(PLURI)POTENTIAL COMPACTIFICATIONS

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Abstract. Using pluricomplex Green functions we introduce a compactification of a complex manifold M invariant with respect to biholomorphisms similar to the Martin compactification in the potential theory.

For this we show the existence of a norming volume form V on M such that all negative plurisubharmonic functions on M are in $L^1(M,V)$. Moreover, the set of such functions with the norm not exceeding 1 is compact. Identifying a point $w \in M$ with the normalized pluricomplex Green function with pole at w we get an imbedding of M into a compact set and the closure of M in this set is the pluripotential compactification.

1. Introduction

In this paper we construct biholomorphically invariant compactifications of complex manifolds. For domains in the complex plane there is the Carathéodory compactification that is invariant with respect to biholomorphisms. It is constructed using prime ends. There are papers that used this notion for higher dimensions but it, seemingly, did not lead to invariant compactifications.

Our construction is similar to the Martin compactification but instead of Green functions that are not biholomorphically invariant we use their analog on complex manifolds, namely, the pluricomplex Green functions.

The classical Martin's approach is to consider the normalized Green functions $\widetilde{G}_D(x,y) = G_D(x,y)/G_D(x_0,y)$ on a domain $D \subset \mathbb{R}^n$, where x_0 is a fixed point in D, and then define the Martin boundary as the set of all sequences $\widetilde{G}_D(x,y_j)$ that converge in L^1_{loc}. Due to Harnack's inequalities the choice of the point x_0 is non-essential. The limit is a harmonic function on D that is called the Martin kernel.

On a complex manifold M this normalization does not work because pluricomplex Green functions $g_M(z,w)$ are only maximal, i. e., $(dd^c_z g_M(z,w))^n = 0$ outside of w. There are no Harnack's inequalities and the existence of a subsequence converging in L^1_{loc} is not guaranteed.

To circumvent this obstacle we show the existence of a norming volume form V on M or D such that all negative (pluri)subharmonic functions on M or D are in $L^1(M,V)$ or $L^1(D,V)$ respectively. Moreover, the set of such functions with the norm not exceeding 1 is compact. Identifying a point $w \in M$ with the normalized pluricomplex Green function with pole at w we get an imbedding of M into a compact set and the closure of M in this set is the pluripotential compactification. The same approach in the real case produces the Martin compactification.

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Unfortunately, we were not able to prove that the limits of pluricomplex Green functions are maximal. It is known due to Lelong [10] that almost any plurisubharmonic function is the limit in L^1_{loc} of maximal functions. Thus the general theory is not applicable. At the last section we compute pluripotential compactification for a ball, smooth strongly convex domains and a bidisk. In all cases the limits are maximal and are scalar multiples of pluriharmonic Poisson kernels computed in [4] and [2]. In the two first cases the pluripotential boundary coincides with the Euclidean boundary while in the case of a bidisk it is the product of a circle and a 2-sphere.

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2. Green functions

We denote by $G_D(x,y)$ the *negative* Green function on a domain D in $\mathbb{R}^n$. It is known that the Green function is symmetric and continuous in $G_D \times G_D$ and if ∂D is C^2-smooth, then (see [16, 24.1]) $G_D(x,y)$ is continuous on $D \times \overline{D}$ and at every point $y \in \partial D$ there is the derivative

$$P_D(x,y) = \frac{\partial G_D(x,y)}{\partial n(y)}, \tag{1}$$

along the outward normal vector $n(y)$ to ∂D at y. The function $P(x,y)$ is harmonic in x, positive on D and is called the Poisson kernel of D.

By [5, Theorem 6.18] every negative subharmonic function u on D can be represented as

$$u(x) = \int\limits_D G_D(x,y)\Delta u(y) + H(x),$$

where Δu is the Riesz mass of u and $H(x)$ is the least harmonic majorant of u. It was proved in [14] that if $H(x)$ is a negative harmonic function on a domain D with C^2-boundary, then

$$H(x) = -\int\limits_{\partial D} P_D(x,y)\, d\mu(y),$$

where μ is a Borel measure on ∂D.

Combining the two previous equations we get the Poisson–Jensen formula for a negative subharmonic function u on D:

$$u(x) = \int\limits_D G_D(x,y)\Delta u(y) - \int\limits_{\partial D} P_D(x,y)\, d\mu(y). \tag{2}$$

Let $\delta(x)$ be the distance from x to ∂D. The following theorem was proved in [9] and [15, Eqns. (5) and (7)] (see also [17] and [18]).

Theorem 2.1. *If $D \subset \mathbb{R}^n$ is a domain with C^2-boundary, then there is a constant $A > 0$ depending only on D such that the Green function satisfies the inequality*

$$G_D(x,y) \ge -\frac{1}{2}\ln\left(1 + A\frac{\delta(x)\delta(y)}{|x-y|^2}\right), \quad n = 2, \tag{3}$$

and

$$G_D(x,y) \ge -A\frac{\delta(x)\delta(y)}{|x-y|^n}, \quad n > 2. \tag{4}$$

It follows from this inequalities (see [14] and [15]) that there is a constant $B > 0$ depending only on D such that

$$P_D(x,y) \le \frac{B\delta(x)}{|x-y|^n} \tag{5}$$

for all $n \ge 2$.

3. Norming functions and Martin compactification

If $\phi > 0$ is a continuous function on a domain $D \subset \mathbb{R}^n$, then we denote by $L^1(D,\phi)$ the space of all Lebesgue measurable functions u on D such that

$$\|u\|_\phi = \int_D |u|\phi\, dx < \infty.$$

Let us call a positive continuous function $\phi(z)$ on D *norming* if for each compact set $F \subset\subset D$ there is a positive constant $C(F)$ and for every increasing sequence of subdomains $D_j \subset\subset D$ with $\cup D_j = D$ there is a sequence of numbers $\varepsilon_j > 0$ converging to zero such that for every negative subharmonic function u on D:

(1) $\|u\|_\phi < \infty$;
(2) $u(x) \le -C(F)\|u\|_\phi$ for every point $x \in F$;
(3) $\|u\|_{L^1(D\setminus D_j,\phi)} \le \varepsilon_j \|u\|_\phi$.

The first important feature of norming functions ϕ on D is the integrability of all negative subharmonic functions. Hence, the cone $SH^-(D)$ of all non-positive subharmonic functions on D lies in the Banach space $L^1(D,\phi)$.

As the following lemma shows this embedding of $SH^-(D)$ into $L^1(D,\phi)$ practically does not depend on the choice of ϕ.

Lemma 3.1. *Norming functions determine equivalent norms on the cone of negative subharmonic functions.*

Proof. Let ϕ_1 and ϕ_2 be norming functions on D. We take a compact set $F \subset D$ of positive Lebesgue measure and find a constant $c > 0$ such that $u(x) \le -c\|u\|_{\phi_1}$ for all $x \in F$ and all negative subharmonic functions u on D. Then

$$-\|u\|_{\phi_2} \le \int_F u\,\phi_2\, dx \le -c\|u\|_{\phi_1} \int_F \phi_2\, dx.$$

This shows that there is a constant $a > 0$ such that $\|u\|_{\phi_2} \ge a\|u\|_{\phi_1}$ on $SH^-(D)$. □

Another important feature of norming functions ϕ is given by the following theorem.

Theorem 3.2. *The set $SH_1^-(D,\phi) = \{u \in SH^-(D) : \|u\|_\phi \le 1\}$ is compact in $L^1(D,\phi)$.*

Proof. Let v_j be a sequence of subharmonic functions in $SH_1^-(D)$. By [6, Theorem 4.1.9] there is a subsequence v_{j_k} converging in $L^1_{loc}(D)$ to a subharmonic function v on D. Moreover,

$$\limsup_{k\to\infty} v_{j_k}(z) \le v(z)$$

and the left and the right side of this inequality are equal a.e.

By the third property of norming functions this subsequence converges to v in $L^1(D,\phi)$ and by Fatou's lemma $\|v\|_\phi \le 1$. $\square$

The following lemma provides estimates of integrals of subharmonic functions on compact sets.

Lemma 3.3. *Let ϕ be a norming function on a domain $D \subset \mathbb{R}^n$ and let F be a compact set in D with $a = \|\chi_F\|_\phi > 0$. Then there is a positive constant c, depending only on F, such that for every negative subharmonic function u on D we have*

$$ca\|u\|_\phi \le -\int\limits_F u\phi\,dx.$$

Proof. By the second property of norming functions

$$\int\limits_F u\phi\,dx \le -C(F)\|u\|_\phi \int\limits_F \phi\,dx = -C(F)\|\chi_F\|_\phi\|u\|_\phi.$$

$\square$

Let $D \subset \mathbb{R}^n$ be a Greenian domain (see [1]), i. e. a domain such that for each $y \in D$ there is the Green function $G_D(x,y)$. Since the Green functions are continuous in both variables, the mapping $\Phi : D \to SH_1^-(D,\phi)$ defined as $\Phi(y) = \widetilde{G}_D(x,y) = l^{-1}(y)G_D(x,y)$, where $l(y) = \|G_D(x,y)\|_\phi$, is a homeomorphism on its image. The closure $\widetilde{D}_\phi$ of $\Phi(D)$ in $SH_1^-(D,\phi)$ is compact and consists of $\Phi(D)$ and the set $\partial_M D$ of the limits in $L^1(D,\phi)$ of sequences of functions $\{\widetilde{G}_D(x,y_j)\}$ such that the sequence $\{y_j\}$ has no accumulation points in D. Since these limits are harmonic on D, the sets $\partial_M D$ and $\Phi(D)$ do not meet. By Lemma 3.1 if ϕ and ψ are norming functions on D, then the sets $\widetilde{D}_\phi$ are homeomorphic to each other.

In [12] R. S. Martin defines the Martin compactification of D by choosing a point $x_0 \in D$ and then adding to D all equivalence classes of converging uniformly on compacta sequences of function $m^{-1}(y_j)G_D(x,y_j)$, where $m(y_j) = G_D(x_0,y_j)$ and the sequence $\{y_j\}$ has no accumulation points. By Harnack's inequality, the third property of a norming function ϕ and Lemma 3.3 the uniform convergence on compacta of harmonic functions is equivalent to convergence in $L^1(D,\phi)$. Therefore, $\widetilde{D}_\phi$ is homeomorphic to the Martin compactification of D and $\partial_M D$ is the Martin boundary of D.

4. The existence of norming functions

Now we will show that every domain $D \subset \mathbb{R}^n$ has a norming function. We start with a lemma. Let $B(a,r)$ be the ball of radius r centered at a in $\mathbb{R}^n$

Lemma 4.1. *If $D \subset \mathbb{R}^n$ is a domain with C^2-boundary, then the function $\phi(x) \equiv 1$ on D is norming.*

Proof. First we prove that both Green and Poisson kernels are uniformly integrable on D. We may assume that $B(0,s) \subset D \subset B(0,1)$ for some $s > 0$. For $\varepsilon > 0$ we let $D_\varepsilon = \{x \in D : \delta(x) < \varepsilon\}$,

$$v_\varepsilon(y) = \int\limits_{D_\varepsilon} G_D(x,y)\,dx \text{ and } h_\varepsilon(y) = \int\limits_{D_\varepsilon} P_D(x,y)\,dx.$$

Since $D \subset B(0,1)$,

$$G_D(x,y) \geq -\frac{1}{|x-y|^{n-2}}, n \geq 3 \text{ and } G_D(x,y) \geq \ln \frac{|x-y|}{2}, n = 2.$$

Thus the function $v_\varepsilon(y)$ is defined and continuous on D.

Let $n \geq 3$. For a point $y \in D_\varepsilon$ take a ball $B = B(y,r)$ of the radius $r = \delta(y)$. Since $\delta(x) \leq \delta(y) + |x-y|$, by Theorem 2.1

$$G_D(x,y) \geq -A\left(\frac{r^2}{|x-y|^n} + \frac{r}{|x-y|^{n-1}}\right).$$

Therefore

$$v_\varepsilon(y) \geq -\int\limits_B \frac{1}{|x-y|^{n-2}}\,dx - A\int\limits_{D_\varepsilon\setminus B} \frac{r^2}{|x-y|^n}\,dx - \int\limits_{D_\varepsilon\setminus B} \frac{r}{|x-y|^{n-1}}\,dx.$$

The first integral is equal to $c_n r^2$. The second integral does not exceed

$$A\int\limits_{B(y,2)\setminus B} \frac{r^2}{|x-y|^n}\,dx = c_n r^2 \ln\frac{2}{r}.$$

The function $|x|^{1-n}$ is integrable on $B(0,2)$. Since the measure of D_ε converges to 0 as $\varepsilon \to 0$, by the absolute continuity of the integral there is $C_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\int\limits_{D_\varepsilon\setminus B} \frac{1}{|x-y|^{n-1}}\,dx \leq C_1(\varepsilon).$$

Thus $v_\varepsilon(y) \geq -C(\varepsilon)\delta(y)$ when $y \in D_\varepsilon$, where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since v_ε is harmonic on $D'_\varepsilon = D \setminus D_\varepsilon$ we see that

$$v_\varepsilon(y) \geq -C(\varepsilon)\varepsilon \geq -C(\varepsilon)\delta(y)$$

on D. In particular, when $\varepsilon = 1$

$$v_1(y) = \int\limits_D G_D(x,y)\,dx \geq -C(1)\delta(y). \tag{6}$$

If $y \in \partial D$, then $\delta(x) \leq |x-y|$ and by (5)

$$h_\varepsilon(y) = \int\limits_{D_\varepsilon} P_D(x,y)\,dx \leq \int\limits_{D_\varepsilon} \frac{B}{|x-y|^{n-1}}\,dx \leq M(\varepsilon),$$

where $M(\varepsilon)$ depends only on D and ε and converges to 0 as $\varepsilon \to 0$.

The function $G_D(x,y)$, considered as a mapping into the extended real line $[-\infty,\infty]$, is continuous on $D \times D$. Hence there is a constant $d_1(\varepsilon) > 0$ such that $G_D(x,y) < -d_1(\varepsilon)$ on $B(0,s)$ when $x,y \in D'_\varepsilon$. Let $h(y)$ be a continuous subharmonic function on D equal to $-d_1(\varepsilon)$ on $B(0,s)$, zero on ∂D and harmonic on $D \setminus \overline{B}(0,s)$. By the maximum principle, the Keldysch–Lavrentiev–Hopf lemma and (6) we have

$$G_D(x,y) \leq h(y) < -K(\varepsilon)\delta(y) \leq d(\varepsilon)v_1(y),$$

where $K(\varepsilon) > 0$, $d(\varepsilon) = C(1)K(\varepsilon)$, $x \in D'_\varepsilon$ and $y \in D$. Also

$$P_D(x,y) = \frac{\partial G_D(x,y)}{\partial n(y)} \geq K(\varepsilon) > 0,$$

when $x \in D'_\varepsilon$.

Let u be a negative subharmonic function on D. By Fubini's theorem and (2)

$$\int_D u\,dy = \int_D v_1 \Delta u - \int_{\partial D} h_1\,d\mu. \tag{7}$$

The second integral in the right side is finite because $0 \le h_1(y) \le M(1)$. Now for $x \in D'_\varepsilon$ we have

$$\int_D v_1 \Delta u \ge \frac{1}{d(\varepsilon)} \int_D G_D(x,y)\Delta u(y) \ge \frac{1}{d(\varepsilon)} u(x).$$

Therefore, if the first integral in the right side of (7) is infinite, then u will be equal to $-\infty$ everywhere. Thus $\|u\|_1 < \infty$ and this proves the first property of norming functions.

If $x \in D'_\varepsilon$, then the estimate for G_D above yields

$$\int_D G_D(x,y)\Delta u(y) \le d(\varepsilon) \int_D v_1 \Delta u.$$

The estimate for P tells us that

$$\int_{\partial D} P_D(x,y)\,d\mu(y) \ge K(\varepsilon) \int_{\partial D} d\mu \ge \frac{K(\varepsilon)}{M(1)} \int_{\partial D} h_1(y)\,d\mu.$$

Let $a(\varepsilon) = \min\{d(\varepsilon), K(\varepsilon)/M(1)\}$. For $x \in D'_\varepsilon$ by (2) and (7) we get

$$u(x) \le a(\varepsilon) \left(\int_D v_1(y)\Delta u(y) - \int_{\partial D} h_1(y) d\mu \right) = a(\varepsilon) \int_D u(y)\,dy$$

and this proves the second property of norming functions.

The function $v_1(y)$ is negative and subharmonic on D. Hence by Keldysch–Lavrentiev–Hopf lemma $v_1(y) < -\alpha\delta(y)$, $\alpha > 0$. Also $h_1(y) > \beta > 0$ on ∂D. Thus

$$\int_{D_\varepsilon} u\,dy = \int_D v_\varepsilon \Delta u - \int_{\partial D} h_\varepsilon\,d\mu \ge -C(\varepsilon) \int_D \delta \Delta u - M(\varepsilon) \int_{\partial D} d\mu \ge$$
$$\frac{C(\varepsilon)}{\alpha} \int_D v_1 \Delta u - \frac{M(\varepsilon)}{\beta} \int_{\partial D} h_1 d\mu \ge b(\varepsilon) \int_D u(y)\,dy,$$

where $b(\varepsilon) = \max\{C(\varepsilon)\alpha^{-1}, M(\varepsilon)\beta^{-1}\}$ converges to 0 as $\varepsilon \to 0$. This shows the third property of norming functions.

The case $n = 2$ has a completely analogous proof. □

This lemma fails for general bounded domains as the following example shows.

Example 4.2. Let $D = \{z = x + iy \in \mathbb{C} : |z| < 1, 0 < \operatorname{Arg} z < \pi/2\}$ and let

$$u(z) = \mathbf{Im}\,\frac{1}{z^2} = -\frac{2xy}{(x^2+y^2)^2}.$$

Then u is a negative harmonic function on D and it is not integrable.

Theorem 4.3. *Every domain $D \subset \mathbb{R}^n$ has a norming function.*

Proof. Let $D_j \subset\subset D_{j+1} \subset\subset D$ be a sequence of subdomains with smooth boundaries such that $\cup D_j = D$ and let $G_j = D_{j+1} \setminus D_j$.

By Lemma 4.1 there are positive constants c_j such that

$$u(x) \le c_j \int_{D_{j+1}} u\, dx \tag{8}$$

for each negative subharmonic function u on D and every point $x \in D_j$. Consequently, there are constants b_j, $0 < b_j < 1$, such that

$$\int_{D_j} u\, dx \le b_j \int_{D_{j+1}} u\, dx$$

and

$$\int_{D_1} u\, dx \le d_j \int_{D_j} u\, dx,$$

where $0 < d_j = b_1 b_2 \dots b_{j-1} < 1$.

Let $G_j = D_{j+1} \setminus D_j$ and let $\phi(x)$ be a positive continuous function on D such that $\phi \le 1$ on D_1 and $\phi(x) \le 2^{-j} d_j$ on G_j.

Now

$$\int_{D\setminus D_j} u\phi\, dx = \sum_{k=j}^{\infty} \int_{G_k} u\phi\, dx \ge \sum_{k=j+1}^{\infty} d_k 2^{-k} \int_{D_k} u\, dx \ge 2^{-j} \int_{D_1} u\, dx. \tag{9}$$

So

$$\int_{D} u\phi\, dx \ge 2 \int_{D_1} u\, dx > -\infty.$$

By (9)

$$\int_{D\setminus D_j} u\phi\, dx \ge \frac{2^{-j+1}}{c} \int_{D} u\phi\, dx, \tag{10}$$

where $c = \inf \phi(x)$ on D_1.

By (8) and (10) for all points $x \in D_j$ we have

$$u(x) \le c_j \int_{D_{j+1}} u\, dx \le c_j \int_{D_1} u\, dx \le \frac{c_j}{2} \int_{D} u\phi\, dx. \tag{11}$$

Formulas (10) and (11) show that the function ϕ is norming. □

5. Norming volume forms on complex manifolds

If V is a positive continuous volume form on a complex manifold M, then $L^1(M,V)$ is the space of all Lebesgue functions u on M such that

$$\|u\|_V = \int_M |u|\, dV < \infty.$$

A positive continuous volume form V on M is *norming* if for each compact set $F \subset\subset M$ there is a positive constant $C(F)$ and for every increasing sequence of open sets $M_j \subset\subset M$ with $\cup M_j = M$ there is a sequence of numbers $\varepsilon_j > 0$ converging to zero such that for every negative plurisubharmonic function u on M:

(1) $\|u\|_V < \infty$;
(2) $u(z) \le -C(F)\|u\|_V$ for every point $z \in F$;
(3) $\|u\|_{L^1(M \setminus M_j, V)} \le \varepsilon_j \|u\|_V$.

Theorem 5.1. *Every connected complex manifold M has a norming volume form.*

Proof. First, we prove this theorem when M is a relatively compact connected open set with smooth boundary in a complex manifold. Let us take a finite open cover of $\overline{M}$ by biholomorphic images $F_j(D'_j)$ of domains $D'_j \subset \mathbb{C}^n$, where $n = \dim M$ and $1 \le j \le m$. We may assume that the sets $D_j = F_j^{-1}(M \cap F_j(D_j))$ are domains. Then the open sets $U_j = F_j(D_j) \subset M$ form a finite open cover of M. Let ϕ_j be a norming function on D_j, $G_j = F_j^{-1}$ and $V_j = G_j^*(\phi_j \, dx)$ be the pull-back of the volume form $\phi_j \, dx$ on D_j to U_j.

Let ψ_j be a partition of unity subordinated to the cover $\{U_j\}$. We let $V = \sum_{j=1}^k \psi_j V_j$. We assume that for all j the sets $\{\psi_j > 0\}$ are non-empty. If u is a negative plurisubharmonic function on M, then

$$\int_M u \, dV = \sum_{j=1}^m \int_{D_j} u(F_j(x)) \phi_j(x) \, dx > -\infty.$$

Suppose that the intersection of U_j and U_k is non-empty. Let us take a compact set $A \subset U_j \cap U_k$ such that

$$V_j(A) = \int_A dV_j > 0.$$

There is a constant $c_k > 0$ such that

$$u(z) < c_k \int_{U_k} u \, dV_k = c_k \int_{D_k} u(F_k(x)) \phi_k(x) \, dx$$

for any point $z \in A$. Hence

$$\int_{U_j} u \, dV_j \le \int_A u \, dV_j < c_k V_j(A) \int_{U_k} u \, dV_k.$$

This means that there are constants $c_{jk} > 0$ such that

$$\int_{U_j} u \, dV_j \le c_{jk} \int_{U_k} u \, dV_k$$

whenever $U_j \cap U_k \ne \emptyset$.

Since M is connected for any $1 \le j, k \le m$ there is a finite chain of sets $U_{i_1}, \dots, U_{i_p}$ such that $U_j = U_{i_1}$, $U_k = U_{i_p}$ and $U_{i_l} \cap U_{i_{l+1}} \ne \emptyset$. Hence for any any $1 \le j, k \le m$ there are constants $c_{jk} > 0$ such that

$$\int_{U_j} u \, dV_j \le c_{jk} \int_{U_k} u \, dV_k.$$

This, in its turn, implies that for any $1 \le j \le m$ there is a constant $d_j > 0$ such that

$$\int_{U_j} u \, dV_j \le \sum_{k=1}^m c_{jk} \int_{U_k} \psi_k u \, dV_k \le d_j \int_M u \, dV.$$

Let F be a compact set in M. For every point $z \in F \cap U_j$ we take relatively compact open sets $F_z \subset U_j$ containing z and then choose a finite cover of F by such sets. Let $A_j \subset\subset U_{k(j)}$ be the elements of this cover. If $z \in A_j \cap F$, then there is a constant $a_j > 0$ such that

$$u(z) < a_j \int\limits_{U_{k(j)}} u\,dV_{k(j)} \leq a_j d_{k(j)} \int\limits_M u\,dV.$$

Taking as $C(F)$ the minimal constant $a_j c_{k(j)}$ we see that V satisfies the second property of norming volume forms.

Let M_k be an increasing sequence of open sets $M_k \subset\subset M$ with $\cup M_k = M$. Then for each j there is a sequence of numbers ε_{jk} converging to zero such that

$$\int\limits_{U_j \setminus M_k} u\,dV \geq \varepsilon_{jk} \int\limits_{U_j} u\,dV_j.$$

Hence

$$\int\limits_{M \setminus M_k} u\,dV = \sum_{j=1}^m \int\limits_{U_j \setminus M_k} \psi_j u\,dV_j \geq \sum_{j=1}^m \int\limits_{U_j \setminus M_k} u\,dV_j \geq \sum_{j=1}^m \varepsilon_{jk} \int\limits_{U_j} u\,dV_j.$$

For each j there is a compact set $B_j \subset U_j$ and a constant $a_j > 0$ such that $V_j(B_j) > 0$ and $\psi_j > a_j$ on B_j. By Lemma 3.3 there are constants $b_j > 0$ such that

$$\int\limits_{U_j} u\,dV_j \geq b_j \int\limits_{B_j} u\,dV_j \geq \frac{b_j}{a_j} \int\limits_{U_j} \psi_j u\,dV_j.$$

Hence

$$\int\limits_{M \setminus M_k} u\,dV \geq \sum_{j=1}^m \frac{\varepsilon_{jk} b_j}{d_j} \int\limits_{U_j} \psi_j u\,dV_j \geq \varepsilon_k \int\limits_M u\,dV,$$

where ε_k is some positive sequence converging to 0. This shows the existence of norming forms on relatively compact connected open sets with smooth boundary in a complex manifold.

In the general case we exhaust M by relatively compact connected open sets M_j with smooth boundary and repeat the proof of Theorem 4.3. □

The first important feature of norming volume forms V on a connected complex manifold M is the fact that every non-positive plurisubharmonic function $u \not\equiv -\infty$ is integrable with respect to the measure dV. Hence, the cone $PSH^-(M)$ of all negative plurisubharmonic functions on M belongs to the Banach space $L^1(M, V)$.

Repeating the proof of Lemma 3.1 we get

Lemma 5.2. *Norming volume forms determine equivalent norms on the cone of negative plurisubharmonic functions.*

Analogously the following plurisubharmonic version of Lemma 3.3 is valid.

Lemma 5.3. *Let V be a norming volume on a complex manifold M and let F be a compact set in M with $V(F) > 0$. Then there is a positive constant c, depending only on F, such that for every negative plurisubharmonic function u on M we have*

$$c\|u\|_V \leq -\int\limits_F u\,dV.$$

Let us denote by B_V the closed unit ball in $L^1(M,V)$.

Theorem 5.4. *The set $PSH_1^-(M,V) = PSH^-(M) \cap B_V$ is compact in $L^1(M,V)$.*

Proof. Let v_j be a sequence of plurisubharmonic functions in $PSH_1^-(M,V)$ and let $U \subset M$ be the biholomorphic image of a domain $D \subset \mathbb{C}^n$ by a mapping F. Then the functions $u_j = v_j \circ F$ are subharmonic on D. By [6, Theorem 4.1.9] there is a subsequence u_{j_k} converging in $L^1_{loc}(D)$ to a subharmonic function u on D. Moreover,

$$w(z) = \limsup_{k\to\infty} u_{j_k}(z) \le u(z)$$

on D and the left and the right side are equal a.e. Thus u is the upper semicontinuous regularization of w and by [8, Prop. 2.9.17] u is plurisubharmonic.

It follows that if $M_i \subset\subset M$ is an increasing sequence of connected open sets such that $\cup M_i = M$, then there is a subsequence $\{v_{j_k}\}$ converging in $L^1(M_i,V)$ for any i to a plurisubharmonic function v on M. By the third property of norming volume forms there is a sequence of numbers $\varepsilon_i > 0$ converging to zero such that

$$\int\limits_{M\setminus M_i} u\,dV \ge \varepsilon_i \int\limits_M u\,dV$$

for all $u \in PSH^-(M)$. Hence

$$\int\limits_M |v_{j_k} - v|\,dV \le \int\limits_{M_i} |v_{j_k} - v|\,dV - \int\limits_{M\setminus M_i} (v_{j_k} + v)\,dV$$
$$\le \int\limits_{M_i} |v_{j_k} - v|\,dV - \varepsilon_i \int\limits_M (v_{j_k} + v)\,dV \le \int\limits_{M_i} |v_{j_k} - v|\,dV + 2\varepsilon_i.$$

Thus this subsequence converges to v in $L^1(M,V)$. □

Proposition 5.5. *If $F : M \to N$ is a proper holomorphic mapping between complex manifolds M and N and V is a norming volume form on N, then $V^* = F^*V$ is a norming volume form on M.*

Proof. Let A be a singular set of F and $A' = F(A)$. Any point in $N \setminus A'$ has the same finite number m of preimages under the mapping F. If u is a negative plurisubharmonic function on M, then $u^*(w) = \sum_{F(z)=w} u(z)$ is a plurisubharmonic function on $N \setminus A'$ that is locally bounded above near any point of A'. Since the set A' is analytic, u^* extends uniquely to N as a plurisubharmonic function. Thus

$$\int\limits_M u\,dV^* = m \int\limits_N u^*\,dV > -\infty,$$

when u is a negative plurisubharmonic function on M.

If G is a compact set in M, then $G' = F(G)$ is a compact set in N and

$$u(z) \le u^*(F(z)) \le c \int\limits_N u^*\,dV = \frac{c}{m} \int\limits_M u\,dV^*.$$

If $M_j \subset\subset M$ is an exhaustion of M and $M'_j = M \setminus M_j$, then taking into account that F is proper we get

$$\int\limits_{M'_j} u\, dV^* \geq m \int\limits_{F(M'_j)} u^*\, dV \geq m\varepsilon_j \int\limits_{N} u^*\, dV = \varepsilon_j \int\limits_{M} u\, dV^*,$$

where $\varepsilon_j > 0$ is some sequence converging to 0. □

6. Pluripotential compactification

Let M be a complex manifold. For $w \in M$ we consider the pluricomplex Green function, introduced in [7],

$$g_M(z,w) = \sup u(z),$$

where the supremum is taken over all negative plurisubharmonic functions u such that the function $u(z) - \log \|z-w\|$ is bounded above near w. It is known that $g_M(z,w)$ is plurisubharmonic in z. (Here we assume that $u \equiv -\infty$ is a plurisubharmonic function.)

The function $g_M(z,w)$ is also maximal in z outside w, i.e., if $G \subset M$ is a domain whose closure does not contain w and v is a plurisubharmonic function on a neighborhood U of $\overline{G}$ such that $v(z) \leq g_M(z,w)$ on ∂G, then $v(z) \leq g_M(z,w)$ on G. Indeed, if v is a negative plurisubharmonic function on M which is less than g_M on a neighborhood of the boundary of a domain $G \subset M$, $w \notin \overline{G}$, then we take the function v_1 equal to g_M on $M \setminus G$ and to $\max\{g_M, v\}$ on G. This function will be negative and plurisubharmonic on M and $v_1 = g_M$ near w_0. Thus $g_M \geq v$ on G.

We introduce *locally uniformly pluri-Greenian* complex manifolds M, where every point $w_0 \in M$ has a coordinate neighborhood U with the following property: there is an open set $W \subset U$ containing w_0 and a constant c such that $g_M(z,w) > \log \|z-w\| + c$ on U whenever $w \in W$;

If M is a ball $B(w_0, r)$ of radius r centered at $w_0 \in \mathbb{C}^n$, then $g_M(z,w_0) = \log(\|z-w_0\|/r)$. Since g_M is monotonic in M, it follows that if M is a bounded domain in $\mathbb{C}^n$, then $g_M(z,w) \geq \log(\|z-w\|/r)$, where r is the radius of circumscribed ball of M centered at w. Hence bounded domains in $\mathbb{C}^n$ are locally uniformly pluri-Greenian.

We will need a version of [8, Lemma 6.2.4].

Lemma 6.1. *If M is a locally uniformly pluri-Greenian complex manifold and $w_0 \in M$, then for any $\varepsilon > 0$ and any neighborhood X of w_0 there is a neighborhood Y of w_0 such that*

$$1 - \varepsilon \leq \frac{g_M(z,w_0)}{g_M(z,w)} < 1 + \varepsilon$$

whenever $w \in Y$ and $z \in M \setminus X$.

Proof. Let U be a coordinate neighborhood of w_0 from the definition of locally uniformly pluri-Greenian manifolds. We may assume that $U \subset X$. By this definition $g_M(z,w) > \log\|z-w\| + c$ on U when $w \in B(w_0, r) \subset U$ for some $r > 0$. On the other hand, if $w \in B(w_0, r/4)$, then $B(w_0, r/2) \subset B(w, 3r/4) \subset B(w_0, r)$ and by monotonicity of pluricomplex Green functions there is a constant c_1 depending only on r such that $g_M(z,w) \leq \log\|z-w\| + c_1$ on $B(w_0, r/2)$.

If $0 < t < r/4$, and $\|w - w_0\| < t/2$, then $\log t + c - 2 \leq g_M(z,w) \leq \log t + c_1 + 2$ on $\partial B(w_0, t)$. Hence there is $0 < t_0 < r/4$ such that $(1+\varepsilon) g_M(z,w) \leq g_M(z,w_0) \leq$

$(1-\varepsilon)g_M(z,w_0)$ on $\partial B(w_0,t_0)$ when $w \in B(w_0,t_0/2)$. Our lemma follows with $Y = B(w_0,t_0/2)$ by the maximality of g_M. $\square$

Let V be a norming volume form on M. Let $c_V(w) = \|g_M(z,w)\|_V$. We define the mapping $\Phi_V : M \to PSH_1^-(M)$ as $\Phi_V(w) = \widetilde{g}_M(z,w) = c_V^{-1}(w)g_M(z,w)$.

Lemma 6.2. *If M is a locally uniformly pluri-Greenian complex manifold, then the mapping Φ_V has the following properties:*

(1) *Φ_V is a continuous bijection onto $\Phi_V(M)$;*
(2) *for every compact set $N \subset M$ the mapping Φ_V is a homeomorphism between N and $\Phi_V(N)$.*

Proof. From properties of locally uniformly pluri-Greenian complex manifolds it follows immediately that Φ_V is a bijection. It follows from Lemma 6.1 and the inequality $g_M(z,w) > \log\|z-w\| + c$ near w that the function $c_V(w)$ is continuous and, consequently, Φ_V is continuous.

If a set $N \subset\subset M$, then Φ_V is continuous and bijective on N. If a sequence $\widetilde{g}_M(z,w_j)$, $w_j \in N$, converges to $\widetilde{g}_M(z,w_0)$ in $L^1(M,V)$, then we take any subsequence w_{j_k} of $\{w_j\}$ converging to $x_0 \in N$. By continuity of Φ_V the sequence $\widetilde{g}_M(z,w_{j_k})$ converges to $\widetilde{g}_M(z,x_0)$ and this implies that $x_0 = w_0$. Thus the sequence $\{w_j\}$ converges to w_0 in M. $\square$

The norm of $\Phi_V(w)$ in $L^1(M,V)$ is equal to 1. Hence, by Theorem 5.4 the closure $\widetilde{M}_V$ of $\Phi_V(M)$ in $L^1(M,V)$ is compact and we call the set $\widetilde{M}_V$ the pluripotential compactification of M. The set $\widetilde{M}_V$ consists of $\Phi_V(M)$ and the set $\partial_M D$ of the limits in $L^1(M,V)$ of sequences of functions $\{\widetilde{G}_D(x,y_j)\}$ such that the sequence $\{y_j\}$ has no accumulation points in D. The closure ∂_P of the set $\widetilde{M}_V \setminus \Phi_V(M)$ is called the *pluripotential* boundary of M.

Lemma 6.3. *Let M and N be locally uniformly pluri-Greenian complex manifolds and $F : M \to N$ be a biholomorphism. Let V and U be norming volume forms on M and N respectively. Then there is a canonical homeomorphism H of $\widetilde{N}_U$ onto $\widetilde{M}_V$ such that $H(\Phi_U(N)) = \Phi_V(M)$.*

Proof. We define $\widetilde{H} : PSH_1^-(N,U) \to PSH_1^-(M,V)$ as $\widetilde{H}(u) = l^{-1}(v)v$, where $v = u \circ F$ and $l(v) = \|v\|_V$. If we prove that $\widetilde{H}$ is a homeomorphism and $\widetilde{H}(\Phi_U(N)) = \Phi_V(M)$, then the restriction H of $\widetilde{H}$ to $\widetilde{N}_U$ will be the required mapping.

First of all, we note that $\widetilde{H}$ is bijective. Secondly, if $U^* = F^*U$ and the mapping $\widetilde{P} : PSH_1^-(N,U) \to PSH_1^-(M,U^*)$ is defined as $\widetilde{P}(u) = u \circ F$, then $\widetilde{P}$ is a bijective isometry. Finally, by Lemma 5.3 the function $l(v)$ is continuous on the compact set $PSH_1^-(M,U^*)$. Hence the mapping $v \to l^{-1}(v)v$ is a homeomorphism of $PSH_1^-(M,V^*)$ onto $PSH_1^-(M,V)$. The composition of two latter mappings is $\widetilde{H}$ and our lemma is proved. $\square$

In particular, all pluripotential compactifications are homeomorphic to each other and we will denote them by $\widetilde{M}$. Another immediate consequence of this lemma is

Theorem 6.4. *Let M and N be locally uniformly pluri-Greenian complex manifolds. Then any biholomorphic mapping $F : M \to N$ extends to a homeomorphism of $\widetilde{M}$ onto $\widetilde{N}$.*

7. Examples

When working with examples it is useful to choose a better normalizing factor for pluricomplex Green function. The factor $\|g_M(z,w)\|_V^{-1}$ was optimal for the proofs but hard to calculate in concrete cases. However, if a sequence $\alpha(w_j)g_M(z,w_j)$ converges in $L^1(M,V)$ to some non-zero function u, then the sequence $\widetilde{g}_M(z,w_j)$ also converges to a scalar multiple of u.

Example 7.1. Let $M=\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$. Evidently, $g_M(z,0)=\log||z||$ and $g_M(z,a)=g_M(f(z),f(a))=\log||f(z)||$, where f is an automorphism of the ball transforming $a=(a_1,a_2)$ into 0. If $w=(w_1,w_2)=f(z)=f(z_1,z_2)$ then

$$w_1=\frac{r-z_1'}{1-z_1'r};\qquad w_2=\frac{\sqrt{1-r^2}}{1-z_1'r}z_2',$$

where $r=||a||$, $z_1'=r^{-1}(z,a)$ and $z_2'=r^{-1}(z,\bar{a})$. Therefore,

$$g_M(z,a)=\log\left|1-\frac{(1-||z||^2)(1-r^2)}{|1-z_1'r|^2}\right|.$$

As a normalizing factor we take $|g(0,a)|=-2\log r$. Then

$$\widetilde{g}_M(z,a)=-\frac{1}{2\log r}\log\left|1-\frac{(1-||z||^2)(1-r^2)}{|1-z_1'r|^2}\right|.$$

If a sequence $\widetilde{g}_M(z,a_j)$ converges and $\|a_j\|\to 1$, then $a_j\to a=(a_1,a_2)\in\partial\mathbb{B}^2$ and the limit is

$$\widetilde{g}_M(z,a)=\frac{||z||^2-1}{|1-(z,a)|^2}.$$

The function $\widetilde{g}_M(z,a)$ is maximal because for mappings $f(\zeta)=(\zeta,C(1-\zeta)):\mathbb{D}\to\mathbb{B}^2$ the functions $\widetilde{g}_M(f(\zeta),a)$ are harmonic.

So in this case $\widetilde{M}=\overline{\mathbb{B}}^2$, $\partial_P M=\partial\mathbb{B}^2$ and the mapping Φ_V is a homeomorphism of $\overline{\mathbb{B}}^2$ onto $\widetilde{M}$. This means that the Euclidean boundary and the pluripotential boundary coincide.

Example 7.2. Let $M=\mathbb{D}^2\subset\mathbb{C}^2$, $z=(z_1,z_2)$ and $w=(w_1,w_2)$. Then

$$g_M(z,w)=\log\max\left\{\left|\frac{z_1-w_1}{1-z_1\bar{w}_1}\right|,\left|\frac{z_2-w_2}{1-z_2\bar{w}_2}\right|\right\}.$$

As a normalizing factor we take $\alpha(w)=|g_M^{-1}(0,w)|=-\log^{-1}\max\{|w_1|,|w_2|\}$.

If a sequence $w_j=(w_{1j},w_{2j})$ in M converges to $w_0=(w_{10},w_{20})$ and $|w_{10}|=1$ while $|w_{20}|<1$, then the sequence $\widetilde{g}_M(z,w_j)$ converges to

$$\widetilde{g}_M(z,w_0)=\frac{|z_1|^2-1}{|1-z_1\bar{w}_{10})|^2}. \tag{12}$$

Similarly, if $|w_{01}|<1$ while $|w_{02}|=1$, then the sequence $\widetilde{g}_M(z,w_j)$ converges to

$$\widetilde{g}_M(z,w_0)=\frac{|z_2|^2-1}{|1-z_2\bar{w}_{20})|^2}. \tag{13}$$

If $|w_{01}|=|w_{02}|$, then the sequence $\widetilde{g}_M(z,w_j)$ converges if and only if the sequence $(\log|w_{1j}|)/\log|w_{2j}|$ has the finite or infinite limit c. If $c=\infty$, then $\widetilde{g}_M(z,w_j)$ converges to the function from (13), while if $c=0$, then the limit of $\widetilde{g}_M(z,w_j)$ is the function from (12).

If $0 < c \le 1$, then $\widetilde{g}_M(z, w_j)$ converges to the function

$$\widetilde{g}_M(z, w_0) = \log \max \left\{ \frac{|z_1|^2 - 1}{|1 - z_1 \bar{w}_{10})|^2}, c^{-1} \frac{|z_2|^2 - 1}{|1 - z_2 \bar{w}_{20})|^2} \right\}.$$

If $1 \le c < \infty$, then $\widetilde{g}_M(z, w_j)$ converges to the function

$$\widetilde{g}_M(z, w_0) = \log \max \left\{ c \frac{|z_1|^2 - 1}{|1 - z_1 \bar{w}_{10})|^2}, \frac{|z_2|^2 - 1}{|1 - z_2 \bar{w}_{20})|^2} \right\}.$$

All limit functions are maximal. The non-distinguished part of ∂M is squeezed into two circles while every point in the distinguished boundary, that is a 2-torus $\mathbb{T}^2$, blows up to an interval $(0, \infty)$ connecting these circles. If we add to every point of $\mathbb{T}^2$ the interval $(0, 1]$ and the circle from (12) we will get a filled torus in $\mathbb{R}^3$. Adding interval $(1, \infty)$ and the circle from (13) we will get another filled torus in $\mathbb{R}^3$. Thus $\partial_P M$ is the double of a filled torus or the product of a circle and a 2-sphere.

Example 7.3. Let M be a smooth strongly convex domain in $\mathbb{C}^n$. A *complex geodesic* is a holomorphic map $\phi : \mathbb{D} \to M$ which is an isometry between the Poincaré metric on $\mathbb{D}$ and the Kobayashi distance k_M on M. According to Lempert (see [11]) on smooth strongly convex domains complex geodesics are injective maps smooth up to the boundary and the Kobayashi and Carathéodory distances coincide. The latter implies (see [13] or [2]) that $g_M(z, w) = \log(\tanh k_M(z, w))$.

In [3, Theorem 3] the authors constructed a continuous mapping $\Phi : \partial M \times M \to \mathbb{B}^n$ that is smooth on $\partial M \times \overline{M}$ outside of the diagonal in $\partial M \times \partial M$ and has the following properties:

1. (1) for every $p \in \partial M$ there is $q \in \partial \mathbb{B}^n$ such that $\Phi(p, \phi(\zeta))$, where ϕ is any complex geodesic in M and $p \in \phi(\mathbb{T})$, is a complex geodesic in $\mathbb{B}^n$ passing through q;
2. (2) for a fixed $p \in \partial M$ the mapping $\Phi(p, z)$ is a homeomorphism of $\overline{M}$ onto $\mathbb{B}^n$ smooth outside of $\{p\}$.

For a complex geodesic ϕ such that $\phi(0) = w$ and $\phi(\zeta) = z$ we choose a point $p \in \phi(\mathbb{T})$. Due to the isometry properties of Φ the value of the function $\log(\tanh k_{\mathbb{B}^n}(\Phi(p, w), \Phi(p, z))$ does not depend on the choice of p and is equal to $\log(\tanh k_M(z, w)) = g_M(z, w)$.

Let $\{w_j\} \subset B$ be a sequence converging to $p \in \partial M$. For a complex geodesic passing through w_j and z we choose as $p_j(z)$ the nearest point to p in $\phi(\mathbb{T})$. By the continuity of Φ the mappings $\Phi(p_j(z), z)$ converge to $\Phi(p, z)$ uniformly on compacta in M. Let $x_j \in M$ be the points such that $\Phi(p_j(x_j), x_j) = 0$ and let $\lambda_j = -2 \log^{-1} \|\Phi(p_j(x_j), w_j)\|$. Then the functions

$$\lambda_j g_M(z, w_j) = \lambda_j \log(\tanh k_{\mathbb{B}^n}(\Phi(p_j(z), z), \Phi(p_j(z), w_j))$$

converge uniformly on compacta to $\widetilde{g}_M(z, p) = \widetilde{g}_{\mathbb{B}^n}(\Phi(p, z), p)$. This function is maximal because it is harmonic on geodesics passing through p, equal to 0 on $\partial M \setminus \{p\}$ and smooth. So by [2, Theorem 7.3] $\widetilde{g}_M(z, p)$ is a scalar multiple of the function $\Omega_{M,p}(z)$. In [2] the latter function is called *pluricomplex Poisson kernel* of M and it is equal to the derivative of $g_M(z, p)$ along the outside normal at p like in the classical formula (1).

In [4] Demailly introduced the notion of pluriharmonic Poisson kernels that depend on the choice of a measure on the boundary and are scalar multiples of each other. It was proved in [2] that $\Omega_{M,p}$ is a pluriharmonic Poisson kernel in the sense of Demailly. He also computed these kernels for the ball and the polydisk and they are scalar multiples of the functions computed in Examples 1 and 2.

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