\left[\tfrac{u(t_{n+1},\cdot)-u(t_n,\cdot)}{\tau}\right]\right\|_{0,h}\\
&\le\tfrac{\mathsf{L}^{\frac32}}{12}\,h^2\max_{[0,T]\times\mathcal{I}}|u_{txxxx}|,\quad n=0,\dots,N-1.
\end{aligned}
\tag{3.23}
$$

4. Convergence Analysis

4.1. **A mollifier.** For $\delta > 0$, let $\mathfrak{n}_\delta \in \mathsf{C}^3(\mathbb{R};\mathbb{R})$ (cf. [7], [9]) be an odd fuction defined by

$$\mathfrak{n}_\delta(x) := \begin{cases} x, & \text{if } x \in [0,\delta], \\ p_\delta(x), & \text{if } x \in (\delta, 2\delta], \\ 2\,\delta, & \text{if } x > 2\delta, \end{cases} \quad \forall\, x \geq 0, \tag{4.1}$$

where p_δ is the unique polynomial of $\mathbb{P}^7[\delta, 2\,\delta]$ that satisfies the following conditions:

$$p_\delta(\delta) = \delta,\;\; p'_\delta(\delta) = 1,\;\; p''_\delta(\delta) = p'''_\delta(\delta) = 0,\;\; p_\delta(2\,\delta) = 2\,\delta,\;\; p'_\delta(2\,\delta) = p''_\delta(2\,\delta) = p'''_\delta(2\,\delta) = 0.$$

4.2. **The (MBRFD) scheme.** The modified version of the (BRFD) method (cf. [1], [7], [9]) is a recursive procedure that, for given $\delta > 0$, derives approximations $(V_\delta^n)_{n=0}^N \subset \mathfrak{X}_h^\circ$ of the solution u performing the steps below.

`Step 1`: Let $V_\delta^0 \in \mathfrak{X}_h^\circ$ be defined by

$$V_\delta^0 := u^0 \tag{4.2}$$

and $V_\delta^{\frac{1}{2}} \in \mathfrak{X}_h^\circ$ be specified by

$$\frac{V_\delta^{\frac{1}{2}} - V_\delta^0}{(\tau/2)} = \Delta_h\left(\frac{V_\delta^{\frac{1}{2}} + V_\delta^0}{2}\right) + g\big(u^0\big) \otimes \left(\frac{V_\delta^{\frac{1}{2}} + V_\delta^0}{2}\right) + \mathsf{I}_h^\circ\left[\frac{f(t^{\frac{1}{2}},\cdot) + f(t_0,\cdot)}{2}\right]. \tag{4.3}$$

`Step 2`: Define $\Phi_\delta^{\frac{1}{2}} \in \mathfrak{X}_h$ by

$$\Phi_\delta^{\frac{1}{2}} := g\big(\mathfrak{n}_\delta\big(V_\delta^{\frac{1}{2}}\big)\big) \tag{4.4}$$

and find $V_\delta^1 \in \mathfrak{X}_h^\circ$ such that

$$\frac{V_\delta^1 - V_\delta^0}{\tau} = \Delta_h\left(\frac{V_\delta^1 + V_\delta^0}{2}\right) + \mathfrak{n}_\delta\big(\Phi_\delta^{\frac{1}{2}}\big) \otimes \left(\frac{V_\delta^1 + V_\delta^0}{2}\right) + \mathsf{I}_h^\circ\left[\frac{f(t_1,\cdot) + f(t_0,\cdot)}{2}\right]. \tag{4.5}$$

`Step 3`: For $n = 1, \dots, N-1$, first define $\Phi_\delta^{n+\frac{1}{2}} \in \mathfrak{X}_h$ by

$$\Phi_\delta^{n+\frac{1}{2}} := 2\,g\big(\mathfrak{n}_\delta\big(V_\delta^n\big)\big) - \Phi_\delta^{n-\frac{1}{2}} \tag{4.6}$$

and, then, find $V_\delta^{n+1} \in \mathfrak{X}_h^\circ$ such that

$$\frac{V_\delta^{n+1} - V_\delta^n}{\tau} = \Delta_h\left(\frac{V_\delta^{n+1} + V_\delta^n}{2}\right) + \mathfrak{n}_\delta\big(\Phi_\delta^{n+\frac{1}{2}}\big) \otimes \left(\frac{V_\delta^{n+1} + V_\delta^n}{2}\right) + \mathsf{I}_h^\circ\left[\frac{f(t_{n+1},\cdot) + f(t_n,\cdot)}{2}\right]. \tag{4.7}$$

4.3. **Existence and uniqueness of the (MBRFD) approximations.**

Proposition 4.1. *Let* $g_\star^0 = \max_{x\in\mathcal{I}} |g(u_0(x))|$, $\delta \geq g_\star^0$ *and* $\mathsf{C}_\delta^{\mathsf{BR,I}} := \frac{1}{4}\sup_{\mathbb{R}} |\mathfrak{n}_\delta|$. *When* $\tau\,\mathsf{C}_\delta^{\mathsf{BR,I}} \leq \frac{1}{2}$, *then the modified* (BRFD) *approximations are well-defined.*

Proof. Let $\zeta \in \mathfrak{X}_h$, $\varepsilon \in (0,1]$ and $\mathsf{T}_{\mathsf{BR}} : \mathfrak{X}_h^\circ \to \mathfrak{X}_h^\circ$ be a linear operator given by

$$\mathsf{T}_{\mathsf{BR}} v := 2\,v - \varepsilon\,\tau\,\Delta_h v - \varepsilon\,\tau\,[\,\mathfrak{n}_\delta(\zeta) \otimes v\,] \quad \forall\, v \in \mathfrak{X}_h^\circ.$$

Since $\delta \geq g_\star^0$, the definition of $\mathfrak{n}_\delta$ yields that $\mathfrak{n}_\delta(g(u^0)) = g(u^0)$. Thus, from (4.3), (4.5) and (4.7) it is easily seen that the well-posedness of $V_\delta^{\frac{1}{2}}$ and $(V_\delta^n)_{n=1}^N$ follows easily by securing the invertibility of T_{BR}. Moving towards to this target, first we use (2.4) to obtain

$$\begin{aligned} (\mathsf{T}_{\mathsf{BR}} v, v)_{0,h} &= 2\,\|v\|_{0,h}^2 + \tau\,\varepsilon\,|v|_{1,h}^2 - \tau\,\varepsilon\,\big(\mathfrak{n}_\delta(\zeta) \otimes v, v\big)_{0,h} \\ &\geq 2\,\|v\|_{0,h}^2 + \tau\,\varepsilon\,|v|_{0,h}^2 - \tau\,\varepsilon\,\|v\|_{0,h}^2\,|\mathfrak{n}_\delta(\zeta)|_{\infty,h} \\ &\geq \tau\,\varepsilon\,|v|_{0,h}^2 + 4\,\|v\|_{0,h}^2\left(\tfrac{1}{2} - \tfrac{\tau}{4}\max_{\mathbb{R}}|\mathfrak{n}_\delta|\right) \\ &\geq \tau\,\varepsilon\,|v|_{1,h}^2 + 4\,\|v\|_{0,h}^2\,\big(\tfrac{1}{2} - \tau\,\mathsf{C}_\delta^{\mathsf{BR,I}}\big) \quad \forall\, v \in \mathfrak{X}_h^\circ. \end{aligned} \tag{4.8}$$

Let us assume that $\tau\,\mathsf{C}_{\delta}^{\mathsf{BR,I}}\le\frac{1}{2}$. When $v\in\mathrm{Ker}(\mathsf{T}_{\mathsf{BR}})$, then $(\mathsf{T}_{\mathsf{BR}}v,v)_{0,h}=0$, which, along with (4.8), yields $|v|_{1,h}=0$, or, equivalently, $v=0$. The latter argument shows that $\mathrm{Ker}(\mathsf{T}_{\mathsf{BR}})=\{0\}$ and, thus, T_{BR} is invertible, since $\mathfrak{X}_h^{\circ}$ has finite dimension. □

Remark 4.1. *Let us assume that $\tau\,\mathsf{C}_{\delta}^{\mathsf{BR,I}}\le\frac{1}{2}$ and $\delta\ge g_{\star}^{0}$. Since $V_{\delta}^{0}:=u^{0}$ and $V_{\delta}^{\frac{1}{2}}$ is well-defined, in view of (4.3) and (1.6), we conclude that $U^{\frac{1}{2}}$ is, also, well-defined and $U^{\frac{1}{2}}=V_{\delta}^{\frac{1}{2}}$.*

4.4. **Convergence of the (MBRFD) scheme.** In the theorem below, we investigate the convergence properties of the modified (BRFD) approximations.

Theorem 4.2. *Let $u_{\star}:=\max\limits_{[0,T]\times\mathcal{I}}|u|$, $g_{\star}:=\max\limits_{[0,T]\times\mathcal{I}}|g(u)|$, $\delta_{\star}\ge\max\{u_{\star},g_{\star}\}$ and $\tau\,\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,I}}\le\frac{1}{2}$, where $\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,I}}$ is the constant specified in Proposition 4.1. Then, there exist constants $\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,1}}\ge\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,I}}$, $\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,2}}>0$, $\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,3}}>0$ and $\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,4}}>0$, independent of τ and h, such that: if $\tau\,\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,1}}\le\frac{1}{2}$, then*

$$|u^{\frac{1}{2}}-V_{\delta_{\star}}^{\frac{1}{2}}|_{1,h}\le\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,2}}\,(\tau^{2}+\tau^{\frac{1}{2}}\,h^{2}),\tag{4.9}$$

$$\max_{0\le m\le N-1}\|g(u^{m+\frac{1}{2}})-\Phi_{\delta_{\star}}^{m+\frac{1}{2}}\|_{0,h}+\max_{0\le m\le N}|u^{m}-V_{\delta_{\star}}^{m}|_{1,h}\le\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,3}}\,(\tau^{2}+h^{2})\tag{4.10}$$

and

$$\max_{0\le m\le N-1}\left|g(u^{m+\frac{1}{2}})-\Phi_{\delta_{\star}}^{m+\frac{1}{2}}\right|_{1,h}\le\mathsf{C}_{\delta_{\star}}^{\mathsf{BCV,4}}\,(\tau^{2}+h^{2}).\tag{4.11}$$

Proof. To simplify the notation, we set $u_{\star}^{0}:=\max_{\mathcal{I}}|u^{0}|$, $\mathbf{e}^{\frac{1}{2}}:=u^{\frac{1}{2}}-V_{\delta_{\star}}^{\frac{1}{2}}$, $\mathbf{e}^{m}:=u^{m}-V_{\delta_{\star}}^{m}$ for $m=0,\dots,N$, and $\boldsymbol{\theta}^{m}:=g(u^{m+\frac{1}{2}})-\Phi_{\delta_{\star}}^{m+\frac{1}{2}}$ for $m=0,\dots,N-1$. In the sequel, we will use the symbol C to denote a generic constant that is independent of τ, h and $\delta_{\star}$, and may changes value from one line to the other. Also, we will use the symbol $C_{\delta_{\star}}$ to denote a generic constant that depends on $\delta_{\star}$ but is independent of τ, h, and may changes value from one line to the other.

`Part 1:` Since $\mathbf{e}^{0}=0$, after subtracting (4.3) from (3.10) we obtain

$$\mathbf{e}^{\frac{1}{2}}=\tfrac{\tau}{4}\,\Delta_{h}\mathbf{e}^{\frac{1}{2}}+\tfrac{\tau}{4}\left[\,g(u^{0})\otimes\mathbf{e}^{\frac{1}{2}}\,\right]+\tfrac{\tau}{2}\,\mathsf{s}^{\frac{1}{4}}.\tag{4.12}$$

Next, take the $(\cdot,\cdot)_{0,h}$–inner product of (4.12) with $\mathbf{e}^{\frac{1}{2}}$, and then use (2.4), the Cauchy-Schwarz inequality, (3.3), (3.6), (3.7), (3.13) and the arithmetic mean inequality to get

$$\begin{aligned}\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}+\tfrac{\tau}{4}\,|\mathbf{e}^{\frac{1}{2}}|_{1,h}^{2}&=\tfrac{\tau}{4}\,(g(u^{0})\otimes\mathbf{e}^{\frac{1}{2}},\mathbf{e}^{\frac{1}{2}})_{0,h}+\tfrac{\tau}{2}\,(\mathsf{s}^{\frac{1}{4}},\mathbf{e}^{\frac{1}{2}})_{0,h}\\&\le\tfrac{\tau}{4}\,|g(u^{0})|_{\infty,h}\,\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}+\tfrac{\tau}{2}\left[\,\|\mathsf{s}^{\frac{1}{4}}-\mathsf{r}^{\frac{1}{4}}\|_{0,h}+\|\mathsf{r}^{\frac{1}{4}}\|_{0,h}\,\right]\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}\\&\le\tfrac{\tau}{4}\,|g(u^{0})|_{\infty,h}\,\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}+C\,(\tau^{2}+\tau\,h^{2})\,\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}\\&\le\tfrac{\tau}{4}\max_{|x|\in[0,u_{\star}^{0}]}|g(x)|\,\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}+C\,(\tau^{2}+\tau\,h^{2})^{2}+\tfrac{1}{2}\,\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}.\end{aligned}$$

Let $\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,II}}:=\max\{\frac{1}{2}\max_{|x|\in[0,u_{\star}^{0}]}|g(x)|,\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,I}}\}$ and $\tau\,\mathsf{C}_{\delta_{\star}}^{\mathsf{BR,II}}\le\frac{1}{2}$. Then, the inequality above yields that

$$\|\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}+\tau\,|\mathbf{e}^{\frac{1}{2}}|_{1,h}^{2}\le C\,(\tau^{2}+\tau\,h^{2})^{2}.\tag{4.13}$$

Taking the $(\cdot,\cdot)_{0,h}$–inner product of (4.12) with $\Delta_{h}\mathbf{e}^{\frac{1}{2}}$, and then using (2.4), we obtain

$$4\,|\mathbf{e}^{\frac{1}{2}}|_{1,h}^{2}+\tau\,\|\Delta_{h}\mathbf{e}^{\frac{1}{2}}\|_{0,h}^{2}=\mathsf{a}^{1}+\mathsf{a}^{2},\tag{4.14}$$

where

$$\begin{aligned}\mathsf{a}^{1}&:=-\,\tau\,(g(u^{0})\otimes\mathbf{e}^{\frac{1}{2}},\Delta_{h}\mathbf{e}^{\frac{1}{2}})_{0,h},\\\mathsf{a}^{2}&:=-\,2\,\tau\,(\eta^{\frac{1}{4}},\Delta_{h}\mathbf{e}^{\frac{1}{2}})_{0,h}.\end{aligned}$$

Now, we use the Cauchy-Schwarz inequality, the arithmetic mean inequality and (4.13), to have

$$
\begin{aligned}
\mathsf{a}^1 &\le \tau \max_{|x|\in[0,u_\star^0]} |g(x)|\, \|\mathbf{e}^{\frac12}\|_{0,h}\, \|\Delta_h \mathbf{e}^{\frac12}\|_{0,h} \\
&\le C\,\tau\, \|\mathbf{e}^{\frac12}\|_{0,h}^2 + \tfrac{\tau}{6}\, \|\Delta_h \mathbf{e}^{\frac12}\|_{0,h}^2 \\
&\le C\,\tau\,(\tau^2+\tau\,h^2)^2 + \tfrac{\tau}{6}\, \|\Delta_h \mathbf{e}^{\frac12}\|_{0,h}^2.
\end{aligned} \tag{4.15}
$$

Also, (3.3), the Cauchy-Schwarz inequality, (2.3), (3.6), (3.9), (3.13) and the arithmetic mean inequality, yield

$$
\begin{aligned}
\mathsf{a}^2 &= -2\,\tau\left(\mathsf{s}^{\frac14}-\mathsf{r}^{\frac14}, \Delta_h\mathbf{e}^{\frac12}\right)_{0,h} - 2\,\tau\left(\mathsf{r}_A^{\frac14}-\mathsf{r}_B^{\frac14}-\mathsf{r}_D^{\frac14}, \Delta_h\mathbf{e}^{\frac12}\right)_{0,h} + 2\,\tau\left(\mathsf{r}_C^{\frac14}, \Delta_h\mathbf{e}^{\frac12}\right)_{0,h} \\
&\le 2\,\tau\left[\,\|\mathsf{s}^{\frac14}-\mathsf{r}^{\frac14}\|_{0,h} + \|\mathsf{r}_A^{\frac14}\|_{0,h} + \|\mathsf{r}_B^{\frac14}\|_{0,h} + \|\mathsf{r}_D^{\frac14}\|_{0,h}\,\right] \|\Delta_h\mathbf{e}^{\frac12}\|_{0,h} - 2\,\tau\,(\!(\delta_h\mathsf{r}_C^{\frac14}, \delta_h\mathbf{e}^{\frac12})\!)_{0,h} \\
&\le C\,\tau\,(\tau^2+h^2)\,\|\Delta_h\mathbf{e}^{\frac12}\|_{0,h} + 2\,\tau\,|\mathsf{r}_C^{\frac14}|_{1,h}\,|\mathbf{e}^{\frac12}|_{1,h} \\
&\le C\,\tau\,(\tau^2+h^2)\,\|\Delta_h\mathbf{e}^{\frac12}\|_{0,h} + C\,\tau^2\,|\mathbf{e}^{\frac12}|_{1,h} \\
&\le C\left[\,\tau\,(\tau^2+h^2)^2+\tau^4\,\right] + \tfrac{\tau}{6}\,\|\Delta_h\mathbf{e}^{\frac12}\|_{0,h}^2 + |\mathbf{e}^{\frac12}|_{1,h}^2.
\end{aligned} \tag{4.16}
$$

In view of (4.14), (4.15) and (4.16), we arrive at

$$
|\mathbf{e}^{\frac12}|_{1,h}^2 + \tau\,\|\Delta_h\mathbf{e}^{\frac12}\|_{0,h}^2 \le C\,(\tau^2+\tau^{\frac12}\,h^2)^2, \tag{4.17}
$$

which, obviously, yields (4.9).

Since $\delta_\star \ge u_\star$, using (4.1), (4.4) and (4.13), we have

$$
\begin{aligned}
\|\boldsymbol{\theta}^0\|_{0,h}^2 &= \left\|g\big(\mathfrak{n}_{\delta_\star}(u^{\frac12})\big) - g\Big(\mathfrak{n}_{\delta_\star}\big(V_{\delta_\star}^{\frac12}\big)\Big)\right\|_{0,h}^2 \\
&\le \sup_{\mathbb{R}} |(g\circ\mathfrak{n}_{\delta_\star})'|^2\, \|\mathbf{e}^{\frac12}\|_{0,h}^2 \\
&\le C_{\delta_\star}\,(\tau^2+\tau\,h^2)^2.
\end{aligned} \tag{4.18}
$$

Also, using Lemma 2.2, (2.2) and (4.17), we get

$$
\begin{aligned}
|\boldsymbol{\theta}^0|_{1,h}^2 &= \left|g\big(\mathfrak{n}_{\delta_\star}(u^{\frac12})\big) - g\Big(\mathfrak{n}_{\delta_\star}\big(V_{\delta_\star}^{\frac12}\big)\Big)\right|_{1,h}^2 \\
&\le 2\sup_{\mathbb{R}} |(g\circ\mathfrak{n}_{\delta_\star})'|^2\, |\mathbf{e}^{\frac12}|_{1,h}^2 + 2\sup_{\mathbb{R}} |(g\circ\mathfrak{n}_{\delta_\star})''|^2\, \|\delta_h g(u^{\frac12})\|_{\infty,h}^2\, \|\mathbf{e}^{\frac12}\|_{0,h}^2 \\
&\le C_{\delta_\star}\,|\mathbf{e}^{\frac12}|_{1,h}^2 \\
&\le C_{\delta_\star}\,(\tau^2+\tau^{\frac12}\,h^2)^2.
\end{aligned} \tag{4.19}
$$

`Part 2:` We subtract (4.5) and (4.7) from (3.11), to obtain the following error equations:

$$
2\,(\mathbf{e}^{n+1}-\mathbf{e}^n) = \tau\,\Delta_h\left(\mathbf{e}^{n+1}+\mathbf{e}^n\right) + \sum_{\kappa=1}^{3}\mathsf{Q}^{\kappa,n}, \quad n=0,\dots,N-1, \tag{4.20}
$$

where

$$
\begin{aligned}
\mathsf{Q}^{1,n} &:= 2\,\tau\,\mathsf{s}^{n+\frac12}, \\
\mathsf{Q}^{2,n} &:= \tau\,\mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{n+\frac12}\big)\otimes\left(\mathbf{e}^{n+1}+\mathbf{e}^n\right), \\
\mathsf{Q}^{3,n} &:= \tau\left[\,g(u^{n+\frac12}) - \mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{n+\frac12}\big)\right]\otimes\left(u^{n+1}+u^n\right).
\end{aligned}
$$

We take the inner product $(\cdot,\cdot)_{0,h}$ of (4.20) with $(\mathbf{e}^{n+1}-\mathbf{e}^n)$, and then, use (2.3), to have

$$
2\,\|\mathbf{e}^{n+1}-\mathbf{e}^n\|_{0,h}^2 + \tau\left[\,|\mathbf{e}^{n+1}|_{1,h}^2 - |\mathbf{e}^n|_{1,h}^2\,\right] = \sum_{\kappa=1}^{3}\mathsf{q}^{\kappa,n}, \quad n=0,\dots,N-1, \tag{4.21}
$$

where

$$\mathsf{q}^{\kappa,n} := (\mathsf{Q}^{\kappa,n}, \mathbf{e}^{n+1} - \mathbf{e}^n)_{0,h}.$$

Let $n \in \{0\dots,N-1\}$. Using the Cauchy-Schwarz inequality, the arithmetic mean inequality, (3.6) and (3.13), we have

$$\begin{aligned}\mathsf{q}^{1,n} &\le 2\,\tau \left[\, \|\mathsf{s}^{n+\frac12} - \mathsf{r}^{n+\frac12}\|_{0,h} + \|\mathsf{r}^{n+\frac12}\|_{0,h}\,\right] \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le 2\,\tau\,(\tau^2 + h^2)\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C\,\tau^2\,(\tau^2 + h^2)^2 + \tfrac16\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}^2.\end{aligned} \tag{4.22}$$

Next, we use the Cauchy-Schwarz inequality, (2.2), (4.1) and the arithmetic mean inequality, to get

$$\begin{aligned}\mathsf{q}^{2,n} &\le \tau\,|\mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac12})|_{\infty,h}\,\|\mathbf{e}^{n+1} + \mathbf{e}^n\|_{0,h}\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C_{\delta_\star}\,\tau\,|\mathbf{e}^{n+1} + \mathbf{e}^n|_{1,h}\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C_{\delta_\star}\,\tau^2\left[\,|\mathbf{e}^{n+1}|_{1,h}^2 + |\mathbf{e}^n|_{1,h}^2\,\right] + \tfrac16\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}^2.\end{aligned} \tag{4.23}$$

Finally, taking into account that $\delta_\star \ge g_\star$, we apply the Cauchy-Schwarz inequality, (4.1) and the arithmetic mean inequality to obtain

$$\begin{aligned}\mathsf{q}^{3,n} &\le 2\,\tau\,u_\star\,\|\mathfrak{n}_{\delta_\star}(g(u^{n+\frac12})) - \mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac12})\|_{0,h}\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C\,\tau\,\max_{\mathbb{R}}|\mathfrak{n}_{\delta_\star}'|\,\left\|g(u^{n+\frac12}) - \Phi_{\delta_\star}^{n+\frac12}\right\|_{0,h}\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C_{\delta_\star}\,\tau\,\|\boldsymbol{\theta}^n\|_{0,h}\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}\\ &\le C_{\delta_\star}\,\tau^2\,\|\boldsymbol{\theta}^n\|_{0,h}^2 + \tfrac16\,\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}^2.\end{aligned} \tag{4.24}$$

From (4.21), (4.22), (4.23) and (4.24), we conclude that there exists a constant $\mathsf{C}_{\delta_\star}^{\mathsf{BR,III}} > 0$, such that

$$\begin{aligned}\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,h}^2 + \tau\,|\mathbf{e}^{n+1}|_{1,h}^2 \le \tau\,|\mathbf{e}^n|_{1,h}^2 + \mathsf{C}_{\delta_\star}^{\mathsf{BR,III}}\,\tau^2\left[\,|\mathbf{e}^{n+1}|_{1,h}^2 + |\mathbf{e}^n|_{1,h}^2 + \|\boldsymbol{\theta}^n\|_{0,h}^2\,\right]\\ + C\,\tau^2\,(\tau^2 + h^2)^2, \quad n = 0,\dots,N-1.\end{aligned} \tag{4.25}$$

Let us find an error equation governing the midpoint error $\|\boldsymbol{\theta}^n\|_{0,h}$. Subtracting (4.6) from (3.14) and using (4.1) and the assumption $\delta_\star \ge u_\star$, we obtain

$$\boldsymbol{\theta}^n + \boldsymbol{\theta}^{n-1} = 2\left[g(\mathfrak{n}_{\delta_\star}(u^n)) - g(\mathfrak{n}_{\delta_\star}(V_{\delta_\star}^n))\right] + 2\,\mathsf{r}^n, \quad n = 1,\dots,N-1, \tag{4.26}$$

which, easily, yields that

$$\boldsymbol{\theta}^n - \boldsymbol{\theta}^{n-2} = 2\,\mathsf{R}^n + 2\,(\mathsf{r}^n - \mathsf{r}^{n-1}), \quad n = 2,\dots,N-1, \tag{4.27}$$

where $\mathsf{R}^n \in \mathfrak{X}_h^\circ$ is defined by

$$\mathsf{R}^n := g(\mathfrak{n}_{\delta_\star}(V_{\delta_\star}^{n-1})) - g(\mathfrak{n}_{\delta_\star}(V_{\delta_\star}^n)) - g(\mathfrak{n}_{\delta_\star}(u^{n-1})) + g(\mathfrak{n}_{\delta_\star}(u^n)). \tag{4.28}$$

Then, we use (2.9), (4.1) and the mean value theorem, to get

$$\begin{aligned}\|\mathsf{R}^n\|_{0,h} &\le \sup_{\mathbb{R}}|(g\circ\mathfrak{n}_{\delta_\star})'|\,\|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{0,h}\\ &\qquad + \sup_{\mathbb{R}}|(g\circ\mathfrak{n}_{\delta_\star})''|\,|u^{n-1} - u^n|_{\infty,h}\left[\|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{0,h} + \|\mathbf{e}^n\|_{0,h}\right]\\ &\le C_{\delta_\star}\left[\,\|\mathbf{e}^n - \mathbf{e}^{n-1}\|_{0,h} + \tau\,\|\mathbf{e}^n\|_{0,h}\,\right], \quad n = 2,\dots,N-1.\end{aligned} \tag{4.29}$$

Taking the $(\cdot,\cdot)_{0,h}$ inner product of both sides of (4.27) with $\tau\big(\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\big)$, and then using the Cauchy-Schwarz inequality, (4.29), (3.17) and (2.2), it follows that

$$\begin{aligned}\tau\,\|\boldsymbol{\theta}^n\|_{0,h}^2-\tau\,\|\boldsymbol{\theta}^{n-2}\|_{0,h}^2 &\leq \big[\,2\,\tau\,\|\mathsf{R}^n\|_{0,h}+2\,\tau\,\|\mathsf{r}^n-\mathsf{r}^{n-1}\|_{0,h}\,\big]\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h}\\ &\leq C_{\delta_\star}\,\big[\,\tau\,\|\mathbf{e}^n-\mathbf{e}^{n-1}\|_{0,h}+\tau^2\,\|\mathbf{e}^n\|_{0,h}\,\big]\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h}\\ &\qquad+C\,\tau^4\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h}\\ &\leq C_{\delta_\star}\,\tau\,\|\mathbf{e}^n-\mathbf{e}^{n-1}\|_{0,h}\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h}\\ &\qquad+C_{\delta_\star}\,\tau^2\,|\mathbf{e}^n|_{1,h}\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h}\\ &\qquad+C\,\tau^4\,\|\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}\|_{0,h},\quad n=2,\dots,N-1,\end{aligned}$$

which, along with the application of the arithmetic mean inequality, yields

$$\begin{aligned}\tau\,\|\boldsymbol{\theta}^n\|_{0,h}^2+\tau\,\|\boldsymbol{\theta}^{n-1}\|_{0,h}^2 \leq&\ \tau\,\|\boldsymbol{\theta}^{n-1}\|_{0,h}^2+\tau\,\|\boldsymbol{\theta}^{n-2}\|_{0,h}^2+\|\mathbf{e}^n-\mathbf{e}^{n-1}\|_{0,h}^2+C\,\tau^6\\ &+C_{\delta_\star}\,\tau^2\,\big[\,|\mathbf{e}^n|_{1,h}^2+\|\boldsymbol{\theta}^n\|_{0,h}^2+\|\boldsymbol{\theta}^{n-2}\|_{0,h}^2\,\big],\quad n=2,\dots,N-1.\end{aligned}\tag{4.30}$$

Thus, from (4.25) and (4.30), we conclude that there exists a constant $\mathsf{C}_{\delta_\star}^{\mathsf{BR,IV}}>0$ such that:

$$(1-\mathsf{C}_{\delta_\star}^{\mathsf{BR,IV}}\,\tau)\,\mathsf{Z}^{n+1}\leq(1+\mathsf{C}_{\delta_\star}^{\mathsf{BR,IV}}\,\tau)\,\mathsf{Z}^n+C\,\tau^2\,(\tau^2+h^2)^2,\quad n=2,\dots,N-1,\tag{4.31}$$

where

$$\mathsf{Z}^m:=\|\mathbf{e}^m-\mathbf{e}^{m-1}\|_{0,h}^2+\tau\,\big[\,|\mathbf{e}^m|_{1,h}^2+\|\boldsymbol{\theta}^{m-1}\|_{0,h}^2+\|\boldsymbol{\theta}^{m-2}\|_{0,h}^2\,\big],\quad n=2,\dots,N.\tag{4.32}$$

Assuming that $\tau\,\mathsf{C}_{\delta_\star}^{\mathsf{BR,V}}\leq\frac{1}{2}$ with $\mathsf{C}_{\delta_\star}^{\mathsf{BR,V}}:=\max\{\mathsf{C}_{\delta}^{\mathsf{BR,III}},\mathsf{C}_{\delta_\star}^{\mathsf{BR,IV}}\}$, a standard discrete Gronwall argument based on (4.31) yields

$$\begin{aligned}\max_{2\leq m\leq N}\mathsf{Z}^m &\leq C_{\delta_\star}\,\big[\,\mathsf{Z}^2+\tau\,(\tau^2+h^2)^2\,\big]\\ &\leq C_{\delta_\star}\,\big[\|\mathbf{e}^2-\mathbf{e}^1\|_{0,h}^2+\tau\,|\mathbf{e}^2|_{1,h}^2+\tau\,\|\boldsymbol{\theta}^1\|_{0,h}^2+\tau\,\|\boldsymbol{\theta}^0\|_{0,h}^2+\tau\,(\tau^2+h^2)^2\,\big].\end{aligned}\tag{4.33}$$

Since $\mathbf{e}^0=0$, after setting $n=0$ in (4.25) and then using (4.18), we obtain

$$\begin{aligned}\|\mathbf{e}^1\|_{0,h}^2+\tau\,|\mathbf{e}^1|_{1,h}^2 &\leq C_{\delta_\star}\,\big[\,\tau^2\,(\tau^2+h^2)^2+\tau^2\,\|\boldsymbol{\theta}^0\|_{0,h}^2\,\big]\\ &\leq C_{\delta_\star}\,\tau^2\,(\tau^2+h^2)^2.\end{aligned}\tag{4.34}$$

Also, setting $n=1$ in (4.26) and then using (4.2), we get

$$\boldsymbol{\theta}^1=-\boldsymbol{\theta}^0+2\,\mathsf{r}^1,\tag{4.35}$$

which, along with (4.18) and (3.16), yields

$$\|\boldsymbol{\theta}^1\|_{0,h}^2\leq C_{\delta_\star}\,(\tau^2+\tau\,h^2)^2.\tag{4.36}$$

Also, setting $n=1$ in (4.25), and then using (4.34) and (4.36), we have

$$\begin{aligned}\|\mathbf{e}^2-\mathbf{e}^1\|_{0,h}^2+\tau\,|\mathbf{e}^2|_{1,h}^2 &\leq C\,\tau^2\,(\tau^2+h^2)^2+C_{\delta_\star}\,\big[\,\tau\,|\mathbf{e}^1|_{1,h}^2+\tau^2\,\|\boldsymbol{\theta}^1\|_{0,h}^2\,\big]\\ &\leq C_{\delta_\star}\,\tau^2\,(\tau^2+h^2)^2.\end{aligned}\tag{4.37}$$

Thus, (4.33), (4.37), (4.36) and (4.18) yield

$$\max_{2\leq m\leq N}\mathsf{Z}^m\leq C_{\delta_\star}\,\tau\,(\tau^2+h^2)^2.\tag{4.38}$$

Since $\mathbf{e}^0=0$, (4.10) follows, easily, from (4.32), (4.38) and (4.34).

`Part 3:` Let us define $\boldsymbol{\rho}^m:=\mathsf{R}_h[u(t^m,\cdot)]-u^m\in\mathfrak{X}_h^\circ$ and $\boldsymbol{\eta}^m:=V_{\delta_\star}^m-\mathsf{R}_h[u(t^m,\cdot)]\in\mathfrak{X}_h^\circ$ for $m=0,\dots,N$. Then, using (4.5), (4.7), (3.2) and (3.18) we get

$$2\,(\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n)=\tau\,\Delta_h\big(\boldsymbol{\eta}^{n+1}+\boldsymbol{\eta}^n\big)+\sum_{\kappa=1}^{4}\mathsf{B}^{\kappa,n},\quad n=0,\dots,N-1,\tag{4.39}$$

where

$$\mathsf{B}^{1,n} := 2\,\tau\left(\tfrac{u^{n+1}-u^n}{\tau} - \mathsf{R}_h\left[\tfrac{u(t^{n+1},\cdot)-u(t^n,\cdot)}{\tau}\right]\right),$$
$$\mathsf{B}^{2,n} := -\,2\,\tau\,\mathsf{r}^{n+\frac{1}{2}},$$
$$\mathsf{B}^{3,n} := -\,\tau\,\mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{n+\frac{1}{2}}\big)\otimes(\mathbf{e}^{n+1}+\mathbf{e}^n),$$
$$\mathsf{B}^{4,n} := \tau\left[\mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{n+\frac{1}{2}}\big) - \mathfrak{n}_{\delta_\star}\big(g(u^{n+\frac{1}{2}})\big)\right]\otimes(u^{n+1}+u^n).$$

Take the $(\cdot,\cdot)_{0,h}$–inner product of (4.39) with $\Delta_h(\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n)$, and then, use (2.4) and (2.3), to have

$$2\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2 + \tau\left[\,\|\Delta_h\boldsymbol{\eta}^{n+1}\|_{0,h}^2 - \|\Delta_h\boldsymbol{\eta}^n\|_{0,h}^2\,\right] = \sum_{\kappa=1}^{4}\mathsf{b}^{\kappa,n},\quad n=0,\dots,N-1, \tag{4.40}$$

where

$$\mathsf{b}^{\kappa,n} := ((\delta_h\mathsf{B}^{\kappa,n},\delta_h(\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n)))_{0,h}.$$

Let $n\in\{0\dots,N-1\}$. Using the Cauchy-Schwarz inequality, the arithmetic mean inequality, (3.8) and (3.23), we have

$$\begin{aligned}\mathsf{b}^{1,n} &\le |\mathsf{B}^{1,n}|_{1,h}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}\\ &\le C\,\tau\,h^2\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}\\ &\le C\,\tau^2\,h^4 + \tfrac{1}{6}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}\mathsf{b}^{2,n} &\le 2\,\tau\,|\mathsf{r}^{n+\frac{1}{2}}|_{1,h}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}\\ &\le C\,\tau^3\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}\\ &\le C\,\tau^6 + \tfrac{1}{6}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2.\end{aligned} \tag{4.42}$$

Using, again, the Cauchy-Schwarz inequality and the arithmetic mean inequality, we get

$$\mathsf{b}^{3,n}+\mathsf{b}^{4,n} \le \tfrac{3}{2}\,\tau^2\left(|\mathsf{c}^{3,n}|_{1,h}^2 + |\mathsf{c}^{4,n}|_{1,h}^2\right) + \tfrac{2}{6}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2 \tag{4.43}$$

where

$$\mathsf{c}^{3,n} := \mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac{1}{2}})\otimes(\mathbf{e}^{n+1}+\mathbf{e}^n),$$
$$\mathsf{c}^{4,n} := (g(u^{n+\frac{1}{2}}) - \mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac{1}{2}}))\otimes(u^{n+1}+u^n).$$

Then, we use (4.1), (2.1), (4.10), (2.2), (2.5) and the assumption $\delta_\star > g_\star$ to get

$$\begin{aligned}|\mathsf{c}^{3,n}|_{1,h} &\le \sup_{\mathbb{R}}|\mathfrak{n}_{\delta_\star}'|\,\big|\Phi_{\delta_\star}^{n+\frac{1}{2}}\big|_{1,h}\,|\mathbf{e}^{n+1}+\mathbf{e}^n|_{\infty,h} + \sup_{\mathbb{R}}|\mathfrak{n}_{\delta_\star}|\,|\mathbf{e}^{n+1}+\mathbf{e}^n|_{1,h}\\ &\le C_{\delta_\star}\left[1+\big|\Phi_{\delta_\star}^{n+\frac{1}{2}}-g(u^{n+\frac{1}{2}})\big|_{1,h} + |g(u^{n+\frac{1}{2}})|_{1,h}\right](|\mathbf{e}^{n+1}|_{1,h}+|\mathbf{e}^n|_{1,h})\\ &\le C_{\delta_\star}\left[1+\big|\Phi_{\delta_\star}^{n+\frac{1}{2}}-g(u^{n+\frac{1}{2}})\big|_{1,h}\right](\tau^2+h^2)\\ &\le C_{\delta_\star}\left[\,|\boldsymbol{\theta}^n|_{1,h}+(\tau^2+h^2)\,\right]\end{aligned} \tag{4.44}$$

and

$$
\begin{aligned}
|\mathsf{c}^{4,n}|_{1,h} &\le C\left[\,\|\mathfrak{n}_{\delta_\star}(g(u^{n+\frac{1}{2}}))-\mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac{1}{2}})\|_{0,h}+\left|\mathfrak{n}_{\delta_\star}(g(u^{n+\frac{1}{2}}))-\mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac{1}{2}})\right|_{1,h}\,\right]\\
&\le C\left|\mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{n+\frac{1}{2}})-\mathfrak{n}_{\delta_\star}(g(u^{n+\frac{1}{2}}))\right|_{1,h}\\
&\le C\left[\,\sup_{\mathbb{R}}|\mathfrak{n}'_{\delta_\star}|\,|\boldsymbol{\theta}^n|_{1,h}+\max_{\mathbb{R}}|\mathfrak{n}''_{\delta_\star}|\,|\!|\!|\delta_h(g(u^{n+\frac{1}{2}}))|\!|\!|_{\infty,h}\,\|\boldsymbol{\theta}^n\|_{0,h}\,\right]\\
&\le C_{\delta_\star}\,\left[\,|\boldsymbol{\theta}^n|_{1,h}+(\tau^2+h^2)\,\right].
\end{aligned}
\tag{4.45}
$$

Thus, (4.43), (4.44) and (4.45) yield

$$
\mathsf{b}^{3,n}+\mathsf{b}^{4,n}\le C_{\delta_\star}\,\tau^2\,\left[\,|\boldsymbol{\theta}^n|_{1,h}^2+(\tau^2+h^2)^2\,\right]+\tfrac{2}{6}\,|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2. \tag{4.46}
$$

From (4.40), (4.41), (4.42) and (4.46), we conclude that there exists a constant $\mathsf{C}_{\delta_\star}^{\mathsf{BR,VI}}>0$, such that

$$
\begin{aligned}
|\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^{n+1}\|_{0,h}^2 \le \tau\,\|\Delta_h\boldsymbol{\eta}^n\|_{0,h}^2&+\mathsf{C}_{\delta_\star}^{\mathsf{BR,VI}}\,\tau^2\,|\boldsymbol{\theta}^n|_{1,h}^2\\
&+\mathsf{C}_{\delta_\star}^{\mathsf{BR,VI}}\,\tau^2\,(\tau^2+h^2)^2,\quad n=0,\dots,N-1.
\end{aligned}
\tag{4.47}
$$

Taking the $(\cdot,\cdot)_{0,h}$ inner product of both sides of (4.27) by $\tau\,\Delta_h(\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2})$, and using (2.3), the Cauchy-Schwarz inequality and (3.17), we have

$$
\begin{aligned}
\tau\,|\boldsymbol{\theta}^n|_{1,h}^2-\tau\,|\boldsymbol{\theta}^{n-2}|_{1,h}^2 &= 2\,\tau\,(\!(\delta_h\mathsf{R}^n,\delta_h(\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}))\!)_{0,h}+2\,\tau\,(\!(\delta_h(\mathsf{r}^n-\mathsf{r}^{n-1}),\delta_h(\boldsymbol{\theta}^n+\boldsymbol{\theta}^{n-2}))\!)_{0,h}\\
&\le 2\,\tau\,\left(\,|\mathsf{R}^n|_{1,h}+|\mathsf{r}^n-\mathsf{r}^{n-1}|_{1,h}\right)\,(|\boldsymbol{\theta}^n|_{1,h}+|\boldsymbol{\theta}^{n-2}|_{1,h})\\
&\le 2\,\tau\,\left(\,|\mathsf{R}^n|_{1,h}+\tau^3\right)\,(|\boldsymbol{\theta}^n|_{1,h}+|\boldsymbol{\theta}^{n-2}|_{1,h}),\quad n=2,\dots,N-1.
\end{aligned}
\tag{4.48}
$$

Using (4.28), (2.10), (2.2), (4.10) and (3.23), we get

$$
\begin{aligned}
|\mathsf{R}^n|_{1,h} &\le C_{\delta_\star}\,\left[\,|\mathbf{e}^n-\mathbf{e}^{n-1}|_{1,h}+\tau\,(\tau^2+h^2)\,\right]\\
&\le C_{\delta_\star}\,\left[\,|\boldsymbol{\eta}^n-\boldsymbol{\eta}^{n-1}|_{1,h}+|\boldsymbol{\rho}^n-\boldsymbol{\rho}^{n-1}|_{1,h}+\tau\,(\tau^2+h^2)\,\right]\\
&\le C_{\delta_\star}\,\left[\,|\boldsymbol{\eta}^n-\boldsymbol{\eta}^{n-1}|_{1,h}+\tau\,(\tau^2+h^2)\,\right],\quad n=2,\dots,N-1.
\end{aligned}
\tag{4.49}
$$

Then, (4.48), (4.49) and the arithmetic mean inequality, yield

$$
\begin{aligned}
\tau\,|\boldsymbol{\theta}^n|_{1,h}^2+\tau\,|\boldsymbol{\theta}^{n-1}|_{1,h}^2 &\le \tau\,|\boldsymbol{\theta}^{n-1}|_{1,h}^2+\tau\,|\boldsymbol{\theta}^{n-2}|_{1,h}^2\\
&\quad+C_{\delta_\star}\,\tau\,\left[\,|\boldsymbol{\eta}^n-\boldsymbol{\eta}^{n-1}|_{1,h}+\tau\,(\tau^2+h^2)\right]\,(|\boldsymbol{\theta}^n|_{1,h}+|\boldsymbol{\theta}^{n-2}|_{1,h})\\
&\le \tau\,|\boldsymbol{\theta}^{n-1}|_{1,h}^2+\tau\,|\boldsymbol{\theta}^{n-2}|_{1,h}^2+|\boldsymbol{\eta}^n-\boldsymbol{\eta}^{n-1}|_{1,h}^2\\
&\quad+C_{\delta_\star}\,\left[\,\tau^2\,(|\boldsymbol{\theta}^n|_{1,h}^2+|\boldsymbol{\theta}^{n-2}|_{1,h}^2)+\tau^2\,(\tau^2+h^2)^2\,\right],\quad n=2,\dots,N-1.
\end{aligned}
\tag{4.50}
$$

Combining (4.47) and (4.50), we conclude that there exists a positive constant $\mathsf{C}_{\delta_\star}^{\mathsf{BR,VII}}$ such that:

$$
(1-\mathsf{C}_{\delta_\star}^{\mathsf{BR,VII}}\,\tau)\,\mathsf{Z}_\star^{n+1}\le\,(1+\mathsf{C}_{\delta_\star}^{\mathsf{BR,VII}}\,\tau)\,\mathsf{Z}_\star^n+C_{\delta_\star}\,\tau^2\,(\tau^2+h^2)^2,\quad n=2,\dots,N-1, \tag{4.51}
$$

where

$$
\mathsf{Z}_\star^m:=|\boldsymbol{\eta}^m-\boldsymbol{\eta}^{m-1}|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^m\|_{0,h}^2+\tau\,|\boldsymbol{\theta}^{m-1}|_{1,h}^2+\tau\,|\boldsymbol{\theta}^{m-2}|_{1,h}^2,\quad m=2,\dots,N. \tag{4.52}
$$

Assuming that $\tau\,\mathsf{C}_{\delta_\star}^{\mathsf{BR,VIII}}\le\frac{1}{2}$, where $\mathsf{C}_{\delta_\star}^{\mathsf{BR,VIII}}:=\max\{\mathsf{C}_{\delta_\star}^{\mathsf{BR,VII}},\mathsf{C}_{\delta_\star}^{\mathsf{BR,VI}}\}$, and using a standard discrete Gronwall argument based on (4.51), we obtain

$$
\begin{aligned}
\max_{2\le m\le N}\mathsf{Z}_\star^m &\le C_{\delta_\star}\,\left[\,\mathsf{Z}_\star^2+\tau\,(\tau^2+h^2)^2\,\right]\\
&\le C_{\delta_\star}\,\left[\,|\boldsymbol{\eta}^2-\boldsymbol{\eta}^1|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^2\|_{0,h}^2+\tau\,|\boldsymbol{\theta}^1|_{1,h}^2+\tau\,|\boldsymbol{\theta}^0|_{1,h}^2+\tau\,(\tau^2+h^2)^2\,\right].
\end{aligned}
\tag{4.53}
$$

After setting $n=0$ in (4.47) and then using (4.19) and (3.21), we obtain

$$\begin{aligned} |\boldsymbol{\eta}^1-\boldsymbol{\eta}^0|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^1\|_{0,h}^2 &\le \tau\,\|\Delta_h\boldsymbol{\eta}^0\|_{0,h}^2+C_{\delta_\star}\,\left[\,\tau^2\,|\boldsymbol{\theta}^0|_{1,h}^2+\tau^2\,(\tau^2+h^2)^2\,\right]\\ &\le C\,\tau\,h^4+C_{\delta_\star}\,\tau^2\,(\tau^2+h^2)^2\\ &\le C_{\delta_\star}\,\tau\,(\tau^2+h^2)^2. \end{aligned} \tag{4.54}$$

Using (4.35), (4.19) and (3.16), we have

$$|\boldsymbol{\theta}^1|_{1,h}^2\le C_{\delta_\star}\,(\tau^2+h^2)^2. \tag{4.55}$$

Set $n=1$ in (4.47) to conclude that

$$|\boldsymbol{\eta}^2-\boldsymbol{\eta}^1|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^2\|_{0,h}^2\le\tau\,\|\Delta_h\boldsymbol{\eta}^1\|_{0,h}^2+C_{\delta_\star}\,\left[\,\tau^2\,(\tau^2+h^2)^2+\tau^2\,|\boldsymbol{\theta}^1|_{1,h}^2\,\right]$$

which, along with, (4.54) and (4.55), yields

$$|\boldsymbol{\eta}^2-\boldsymbol{\eta}^1|_{1,h}^2+\tau\,\|\Delta_h\boldsymbol{\eta}^2\|_{0,h}^2\le\,C_{\delta_\star}\,\tau\,(\tau^2+h^2)^2. \tag{4.56}$$

Thus, from (4.53), (4.56), (4.55) and (4.19), we obtain

$$\max_{2\le m\le N}\mathsf{Z}_\star^m\le\,C_{\delta_\star}\,\tau\,(\tau^2+h^2)^2. \tag{4.57}$$

Finally, (4.11) follows, easily, from (4.52) and (4.57). □

4.5. **Convergence of the (BRFD) method.**

Theorem 4.3. *Let* $u_\star:=\max_{[0,T]\times\mathcal{I}}|u|$, $g_\star:=\max_{[0,T]\times\mathcal{I}}|g(u)|$, $\delta_\star\ge 2\,\max\{u_\star,g_\star\}$, $\mathsf{C}_{\delta_\star}^{\mathsf{BR,I}}$ *be the constant determined in Proposition 4.1,* $\mathsf{C}_{\delta_\star}^{\mathsf{BCV},1}$, $\mathsf{C}_{\delta_\star}^{\mathsf{BCV},2}$, $\mathsf{C}_{\delta_\star}^{\mathsf{BCV},3}$ *and* $\mathsf{C}_{\delta_\star}^{\mathsf{BCV},4}$ *be the constants specified in Theorem 4.2, where* $\mathsf{C}_{\delta_\star}^{\mathsf{BCV},1}\ge\mathsf{C}_{\delta_\star}^{\mathsf{BR,I}}$. *If*

$$\tau\,\mathsf{C}_{\delta_\star}^{\mathsf{BCV},1}\le\tfrac12,\quad \mathsf{C}_{\delta_\star}^{\mathsf{BCV},2}\sqrt{\mathsf{L}}\,(\tau^2+\tau^{\frac12}\,h^2)\le\tfrac{\delta_\star}{2},\quad \mathsf{C}_{\delta_\star}^{\mathsf{BCV},4}\sqrt{\mathsf{L}}\,(\tau^2+h^2)\le\tfrac{\delta_\star}{2}, \tag{4.58}$$

then, the method (BRFD) is well-defined and the following error estimates hold

$$|u^{\frac12}-U^{\frac12}|_{1,h}\le\,\mathsf{C}_{\delta_\star}^{\mathsf{BCV},2}\,(\tau^2+\tau^{\frac12}\,h^2) \tag{4.59}$$

and

$$\max_{0\le m\le N-1}|g(u^{m+\frac12})-\Phi^{m+\frac12}|_{1,h}+\max_{0\le m\le N}|u^m-U^m|_{1,h}\le\,\max\{\mathsf{C}_{\delta_\star}^{\mathsf{BCV},3},\mathsf{C}_{\delta_\star}^{\mathsf{BCV},4}\}\,(\tau^2+h^2). \tag{4.60}$$

Proof. Since $\delta_\star\ge 2\,\max\{g_\star,u_\star\}$, the convergence estimates (4.9) and (4.11), the discrete Sobolev inequality (2.1) and the mesh size conditions (4.58) imply that the (MBRFD) are well-defined and

$$\begin{aligned} \left|\Phi_{\delta_\star}^{n+\frac12}\right|_{\infty,h} &\le\left|g(u^{n+\frac12})-\Phi_{\delta_\star}^{n+\frac12}\right|_{\infty,h}+\left|g(u^{n+\frac12})\right|_{\infty,h}\\ &\le\sqrt{\mathsf{L}}\left|g(u^{n+\frac12})-\Phi_{\delta_\star}^{n+\frac12}\right|_{1,h}+g_\star\\ &\le\mathsf{C}_{\delta_\star}^{\mathsf{BCV},4}\sqrt{\mathsf{L}}\,(\tau^2+h^2)+\tfrac{\delta_\star}{2}\\ &\le\delta_\star,\quad n=0,\dots,N-1, \end{aligned}$$

and

$$\begin{aligned} \left|V_{\delta_\star}^{\frac12}\right|_{\infty,h} &\le\left|u^{\frac12}-V_{\delta_\star}^{\frac12}\right|_{\infty,h}+\left|u^{\frac12}\right|_{\infty,h}\\ &\le\sqrt{\mathsf{L}}\left|u^{n+\frac12}-V_{\delta_\star}^{\frac12}\right|_{1,h}+u_\star\\ &\le\mathsf{C}_{\delta_\star}^{\mathsf{BCV},2}\sqrt{\mathsf{L}}\,(\tau^2+\tau^{\frac12}\,h^2)+\tfrac{\delta_\star}{2}\\ &\le\delta_\star, \end{aligned}$$

which, along with (4.1), yield

$$\mathfrak{n}_{\delta_\star}(V_{\delta_\star}^{\frac12})=V_{\delta_\star}^{\frac12},\quad \mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{n+\frac12}\big)=\Phi_{\delta_\star}^{n+\frac12},\quad n=0,\dots,N-1. \tag{4.61}$$

Thus, the (MBRFD) approximations are (BRFD) approximations when $\delta = \delta_\star$, i.e. (1.5)-(1.10) hold after replacing $U^{\frac{1}{2}}$ by $V_{\delta_\star}^{\frac{1}{2}}$, U^n by $V_{\delta_\star}^n$ for $n = 0,\dots,N$, and $\Phi^{n+\frac{1}{2}}$ by $\Phi_{\delta_\star}^{n+\frac{1}{2}}$ for $n = 0,\dots,N-1$.

Let $U^{\frac{1}{2}}$, $(U^n)_{n=0}^N$ and $(\Phi^{n+\frac{1}{2}})_{n=0}^{N-1}$ be approximations derived by the (BRFD) method. Then, we introduce the errors $\mathfrak{q}^{\frac{1}{2}} := V_{\delta_\star}^{\frac{1}{2}} - W^{\frac{1}{2}}$, $\mathfrak{q}^n := V_{\delta_\star}^n - W^n$ for $n = 0,\dots,N$, and $\mathfrak{q}_\star^n := \Phi_{\delta_\star}^{n+\frac{1}{2}} - \Phi^{n+\frac{1}{2}}$ for $n = 0,\dots,N-1$. Since $\tau\,\mathsf{C}_{\delta_\star}^{\mathsf{BR,I}} \le \frac{1}{2}$ and $\delta_\star \ge g_\star \ge g_\star^0$, Remark 4.1 and (1.5) yield $\mathfrak{q}^0 = 0$, $\mathfrak{q}^{\frac{1}{2}} = 0$ and $\mathfrak{q}_\star^0 = 0$. Now, we assume that for a given $m \in \{0,\dots,N-1\}$ it holds that $\mathfrak{q}^m = 0$ and $\mathfrak{q}_\star^m = 0$. Subracting (1.10) from (4.7) (or (1.8) from (4.5) when $m = 0$), and then using (4.61), we obtain

$$\mathfrak{q}^{m+1} = \tfrac{\tau}{2}\,\Delta_h \mathfrak{q}^{m+1} + \tfrac{\tau}{2}\left[\mathfrak{n}_{\delta_\star}\big(\Phi_{\delta_\star}^{m+\frac{1}{2}}\big) \otimes \mathfrak{q}^{m+1}\right]. \tag{4.62}$$

Next, taking the inner product $(\cdot,\cdot)_{0,h}$ with $\mathfrak{q}^{m+1}$ and then using (2.4), the Cauchy-Schwarz inequality, (4.1) and the definion of $\mathsf{C}_{\delta_\star}^{\mathsf{BR,I}}$, we get

$$\begin{aligned}
0 &= \|\mathfrak{q}^{m+1}\|_{0,h}^2 + \tfrac{\tau}{2}\,|\mathfrak{q}^{m+1}|_{1,h}^2 - \tfrac{\tau}{2}\left(\mathfrak{n}_{\delta_\star}(\Phi_{\delta_\star}^{m+\frac{1}{2}}) \otimes \mathfrak{q}^{m+1}, \mathfrak{q}^{m+1}\right)_{0,h}\\
&\ge \tfrac{\tau}{2}\,|\mathfrak{q}^{m+1}|_{1,h}^2 + 2\,\|\mathfrak{q}^{m+1}\|_{0,h}^2\left(\tfrac{1}{2} - \tfrac{\tau}{4}\sup_{\mathbb{R}}|\mathfrak{n}_{\delta_\star}|\right)\\
&\ge \tfrac{\tau}{2}\,|\mathfrak{q}^{m+1}|_{1,h}^2 + 2\,\|\mathfrak{q}^{m+1}\|_{0,h}^2\left(\tfrac{1}{2} - \tau\,\mathsf{C}_{\delta_\star}^{\mathsf{BR,I}}\right)\\
&\ge \tfrac{\tau}{2}\,|\mathfrak{q}^{m+1}|_{1,h}^2,
\end{aligned}$$

which, obviously, yields that $\mathfrak{q}^{m+1} = 0$. When $m \le N-2$, observing that

$$\mathfrak{q}_\star^{m+1} = 2\left[g(V_{\delta_\star}^{m+1}) - g(U^{m+1})\right] - \mathfrak{q}_\star^m,$$

we arrive at $\mathfrak{q}_\star^{m+1} = 0$. The induction argument above, shows that, under our assumptions the (BRFD) approximations are those derived from of the (MBRFD) scheme when $\delta = \delta_\star$, and thus the error estimates (4.59) and (4.60) follow as a natural outcome of (4.9), (4.10) and (4.11). □

Remark 4.2. *Let us make the choice $\Phi^{\frac{1}{2}} = g(u^0)$ (see [4], [5]) instead of (1.7). Then, we obtain $\|\boldsymbol{\theta}^0\|_{0,h} = \mathcal{O}(\tau)$, $\|\boldsymbol{\theta}^1\|_{0,h} = \mathcal{O}(\tau)$ and $\mathsf{Z}^2 = \mathcal{O}(\tau(\tau+h^2)^2)$. Thus, from (4.33) we arrive at a suboptimal error estimate of the form $\mathcal{O}(\tau+h^2)$. Here, we skip the problem by introducing (1.6) (cf. [9]) that derives a higher order approximation $\Phi^{\frac{1}{2}}$ of $g(u(t_1,\cdot))$.*

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