

# Decay of the Local Energy for the Charged Klein-Gordon Equation in the Exterior De Sitter-Reissner-Nordström Spacetime

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## Abstract

We show decay of the local energy of solutions of the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime by means of a resonance expansion of the local propagator.

## 1 Introduction

There has been enormous progress in our understanding of scattering properties of solutions of hyperbolic equations on black hole type backgrounds over the last years. The aim of such studies is multifold. First of all these equations and thus their scattering properties are very important in their own right. Secondly, the understanding of dispersive properties of the solutions of these equations is a first step in the understanding of stability properties of the underlying spacetime. Eventually understanding the classical equation is also a first step in understanding the quantization of the field. The most important spacetime in this context is the (De Sitter-)Kerr spacetime, which is conjectured to be the unique solution of the Einstein equations describing a rotating black hole (for uniqueness results see [AIK10] and references therein, and also [Hi17] for the charged case). In the case of positive cosmological constant, nonlinear stability of the De Sitter-Kerr spacetime is now known for small angular velocity  $|a|$  thanks to the seminal result of Hintz and Vasy [HV16]. The case of zero cosmological constant is still open, but see [KS18] for recent progress in this direction. Scattering theories for classical equations are also at the origin of many results in quantum field theory, see *e.g.* the mathematically rigorous description of the Hawking effect in [Ha09].

When studying linear waves on a black hole type spacetime, one encounters several difficulties. The first is linked to trapping and it is already present in the case of the (De Sitter-)Schwarzschild spacetime, which describes spherically symmetric black holes. The second is superradiance, which means that there is no positive conserved quantity for spin 1 equations on the (De Sitter-)Kerr metric. Whereas this difficulty is not present for the wave equation on the (De Sitter-)Schwarzschild metric, it also appears when one considers a charged Klein-Gordon field on the (De Sitter-)Reissner-Nordström metric which describes a spherically symmetric charged black hole. In this context the phenomenon is linked to the charge of the black hole and the test particle and thus different from the Kerr case, where it is linked to the geometry of the spacetime. Superradiance already appears in flat spacetime when one considers a charged Klein-Gordon field which evolves in a strong electric field. In this context the natural setting seems to be the one of Krein spaces, see *e.g.* [GGH13]. This setting however is not available in the context of black holes, see [GGH17].

In the present paper we show a resonance expansion for the solutions of the charged Klein-Gordon equation on the De Sitter-Reissner-Nordström metric. As a corollary we obtain exponential decay of the local energy for these solutions. We restrict our study to the case where the product of black hole charge and the scalar field charge is small. Such a resonance expansion for the solutions of the wave equation has been obtained first by Bony-Häfner for the wave equation on the De Sitter-Schwarzschild metric [BoHa08]. This result has been generalized to much more complicated situations which include

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perturbations of the De Sitter-Kerr metric by Vasy [V13]. This last paper has developed new methods including a Fredholm theory for non elliptic problems. These methods could probably also be applied to the present case. In the present paper however we use the more elementary methods of Bony-Häfner [BoHa08] and Georgescu-Gérard-Häfner [GGH17].

The smallness of the charge product is non-quantitative as it is determined from a compactness argument. It allows at many places of the paper to use perturbation arguments with respect to the non charged case. As far as we are aware the present system has not been studied for general charge products. We shall however stress out that exponentially finite energy growing modes appear when the charge product becomes large with respect to the mass of the field, see [BeHa20]. The origin of this phenomenon is the existence of a resonance at 0 for the wave equation which moves to the upper half plane in the latter case; in contrast, the present paper shows that if the charge product is small enough with respect to the mass of the field then all resonances (including the resonance 0) for the wave equation lie in the lower half plane, yielding an exponential decay of the local energy as explained in the introductory example of Section 3 below. Let us mention that the resonance 0 still exists for the wave equation on the De Sitter-Kerr metric for small angular momentum of the black hole, see [D11]. In absence of cosmological constant, absence of growing modes is known for the wave equation on the Kerr metric for general angular momentum of the black hole, see [W89], but growing modes appear for the Klein-Gordon equation on the Kerr metric, see [SR14]. The question of the existence or not of growing modes is therefore a very subtle question.

Let us also mention that it is crucial for our results that the cosmological constant is strictly positive in order to define resonances as the poles of the meromorphic extension of the weighted resolvent. The low frequency behaviour is more complicated in the zero cosmological constant case and only polynomial decay of the local energy is expected then. Using different techniques, Giorgi has recently shown a linear stability result for the sub-extremal Reissner-Nordström spacetime, see [Gi19].

**Organization of the paper.** The paper is organized as follows. In Section 2 we give an introduction to the De Sitter-Reissner-Nordström metric and the charged Klein-Gordon equation on it. In Section 3, a meromorphic extension result is shown for the cut-off resolvent and resonances are introduced. The resonance expansion is presented in Section 4. Suitable resolvent-type estimates are obtained in Section 5. In section 6 we prove the main theorem by a suitable contour deformation and using the resolvent-type estimates of Section 5. The appendix contains a semiclassical limiting absorption principle for a class of generalized resolvents which might have some independent interest.

**Notations.** The set  $\{z \in \mathbb{C} \mid \Im z \geq 0\}$  will be denoted by  $\mathbb{C}^\pm$ . For any complex number  $\lambda \in \mathbb{C}$ , we will write  $\langle \lambda \rangle := (1 + |\lambda|^2)^{1/2}$ ,  $D(\lambda, R)$  will be the disc centered at  $\lambda \in \mathbb{C}$  of radius  $R > 0$  and  $D(\lambda, R)^c$  its complementary set. For all  $\omega = |\omega|e^{i\theta} \in \mathbb{C} \setminus ]-\infty, 0]$ ,  $\theta \in \mathbb{R}$ , we will use the branch of the square root defined by  $\sqrt{\omega} := \sqrt{|\omega|}e^{i\theta/2}$ .

The notation  $\mathcal{C}_c^k$  will be used to denote the space of compactly supported  $\mathcal{C}^k$  functions. Also, the Schwartz space on  $\mathbb{R}$  will be denoted by  $\mathcal{S}$ . If  $V, W$  are complex vector spaces, then  $\mathcal{L}(V, W)$  will be the space of bounded linear operators  $V \rightarrow W$ . All the scalar products  $\langle \cdot, \cdot \rangle$  will be antilinear with respect to their first component and linear with respect to their second component. For any function  $f$ , the support of  $f$  will be denoted by  $\text{Supp } f$ . If  $A$  is an operator, we will denote by  $\mathcal{D}(A)$  its domain,  $\sigma(A)$  its spectrum and  $\rho(A)$  its resolvent set.  $A \geq 0$  will mean that  $\langle Au, u \rangle \geq 0$  for all  $u \in \mathcal{D}(A)$ , and  $A > 0$  will mean that  $A \geq 0$  and  $\ker(A) = \{0\}$ .

Now we define the symbol classes on  $\mathbb{R}^{2d}$

$$S^{m,n} := \left\{ a \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{C}) \mid \forall (\alpha, \beta) \in \mathbb{N}^{2d}, \exists C_{\alpha,\beta} > 0, |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} \langle x \rangle^{n-|\beta|} \right\}$$

for any  $(m, n) \in \mathbb{Z}^{2d}$  (here  $\mathbb{N}$  and  $\mathbb{Z}$  both include 0). We then define the semiclassical pseudodifferential operators classes

$$\Psi^{m,n} := \{a^w(x, hD) \mid a \in S^{m,n}\}, \quad \Psi^{-\infty,n} := \bigcap_{m \in \mathbb{Z}} \Psi^{m,n}$$

with  $a^w(x, hD)$  the Weyl quantization of the symbol  $a$ . For any  $c > 0$ , the notation  $P \in c\Psi^{m,n}$  means that  $P \in \Psi^{m,n}$  and the norm of  $P$  is bounded by a positive multiple of  $c$ .

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## 2 Functional framework

### 2.1 The charged Klein-Gordon equation on the De Sitter-Reissner-Nordström metric

Let

$$F(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}$$

with  $M > 0$  the mass of the black hole,  $Q \in \mathbb{R}$  its electric charge and  $\Lambda > 0$  the cosmological constant. We assume that the parameters

$$\Delta := 1 - 4\Lambda Q^2, \quad m_k := \sqrt{\frac{1 + (-1)^k \sqrt{\Delta}}{2\Lambda}}, \quad M_k := m_k - \frac{2}{3}\Lambda m_k^3$$

satisfy for any  $k \in \{1, 2\}$  the relations

$$4\Lambda Q^2 < 1, \quad M_1 < M < M_2 \quad (1)$$

so that  $F$  has four distinct zeros  $-\infty < r_n < 0 < r_c < r_- < r_+ < +\infty$  and is positive for all  $r \in ]r_-, r_+[$  (see [Mok17, Prop. 1] with  $\Lambda$  replaced by  $\Lambda/3$  in our setting; see also [Hi18, Prop. 3.2] for another statement). We also assume that  $9\Lambda M^2 < 1$  so that we can use the work of Bony-Häfner [BoHa08] (the condition (1) only ensures that  $9\Lambda M^2 < 2$ ). The exterior De Sitter-Reissner-Nordström spacetime is the Lorentzian manifold  $(\mathcal{M}, g)$  with

$$\mathcal{M} = \mathbb{R}_t \times ]r_-, r_+[ \times \mathbb{S}_\omega^2, \quad g = F(r) dt^2 - F(r)^{-1} dr^2 - r^2 d\omega^2$$

where  $d\omega^2$  is the standard metric on the unit sphere  $\mathbb{S}^2$ .

Let  $A := \frac{Q}{r} dt$ . Then the charged wave operator on  $(\mathcal{M}, g)$  is

$$\not\partial_g = (\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu) = \frac{1}{F(r)} \left( \left( \partial_t - i\frac{qQ}{r} \right)^2 - \frac{F(r)}{r^2} \partial_r r^2 F(r) \partial_r - \frac{F(r)}{r^2} \Delta_{\mathbb{S}^2} \right)$$

with  $q \in \mathbb{R}$  and the corresponding charged Klein-Gordon equation reads

$$\not\partial_g u + m^2 u = 0, \quad m > 0.$$

We set  $s := qQ \in \mathbb{R}$  the charge product (which appears in the perturbation term of the standard wave operator),  $X := ]r_-, r_+[ \times \mathbb{S}_\omega^2$  and  $V(r) := r^{-1}$  so that the above equation reads

$$(\partial_t - isV)^2 u + \hat{P}u = 0 \quad (2)$$

with

$$\begin{aligned} \hat{P} &= -\frac{F(r)}{r^2} \partial_r (r^2 F(r) \partial_r) - \frac{F(r)}{r^2} \Delta_{\mathbb{S}^2} + m^2 F(r) \\ &= -F(r)^2 \partial_r^2 - F \left( \frac{2F(r)}{r} + \frac{\partial F}{\partial r}(r) \right) \partial_r - \frac{F(r)}{r^2} \Delta_{\mathbb{S}^2} + m^2 F(r) \end{aligned} \quad (3)$$

defined on  $\mathcal{D}(\hat{P}) := \{u \in L^2(X, F(r)^{-1}r^2 dr d\omega) \mid \hat{P}u \in L^2(X, F(r)^{-1}r^2 dr d\omega)\}$  (this is the spatial operator in [BoHa08] with the additional mass term  $m^2 F(r)$ ). In the sequel, we will use the following notations:

$$V_{\pm} := \lim_{r \rightarrow r_{\pm}} V(r) = r_{\pm}^{-1}.$$

It turns out that the positive mass makes the study of the equation easier. Besides the fact that massless charged particles do not exist in physics, it is not excluded that the resonance 0 for the case  $s = 0$  (see [BoHa08]) can move to  $\mathbb{C}^+$  in the case  $s \neq 0$  and  $m = 0$ .

## 2.2 The Regge-Wheeler coordinate

We introduce the Regge-Wheeler coordinate  $x \equiv x(r)$  defined by the differential relation

$$\frac{dx}{dr} := \frac{1}{F(r)}. \quad (4)$$

Using the four roots  $r_{\alpha}$  of  $F$ ,  $\alpha \in I := \{n, c, -, +\}$ , we can write

$$\frac{1}{F(r)} = -\frac{3r^2}{\Lambda} \sum_{\alpha \in I} \frac{A_{\alpha}}{r - r_{\alpha}}$$

where  $A_{\alpha} = \prod_{\beta \in I \setminus \{\alpha\}} (r_{\alpha} - r_{\beta})^{-1}$  for all  $\alpha \in I$ , and  $\pm A_{\pm} > 0$ . Integrating (4) then yields

$$x(r) = -\frac{3}{\Lambda} \sum_{\alpha \in I} A_{\alpha} r_{\alpha}^2 \ln \left| \frac{r - r_{\alpha}}{\mathfrak{r} - r_{\alpha}} \right| \quad (5)$$

with  $\mathfrak{r} := \frac{1}{2} \left( 3M + \sqrt{9M^2 - 8Q^2} \right)$  (we will explain this choice below); observe that  $|Q| < \frac{3}{\sqrt{8}}M$  if (1) holds (see the discussion below (17) in [Mok17]). Therefore, we have

$$|r - r_{\alpha}| = |\mathfrak{r} - r_{\alpha}| \prod_{\beta \in I \setminus \{\alpha\}} \left| \frac{r - r_{\beta}}{\mathfrak{r} - r_{\beta}} \right|^{-A_{\beta} r_{\beta}^2 / (A_{\alpha} r_{\alpha}^2)} \exp \left( -\frac{\Lambda}{3A_{\alpha} r_{\alpha}^2} x \right) \quad \forall \alpha \in I$$

which entails the asymptotic behaviours

$$F(r(x)) + |r(x) - r_{\pm}| \lesssim \exp \left( -\frac{\Lambda}{3A_{\pm} r_{\pm}^2} x \right) \quad x \rightarrow \pm\infty. \quad (6)$$

Note here that

$$-\frac{\Lambda}{3A_{\pm} r_{\pm}^2} = F'(r_{\pm}) = 2\kappa_{\pm} \quad (7)$$

where  $\kappa_- > 0$  is the surface gravity at the event horizon and  $\kappa_+ < 0$  is the surface gravity at the cosmological horizon. Recall that  $\kappa_{\pm}$  is defined by the relation

$$X^{\mu} \nabla_{\mu} X^{\nu} = -2\kappa_{\pm} X^{\nu} \quad X = \partial_t$$

where the above equation is to be considered at the corresponding horizon.

In the appendix A, we follow [BM93, Prop. IV.2] to show the extension result:

**Proposition 2.1.** *There exists a constant  $\mathcal{A} > 0$  such that the function  $x \mapsto r(x)$  extends analytically to  $\{\lambda \in \mathbb{C} \mid |\Re \lambda| > \mathcal{A}\}$ .*

On  $L^2(X, dx d\omega)$ , define the operator  $P := r\hat{P}r^{-1}$ , given in the coordinates  $(x, \omega)$  by the expression

$$P = -r^{-1}\partial_x r^2 \partial_x r^{-1} - \frac{F(r)}{r^2} \Delta_{\mathbb{S}^2} + m^2 F(r) = -\partial_x^2 - W_0 \Delta_{\mathbb{S}^2} + W_1$$

where

$$W_0(x) := \frac{F(r(x))}{r(x)^2}, \quad W_1(x) := \frac{F(r(x))}{r(x)} \frac{\partial F}{\partial r}(r(x)) + m^2 F(r(x)). \quad (8)$$

It will happen in the sequel that we write  $F(x)$  for  $F(r(x))$  and also  $V(x)$  for  $V(r(x))$ . Observe that the potentials  $W_0$  and  $W_1$  satisfy the same estimate as in (6).

As

$$\frac{dW_0}{dx} = F(r) \frac{dW_0}{dr} = \frac{2F(r)}{r^5} (3Mr - 2Q^2 - r^2),$$

we see that the (unstable) maximum of  $W_0$  occurs when  $x = 0$ , *i.e.*  $r = \mathfrak{r} = \frac{1}{2} (3M + \sqrt{9M^2 - 8Q^2})$ : this is the radius of the photon sphere. It is the only trapping set outside the black hole for null geodesics (see [Mok17]). The trapping will have a consequence on some resolvent type estimates, see the paragraph 5.2.

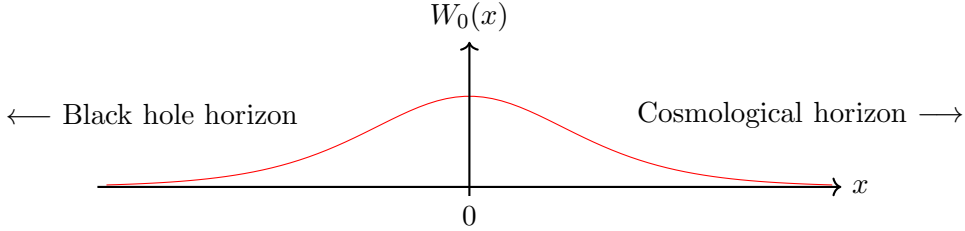


Figure 1: The potential  $W_0$  in the Regge-Wheeler coordinates.

### 2.3 The charge Klein-Gordon operator

Taking advantage of the spherical symmetry, we write

$$L^2(\mathbb{R} \times \mathbb{S}^2, dx d\omega) \simeq \bigoplus_{\ell \in \mathbb{N}} (L^2(\mathbb{R}, dx) \otimes Y_\ell) =: \bigoplus_{\ell \in \mathbb{N}} \mathcal{V}_\ell$$

where for all  $\ell \in \mathbb{N}$ ,  $Y_\ell$  is the  $(2\ell + 1)$ -dimensional eigenspace of the operator  $(-\Delta_{\mathbb{S}^2}, H^2(\mathbb{S}^2, d\omega))$  associated to the eigenvalue  $\ell(\ell + 1)$ . On each  $\mathcal{V}_\ell$ , we define  $P_\ell$  as the restriction of  $P$  onto  $\mathcal{V}_\ell$  which will be identified with an operator acting on  $L^2(\mathbb{R}, dx)$ , *i.e.*

$$P_\ell = -\partial_x^2 + \ell(\ell + 1)W_0 + W_1 \quad (9)$$

and we set  $\mathcal{D}(P_\ell) := H^2(\mathbb{R}, dx)$  so that  $P_\ell$  is self-adjoint. In the sequel, we will use the following (self-adjoint) realization of the total operator  $P$ :

$$P := \bigoplus_{\ell \in \mathbb{N}} P_\ell, \quad \mathcal{D}(P) := \left\{ u = (u_\ell)_{\ell \in \mathbb{N}} \in \bigoplus_{\ell \in \mathbb{N}} \mathcal{V}_\ell \mid \forall \ell \in \mathbb{N}, u_\ell \in \mathcal{D}(P_\ell) \right\}.$$

Now the charged Klein-Gordon equation reads

$$(\partial_t - isV)^2 u + Pu = 0. \quad (10)$$

The point is to see that if  $u$  is a solution of (10), then  $v := (u, -i\partial_t u - sVu)$  solves the first order equation

$$-i\partial_t v = \hat{K}(s)v \quad (11)$$

where

$$\hat{K}(s) := \begin{pmatrix} sV & \text{Id} \\ P & sV \end{pmatrix} \quad (12)$$

is the charge Klein-Gordon operator. Conversely, if  $v = (v_0, v_1)$  solves (11), then  $v_0$  solves (10). We also define  $\hat{K}_\ell \equiv \hat{K}_\ell(s)$ <sup>1</sup> with  $P_\ell$  in place of  $P$  for any  $\ell \in \mathbb{N}$ . Following [GGH17], we realize  $\hat{K}_\ell$  with the domain

$$\mathcal{D}(\hat{K}_\ell) := \left\{ u \in P_\ell^{-1/2}L^2(\mathbb{R}, dx) \oplus L^2(\mathbb{R}, dx) \mid \hat{K}_\ell u \in P_\ell^{-1/2}L^2(\mathbb{R}, dx) \oplus L^2(\mathbb{R}, dx) \right\}$$

and realize the operator  $\hat{K}$  as the direct sum on  $\mathbb{N} \ni \ell$  of the  $\hat{K}_\ell$ .

Let  $\dot{\mathcal{E}}_\ell$  be the completion of  $P_\ell^{-1/2}L^2(\mathbb{R}, dx) \oplus L^2(\mathbb{R}, dx)$  for the norm<sup>2</sup>

$$\|u\|_{\dot{\mathcal{E}}_\ell}^2 := \langle u_0, P_\ell u_0 \rangle_{L^2(\mathbb{R}, dx)} + \|u_1 - sV u_0\|_{L^2(\mathbb{R}, dx)}^2 \quad u = (u_0, u_1) \in \dot{\mathcal{E}}_\ell$$

and define  $(\dot{\mathcal{E}}, \|\cdot\|_{\dot{\mathcal{E}}})$  as the direct sum of the spaces  $\dot{\mathcal{E}}_\ell$ . [GGH17, Lem. 3.19] shows that  $\hat{K}_\ell$  generates a continuous one-parameter group  $(e^{-it\hat{K}_\ell})_{t \in \mathbb{R}}$  on  $(\dot{\mathcal{E}}_\ell, \|\cdot\|_{\dot{\mathcal{E}}_\ell})$ . We similarly construct the spaces  $(\mathcal{E}_\ell, \|\cdot\|_{\mathcal{E}_\ell})$  and  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  with  $\langle P_\ell \rangle$  instead of  $P_\ell$ . Let us mention here that for any  $n \in \mathbb{R}$  the quantity

$$\langle v | v \rangle_n := \langle v_1 - nv_0, v_1 - nv_0 \rangle_{L^2(\mathbb{R}, dx)} + \langle (P - (sV - n)^2)v_0, v_0 \rangle_{L^2(\mathbb{R}, dx)} \quad (13)$$

is formally conserved if  $v = (u, -i\partial_t u)$  with  $u$  solution of (10) and is continuous with respect to the norm  $\|\cdot\|_{\mathcal{E}}$ . However, it is in general not positive nor continuous with respect to the norm  $\|\cdot\|_{\dot{\mathcal{E}}}$  (see [GGH17, §3.4.3] for more details): this is superradiance. When  $\Lambda = 0$  (that is, when the cosmological horizon is at infinity), the natural energy  $\langle \cdot | \cdot \rangle_{sV_-}$  is positive for  $s$  small enough and it can be used to define a Hilbert space framework.

An important observation is the fact that the norms  $\|\cdot\|_{\dot{\mathcal{E}}_\ell}$  and  $\|\cdot\|_{\mathcal{E}_\ell}$  are locally equivalent, meaning that for any  $v \in \dot{\mathcal{E}}$  and any cut-off  $\chi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ , we have

$$\|\chi v\|_{\dot{\mathcal{E}}} \lesssim \|\chi v\|_{\mathcal{E}} \lesssim \|\chi v\|_{\dot{\mathcal{E}}}. \quad (14)$$

The first inequality is obvious, and the second one is established with the Hardy type estimate  $\|\chi v\|_{L^2} \lesssim \|P^{1/2}v\|_{L^2}$  (see [GGH17, Lem. 9.5]; the validity of this result in our setting is discussed in the subsection 3.4 below).

## 2.4 The quadratic pencil

Let  $u$  be a solution of (10). If we look for  $u$  of the form  $u = e^{izt}v$  with  $z \in \mathbb{C}$  for some  $v$ , then  $v$  satisfies the equation  $(P - (z - sV)^2)v = 0$ . We define the harmonic quadratic pencil

$$p_\ell(z, s) := P_\ell - (z - sV)^2, \quad \mathcal{D}(p_\ell(z, s)) := \langle P_\ell \rangle^{-1}L^2(\mathbb{R}, dx) = H^2(\mathbb{R}, dx)$$

and realize the total quadratic pencil as

$$p(z, s) := \bigoplus_{\ell \in \mathbb{N}} p_\ell(z, s),$$

$$\mathcal{D}(p(z, s)) := \left\{ u = (u_\ell)_{\ell \in \mathbb{N}} \in \bigoplus_{\ell \in \mathbb{N}} \mathcal{V}_\ell \mid \forall \ell \in \mathbb{N}, u_\ell \in \mathcal{D}(p_\ell(z, s)), \sum_{\ell \in \mathbb{N}} \|p_\ell(z, s)u_\ell\|_{\mathcal{V}_\ell}^2 < +\infty \right\}.$$

[GGH17, Prop. 3.15] sets the useful relations

$$\rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R} = \{z \in \mathbb{C} \setminus \mathbb{R} \mid p_\ell(z, s) : H^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx) \text{ is bijective}\} \quad (15)$$

<sup>1</sup>We will often drop the dependence in  $s$ .

<sup>2</sup>Note that the norm  $\|\cdot\|_{\dot{\mathcal{E}}_\ell}^2$  is conserved if  $[P_\ell, sV] = 0$ ; it is the case if  $s = 0$ .

and

$$\hat{R}_\ell(z, s) := (\hat{K}_\ell(s) - z)^{-1} = \begin{pmatrix} p_\ell(z, s)^{-1}(z - sV) & p_\ell(z, s)^{-1} \\ \text{Id} + (z - sV)p_\ell(z, s)^{-1}(z - sV) & (z - sV)p_\ell(z, s)^{-1} \end{pmatrix} \quad (16)$$

for all  $z \in \rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R}$ . In comparison, the relation (1.7) in [BoHa08] involves the resolvent of  $P_\ell$ , which corresponds to the case  $s = 0$  for us. [GGH17, Prop. 3.12] shows that (16) is also valid for  $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$  when we work on  $(\mathcal{E}_\ell, \|\cdot\|_{\mathcal{E}_\ell})$ ; by using the local equivalence (14) of the norms  $\|\cdot\|_{\mathcal{E}_\ell}$  and  $\|\cdot\|_{\mathcal{E}_\ell}$ , we can use (16) for  $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$  if we consider the cut-off resolvent  $\chi \hat{R}_\ell(z, s) \chi$  with  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ . In the sequel, we will simply call  $p_\ell(z, s)$  the quadratic pencil when  $\ell \in \mathbb{N}$  will be fixed.

### 3 Meromorphic extension and resonances

We construct in this Section a meromorphic extension for the weighted resolvent of  $\hat{K}(s)$ . The main Theorem 4.2, which provides asymptotic decay (in time) for solutions of the charged Klein-Gordon equation (2), relies on such a construction. It will be established by means of the *theory of resonances*. Let us briefly introduce the basic idea of this theory.

**An introductory example to the theory of resonances.** An explicit and comprehensive example is given by the one-dimensional wave equation on  $\mathbb{R}$ :

$$(\partial_t^2 - \partial_x^2)u = f \in L^2(\mathbb{R}, dx). \quad (17)$$

Here  $-\partial_x^2$  is the self-adjoint realization of the one-dimensional Laplacian on  $H^2(\mathbb{R}, dx)$ . Taking the time-dependent Fourier transform (denoted by the symbol  $\hat{\cdot}$ ), we get

$$(-\partial_x^2 - z^2)\hat{u} = \hat{f}$$

which is the one-dimensional Helmholtz equation. Call  $R(z)$  the resolvent  $(-\partial_x^2 - z^2)^{-1}$ . Then  $R(z)$  is well-defined for  $z \in \mathbb{C}^+$  and we have for such spectral parameters

$$\hat{u} = R(z)\hat{f}.$$

In this very simple example, we have an explicit representation formula:

$$(R(z)\hat{f})(x) = \frac{i}{2z} \int_{-\infty}^{+\infty} e^{iz|x-y|} \hat{f}(y) dy \quad \forall z \in \mathbb{C}^+, \forall x \in \mathbb{R}.$$

Let us set  $R(z; x, y) := \frac{i}{2z} e^{iz|x-y|}$  the kernel of  $R(z)$ . By Schur's estimate,

$$\|R(z)\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)} \leq \left( \sup_{x \in \mathbb{R}} \int_{-\infty}^{+\infty} |R(z; x, y)| dy \right)^{1/2} \left( \sup_{y \in \mathbb{R}} \int_{-\infty}^{+\infty} |R(z; x, y)| dx \right)^{1/2} = \frac{1}{|z| |\Im z|}.$$

Now take an *exponential* weight  $w(x) = \mathcal{O}_{|x| \rightarrow +\infty}(e^{-\kappa|x|})$  for some  $\kappa > 0$ . Then Schur's estimate gives

$$\|w^\delta R(z) w^\delta\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)} \lesssim \frac{1}{|z|(\kappa - |\Im z|)}$$

for all  $\delta > 0$  and  $z \in \overline{\mathbb{C}^-} \setminus \{0\}$  such that  $\Im z > -\kappa\delta$ . We can thus extend the weighted resolvent  $w^\delta R(z) w^\delta$  from  $\mathbb{C}^+$  to  $\{\lambda \in \mathbb{C} \setminus \{0\} \mid \Im \lambda > -\kappa\delta\}$  as an analytic family of bounded operators acting on  $L^2(\mathbb{R}, dx)$ . We can easily check that

$$\lim_{z \rightarrow 0} \|z w^\delta R(z) w^\delta\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)} < +\infty$$



for any  $\delta > 0$ . Therefore  $z = 0$  is a pole of the extension: it is called *resonance*. The operator  $w^\delta R(z)w^\delta : L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)$  is then the *meromorphic extension* of the original weighted resolvent. Notice that we can replace any exponential weight by a cut-off  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ , for

$$\|\chi R(z)\chi\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)} \leq \|\chi w^{-\delta}\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)}^2 \|w^\delta R(z)w^\delta\|_{L^2(\mathbb{R}, dx) \rightarrow L^2(\mathbb{R}, dx)}.$$

Collecting estimate for the cut-off resolvent is generally easier. Putting aside the regularity issue, we have the inversion formula

$$u(t, x) = \frac{1}{2\pi} \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-izt} R(z) \hat{f}(z, x) dz$$

for some  $\nu > 0$ . If  $f$  is compactly supported in  $x \in \mathbb{R}$ , then

$$\chi u(t, x) = \frac{1}{2\pi} \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-izt} \chi R(z) \chi \hat{f}(z, x) dz \quad (18)$$

for any cut-off  $\chi$  such that  $\chi \equiv 1$  on  $\text{Supp } f(t, \cdot)$ . Provided that one has some nice estimates on  $\chi R(z)\chi$ , we can use a contour deformation to obtain integrals in  $\mathbb{C}^-$  which provides exponentially decaying terms. In the meanwhile, the residue theorem makes appear the resonances; in the present example, this yields the following local energy estimate: for  $t \gg 0$ ,

$$\|\chi u(t, \cdot)\|_{H^2(\mathbb{R}, dx)} = \frac{1}{2} \chi(1, \chi \hat{f}(0))_{L^2(\mathbb{R}, dx)} + E(t), \quad E(t) \lesssim e^{-\nu t} \|f\|_{H^2(\mathbb{R}, dx)} \quad (19)$$

for any  $\nu > 0$ . For a more detailed and rigorous derivation of the above result in more general setting, we refer to the book of Dyatlov-Zworski [DZ19]. Let us make some comments on (19):

1. The non-vanishing term in (19) is the projection on the *resonant state*  $x \mapsto 1$ . This function indeed solves (17) but is not integrable so it is not an eigenvalue of  $-\partial_x^2$  in  $H^2(\mathbb{R}, dx)$  (it is in some weighted Sobolev spaces). The resonance 0 and the resonant state 1 for the wave equation exist also on the De Sitter-Schwarzschild metric (see Bony-Häfner [BoHa08], formula (1.9); observe that the resonant state is  $r$  therein because of the transformation  $r\hat{P}r^{-1}$  below equation (1.3)) as well as on De Sitter-Kerr metric (see Dyatlov [D11], formula (1.5)).
2. One may notice that there is no loss of derivatives in (19), that is the norms are the same both on the left-hand and right-hand sides. We know that decay estimates without loss of derivatives become false in presence of an obstacle by the work of Ralston [R69]. And indeed, for the wave equation on the De Sitter-Schwarzschild and De Sitter-Kerr metrics, where there exist trapping sets (the so-called photon sphere in the spherically symmetric case, as the one we have discussed about in Subsection 2.2), there is a loss of angular derivatives in the estimates in [BoHa08] and [D11].
3. We see in (18) the importance of the meromorphic extension of the cut-off resolvent as well as the localization of resonances in the complex plane. Any resonance in  $\mathbb{C}^+$  gives exponentially growing terms (the corresponding resonant state is called *growing mode*); conversely, any resonance in  $\mathbb{C}^-$  gives exponentially decaying terms. The existence of real resonances has more subtle consequences: it leads to polynomially growing terms if the multiplicity of the resonance (as a pole of the meromorphic extension) is greater than 1, or to a stationary term with no growth or decay in time for a resonance of multiplicity equal to 1 (this happens in [BoHa08] and [D11]).
4. In the non-superradiant case [BoHa08], the meromorphic extension of the weighted resolvent  $w^\delta(P - z^2)^{-1}w^\delta$  with  $w := \sqrt{(r - r_-)(r_+ - r)}$  is a direct consequence of the work of Mazzeo-Melrose [MM87]. In the superradiant case however, as in [D11] or in the present paper, the resolvent  $(P - z^2)$  is replaced by the quadratic pencil  $p(z, s)$ , and [MM87] can not be directly applied anymore.



**The charged Klein-Gordon equation on the De Sitter-Reissner-Nordström metric.** In our setting, we know that the presence of the photon sphere will eventually lead to a loss of derivative in the estimates. Moreover, because of the superradiance, the homogeneous norm  $\|\cdot\|_{\varepsilon}$  is not conserved and may grow (possibly exponentially fast) in time. Yet, it will turn out that no pole lies on and above the real axis provided that  $s$  remains small, entailing an exponential decay of the solutions of equation (2).

As motivated above, we first try to construct a meromorphic extension of the weighted resolvent of  $\hat{K}(s)$ . The presence of the mixed term  $sV\partial_t$  in (2) prevents us to directly use Mazzeo-Melrose result [MM87]. However,  $p(z, s)^{-1}$  formally tends as  $s \rightarrow 0$  to  $(P - z^2)^{-1}$  for which [MM87] applies; moreover, the case  $s = 0$  is very similar (even easier) to the case treated in [BoHa08]. We will therefore try to obtain results for small  $s$  using perturbation arguments. Our strategy is the following one:

- (i) Define first suitable "asymptotic" energy spaces by removing the troublesome negative contributions from the electromagnetic potential  $sV$  near  $r_{\pm}$  and define "asymptotic" selfadjoint Hamiltonians  $\hat{H}_{\pm}(s)$  (see the paragraph 3.1 below).
- (ii) For  $s = 0$ , the situation is really similar to the Klein-Gordon equation on De Sitter-Schwarzschild metric: using the standard results [BoHa08] and [MM87], we can meromorphically extend the weighted resolvent of  $\hat{H}_{\pm}(0)$  from  $\mathbb{C}^+$  to  $\mathbb{C}$  with no poles on and above the real axis (see Lemma 3.1 and Lemma 3.2).
- (iii) If  $s$  remains small, we can use analytic Fredholm theory to get a meromorphic extension for the weighted resolvents of the asymptotic Hamiltonians  $\hat{H}_{\pm}(s)$  into a strip in  $\mathbb{C}^-$  (the perturbation argument entails a bound on the width of this strip which is directly linked to the rate of decay of the potentials  $W_0$  and  $W_1$  in  $P$  near  $r_{\pm}$ ). We will also get the absence of poles near the real axis (see Lemma 3.1 and Lemma 3.3).
- (iv) Finally, we construct a parametrix for the resolvent of an equivalent operator to  $\hat{K}(s)$  by gluing together the resolvent of  $\hat{H}_{\pm}(s)$  (see (27)). Using again the analytic Fredholm theory for  $s$  sufficiently small, we show the existence of the weighted resolvent and also that the poles can only lie below the real axis (see Theorem 3.8).

The sequel of this Section is organized as follows: Subsection 3.1 introduces notations and tools (operators, functional spaces) from [GGH17] which will be used for the construction of the meromorphic extension of the weighted resolvent of  $\hat{K}(s)$ . Subsection 3.2 aims to show that results obtained in [GGH17] are available for us. Then Subsection 3.3 establishes the announced results for the asymptotic Hamiltonians  $\hat{H}_{\pm}(s)$ . Subsection 3.4 eventually gives the proof of the existence of the meromorphic extension of the weighted resolvent of  $\hat{K}(s)$  and also shows that the poles in any compact neighbourhood of 0 lie below the real axis.

### 3.1 Notations

We introduce some notations following [GGH17, §2.1]. First observe that if  $u$  solves (2), then  $v := e^{-isV_+t}u$  satisfies

$$(\partial_t^2 - 2is(V(r) - V_+)\partial_t - s^2(V(r) - V_+)^2 + P)v = 0.$$

We can therefore work with the potential<sup>3</sup>  $\tilde{V} := V - V_+ = \mathcal{O}_{r \rightarrow r_+}(r_+ - r)$  in this Section. In order not to overload notations, we will still denote  $\tilde{V}$  by  $V$  and  $\lim_{r \rightarrow r_{\pm}} \tilde{V}(r) = V_{\pm}$ .

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<sup>3</sup>From a geometrical point of view, we are changing the gauge. Namely,  $\frac{\mathcal{Q}}{r}dt$  is replaced by  $(\frac{\mathcal{Q}}{r} - \frac{\mathcal{Q}}{r_+})dt$  which does not degenerate anymore at  $r = r_+$ . To see this, we use the standard Eddington-Finkelstein advanced and retarded coordinates  $u = t - x$ ,  $v = t + x$  to define the horizons: we have locally near the cosmological horizon  $dt = du + dx$ ,  $dt = dv - dx$  and then  $\frac{dt}{dr} = \pm F(r)^{-1}$ . We eventually use that  $(\frac{1}{r} - \frac{1}{r_+})F(r)^{-1}$  remains bounded and does not vanish at  $r = r_+$ .

Let us define  $\mathcal{H} := L^2(X, drd\omega)$  and

$$\mathcal{P} := rF(r)^{-1/2}\hat{P}r^{-1}F(r)^{1/2} = -r^{-1}F(r)^{1/2}\partial_r r^2 F(r)\partial_r r^{-1}F(r)^{1/2} - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2 F(r) \quad (20)$$

with  $\hat{P}$  given by (3). Since  $u \mapsto r^{-1}F^{1/2}u$  is an unitary isomorphism from  $\mathcal{H}$  to  $L^2(X, F^{-1}r^2 drd\omega)$ , the results obtained below on  $\mathcal{P}$  will also apply to  $\hat{P}$  (and thus to  $P$ ). Observe that the space  $\dot{\mathcal{E}}$  has been defined in our setting with the operator  $P$  which is  $r\hat{P}r^{-1}$  expressed with the Regge-Wheeler coordinate, and  $\hat{P}$  is equivalent to  $\mathcal{P}$  as explained above; in the sequel, we will denote by  $\dot{\mathcal{E}}$  the completion of  $\mathcal{P}^{-1/2}\mathcal{H} \oplus \mathcal{H}$  for the norm  $\|(u_0, u_1)\|_{\dot{\mathcal{E}}}^2 := \langle u_0, \mathcal{P}u_0 \rangle_{\mathcal{H}} + \|u_1 - sVu_0\|_{\mathcal{H}}^2$ . Let  $i_{\pm}, j_{\pm} \in C^\infty(]r_-, r_+[ , \mathbb{R})$  such that

$$\begin{aligned} i_{\pm} = j_{\pm} = 0 \text{ close to } r_{\mp}, \quad i_{\pm} = j_{\pm} = 1 \text{ close to } r_{\pm}, \\ i_-^2 + i_+^2 = 1, \quad i_{\pm}j_{\pm} = j_{\pm}, \quad i_-j_+ = i_+j_- = 0. \end{aligned}$$

We then define the operators

$$k_{\pm} := s(V \mp j_{\mp}^2 V_-), \quad \mathcal{P}_{\pm} := \mathcal{P} - k_{\pm}^2, \quad \tilde{\mathcal{P}}_- := \mathcal{P} - (sV_- - k_-)^2.$$

We now define the isomorphism on  $\dot{\mathcal{E}}$  (see comments above Lemma 3.13 in [GGH17])

$$\Phi(sV) := \begin{pmatrix} \text{Id} & 0 \\ sV & \text{Id} \end{pmatrix}$$

and we introduce the energy Klein-Gordon operator

$$\hat{H}(s) = \Phi(sV)\hat{K}(s)\Phi^{-1}(sV) = \begin{pmatrix} 0 & \text{Id} \\ \mathcal{P} - s^2V^2 & 2sV \end{pmatrix}$$

with domain

$$\mathcal{D}(\hat{H}(s)) = \left\{ u \in \mathcal{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \mid \hat{H}(s)u \in \mathcal{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \right\}$$

and the asymptotic Hamiltonians

$$\hat{H}_{\pm}(s) = \begin{pmatrix} 0 & \text{Id} \\ \mathcal{P}_{\pm} & 2k_{\pm} \end{pmatrix}$$

with domains

$$\begin{aligned} \mathcal{D}(\hat{H}_+(s)) &= \left( \mathcal{P}_+^{-1/2}\mathcal{H} \cap \mathcal{P}_+^{-1}\mathcal{H} \right) \oplus \langle \mathcal{P}_+ \rangle^{-1/2}\mathcal{H}, \\ \mathcal{D}(\hat{H}_-(s)) &= \Phi(sV_-) \left( \tilde{\mathcal{P}}_-^{-1/2}\mathcal{H} \cap \tilde{\mathcal{P}}_-^{-1}\mathcal{H} \right) \oplus \langle \tilde{\mathcal{P}}_- \rangle^{-1/2}\mathcal{H}. \end{aligned}$$

These operators are self-adjoint on the following spaces (see the beginning of the paragraph 5.2 in [GGH17]):

$$\begin{aligned} \dot{\mathcal{E}}_+ &:= \mathcal{P}_+^{-1/2}\mathcal{H} \oplus \mathcal{H}, \\ \dot{\mathcal{E}}_- &:= \Phi(sr_-^{-1}) \left( \tilde{\mathcal{P}}_-^{-1/2}\mathcal{H} \oplus \mathcal{H} \right). \end{aligned}$$

In the sequel, we will also use the spaces  $\dot{\mathcal{E}}_{\pm}$  defined as above but with the operators  $\langle \mathcal{P}_{\pm} \rangle$  instead of  $\mathcal{P}_{\pm}$ . Finally, we define the weight  $w(r) := \sqrt{(r - r_-)(r_+ - r)}$ .

### 3.2 Abstract setting

Meromorphic extensions in our setting follow from the works of Mazzeo-Melrose [MM87] and Guillarmou [Gu04], as stated in [GGH17, Prop. 5.3]. The abstract setting in which this result can be used is recalled in this paragraph.

We first recall for the reader convenience the Abstract assumptions (A1)-(A3), the Meromorphic Extensions assumptions (ME1)-(ME2) as well as the "Two Ends" assumptions (TE1)-(TE3) of [GGH17]:

$$\mathcal{P} > 0, \tag{A1}$$

$$\left\{ \begin{array}{l} sV \in \mathcal{B}(\mathcal{P}^{-1/2}L^2) > 0, \\ \text{if } z \neq \mathbb{R} \text{ then } (z - sV)^{-1} \in \mathcal{B}(\mathcal{P}^{-1/2}L^2) \text{ and there exists } n > 0 \\ \text{such that } \|(z - sV)^{-1}\|_{\mathcal{B}(\mathcal{P}^{-1/2}L^2)} \lesssim |\Im z|^{-n}, \\ \text{there exists } c > 0 \text{ such that } \|(z - sV)^{-1}\|_{\mathcal{B}(\mathcal{P}^{-1/2}L^2)} \lesssim \|z - \|sV\|_{L^\infty} \\ \text{if } |z| \geq c\|sV\|_{L^\infty} \end{array} \right., \tag{A2}$$

$$\left\{ \begin{array}{l} \text{(a) } wVw \in L^\infty, \\ \text{(b) } [V, w] = 0 \\ \text{(c) } \mathcal{P}^{-1/2}[\mathcal{P}, w^{-\epsilon}]w^{\epsilon/2} \in \mathcal{B}(L^2) \text{ for all } 0 < \epsilon \leq 1, \\ \text{(d) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|\mathcal{P}^{1/2}u\|_{L^2} \text{ for all } u \in \mathcal{P}^{-1/2}L^2, \\ \text{(e) } w^{-1}\langle \mathcal{P} \rangle^{-1} \in \mathcal{B}(L^2) \text{ is compact} \end{array} \right., \tag{ME1}$$

$$\left\{ \begin{array}{l} \text{For all } \epsilon > 0 \text{ there exists } \delta_\epsilon > 0 \text{ such that } w^{-\epsilon}(\mathcal{P} - z^2)^{-1}w^{-\epsilon} \text{ extends from } \mathbb{C}^+ \\ \text{to } \{z \in \mathbb{C} \mid \Im z > -\delta_\epsilon\} \text{ as a finite meromorphic function with values} \\ \text{in compact operators acting on } L^2 \end{array} \right., \tag{ME2}$$

$$\left\{ \begin{array}{l} [x, sV] = 0, \\ x \mapsto w(x) \in C^\infty(\mathbb{R}, \mathbb{R}), \\ \chi_1(x)\mathcal{P}\chi_2(x) = 0 \text{ for all } \chi_1, \chi_2 \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ bounded with all their derivatives} \\ \text{and such that } \text{Supp } \chi_1 \cap \text{Supp } \chi_2 = \emptyset \end{array} \right., \tag{TE1}$$

$$\text{There exists } \ell_- \in \mathbb{R} \text{ such that } (\mathcal{P}_+, k_+) \text{ and } (\tilde{\mathcal{P}}_-, (k_- - \ell_-)) \text{ satisfy (A2)}, \tag{TE2}$$

$$\left\{ \begin{array}{l} \text{(a) } wi_+sVi_+w, wi_-(sV - \ell_-)i_-w \in L^\infty, \\ \text{(b) } [\mathcal{P} - s^2V^2, i_\pm] = \tilde{i}[\mathcal{P} - s^2V^2, i_\pm]\tilde{i} \text{ for some } \tilde{i} \in C_c^\infty(]-2, 2[, \mathbb{R}) \text{ such that } \tilde{i}|_{[-1,1]} \equiv 1 \\ \text{(c) } (\mathcal{P}_+, k_+, w) \text{ and } (\mathcal{P}_-, (k_- - \ell_-), w) \text{ fulfill (ME1) and (ME2)}, \\ \text{(d) } \mathcal{P}_\pm^{1/2}i_\pm\mathcal{P}_\pm^{-1/2}, \mathcal{P}^{1/2}i_\pm\mathcal{P}^{-1/2} \in \mathcal{B}(L^2), \\ \text{(e) } w[(\mathcal{P} - s^2V^2), i_\pm]w\mathcal{P}_\pm^{-1/2}, w[(\mathcal{P} - s^2V^2), i_\pm]w\mathcal{P}^{-1/2}, [(\mathcal{P} - s^2V^2), i_\pm]\mathcal{P}_\pm^{-1/2}, \\ [(\mathcal{P} - s^2V^2), i_\pm]\mathcal{P}^{-1/2}, \mathcal{P}^{-1/2}[w^{-1}, \mathcal{P}]w \text{ are bounded operators on } L^2, \\ \text{(e) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|\mathcal{P}^{1/2}u\|_{L^2} \text{ for all } u \in \mathcal{P}^{-1/2}L^2 \end{array} \right. . \tag{TE3}$$

[GGH17, §9] shows that all the above hypotheses actually follow from some geometric assumptions (the assumptions (G1)-(G7) in [GGH17, §2.1.1]). We show here that the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime can be dealt within this geometric setting:

(G1) The operator  $P$  in [GGH17] is  $-\Delta_{\mathbb{S}^2}$  for us, and satisfies of course  $[\Delta_{\mathbb{S}^2}, \partial_\phi] = 0$ .

(G2) The operator  $h_{0,s}$  in [GGH17] is  $\mathcal{P}$  for us, that is  $\alpha_1(r) = \alpha_3(r) = r^{-1}F(r)^{1/2}$ ,  $\alpha_2(r) = rF(r)^{1/2}$  and  $\alpha_4(r) = mF(r)^{1/2}$ . These last coefficients are clearly smooth in  $r$ . Furthermore, since we can write  $F(r) = g(r)w(r)^2$  with  $g(r) = \frac{\Lambda}{3r^2}(r - r_n)(r - r_c) \gtrsim 1$  for all  $r \in ]r_-, r_+[$ , it comes for

all  $j \in \{1, 2, 3, 4\}$  as  $r \rightarrow r_{\pm}$

$$\begin{aligned}\alpha_j(r) - w(r) \left( i_-(r) \alpha_j^- + i_+(r) \alpha_j^+ \right) &= w(r) \left( g(r)^{1/2} - \alpha_j^{\pm} \right) = \mathcal{O}_{r \rightarrow r_{\pm}}(w(r)^2), \\ \alpha_1^{\pm} &= \alpha_3^{\pm} = \frac{1}{r_{\pm}^2} \sqrt{\frac{\Lambda(r_{\pm} - r_n)(r_{\pm} - r_c)}{3}}, \\ \alpha_2^{\pm} &= \sqrt{\frac{\Lambda(r_{\pm} - r_n)(r_{\pm} - r_c)}{3}}, \\ \alpha_4^{\pm} &= \frac{m}{r_{\pm}} \sqrt{\frac{\Lambda(r_{\pm} - r_n)(r_{\pm} - r_c)}{3}}.\end{aligned}$$

Also, we clearly have  $\alpha_j(r) \gtrsim w(r)$ . Direct computations show that

$$\partial_r^m \partial_{\omega}^n \left( \alpha_j - w \left( i_- \alpha_j^- + i_+ \alpha_j^+ \right) \right) (r) = \mathcal{O}_{r \rightarrow r_{\pm}}(w(r)^{2-2m})$$

for all  $m, n \in \mathbb{N}$ .

- (G3) The operator  $k_s$  in [GGH17] is  $sV(r)$  for us, so  $k_s = k_{s,v}$  and  $k_{s,r} = 0$ . We have  $V(r) - V_{\pm} = \mathcal{O}_{r \rightarrow r_{\pm}}(|r_{\pm} - r|) = \mathcal{O}_{r \rightarrow r_{\pm}}(w(r)^2)$  (with  $V_+ = 0$ , recall the discussion at the beginning of Subsection 3.1) and  $\partial_r^m \partial_{\omega}^n V(r)$  is bounded for any  $m, n \in \mathbb{N}$ .
- (G4) The perturbation  $k$  in [GGH17] is simply  $k = k_s = sV$  for us, so that this assumption is trivially verified.
- (G5) The operator  $h_0$  in [GGH17] is simply  $h_0 = h_{0,s} = \mathcal{P}$  for us, and we have

$$\begin{aligned}\mathcal{P} &= -\alpha_1(r) \partial_r w(r)^2 r^2 g(r) \partial_r \alpha_1(r) - \alpha_1(r)^2 \Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 \\ &= \alpha_1(r) \left( -\partial_r w(r)^2 r^2 g(r) \partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2 \right) \alpha_1(r) \\ &\gtrsim \alpha_1(r) \left( -\partial_r w(r)^2 \partial_r - \Delta_{\mathbb{S}^2} + 1 \right) \alpha_1(r).\end{aligned}$$

- (G6) This assumption is trivial in our setting.
- (G7) We check that  $(\mathcal{P}_+, k_+)$  and  $(\tilde{\mathcal{P}}_-, k_- - sV_-)$  satisfy (G5). Since  $\alpha_1(r), k_+(r) = \mathcal{O}_{r \rightarrow r_{\pm}}(|r_{\pm} - r|)$ , we can write for  $|s| < mr_-$

$$\begin{aligned}\mathcal{P}_+ &= -\alpha_1(r) \partial_r w(r)^2 r^2 g(r) \partial_r \alpha_1(r) - \alpha_1(r)^2 \Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 - k_+(r)^2 \\ &= \alpha_1(r) \left( -\partial_r w(r)^2 r^2 g(r) \partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2 - \frac{k_+(r)^2}{\alpha_1(r)^2} \right) \alpha_1(r) \\ &\gtrsim \alpha_1(r) \left( -\partial_r w(r)^2 \partial_r - \Delta_{\mathbb{S}^2} + 1 \right) \alpha_1(r).\end{aligned}$$

As  $k_-(r) - sV_- = \mathcal{O}_{r \rightarrow r_{\pm}}(|r_{\pm} - r|)$  too, we get the same conclusion with  $\tilde{\mathcal{P}}_-$ .

To end this Subsection, we recall from [GGH17, §9] that

$$(G3) \implies (A1)-(A3), \quad (G3) \implies (ME1), \quad (G3)-(G5) \implies (TE1)-(TE3)$$

and (ME2) is satisfied by assumptions (G1), (G2) and (G7) on the form of the operator  $\mathcal{P}$  using Mazzeo-Melrose standard result (see [GGH17, §9.2.2] and also [MM87] for the original work of Mazzeo-Melrose).

### 3.3 Study of the asymptotic Hamiltonians

The aim of this paragraph is to show the existence of a meromorphic continuation of the weighted resolvent  $w^{\delta}(\hat{H}_{\pm}(s) - z)^{-1}w^{\delta}$  from  $\mathbb{C}^+$  into a strip in  $\mathbb{C}^-$  which is analytic in  $z$  in a tight box near 0. We start with the meromorphic extension.

**Lemma 3.1.** For all  $\delta > \delta' > 0$  and all  $s \in \mathbb{R}$ ,  $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$  has a meromorphic extension from  $\mathbb{C}^+$  to  $\{\omega \in \mathbb{C} \mid \Im\omega > -\delta'\}$  with values in compact operators acting on  $\hat{\mathcal{E}}_\pm$ .

*Proof.* Since hypotheses (G) are satisfied, we can apply [GGH17, Lem. 9.3] which shows that we can apply Mazzeo-Melrose result: the meromorphic extension of  $w^\delta(\mathcal{P}_\pm - z^2)^{-1}w^\delta$  exists from  $\mathbb{C}^+$  to a strip  $\mathcal{O}_\delta$ . This strip is explicitly given in the work of Guillarmou (cf. [Gu04, Thm. 1.1]):

$$\mathcal{O}_\delta = \left\{ z \in \mathbb{C} \mid z^2 = \lambda(3 - \lambda), \Re\lambda > \frac{3}{2} - \delta \right\}.$$

The absence of essential singularity is due to the fact that the metric  $g$  is even (cf. [Gu04, Def. 1.2 & Thm. 1.4]). We have to check that the set  $\mathcal{O}_\delta$  contains a strip in  $\overline{\mathbb{C}^-}$ . To see this, write  $\lambda = \alpha + i\beta$  and  $z = a + ib$  with  $\alpha, \beta, a, b \in \mathbb{R}$ ,  $b \leq 0$  and  $z^2 = \lambda(3 - \lambda)$ . Solving for

$$\begin{cases} a^2 - b^2 = \alpha(3 - \alpha) + \beta^2 \\ 2ab = (3 - 2\alpha)\beta \end{cases} \quad (21)$$

we find

$$\begin{cases} \beta = \pm \sqrt{\frac{1}{2}(a^2 - b^2 - 9/4) + \frac{1}{2}\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2b^2}} \\ \alpha = \frac{3}{2} - \frac{ab}{\beta} \end{cases}$$

and these expressions make sense since  $\beta = 0$  can happen only if  $ab = 0$ , and

$$\beta = \pm \frac{|a||b|}{\sqrt{b^2 + 9/4}} + \mathcal{O}_{a \rightarrow 0}(a), \quad \beta = \mathcal{O}_{b \rightarrow 0}(b).$$

If  $b = 0$  then  $\alpha = 3/2$  and  $\beta$  solves  $a^2 = 9/4 + \beta^2$ , and conversely  $\alpha = 3/2$  implies  $b = 0$ . Hence  $\alpha = 3/2$  allows all  $z \in \mathbb{R}$ . We may now assume  $b < 0$  (hence  $\alpha \neq 0$ ). The condition  $\Re\lambda = \alpha > 3/2 - \delta$  reads  $\frac{ab}{\beta} < \delta$ , and this condition is trivially satisfied if  $\alpha \geq 3/2$  since (21) implies that  $\frac{ab}{\beta} \leq 0 < \delta$ . Otherwise, if  $\alpha < 3/2$  then (21) implies that  $\frac{ab}{\beta} > 0$  and

$$b > -\left|\frac{\beta}{a}\right|\delta.$$

We compute

$$\left(\frac{\beta}{a}\right)' = \frac{a\beta' - \beta}{a^2}$$

where  $'$  denotes here the derivative with respect to  $a$ , and

$$\beta' = \frac{a}{2\beta} \left( 1 + \frac{(a^2 - b^2 - 9/4) + 2b^2}{\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2b^2}} \right)$$

so that

$$\begin{aligned} a\beta' - \beta = 0 &\iff a^2 \left( 1 + \frac{(a^2 - b^2 - 9/4) + 2b^2}{\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2b^2}} \right) = 2\beta^2 \\ &\iff a^2(a^2 - b^2 - 9/4) + 2a^2b^2 \\ &= -(b^2 + 9/4)\sqrt{(a^2 - b^2 - 9/4)^2 + 4a^2b^2} + (a^2 - b^2 - 9/4)^2 + 4a^2b^2 \\ &\iff (b^2 + 9/4)^2((a^2 - b^2 - 9/4)^2 + 4a^2b^2) = (b^4 + 81/16 + a^2b^2 - 9a^2/4 + 9b^2/2)^2. \end{aligned}$$

After some tedious simplifications, we obtain the very simple condition

$$a\beta' - \beta = 0 \iff 9a^4b^2 = 0.$$

Thus  $a = 0$  is the only possible extremum of  $\beta$  when  $b < 0$ . One can check that  $\beta \rightarrow 1$  as  $a \rightarrow \pm\infty$ , whence

$$\{z \in \mathbb{C} \mid 0 \geq \Im z > -\delta\} \subset \mathcal{O}_\delta.$$

From there, we deduce the existence of the meromorphic extension of  $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$  for  $z \in \{\omega \in \mathbb{C} \mid \Im \omega > -\delta'\}$  thanks to [GGH17, Lem. 4.3 & Prop. 4.4] (the parameters  $\epsilon$  and  $\delta_\epsilon$  therein are identical in our situation, and  $\delta_{\epsilon/2}$  can be replaced by any  $\delta' < \delta_\epsilon$ ).  $\square$

Before proving the analyticity near 0 of the weighted resolvent, we need to prove the following result:

**Lemma 3.2.** *For all  $\delta > 0$ ,  $w^\delta(\mathcal{P} - z^2)^{-1}w^\delta$  has no pole in  $\mathbb{R}$ .*

*Proof.* We can work with the operator  $P$  expressed in the Regge-Wheeler coordinate since  $P \mapsto \mathcal{P}$  is an unitary transform (as explained at the beginning of Subsection 3.1).

For all  $\ell \in \mathbb{N}$ ,  $P_\ell$  is selfadjoint and the potential  $W_0 + \ell(\ell + 1)W_1$  (with  $W_0$  and  $W_1$  as in (8)) is bounded on  $\mathcal{D}(P_\ell)$  and tends to 0 at infinity exponentially fast; as a result, the Kato-Agmon-Simon theorem (cf. [RS78, Thm. XIII.57]) implies that  $P_\ell$  has no positive eigenvalue. As  $P_\ell \geq 0$ , we deduce that there is no eigenvalue on  $\mathbb{R} \setminus \{0\}$ . Furthermore, [BM93, Prop. II.1] shows that 0 is not an eigenvalue for  $P_\ell$  thanks to the exponential decay of  $W_0 + \ell(\ell + 1)W_1$ . Finally,  $P_\ell$  verifies the limiting absorption principle

$$\sup_{\mu > 0} \|\langle x \rangle^{-\alpha} (P_\ell - (\lambda + i\mu))^{-1} \langle x \rangle^{-\alpha}\|_{L^2} < +\infty \quad \forall \lambda \in \mathbb{R} \setminus \{0\}, \forall \alpha > 1,$$

see Mourre [Mou80]. The only issue then is  $z = 0$  which could be a pole.

We introduce then the Jost solutions following [B04] (as  $s = 0$  in this Lemma, the potential  $sV$  vanishes). Fix  $\ell \in \mathbb{N}$  and set  $\tilde{W}_\ell := \ell(\ell + 1)W_0 + W_1$ . Set

$$\kappa := \min\{\kappa_-, |\kappa_+|\} \tag{22}$$

where  $\kappa_\pm$  are the surface gravity at the event and cosmological horizons (cf. (7)). For any  $\alpha \in ]0, 2\kappa[$ ,

$$\int_{-\infty}^{+\infty} |\tilde{W}_\ell(x)| e^{\alpha|x|} dx < +\infty.$$

The convergence of the above integral comes from the exponential decay of  $\tilde{W}_\ell$  at infinity. For all  $z \in \mathbb{C}$  such that  $\Im z > -\kappa$ , [B04, Prop. 2.1] shows that there exist two unique  $\mathcal{C}^2$  functions  $x \mapsto e^\pm(x, z, \ell)$ , that we will simply write  $e_\pm(x)$  or  $e_\pm(x, z)$ , satisfying the Schrödinger equation

$$(\partial_x^2 + z^2 - \tilde{W}_\ell(x))e_\pm(x) = 0 \quad \forall x \in \mathbb{R}$$

with  $\partial_x e_\pm \in L_{loc}^\infty(\mathbb{R}_x, \mathbb{C})$ , and such that if  $\Im z > -\kappa$ , then  $\partial_x^j e_\pm$  is analytic in  $z$  for all  $0 \leq j \leq 1$ . Moreover, they satisfy

$$\lim_{x \rightarrow \pm\infty} \left( |e_\pm(x) - e^{\pm izx}| + |\partial_x e_\pm(x) \mp iz e^{\pm izx}| \right) = 0. \tag{23}$$

By checking the formula on  $\mathcal{C}_c^2(\mathbb{R}, \mathbb{C})$  first and then extending it on  $H^2(\mathbb{R}, dx)$  by density, one easily shows that the kernel  $K$  of  $(P_\ell - (z - sV)^2)^{-1} : H_c^2(\mathbb{R}, dx) \rightarrow L_{loc}^2(\mathbb{R}, dx)$  for  $\Im z > -\kappa$  is given by<sup>4</sup>

$$K(z; x, y) = \frac{1}{\mathcal{W}(z)} (e_+(x, z)e_-(y, z)\mathbf{1}_{x \geq y}(x, y) + e_+(y, z)e_-(x, z)\mathbf{1}_{y \geq x}(x, y))$$

where  $\mathcal{W}(z) = e_+(x)(e_-)'(x) - (e_+)'(x)e_-(x)$  is the Wronskian between  $e_+$  and  $e_-$ . Since  $\mathcal{W}$  is independent of  $x \in \mathbb{R}$  (as shown at the very beginning of the proof of [B04, Prop. 2.1]), we see using

<sup>4</sup>Observe that in the example at the beginning of Section 3, the kernel  $R(z; x, y)$  is also given in terms of the Jost functions  $e_\pm(x, z) = e^{\pm izx}$ .

the non trivial limits for  $e_{\pm}$  in (23) that a pole  $z$  of order  $n > 0$  for  $w^{\delta}(P_{\ell} - (z - sV)^2)^{-1}w^{\delta}$  with  $\Im z > -\kappa$  is a zero of order  $n$  of the Wronskian  $\mathscr{W}$ , and  $e_+(\cdot, z)$  and  $e_-(\cdot, z)$  are then collinear.

We now reproduce the computation (2.14) in [BoHa08]. Assume that  $z = 0$  is a pole; for all  $\ell \in \mathbb{N}$ ,  $e_+(\cdot, 0, \ell) \in L^2_{\text{loc}}(\mathbb{R}, dx)$  and satisfies  $P_{\ell}e_+ = 0$ , so that

$$\begin{aligned} 0 &= \int_{-R_0}^{R_0} (P_{\ell}e_+) \overline{e_+} dx \\ &= [r \overline{e_+} \partial_x (r^{-1}e_+)]_{-R_0}^{R_0} + \int_{-R_0}^{R_0} |r \partial_x (r^{-1}e_+)|^2 dx + \ell(\ell+1) \int_{-R_0}^{R_0} F(r) |r^{-1}e_+|^2 dx \\ &\quad + m^2 \int_{-R_0}^{R_0} F(r) |e_+|^2 dx. \end{aligned}$$

Letting  $R_0 \rightarrow +\infty$  and using the decay of the derivative of  $e_+$  in (23) for  $z = 0$  show that  $e_+ = 0$ , a contradiction<sup>5</sup>.  $\square$

We are now ready to prove the analyticity. In the next result, we will consider  $s$  as a complex number.

**Proposition 3.3.** *Let  $0 < \delta < \kappa$  and  $R > 0$ . There exist  $\varepsilon_0 \equiv \varepsilon_0(\delta) > 0$  and  $\sigma \equiv \sigma(\mathscr{P}, \tilde{k}_{\pm})$  such that the extension of  $w^{\delta}(\hat{H}_{\pm}(s) - z)^{-1}w^{\delta}$  is holomorphic in  $(s, z)$  for  $s \in D(0, \sigma)$  and  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$ .*

The restriction  $\delta < \kappa$  comes from the fact that the extension of  $(\hat{H}_{\pm}(s) - z)^{-1}$  depends itself on  $\delta$  (see formula (24) in the proof). One can increase the exponent of the weight  $w$  but the width of the strip in  $\mathbb{C}^-$  in which the result of Proposition 3.3 holds is bounded by  $\kappa$ .

*Proof.* Observe that  $(\hat{H}_-(s) - z)^{-1} = (\Phi(-sV_-)\hat{H}_-(s)\Phi(sV_-) - z)^{-1}$  on  $\dot{\mathcal{E}}_-$ , so it is sufficient to prove the announced results for  $w^{\delta}(\hat{H}_+(s) - z)^{-1}w^{\delta}$  and  $w^{\delta}(\tilde{H}_-(s) - z)^{-1}w^{\delta}$  with

$$\tilde{H}_-(s) := \Phi(-sV_-)\hat{H}_-(s)\Phi(sV_-) = \begin{pmatrix} sV_- & \text{Id} \\ \tilde{\mathscr{P}}_- & 2k_- - sV_- \end{pmatrix}.$$

Proceeding as in the proof of [GGH17, Prop. 4.4], we can work on the operators  $\mathscr{P} - (z - \tilde{k}_{\pm})^2$  with

$$\tilde{k}_+ := k_+, \quad \tilde{k}_- := k_- - sV_-$$

so that  $\tilde{k}_{\pm}$  are now exponentially decaying potentials at infinity (and are polynomial in  $s$ ).

We reproduce the perturbation argument of [GGH17, Lem. 4.3]. Choose  $\varepsilon_0 \in ]0, \varepsilon_{\delta}[$ ,  $\varepsilon_{\delta}$  as in Lemma 3.1, and pick  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$  so that  $w^{\delta}(\mathscr{P} - z^2)^{-1}w^{\delta}$  is holomorphic (it is possible since there is no pole in  $\mathbb{R}$  by Lemma 3.2) and  $w^{\delta}(\mathscr{P} - (z - \tilde{k}_{\pm})^2)^{-1}w^{\delta}$  is meromorphic<sup>6</sup> in  $z$ . Write then in  $\mathcal{H}$

$$w^{\delta}(\mathscr{P} - z^2)^{-1}w^{\delta} = w^{\delta}(\mathscr{P} - (z - \tilde{k}_{\pm})^2)^{-1}w^{\delta}(\text{Id} + \mathcal{K}_{\pm}(s, z)) \quad (24)$$

with

$$\mathcal{K}_{\pm}(s, z) := w^{-\delta}\tilde{k}_{\pm}(2z - \tilde{k}_{\pm})w^{-\delta}w^{\delta}(\mathscr{P} - z^2)^{-1}w^{\delta}.$$

$\mathcal{K}_{\pm}(s, z)$  is clearly analytic in  $s \in D(0, 1)$  and in  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$ . Since  $\tilde{k}_{\pm} = \mathcal{O}_{r \rightarrow r_{\pm}}(w^{2\kappa})$  by (6) and (22),  $\delta < \kappa$  and  $w^{\delta}(\mathscr{P} - z^2)^{-1}w^{\delta}$  is compact by Lemma 3.1, we see that  $\mathcal{K}_{\pm}(s, z)$  is compact. By two-dimensional analytic Fredholm theory, there exists a subvariety  $S \subset D(0, 1) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$  such that  $(\text{Id} + \mathcal{K}_{\pm}(s, z))^{-1}$  exists and is holomorphic in  $(s, z) \in (D(0, 1) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)) \setminus S$ . We then get the representation formula for the extension:

$$w^{\delta}(\mathscr{P} - (z - \tilde{k}_{\pm})^2)^{-1}w^{\delta} = w^{\delta}(\mathscr{P} - z^2)^{-1}w^{\delta}(\text{Id} + \mathcal{K}_{\pm}(s, z))^{-1}. \quad (25)$$

<sup>5</sup>We may notice here that the positive mass term  $m^2$  allowed us to conclude that  $z = 0$  is not a pole. For the wave equation as in [BoHa08], we do not have any positivity for  $\ell = 0$  and  $z = 0$  is shown to be a pole.

<sup>6</sup>Lemma 3.1 of course applies if we replace  $k_-$  by  $\tilde{k}_-$ .



We claim that for  $\sigma > 0$  sufficiently small, we have

$$\left( D(0, \sigma) \times (]-R, R[ + i]^{-\varepsilon_0, \varepsilon_0[}) \right) \cap S = \emptyset. \quad (26)$$

Otherwise, for every  $n \in \mathbb{N} \setminus \{0\}$ , there is a couple  $(s_n, z_n) \in D(0, 1/n) \times (]-R, R[ + i]^{-\varepsilon_0, \varepsilon_0[})$  such that  $\text{Id} + \mathcal{K}(s_n, z_n)$  is not invertible. By compactness, we can assume that  $(s_n, z_n) \rightarrow (0, z_0)$  as  $n \rightarrow +\infty$  for some  $z_0 \in [-R, R] + i[-\varepsilon_0, \varepsilon_0]$ . But  $\text{Id} + \mathcal{K}(0, z_0) = \text{Id}$  is invertible for all  $z \in \mathbb{C}$ , so  $\text{Id} + \mathcal{K}_\pm(s, z)$  must be invertible too for all  $(s, z)$  in a small neighbourhood of  $(0, z_0)$ , a contradiction.

Assuming now that  $s$  is sufficiently small so that (26) is true, we deduce by (25) that the poles of  $w^\delta(\mathcal{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta$  are exactly the poles of  $w^\delta(\mathcal{P} - z^2)^{-1}w^\delta$ . Since for  $z \in ]-R, R[ + i]^{-\varepsilon_0, \varepsilon_0[}$ ,  $w^\delta(\mathcal{P} - z^2)^{-1}w^\delta$  has no pole, the same conclusion applies for  $w^\delta(\mathcal{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta$ .  $\square$

### 3.4 Construction of the meromorphic extension of the weighted resolvent

The aim of this paragraph is to show the existence of a meromorphic extension for  $w^\delta(\hat{K}(s) - z)^{-1}w^\delta$  in a strip near 0 of width uniform in  $s$ . Since the operators  $\hat{K}(s)$  and  $\hat{H}(s)$  are equivalent modulo the isomorphism  $\Phi(sV)$  (by (3.19) in [GGH17]), we will work with the latter one.

We first need some preliminary results. The starting point is the following result:

**Proposition 3.4** (Prop. 5.5 in [GGH17]). *There is a finite set  $Z \in \mathbb{C} \setminus \mathbb{R}$  with  $\bar{Z} = Z$  such that the spectrum of  $\hat{H}(s)$  is included in  $\mathbb{R} \cup Z$  and the resolvent has a meromorphic extension to  $\mathbb{C} \setminus \mathbb{R}$ . Moreover, the set  $Z$  consists of eigenvalues of finite multiplicity of  $\hat{H}(s)$ .*

An important fact is that [GGH17, Prop. 3.6] shows that  $Z \equiv Z(s)$  is contained in the disc  $D(0, C|s|)$  for some constant  $C > 0$  (we can take  $C = 2\|V\|_{L^\infty}$ ). We show below that  $Z(s) \cap \mathbb{C}^+ = \emptyset$  for  $s$  sufficiently small.

We henceforth use the Regge-Wheeler coordinate  $x$  introduced in Subsection 2.2. We will still denote by  $\mathcal{P}$  the operator defined in (20) expressed in the coordinates  $(x, \omega)$ :

$$\mathcal{P} = -F(r(x))^{-1/2} \partial_x^2 F(r(x))^{-1/2} - W_0(x) \Delta_{\mathbb{S}^2} + W_1(x)$$

Let also  $\mathcal{H} = L^2(\mathbb{R} \times \mathbb{S}_\omega^2, F(r(x)) dx d\omega)$ .

**Lemma 3.5.** *For all  $\delta > 0$ ,  $w^\delta$  sends  $\dot{\mathcal{E}}$  into  $\dot{\mathcal{E}}_\pm$  and  $\dot{\mathcal{E}}$  into itself.*

*Proof.* Let  $u = (u_0, u_1) \in \dot{\mathcal{E}}$ . We only show that  $w^\delta \dot{\mathcal{E}} \subset \dot{\mathcal{E}}_-$ , the proof of the other statements being slightly easier. We thus look for  $v = (v_0, v_1) \in \tilde{\mathcal{P}}_-^{-1/2} \mathcal{H} \oplus \mathcal{H}$  such that  $(w^\delta u_0, w^\delta u_1) = (v_0, sV_- v_0 + v_1)$ . Since  $w^\delta$  is bounded on  $\mathbb{R}$ ,  $w^\delta u_1 \in \mathcal{H}$ . Next, using the facts that  $(w^\delta)'u_0$ ,  $V_+ w^\delta u_0$  and  $W_j^{1/2} w^\delta u_0$  are in  $\mathcal{H}$  thanks to (ME1) (d) ( $0 \leq j \leq 1$ ), we compute

$$\|\tilde{\mathcal{P}}_-^{1/2} w^\delta u_0\|_{\mathcal{H}}^2 = \langle \mathcal{P} w^\delta u_0, w^\delta u_0 \rangle_{\mathcal{H}} - \underbrace{\|s^2 (V_- - k_-)^2 w^\delta u_0\|_{\mathcal{H}}^2}_{\lesssim \|\mathcal{P}^{1/2} u_0\|_{\mathcal{H}}^2}$$

and working with the operators  $\mathcal{P}_\ell$  defined as  $P_\ell$  ( $\ell \in \mathbb{N}$ ), we get

$$\begin{aligned} \langle \mathcal{P}_\ell w^\delta u_0, w^\delta u_0 \rangle_{\mathcal{H}} &= \|\partial_x w^\delta u_0\|_{\mathcal{H}}^2 + \underbrace{\langle (-\ell(\ell+1)W_0 + W_1) w^\delta u_0, w^\delta u_0 \rangle_{\mathcal{H}}}_{\lesssim \|\mathcal{P}_\ell^{1/2} u_0\|_{\mathcal{H}}^2} \\ \|\partial_x w^\delta u_0\|_{\mathcal{H}}^2 &\lesssim \|(w^\delta)'u_0\|_{\mathcal{H}}^2 + \|w^\delta u_0\|_{\mathcal{H}}^2 \lesssim \|\mathcal{P}_\ell^{1/2} u_0\|_{\mathcal{H}}^2. \end{aligned}$$

This proves that  $w^\delta u_0 \in \tilde{\mathcal{P}}_-^{-1/2} \mathcal{H}$ . Hence  $v_0 := w^\delta u_0 \in \tilde{\mathcal{P}}_-^{-1/2} \mathcal{H}$ , and the problem boils down to show that  $v_1 := w^\delta u_1 - sV_- v_0 = w^\delta u_1 - sV_- w^\delta u_0$  is in  $\mathcal{H}$ ; this is a consequence of (ME1) (d) which implies that  $sV_- w^\delta u_0 \in \mathcal{H}$ .  $\square$

For all  $z \in \mathbb{C}^+$  and  $s \in \mathbb{R}$ , we introduce the operator

$$Q(s, z) := i_-^2 (\hat{H}_-(s) - z)^{-1} + i_+^2 (\hat{H}_+(s) - z)^{-1} = \sum_{\pm} i_{\pm}^2 (\hat{H}_{\pm}(s) - z)^{-1}.$$

In Subsection 3.3 above, we have studied the resolvents of the asymptotic Hamiltonians. In particular, we know that  $Q(s, z)$  meromorphically extends into a strip in  $\mathbb{C}^-$  and is analytic in a small neighbourhood of  $\mathbb{R}$ . We wish to show that  $(\hat{H}(s) - z)^{-1}$  has the same properties. To do this, we show that  $Q(s, z)$  is a parametrix for the resolvent on the energy space.

By Lemma 3.5,  $Q(s, z)w^\delta$  is well-defined in  $\dot{\mathcal{E}}$ . Using the potentials  $k_{\pm} = s(V \mp j_{\mp}^2 V_-)$  introduced in Subsection 3.1 as well as the relations  $i_{\pm} j_{\mp} = 0$ , we compute:

$$i_{\pm}^2 (\hat{H}(s) - z) (\hat{H}_{\pm}(s) - z)^{-1} = i_{\pm}^2 \text{Id} - i_{\pm}^2 \begin{pmatrix} 0 & 0 \\ s^2(V^2 - k_{\pm}^2) & 2s(V - k_{\pm}) \end{pmatrix} (\hat{H}_{\pm}(s) - z)^{-1} = i_{\pm}^2 \text{Id}.$$

Since  $i_+^2 + i_-^2 = 1$ , we obtain in  $\dot{\mathcal{E}}$ :

$$(\hat{H}(s) - z)Q(s, z)w^\delta = \left( \text{Id} + \sum_{\pm} [\hat{H}(s), i_{\pm}^2] (\hat{H}_{\pm}(s) - z)^{-1} \right) w^\delta. \quad (27)$$

It follows that for all  $z \notin Z$  (see Proposition 3.4)

$$w^\delta Q(s, z)w^\delta = w^\delta (\hat{H}(s) - z)^{-1} w^\delta (\text{Id} + \hat{\mathcal{K}}_{\pm}(s, z)) \quad (28)$$

with

$$\hat{\mathcal{K}}_{\pm}(s, z) := w^{-\delta} \sum_{\pm} [\hat{H}(s), i_{\pm}^2] (\hat{H}_{\pm}(s) - z)^{-1} w^\delta.$$

**Lemma 3.6.** *The operators on the left and right-hand sides of (28) send  $\dot{\mathcal{E}}$  into itself.*

*Proof.* For the left-hand side of (28), we successively use Lemma 3.5, the facts that  $(\hat{H}_{\pm}(s) - z)^{-1}$  sends  $\dot{\mathcal{E}}_{\pm}$  into  $\mathcal{D}(\hat{H}_{\pm}(s)) \subset \dot{\mathcal{E}}_{\pm}$  and  $i_{\pm}$  sends  $\dot{\mathcal{E}}_{\pm}$  into  $\dot{\mathcal{E}}$  (cf. [GGH17, Lem. 5.4]), and again Lemma 3.5.

We now deal with the right-hand side of (28). By Lemma 3.5, we only have to show that  $w^{-\delta} [\hat{H}(s), i_{\pm}] (\hat{H}_{\pm}(s) - z)^{-1}$  sends  $\dot{\mathcal{E}}_{\pm}$  into  $\dot{\mathcal{E}}$ . Let  $u \in \dot{\mathcal{E}}_{\pm}$  and write  $v = (v_0, v_1) : (\hat{H}_{\pm}(s) - z)^{-1} u \in \dot{\mathcal{E}}_{\pm}$ . We have

$$w^{-\delta} [\hat{H}(s), i_{\pm}] (\hat{H}_{\pm}(s) - z)^{-1} u = w^{-\delta} \begin{pmatrix} 0 & 0 \\ [\mathcal{P}, i_{\pm}] & 0 \end{pmatrix} (v_0, v_1) = \begin{pmatrix} 0 \\ w^{-\delta} [\mathcal{P}, i_{\pm}] v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ w^{-\delta} [\mathcal{P}, i_{\pm}] w^{-\delta} w^\delta v_0 \end{pmatrix}.$$

Since  $w^\delta \dot{\mathcal{E}}_{\pm} \subset \dot{\mathcal{E}}_{\pm}$ , we can use (TE3) (e) to conclude that the second component is in  $\mathcal{H}$ , whence  $w^{-\delta} [\hat{H}(s), i_{\pm}] (\hat{H}_{\pm}(s) - z)^{-1} \dot{\mathcal{E}}_{\pm} \subset \dot{\mathcal{E}}$  (when  $\pm = -$ , we use that  $\mathcal{P}^{1/2} \tilde{\mathcal{P}}_-^{-1/2}$  is bounded on  $\mathcal{H}$ ).  $\square$

In the following, we will consider  $s$  as a complex number lying in a small neighbourhood of 0.

**Lemma 3.7.** *Let  $0 < \delta < \kappa$  and  $R > R_0$ .  $\text{Id} + \hat{\mathcal{K}}_{\pm}(s, z)$  is a holomorphic family of Fredholm operators acting on  $\dot{\mathcal{E}}$  for  $(s, z) \in D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$ , with  $\sigma > 0$  sufficiently small and  $\varepsilon_0 > 0$  as in Proposition 3.3.*

*Proof.* Write

$$\hat{\mathcal{K}}_{\pm}(s, z) = \sum_{\pm} w^{-\delta} [\hat{H}(s), i_{\pm}^2] w^{-\delta} w^\delta (\hat{H}_{\pm}(s) - z)^{-1} w^\delta.$$

By Lemma 3.1,  $w^\delta (\hat{H}_{\pm}(s) - z)^{-1} w^\delta$  is compact on  $\dot{\mathcal{E}}_{\pm}$  and Proposition 3.3 shows that the extension is holomorphic in  $(s, z)$ . Furthermore,

$$w^{-\delta} [\hat{H}(s), i_{\pm}^2] w^{-\delta} = \begin{pmatrix} 0 & 0 \\ w^{-\delta} [\mathcal{P}, i_{\pm}^2] w^{-\delta} & 0 \end{pmatrix}$$

is bounded on  $\dot{\mathcal{E}}_{\pm}$  (as a consequence of (TE3) (e), see the end of the proof of Lemma 3.6). Hence  $\hat{\mathcal{K}}_{\pm}(s, z)$  is compact and thus  $\text{Id} + \hat{\mathcal{K}}_{\pm}(s, z)$  is Fredholm.  $\square$

We are now ready to construct the meromorphic extension of the weighted resolvent. For all  $s_0 > 0$ , define  $R_0 := 2Cs_0$ . Proposition 3.4 and the remark below then show that  $Z(s) \subset D(0, R/2)$  for all  $s \in ]-s_0, s_0[$  and all  $R > R_0$ .

**Theorem 3.8.** *Let  $0 < \delta < \kappa$  and  $s \in ]-s_0, s_0[$ .*

1. *For  $s$  small enough,  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  has a meromorphic extension from  $\mathbb{C}^+ \setminus Z$  to  $\{\omega \in \mathbb{C} \mid \Im\omega > -\delta'\}$  for all  $0 < \delta' < \delta$  with values in compact operators acting on  $\hat{\mathcal{E}}$ .*
2. *For all  $R > R_0$ , there exists  $0 < s_1 < s_0$  such that for all  $s \in ]-s_1, s_1[$ , the extension of  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  is analytic in  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$  with  $\varepsilon_0 > 0$  as in Proposition 3.3.*

*Proof.* We first show Part 1. Let  $s \in \mathbb{C}$  small enough and let  $z \in ]-R, R[ + i] - \delta', \delta'[$ . Since  $\hat{H}_\pm(0) = \hat{H}(0)$ , we observe that  $\hat{\mathcal{K}}_\pm(0, z) = 0$  and  $Q(0, z) = (\hat{H}(0) - z)^{-1}$ . Hence the operator  $\text{Id} + \hat{\mathcal{K}}_\pm(0, z) = \text{Id}$  is invertible for all  $z \in \mathbb{C}$ . Finally, Lemma 3.1 shows that  $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$  is meromorphic in  $z$ . We can therefore use the meromorphic Fredholm theory to invert  $\text{Id} + \hat{\mathcal{K}}_\pm(s, z)$  on  $\hat{\mathcal{E}}$ . Using (28), we have the representation formula

$$w^\delta(\hat{H}(s) - z)^{-1}w^\delta = w^\delta Q(s, z)w^\delta (\text{Id} + \hat{\mathcal{K}}_\pm(s, z))^{-1} \quad (29)$$

which is valid for  $z \in ]-R, R[ + i] - \delta', \delta'[$ . This shows that  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  has a meromorphic extension in this strip and Part 1 is settled.

Let us show Part 2. of the theorem. We pick this time  $(s, z) \in D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$  with  $\sigma, \varepsilon_0 > 0$ . Lemma 3.7 shows that, if  $\sigma$  is very small,  $\text{Id} + \hat{\mathcal{K}}_\pm(s, z)$  is a holomorphic family of Fredholm operators acting on  $\hat{\mathcal{E}}$ . We can thus use the two-dimensional analytic Fredholm theory which implies that there is a meromorphic extension  $D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[) \ni (s, z) \mapsto (\text{Id} + \hat{\mathcal{K}}_\pm(s, z))^{-1}$ , and (29) is valid for  $(s, z) \in D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$  with  $\sigma$  small. This shows that the poles of  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  are the poles of  $(\text{Id} + \hat{\mathcal{K}}_\pm(s, z))^{-1}$  and  $w^\delta Q(s, z)w^\delta$ , the last ones being the poles of  $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ .

The multidimensional analytic Fredholm theory also implies that there exists a (possibly empty) subvariety  $S \subset D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$  such that  $\text{Id} + \hat{\mathcal{K}}_\pm(s, z)$  is invertible for  $(s, z) \notin S$ . We claim that we can take  $\sigma > 0$  small enough so that

$$\left( D(0, \sigma) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[) \right) \cap S = \emptyset.$$

Otherwise, for every  $n \in \mathbb{N} \setminus \{0\}$ , there is a couple  $(s_n, z_n) \in D(0, 1/n) \times (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[)$  such that  $\text{Id} + \hat{\mathcal{K}}_\pm(s_n, z_n)$  is not invertible. By compactness, we can assume that  $(s_n, z_n) \rightarrow (0, z_0)$  as  $n \rightarrow +\infty$  for some  $z_0 \in [-R, R] + i[-\varepsilon_0, \varepsilon_0]$ . But  $\text{Id} + \hat{\mathcal{K}}_\pm(0, z_0) = \text{Id}$  is invertible for all  $z \in \mathbb{C}$ , so  $\text{Id} + \hat{\mathcal{K}}_\pm(s, z)$  must be invertible too for all  $(s, z)$  in a small neighbourhood of  $(0, z_0)$ , a contradiction.

We now assume  $|s| < s_1$  where  $s_1$  is so small that  $\text{Id} + \hat{\mathcal{K}}_\pm(s, z)$  is invertible on  $\hat{\mathcal{E}}$  for  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$ . Using then the formula (29), we conclude that the poles of  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  are precisely the poles of  $w^\delta Q(s, z)w^\delta$ , which are the poles of  $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ . We then use Proposition 3.3 to conclude that there is no pole for  $z \in ]-R, R[ + i] - \varepsilon_0, \varepsilon_0[$ . This completes the proof.  $\square$

As a first consequence, we deduce a holomorphy result for the resolvent.

**Corollary 3.9.** *Let  $\varepsilon_0 > 0$  as in Proposition 3.3. Then for all  $s \in \mathbb{R}$  such that  $|s| < s_1$  with  $Cs_1 < \varepsilon_0$ , the resolvent  $(\hat{H}(s) - z)^{-1}$  is holomorphic in  $z \in \mathbb{C}^+$ . Furthermore, the spectrum of  $\hat{H}(s)$  is contained in  $\mathbb{R}$ .*

*Proof.* We know by Theorem 3.8 that the weighted resolvent  $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$  is holomorphic in  $z \in (]-R, R[ + i] - \varepsilon_0, \varepsilon_0[) \cap \mathbb{C}^+ \subset D(0, R/2) \subset \mathbb{C}^+$  if we assume  $s < s_1$  by Part 2. of Theorem 3.8. By Proposition 3.4,  $(\hat{H}(s) - z)^{-1}$  is holomorphic in  $\mathbb{C}^+ \setminus Z$ . Assume then that  $z_0 \in \mathbb{C}^+ \cap Z$  is a pole of order  $m_0 \in \mathbb{N}$ : there exist some finite rank operators  $A_1, \dots, A_{m_0} : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$(\hat{H}(s) - z)^{-1} = \sum_{j=1}^{m_0} \frac{A_j}{(z - z_0)^j} + \text{holomorphic term} \quad \forall z \in \mathbb{C}^+ \text{ near } z_0.$$

Since  $R > R_0$ ,  $Z \cap \mathbb{C}^+ \subset D(0, R/2) \cap \mathbb{C}^+$  and then

$$w^\delta (\hat{H}(s) - z)^{-1} w^\delta = \sum_{j=1}^{m_0} \frac{w^\delta A_j w^\delta}{(z - z_0)^j} + \text{holomorphic term}$$

is holomorphic in  $z$  near  $z_j$ , so that  $A_1 = \dots = A_{m_0} = 0$  and  $(\hat{H}(s) - z)^{-1}$  is holomorphic in  $z \in \mathbb{C}^+$ . By Proposition 3.4, this implies that the spectrum of  $\hat{H}(s)$  in  $Z \cap \mathbb{C}^+$  is empty; by symmetry, we deduce that  $\bar{Z} \cap \mathbb{C}^- = \emptyset$  too.  $\square$

**Remark 3.10.** *Theorem 3.8 and Corollary 3.9 answer Bachelot's open question of the nature of the sets  $\sigma_p$  (the eigenvalues in  $\mathbb{C}^+$ ) and  $\sigma_{ss}$  (the real resonances, also called hyperradiant modes) defined in [B04] by equations (2.35) and (2.36), when the charge product  $s$  is sufficiently small: both are empty as he conjectured at the end of his paper.*

We finally deduce the existence of the cut-off inverse of the quadratic pencil and define resonances.

**Corollary 3.11.** *Let  $s \in ]-s_0, s_0[$  small enough. The operator  $\chi p(z, s)^{-1} \chi : L^2(\mathbb{R}, dx) \rightarrow H^2(\mathbb{R}, dx)$  defines for any  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$  a meromorphic function of  $z \in \{\omega \in \mathbb{C} \mid \Im \omega > -\kappa\}$  and analytic if  $\Im z > -\varepsilon_0$  with  $\varepsilon_0 > 0$  given by Proposition 3.3.*

*If  $\chi$  is not identically 0, then the poles  $z$  of this extension are exactly the poles of the cut-off resolvent  $\chi(\hat{H}(s) - z)^{-1} \chi$  and are independent of the choice of  $\chi$ . We call them resonances of  $p$  and write  $z \in \text{Res}(p)$ . Similarly, we define  $\text{Res}(p_\ell)$  as the poles of  $\chi p_\ell(z, s)^{-1} \chi$  for all  $\ell \in \mathbb{N}$ .*

*Proof.* Let  $R > 0$  and let  $z \in \mathbb{C}$  with  $-R \leq \Re z \leq R$  and  $\Im z > -\kappa$ . The meromorphic extension  $w^\delta (\hat{H}(s) - z)^{-1} w^\delta : \mathcal{E} \rightarrow \mathcal{E}$  (with  $0 < \delta < \kappa$ ) entails the meromorphic extension  $w^\delta (\hat{K}(s) - z)^{-1} w^\delta : \mathcal{E} \rightarrow \mathcal{E}$  since  $\hat{H}(s)$  and  $\hat{K}(s)$  are equivalent on  $\mathcal{E}$  modulo the isomorphism  $\Phi(sV)$  introduced in Subsection 3.1. Since  $w(x)$  is exponentially decaying by (6), we can write for any cut-off  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$

$$\chi(\hat{K}(s) - z)^{-1} \chi = (\chi w^{-\delta}) w^\delta (\hat{K}(s) - z)^{-1} w^\delta (w^{-\delta} \chi) : L^2(\mathbb{R}, dx) \rightarrow H^2(\mathbb{R}, dx)$$

using that  $\chi w^{-\delta} \in C_c^\infty(\mathbb{R}, \mathbb{R})$ . In particular, if  $\chi$  is not identically 0, the poles of  $\chi(\hat{K}(s) - z)^{-1} \chi$  and  $w^\delta (\hat{K}(s) - z)^{-1} w^\delta$  coincide.

By formula (16) and the discussion below, we see that we can define the operator  $\chi p(z, s)^{-1} \chi : L^2(\mathbb{R}, dx) \rightarrow H^2(\mathbb{R}, dx)$  for any  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$  as a meromorphic function of  $z$ , and its poles are precisely the poles of  $\chi(\hat{K}(s) - z)^{-1} \chi$ .

To conclude the proof, it remains to prove the analyticity in the whole strip  $\{z \in \mathbb{C} \mid \Im z > -\varepsilon_0\}$  (which excludes a possible accumulation of resonances to  $\mathbb{R}$  at infinity): this follows from Theorem 4.1 below.  $\square$

## 4 Resonance expansion for the charged Klein-Gordon equation

We present in this section the main result of this chapter which is an extension of [BoHa08, Theorem 1.3]. By using the formula (15) and (16) as well as (3.21) in [GGH17] and the local equivalence (14) of the norms  $\|\cdot\|_{\mathcal{E}_\ell}$  and  $\|\cdot\|_{\mathcal{E}_\ell}$  if  $z \in \mathbb{R}$ , we can define for  $\Im z > -\kappa$  the meromorphic extension of the cut-off resolvent  $\hat{R}_{\chi, \ell}(z) := \chi(\hat{K}_\ell - z)^{-1} \chi$ . For all resonance  $z_0 \in \text{Res}(p_\ell)$ , denote by  $m(z_0) \in \mathbb{N}$  its multiplicity and set

$$\Pi_{j, k}^{\chi, \ell} := \frac{1}{2\pi i} \oint_{\partial\gamma} \frac{(-i)^k}{k!} \hat{R}_{\chi, \ell}(z) (z - z_0)^k dz$$

defined for all integer  $k \geq -(m(z_0) + 1)$  with  $\gamma$  a small positively oriented circle enclosing  $z_0$  and no other resonance. We will denote by  $\hat{R}_\chi(z)$  and  $\Pi_{j, k}^\chi$  the cut-off resolvent of the full operator  $\hat{K}$  and the corresponding generalized projector, respectively. Recall that  $\text{Res}(p)$  is introduced in Corollary 3.11.

We first introduce the set of pseudo-poles of  $P$  whose points approximate high frequency resonances. The proof is given in Appendix B.

**Theorem 4.1.** *There exist  $K > 0$  and  $\theta > 0$  such that, for any  $C > 0$ , there exists an injective map  $\tilde{b} : \Gamma \rightarrow \text{Res}(p)$  with*

$$\Gamma = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left( \pm \mathbb{N} \setminus \{0\} \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}} - \frac{i}{2} \sqrt{\left| 3 - \frac{12M}{\mathfrak{r}} + \frac{10Q^2}{\mathfrak{r}^2} \right|} \left( \mathbb{N} + \frac{1}{2} \right) \right)$$

the set of pseudo-poles, such that all the poles in

$$\Omega_C = \{ \lambda \in \mathbb{C} \mid |\lambda| > K, \Im \lambda > -\max\{C, \theta|\Re \lambda|\} \}$$

are in the image of  $\tilde{b}$ . Furthermore, if  $\mu \in \Gamma$  and  $\tilde{b}(\mu) \in \Omega_C$ , then

$$\lim_{|\mu| \rightarrow +\infty} (\tilde{b}(\mu) - \mu) = 0.$$

If  $\Re \mu = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left( \pm \ell \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}} \right)$  for  $\ell \in \mathbb{N} \setminus \{0\}$ , then the corresponding pole  $\tilde{b}(\mu)$  has multiplicity  $2\ell + 1$ .

We can now state our main result (the proof is given in Subsection 6):

**Theorem 4.2** (Decay of the local energy). *Let  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ .*

(i) *Let  $\nu > 0$  such that  $\nu \notin \Gamma$  ( $\Gamma$  is the set of pseudo-poles as in Theorem 4.1),  $\nu < \kappa$  and  $\text{Res}(p) \cap \{ \lambda \in \mathbb{C} \mid \Im \lambda = -\nu \} = \emptyset$ . There exists  $N > 0$  such that, for all  $u \in \dot{\mathcal{E}}$  with  $\langle -\Delta_{\mathbb{S}^2} \rangle^N u \in \dot{\mathcal{E}}$  and  $s$  small enough, we have*

$$\chi e^{-it\hat{K}} \chi u = \sum_{\substack{z_j \in \text{Res}(p) \\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} e^{-iz_j t} t^k \Pi_{j,k}^\chi u + E(t)u \quad (30)$$

for  $t > 0$  sufficiently large, with

$$\|E(t)u\|_{\dot{\mathcal{E}}} \lesssim e^{-\nu t} \|\langle -\Delta_{\mathbb{S}^2} \rangle^N u\|_{\dot{\mathcal{E}}}$$

and the sum is absolutely convergent in the sense that

$$\sum_{\substack{z_j \in \text{Res}(p) \\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} \|\Pi_{j,k}^\chi \langle -\Delta_{\mathbb{S}^2} \rangle^{-N}\|_{\dot{\mathcal{E}} \rightarrow \dot{\mathcal{E}}} < +\infty.$$

(ii) *There exists  $\varepsilon > 0$  such that, for any increasing positive function  $g$  with  $\lim_{x \rightarrow +\infty} g(x) = +\infty$  and  $g(x) \leq x$  for  $x \gg 0$ , for all  $u \in \dot{\mathcal{E}}$  with  $g(-\Delta_{\mathbb{S}^2})u \in \dot{\mathcal{E}}$  and  $s$  small enough, we have*

$$\|\chi e^{-it\hat{K}} \chi u\|_{\dot{\mathcal{E}}} \lesssim (g(e^{\varepsilon t}))^{-1} \|g(-\Delta_{\mathbb{S}^2})u\|_{\dot{\mathcal{E}}}$$

for  $t > 0$  sufficiently large.

**Remark 4.3.** 1. Formula (30) provides a physical interpretation of resonances: they are the frequencies and dumping rates of charged Klein-Gordon field in presence of the charged black hole (see Chapter 4.35 in [C92] for a discussion on the interpretation of resonances).

2. Part (ii) of Theorem 4.2 shows that a logarithmic derivative loss in the angular direction  $(\ln \langle -\Delta_{\mathbb{S}^2} \rangle)^\alpha u \in \dot{\mathcal{E}}$  with  $\alpha > 1$  entails the integrability of the local energy:

$$\left\| \int_0^{+\infty} \chi e^{-it(\hat{K}-z)} \chi u \, dt \right\|_{\dot{\mathcal{E}}} \lesssim \|(\ln \langle -\Delta_{\mathbb{S}^2} \rangle)^\alpha u\|_{\dot{\mathcal{E}}}.$$

3. In the limits  $9\Lambda M^2 \rightarrow 1^-$  and  $Q \rightarrow 0$ , the expansion in part (i) of Theorem 4.2 is not empty has (infinitely many) pseudo-poles of  $\Gamma$  (introduced in Theorem 4.1) lie in the strip  $\{z \in \mathbb{C} \mid \Im(z) > -\kappa\}$ . To see this, it suffices to consider the case  $Q = 0$ . Then  $\tau = 3M$  and

$$\min \{|\Im \lambda| \mid \lambda \in \Gamma\} = \frac{\sqrt{F(\tau)}}{4\tau} \sqrt{\left|3 - \frac{12M}{\tau}\right|} = \frac{\sqrt{1 - 9\Lambda M^2}}{12\sqrt{3}M}.$$

We show that

$$\frac{\sqrt{1 - 9\Lambda M^2}}{12\sqrt{3}M} < |\kappa_{\pm}|. \quad (31)$$

Observe that for  $Q = 0$ , we have

$$F'(r) = \frac{2M}{r^2} - \frac{2\Lambda r}{3} = \frac{1 - F(r)}{r} - \Lambda r$$

so that

$$F'(r_{\pm}) = \frac{1}{r_{\pm}} - \Lambda r_{\pm}.$$

Thus (31) becomes

$$\frac{\sqrt{1 - 9\Lambda M^2}}{6\sqrt{3}M} < \frac{|1 - \Lambda r_{\pm}^2|}{r_{\pm}}. \quad (32)$$

Set  $\alpha := 3\sqrt{\Lambda}M < 1$ . The footnote page 6 in [SZ97] shows that

$$r_{\pm} = \frac{1}{\sqrt{\Lambda}} \Im \left( \left( \mp \sqrt{1 - \alpha^2} + i\alpha \right)^{1/3} \right). \quad (33)$$

As  $\mp \sqrt{1 - \alpha^2} + i\alpha$  has modulus one, we can write  $r_{\pm} = \frac{\sin \theta_{\pm}}{\sqrt{\Lambda}}$  for some  $\theta_{\pm} \in ]0, \pi[$  (the roots are positive) and thus (32) reads

$$\frac{\sqrt{1 - \alpha^2}}{6\sqrt{3}M} < \frac{\cos^2 \theta_{\pm}}{\sin \theta_{\pm}} \sqrt{\Lambda}.$$

We eventually show that

$$\frac{\sqrt{1 - \alpha^2}}{2\sqrt{3}\alpha} < \frac{\cos^2 \theta_{\pm}}{\sin \theta_{\pm}}.$$

When  $\alpha \rightarrow 1^-$ , the left-hand side above goes to 0 whereas the right-hand side remains positive: this last assertion can be checked in (33) as then

$$\Im \left( \left( \mp \sqrt{1 - \alpha^2} + i\alpha \right)^{1/3} \right) = \Im \left( i^{1/3} \right) = \Im \left( e^{i\pi/6} \right) = \frac{1}{2} \neq 0.$$

## 5 Estimates for the cut-off inverse of the quadratic pencil

In this section, we show some estimates on the cut-off inverse of the quadratic pencil. We can work with  $\ell \in \mathbb{N}$  fixed but our estimates have to be uniform in  $\ell$ . Since  $(\chi p(-\bar{z} + 2sV, s)\chi)^* = \chi p(z, s)\chi$ , we can restrict ourselves to consider  $z \in \mathbb{C}$  with  $\Re z > -2s_0 \|V\|_{L^\infty}$  for some fixed  $s_0 > 0$  such that  $0 < |s| < s_0$ . In the following, we are simply denoting by  $L^2$  the space  $L^2(\mathbb{R}, dx)$ . For some real numbers  $R, C_0, C_1 > 0$  (determined by Theorem 5.1 below), we define the

- zone I as  $[-R, R] + i[-C_0, C_0]$ ,

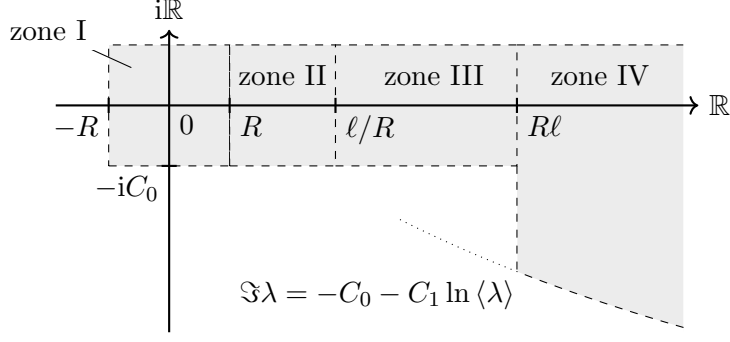


Figure 2: The four zones.

- zone II as  $[R, \ell/R] + i[-C_0, C_0]$ ,
- zone III as  $[\ell/R, R\ell] + i[-C_0, C_0]$ ,
- zone IV as  $([R\ell, +\infty[ + i]-\infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im \lambda \geq -C_0 - C_1 \ln \langle \lambda \rangle\} \cap \Omega_\kappa$

with  $\Omega_\kappa := \{\omega \in \mathbb{C} \mid \Im \omega > -\kappa\}$  (recall that  $\kappa := \min\{\kappa_-, |\kappa_+|\}$ ).

We quote here all the estimates that we are going to show in this Section in the following theorem (which is similar to [BoHa08, Thm. 2.1]):

**Theorem 5.1.** *Let  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ ,  $s \in \mathbb{R}$  and  $\varepsilon, \Omega_\kappa$  as above. If  $s$  is small enough, then the following estimates hold uniformly in  $\ell \in \mathbb{N}$ :*

1. For all  $R > 0$ ,  $C > 0$  and  $0 < C_0 < \varepsilon$ ,  $\text{Res}(p) \cap ([-R, R] + i[-C_0, C]) = \emptyset$  and the operator

$$\chi p(z, s)^{-1} \chi : L^2 \rightarrow L^2 \quad (34)$$

exists and is bounded uniformly in  $z \in [-R, R] + i[-C_0, C]$ . Moreover, we have

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \leq \|\chi p(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim \prod_{\substack{z_j \in \text{Res}(p) \\ |z_j| < 2R}} \frac{1}{|z - z_j|}. \quad (35)$$

2. There exist  $R > 0$  and  $0 < C_0 < \varepsilon$  such that there is no resonance in  $[R, \ell/R] + i[-C_0, C_0]$ . Furthermore, for all  $z \in [R, \ell/R] + i[-C_0, C_0]$ , we have

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim \frac{1}{\langle z \rangle^2}. \quad (36)$$

3. Let  $R > 0$  and  $0 < C_0 < \varepsilon$  be fixed and suppose that  $\ell \gg 0$ . The number of resonances of  $p_\ell$  in  $[\ell/R, R\ell] + i[-C_0, C_0]$  is bounded uniformly in  $\ell$  and there exists  $C > 0$  such that, for all  $z \in [\ell/R, R\ell] + i[-C_0, C_0]$ ,

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim \langle z \rangle^C \prod_{\substack{z_j \in \text{Res}(p_\ell) \\ |z - z_j| < 1}} \frac{1}{|z - z_j|}. \quad (37)$$

Furthermore, there exists  $\varepsilon > 0$  such that there is no resonance in  $[\ell/R, R\ell] + i[-\varepsilon, 0]$  and we have for all  $z \in [\ell/R, R\ell] + i[-\varepsilon, 0]$

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim \frac{\ln \langle z \rangle}{\langle z \rangle} e^{|\Im z| \ln \langle z \rangle}. \quad (38)$$



4. Let  $R \gg 0$ ,  $C_0 > 0$  and  $C_1 > 0$ . Set

$$\tilde{\Omega}_\ell := ([R\ell, +\infty[ + i] - \infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im \lambda \geq -C_0 - C_1 \ln \langle \lambda \rangle\} \cap \Omega_\kappa.$$

There is no resonance in  $\tilde{\Omega}_\ell$  and there exists  $C > 0$  such that for all  $z$  in this set,

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \leq C \langle z \rangle^{-1} e^{C|\Im z|}. \quad (39)$$

**Remark 5.2.** High frequency resonances of the zone III (i.e. resonances whose real part are of order  $\ell \gg 0$ ) are localized in Theorem 4.1.

The announced estimate in the zone I is a direct application of results of Section 3 (see Theorem 3.8). We thus show the estimates for the zones II, III and IV.

## 5.1 Estimates in the zone II

We prove part 2. of Theorem 5.1 using the complex scaling introduced in [Zw99, §4]. Observe that the zone II does not exist if  $\ell = 0$ , so that we can assume that  $\ell \geq 1$ . Let  $z \in [R, \ell/R] + i[-C_0, C_0]$  and choose  $N \in [R, \ell/R]$  such that  $z \in [N, 2N] + i[-C_0, C_0]$ . We introduce the semiclassical parameter

$$h := N^{-1}$$

and the new spectral parameter

$$\lambda := h^2 z^2 \in [1/4, 4] + i[-4C_0 h, 4C_0 h].$$

In this setting, we define the operator

$$\tilde{p}_h(\sqrt{\lambda}, s) := h^2 p_\ell(z, s) = \underbrace{-h^2 \partial_x^2 + \alpha^2 W_0(x)}_{=: Q_h} - \lambda + \underbrace{h^2 W_1(x) + 2h\sqrt{\lambda} s V(x) - h^2 s^2 V(x)^2}_{=: R_h(\lambda)} \quad (40)$$

where  $\alpha := h(\ell(\ell+1))^{1/2} \gg 2\mathcal{A} > 0$ ,  $\mathcal{A}$  as in Proposition 2.1.

We now use the ( $\alpha$ -dependent) contour  $\Gamma_\theta := \Gamma_\theta^- \cup \Gamma_\theta^+$  for  $0 < \theta < \pi/2$ , with<sup>7</sup>

$$\Gamma_\theta^\pm := \{x + i f_\theta^\pm(x, \ln(g_\infty^\pm)/\kappa_\pm) \mid x \in \mathbb{R}_\pm\}$$

where (using estimate (6) for  $W_0$ )

$$g_\infty^\pm := \lim_{x \rightarrow \pm\infty} e^{2\kappa_\pm x} W_0(x)$$

and

$$f_\theta^\pm(x, \beta) := \begin{cases} 0 & \text{if } |x| \leq \beta/2 - C_1 \\ \theta(x - \beta/2) & \text{if } |x| \geq \beta/2 + C_2 \end{cases}$$

with constants  $C_1, C_2 > 0$  as in (4.4) in [Zw99, §4] (see Figure 3 for the behaviour of  $\Gamma_\theta$ ). Define next  $L^2(\Gamma_\theta)$  and  $H^2(\Gamma_\theta)$  as the associated Lebesgue and Sobolev spaces. Using the analytic extension of  $x \mapsto r(x)$  on the set  $\Sigma := \{\eta \in \mathbb{C} \mid |\Re \eta| > \mathcal{A}\}$ , we extend  $V$ ,  $W_0$  and  $W_1$  on  $\Sigma$  (and still denote them  $V$ ,  $W_0$  and  $W_1$ ). We then define the distorted operators

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s) = \tilde{p}_h(\sqrt{\lambda}, s) \upharpoonright_{\Gamma_\theta}, \quad Q_{h,\theta} := Q_h \upharpoonright_{\Gamma_\theta}, \quad R_{h,\theta}(\lambda) = R_h(\lambda) \upharpoonright_{\Gamma_\theta}.$$

If  $q_{h,\theta}$  denotes the symbol of  $Q_{h,\theta}$ , then [Zw99, Lem. 4.3] shows that there exists  $0 < c < 1$  and  $\theta_0 > 0$  such that

$$|q_{h,\theta}(x, \xi) - \lambda| \gtrsim \theta(\langle \xi \rangle^2 + e^{\kappa_\pm x} \langle \alpha \rangle^2) \quad \pm x \geq 0$$

<sup>7</sup>The factor  $1/\kappa_\pm$  in the second argument of  $f_\theta^\pm$  comes from the fact that  $\kappa_\pm x$  corresponds to Zworski's variable  $r$ .

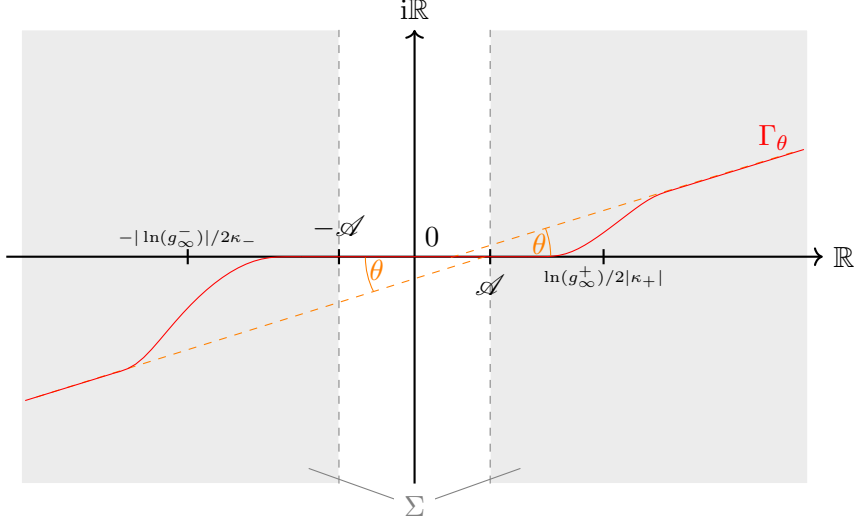


Figure 3: The contour  $\Gamma_\theta$ .

provided that  $\Re(\lambda) > 0$ ,  $\Im(\lambda) < c$  and  $0 < \theta < \theta_0$  (recall that  $\pm\kappa_\pm < 0$ ). For  $h$  small enough (that is  $R$  large enough), we can apply [Zw99, Prop. 4.1] to get

$$\|(Q_{h,\theta} - \lambda)^{-1}\|_{L^2(\Gamma_\theta) \rightarrow H^2(\Gamma_\theta)} = \mathcal{O}(\theta^{-1}).$$

In order to invert the distorted quadratic pencil, we use a Neumann series argument by showing that  $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$  (as a multiplication operator). In view of the form of  $R_{h,\theta}$  in (40) and because the extension of  $r$  is analytic, it is enough to bound  $x \mapsto r(x + if_\theta^\pm(x))$  from below and above for  $|x| > 2\mathcal{A}$ . By Lagrange inversion formula (59), we can write

$$|r(x + if_\theta^\pm(x)) - r_\pm| \leq \sum_{k=1}^{+\infty} c_k \frac{e^{2k\kappa_\pm x}}{k!}$$

for some coefficients  $c_k > 0$  (recall from Subsection 2.2 that  $2\kappa_\pm = \frac{\Lambda}{3A_\pm r_\pm^2}$ ), and the series converges (because (59) converges uniformly when  $|\Re x| > \mathcal{A}$ ). Since the sum is decreasing with respect to  $x$ , we deduce  $r < +\infty$  on  $\Gamma_\theta$ . On the other hand,

$$|r - r_\pm| = C(r - r_n)^{-\frac{A_n r_n^2}{A_\pm r_\pm^2}} (r - r_c)^{-\frac{A_c r_c^2}{A_\pm r_\pm^2}} |r - r_\mp|^{-\frac{A_\mp r_\mp^2}{A_\pm r_\pm^2}} e^{2\kappa_\pm x}$$

with  $C \in \mathbb{R}$ . Since no terms on the right-hand side can blow up when restricted on  $\Gamma_\theta$  and since the exponential goes to zero when  $|x| \rightarrow +\infty$ , it follows that  $r \rightarrow r_\pm > 0$  as  $x \rightarrow \pm\infty$ . We therefore conclude that the restriction of  $r$  on  $\Gamma_\theta \cap D(0, R_0)^\mathbb{C}$  is bounded from below and above for  $R_0 \gg 0$ , giving  $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$ . Thus,

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} = (1 + (Q_{h,\theta} - \lambda)^{-1} R_{h,\theta}(\lambda))^{-1} (Q_{h,\theta} - \lambda)^{-1}.$$

We finally choose  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$  and increase if necessary the value of the number  $\mathcal{A}$  so that  $\text{Supp} \chi \subset [-\mathcal{A}, \mathcal{A}]$ . From [SZ91, Lem. 3.5], we have in the  $L^2$  sense

$$\chi \tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} \chi = \chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi$$

whence

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} = h^2 \|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim \langle z \rangle^{-2}.$$

## 5.2 Estimates in the zone III

We turn to the proof of part 3. of Theorem 5.1. We define the semiclassical parameter

$$h := (\ell(\ell + 1))^{-1/2}$$

with again  $\ell > 0$  since the zone does not exist for  $\ell = 0$ . For  $z \in [\ell/R, R\ell] + i[-C_0, C_0]$ , we define a new spectral parameter

$$\lambda := h^2 z^2 \in \left[ \frac{1}{3R^2}, R^2 \right] + i[-\sqrt{2} C_0 R h, \sqrt{2} C_0 R h] \subset [a, b] + i[-ch, ch]$$

for some  $0 < a < b$  and  $c > 0$ . Finally, we set

$$\tilde{P}_h := h^2 P_\ell = -h^2 \partial_x^2 + W_0 + h^2 W_1, \quad \tilde{p}_h(\sqrt{\lambda}, s) := \tilde{P}_h - (\sqrt{\lambda} - hsV)^2$$

and write  $\tilde{P}_\theta$  and  $\tilde{p}_\theta$  for the corresponding distorted operators on the contour  $\Gamma_\theta$  as we did in the paragraph 5.1. We are still using the subscript  $L^2$  when we work with the distorted operators.

As  $W_0$  admits a non-degenerate maximum at  $x = 0$  (see Section 2, Figure 1),  $(x, \xi) = (0, 0)$  is a trivial solution of the Hamilton equations associated to the principal symbol of  $\tilde{P}_h$ :

$$\begin{cases} \dot{x} &= 2\xi \\ \dot{\xi} &= -W'_0(x) \end{cases}.$$

Therefore the energy level  $\mathfrak{E}_0 := W_0(0)$  is trapping. For this reason, the zone III is called the trapping zone.

We first show an adaptation of [BM04, Lem. 6.5] to our setting.

**Proposition 5.3.** *For  $\theta = Nh$  with  $N > 0$  large enough and  $s \in \mathbb{R}$  sufficiently small, there exist  $C \equiv C(N) > 0$  and  $\varepsilon > 0$  such that, for all  $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$  and  $|\lambda - E| \leq \varepsilon\theta/2$ , it holds*

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \rightarrow L^2} = \mathcal{O}(h^{-C}) \prod_{\substack{\lambda_j \in \text{Res}(\tilde{p}) \\ |\lambda - \lambda_j| < \varepsilon\theta}} \frac{h}{|\lambda - \lambda_j|}.$$

*Proof.* The announced estimate is known for the resolvent  $(\tilde{P}_\theta - \lambda)^{-1} = \tilde{p}_\theta(\sqrt{\lambda}, 0)^{-1}$  corresponding to the case  $s = 0$ . The argument can be found in [TZ98] which uses techniques developed in [S96], and the authors of [BM04] adapted it for the one dimensional case of a non degenerate trapping energy level  $\mathfrak{E}_0$ . More precisely, for  $\theta = Nh$  with  $N \gg 0$  large enough, one can construct a bounded operator  $\tilde{K} \in \mathcal{L}(L^2, L^2)$  (see (6.15) in [BM04]) satisfying the following properties:

- (i)  $\|\tilde{K}\|_{L^2 \rightarrow L^2} = \mathcal{O}(1)$ ,
- (ii)  $r := \text{rank } \tilde{K} \leq \mathcal{O}(\theta h^{-1} \ln(1/\theta))$ ,
- (iii) for  $h$  small enough, there exists  $\varepsilon > 0$  such that, for all  $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$  and  $\lambda \in [E - \varepsilon\theta, E + \varepsilon\theta]$ ,

$$\|(\tilde{P}_\theta - i\theta\tilde{K} - \lambda)^{-1}\|_{L^2 \rightarrow \mathcal{D}} \leq \mathcal{O}(\theta^{-1}), \quad \mathcal{D} := \mathcal{D}(\tilde{P}_\theta).$$

In [SZ91, Lem. 3.2], it is shown that  $\tilde{P}_\theta - \lambda$  is a Fredholm operator from its domain  $\mathcal{D}$  to  $L^2$ , so we can construct a well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{P}_\theta - \lambda & R_- \\ R_+ & 0_{\mathbb{C}^r \rightarrow \mathbb{C}^r} \end{pmatrix} : \mathcal{D} \oplus \mathbb{C}^r \rightarrow L^2 \oplus \mathbb{C}^r \quad (41)$$

where  $R_-$  and  $R_+$  are constructed with  $\tilde{P}_\theta - i\theta\tilde{K} - \lambda$  (see [S96], page 401, below (6.12) for the construction).

Now consider  $s \neq 0$ . If  $s$  is small enough,  $(\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}$  is invertible by pseudodifferential calculus<sup>8</sup> as for the case  $s = 0$ . By the resolvent identity, one can show that

$$\|(\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \rightarrow \mathcal{D}} \leq \mathcal{O}(\theta^{-1}) + \mathcal{O}(h|s|)\|(\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \rightarrow \mathcal{D}} \mathcal{O}(\theta^{-1})$$

since  $|\lambda| \leq h(|\Re z| + |\Im z|) \leq \mathcal{O}(1) + \mathcal{O}(h)$ . Hence for  $s$  sufficiently small, we have

$$\|(\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \rightarrow \mathcal{D}} \leq \mathcal{O}(\theta^{-1}) \quad \lambda \in [E - \varepsilon\theta, E + \varepsilon\theta].$$

Because the quadratic pencil remains a Fredholm operator provided that  $\|hsV\|_{L^\infty}$  is sufficiently small<sup>9</sup>, we can write a new well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{p}_\theta(\sqrt{\lambda}, s) & R_- \\ R_+ & 0_{\mathbb{C}^r \rightarrow \mathbb{C}^r} \end{pmatrix} : \mathcal{D} \oplus \mathbb{C}^r \rightarrow L^2 \oplus \mathbb{C}^r$$

where this time  $R_+$  and  $R_-$  are constructed with  $\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2$ . If we set

$$\mathcal{E}(\lambda) := \mathcal{P}(\lambda)^{-1} = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_0(\lambda) \end{pmatrix} : L^2 \oplus \mathbb{C}^r \rightarrow \mathcal{D} \oplus \mathbb{C}^r,$$

then the relations  $\mathcal{E}(\lambda)\mathcal{P}(\lambda) = \mathcal{P}(\lambda)\mathcal{E}(\lambda) = \text{Id}$  as well as the following estimate (which is a consequence of properties (i) and (iii) above)

$$\|(\tilde{P}_\theta - i\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)\|_{L^2 \rightarrow L^2} = \mathcal{O}(1)$$

imply as in [BM04] that  $\|E(\lambda)\|_{L^2 \rightarrow \mathcal{D}}, \|E_-(\lambda)\|_{L^2 \rightarrow \mathbb{C}^r} = \mathcal{O}(\theta^{-1})$  and  $\|E_+(\lambda)\|_{\mathbb{C}^r \rightarrow \mathcal{D}}, \|E_0(\lambda)\|_{\mathbb{C}^r \rightarrow \mathbb{C}^r} = \mathcal{O}(1)$ . Applying formula (8.11) in [S96], we obtain

$$(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1} = E(\lambda) - E_+(\lambda)E_0(\lambda)^{-1}E_-(\lambda)$$

which implies

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \rightarrow \mathcal{D}} = \mathcal{O}(\theta^{-1})(1 + \|E_0(\lambda)^{-1}\|_{\mathbb{C}^r \rightarrow \mathbb{C}^r})$$

as in [BM04, Lem. 6.5], and we then follow the end of its proof to conclude.  $\square$

We can now follow the arguments below [BoHa08, Lem. 2.2]. The set of pseudo-poles (2.28) and the injective map (2.29) in this reference exist in our setting by Theorem 4.1 (but are quite different). This implies that there is no resonance in  $\Omega(h) := [a/2, 2b] + i[-\varepsilon h, ch]$  provided that  $h$  and  $s$  are small enough. As a result, (37) holds true. As for the estimate (38), we use Burq's Lemma:

**Lemma 5.4** (Lemma 2.3 in [BoHa08]). *Suppose that  $f(\lambda, h)$  is a family of holomorphic functions defined for  $0 < h < 1$  in a neighbourhood of  $\Omega(h) := [a/2, 2b] + i[-ch, ch]$  with  $0 < a < b$  and  $c > 0$ , such that*

$$|f(\lambda, h)| \lesssim \begin{cases} h^{-C'} & \text{in } \Omega(h) \\ \frac{1}{|\Im \lambda|} & \text{in } \Omega(h) \cap \mathbb{C}^+ \end{cases}$$

for some  $C' > 0$ . Then there exists  $h_0, C > 0$  such that, for any  $0 < h < h_0$  and any  $\lambda \in [a, b] + i[-ch, 0]$ ,

$$|f(\lambda, h)| \leq C \frac{|\ln h|}{h} e^{C|\Im \lambda| |\ln h|/h}.$$

We apply this result to the function  $f(\lambda, h) := \|\chi(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\chi\|$ , observing that for all  $\lambda \in \Omega(h) \cap \mathbb{C}^+$  the resolvent identity gives

$$\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\| \leq \frac{1}{|\Im \lambda|} + \|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\| \frac{\mathcal{O}(h|s|)}{|\Im \lambda|} \lesssim \frac{1}{|\Im \lambda|}$$

because  $\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\|$  is uniformly bounded on this set for  $h$  and  $s$  small enough.

<sup>8</sup>See the definition of the operator  $K$  at the beginning of the proof of [BM04, Lem. 6.5].

<sup>9</sup>Recall that the set of Fredholm operators in  $\mathcal{L}(\mathcal{D}, L^2)$  is open for the norm topology.

### 5.3 Estimates in the zone IV

This last paragraph is devoted to the proof of part 4. of Theorem 5.1. For  $z \in ([R\ell, +\infty[ + i] - \infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im \lambda \geq -C_0 - C_1 \ln \langle \lambda \rangle\}$ , there exists a number  $N > R\ell > 0$  such that  $z \in [N, 2N] + i[-C \ln N, C_0]$ . We introduce the semiclassical parameters

$$h := \frac{1}{N}, \quad \mu := \ell(\ell + 1)h^2, \quad \nu := h^2.$$

Observe that these parameters are very small when  $N \gg 0$ . Moreover, we can consider that  $h \leq 1$  even if  $\ell = 0$ , simply by taking  $R \geq 1$  in the zone I if it was not the case ( $R$  as in Theorem 5.1). We then define a new spectral parameter

$$\lambda := z^2 h^2 \in [1, 2] + i[Ch \ln h, C_0 h] \subset [a, b] + i[-ch |\ln h|, ch]$$

where  $0 < a \leq 1 < 2 \leq b < +\infty$  and  $\max\{C, C_0\} < c < +\infty$  (observe that  $a$  and  $b$  do not depend on  $h$ ). Let  $J := [a, b]$  and set

$$J^+ := \{\eta \in \mathbb{C}^+ \mid \Re(\eta) \in J\}.$$

Define then

$$\tilde{P}_h := h^2 P_\ell = -h^2 \partial_x^2 + \mu W_0 + \nu W_1, \quad \tilde{p}_h(\sqrt{\lambda}, s) := h^2 p_\ell(z, s) = \tilde{P}_h - (\sqrt{\lambda} - hsV)^2.$$

**Semiclassical limiting absorption principle for the quadratic pencil.** As in [BoHa08], we first get a control until the real line by using a semiclassical limiting absorption principle for the semiclassical quadratic pencil. The appendix C provides a proof, close to the idea developed by Gérard [Ge08], of such a result for a class of perturbed resolvents, so we only have to check if the required abstract assumptions are satisfied.

Introduce the generator of dilations  $A := -ih(x\partial_x + \partial_x x)$  with domain  $\mathcal{D}(A) := \{u \in L^2 \mid Au \in L^2\}$ . We then pick  $\rho \in C_c^\infty(\mathbb{R}, [0, 1])$  such that  $\text{Supp } \rho \subset [a/3, 3b]$  and  $\rho \equiv 1$  on  $I := [a/2, 2b]$ , and we define  $\mathcal{A}$  as the closure of the operator  $\rho(P)A\rho(P)$ . In this setting,  $\rho(P)A\rho(P)$  is well-defined on  $\mathcal{D}(A)$ ,  $\mathcal{A}$  is self-adjoint and we have  $P \in \mathcal{C}^2(\mathcal{A})$  (cf. [BoHa08, §2.4]) so that (I) holds. A direct computation shows that

$$ih^{-1}[P, A] = 4P - 4\mu W_0 - 4\nu W_1 - 2\mu x W_0' - 2\nu x W_1'$$

so that, for  $\mu$  and  $\nu$  sufficiently small, we get the Mourre estimate (M) (uniform in  $\mu, \nu$ )

$$\mathbf{1}_I(P)[P, iA]\mathbf{1}_I(P) \geq ah\mathbf{1}_I(P).$$

Since  $V \in \mathcal{B}(\mathcal{D}(P), L^2)$  it is clear that  $V \in L_{loc}^\infty(\tilde{P}_h)$ . Moreover, assumption (C) is fulfilled for  $f(z, B) := (\sqrt{z} - sB)^2$ .

It remains to show that assumption (A) is satisfied for  $B = hV$ . Observe that this abstract assumption is particularly well adapted to semiclassical pseudodifferential calculus framework, especially the commutator estimate which provides the supplementary term  $h$ . In [Ha01], it is shown that  $\mathcal{A} \in \Psi^{-\infty, 1}$  ( $\mathcal{A}$  is the operator  $c_{\tilde{\chi}}$  in [Ha01], see above Lemma 3.3). We will use it to show the following result:

**Lemma 5.5.** *Let  $\sigma \in [0, 1]$ . Then  $V \in \mathcal{B}(\mathcal{D}(\langle \mathcal{A} \rangle^\sigma))$  and  $[V, \chi(\tilde{P}_h)] \in h\mathcal{B}(\mathcal{D}(\langle \mathcal{A} \rangle^\sigma))$ .*

*Proof.* Let  $\Omega := [0, 1] + i\mathbb{R}$  and let  $z \in \Omega$ . On  $\mathcal{D}(\langle \mathcal{A} \rangle^2) \times \mathcal{D}(\langle \mathcal{A} \rangle^2)$ , we define the sesquilinear form

$$Q_z(\varphi, \psi) := \langle V \langle \mathcal{A} \rangle^{-2z} \varphi, \langle \mathcal{A} \rangle^{2z} \psi \rangle \quad \forall \varphi, \psi \in \mathcal{D}(\langle \mathcal{A} \rangle^2).$$

By functional calculus,  $Q_z$  is well-defined and analytic in  $z \in \Omega$ . When  $z \in \{0\} + i\mathbb{R}$ ,  $|(1 + \lambda^2)^{z/2}| = 1$  for all  $\lambda \in \mathbb{R}$  so that functional calculus first applied to  $\langle \mathcal{A} \rangle^{2z}$  and then to  $\langle \mathcal{A} \rangle^{-2z}$  gives

$$\begin{aligned} |Q_z(\varphi, \psi)| &\leq |\langle V \langle \mathcal{A} \rangle^{-2z} \varphi, \psi \rangle| \\ &= |\langle \langle \mathcal{A} \rangle^{-2z} \varphi, V \psi \rangle| \\ &\leq |\langle \varphi, V \psi \rangle| \\ &\leq \|V\|_{L^\infty} \|\varphi\|_{L^2} \|\psi\|_{L^2}. \end{aligned}$$

When  $z = 1$ , pseudodifferential calculus shows that  $\langle \mathcal{A} \rangle^2 V \langle \mathcal{A} \rangle^{-2} \in \Psi^{0,0}$ , so that for all  $z \in \{1\} + i\mathbb{R}$  (using again functional calculus for  $\langle \mathcal{A} \rangle^{\pm 2i\Im z}$ ),

$$\begin{aligned} |Q_z(\varphi, \psi)| &\leq |\langle \langle \mathcal{A} \rangle^2 V \langle \mathcal{A} \rangle^{-2} \langle \mathcal{A} \rangle^{-2i\Im z} \varphi, \psi \rangle| \\ &\leq \| \langle \mathcal{A} \rangle^2 V \langle \mathcal{A} \rangle^{-2} \|_{L^2 \rightarrow L^2} \| \langle \mathcal{A} \rangle^{-2i\Im z} \varphi \|_{L^2} \| \psi \|_{L^2} \\ &\leq \| \langle \mathcal{A} \rangle^2 V \langle \mathcal{A} \rangle^{-2} \|_{L^2 \rightarrow L^2} \| \varphi \|_{L^2} \| \psi \|_{L^2}. \end{aligned}$$

By the maximum principle, there exists a constant  $C > 0$  such that  $Q_z$  is bounded by  $C$  for all  $0 \leq \Re z \leq 1$ . In particular, we can extend  $Q_{\sigma/2}$  on  $L^2 \times \mathcal{D}(\langle \mathcal{A} \rangle^2)$  as a bounded sesquilinear form and for  $\sigma \in [0, 1]$  and  $\varphi \in L^2$ , we have

$$|Q_{\sigma/2}(\varphi, \psi)| \leq C \| \varphi \|_{L^2} \| \psi \|_{L^2}.$$

This means that the map  $\mathcal{D}(\langle \mathcal{A} \rangle^\sigma) \ni \psi \mapsto \langle V \langle \mathcal{A} \rangle^{-\sigma} \varphi, \langle \mathcal{A} \rangle^\sigma \psi \rangle$  is continuous. By definition of the adjoint operator and because  $\langle \mathcal{A} \rangle^\sigma$  is self-adjoint, this implies that  $V \langle \mathcal{A} \rangle^{-\sigma} \varphi \in \mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$  for all  $\varphi \in L^2$ .

Consider now the sesquilinear form

$$\tilde{Q}_z(\varphi, \psi) := \langle [V, \chi(\tilde{P}_h)] \langle \mathcal{A} \rangle^{-2z} \varphi, \langle \mathcal{A} \rangle^{2z} \psi \rangle \quad \forall \varphi, \psi \in \mathcal{D}(\langle \mathcal{A} \rangle^2).$$

By semiclassical pseudodifferential calculus, we have (see *e.g.* (4.4.19) in [Zw12])

$$\begin{aligned} [V, \chi(\tilde{P}_h)] &= \frac{h}{i} \{V(x), \chi(\xi^2 + \mu W_0(x) + \nu W_1(x))\}^w + h^3 \Psi^{-\infty, 0} \\ &= h \Psi^{-\infty, -\infty} + h^3 \Psi^{-\infty, 0} \end{aligned}$$

because  $V(x) \in \Psi^{0,0}$ ,  $V'(x) \in \Psi^{0,-\infty}$  and  $\chi(\tilde{P}_h) \in \Psi^{-\infty,0}$ . Despite the fact that the error term above looks less regular than the main term, it is in fact more regular as it can be shown using expansion (4.4.15) in [Zw12] (but we will not need such a regularity). Now we can proceed as above with  $Q_z$  and  $V$  to conclude.  $\square$

Now that all assumptions in appendix C have been checked, we can use Theorem C.1 as well as the fact that  $\| \langle x \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma \| \lesssim 1$  for all  $\sigma \leq 1$ <sup>10</sup>: for  $\sigma \in ]1/2, 1]$  and  $h$  small enough, we have uniformly in  $\mu, \nu$

$$\sup_{\lambda \in J^+} \left\| \langle x \rangle^{-\sigma} \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \langle x \rangle^{-\sigma} \right\| \leq \| \langle x \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma \| \left( \sup_{\lambda \in J^+} \left\| \langle \mathcal{A} \rangle^{-\sigma} \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \langle \mathcal{A} \rangle^{-\sigma} \right\| \right) \| \langle \mathcal{A} \rangle^\sigma \langle x \rangle^{-\sigma} \| \lesssim h^{-1}.$$

**Estimates below the real axis.** Next, we can use the work of Martinez [Ma02] to get a bound under the real line. Indeed, Section 4 of the last reference applies in our setting because  $\tilde{p}_h(\sqrt{\lambda}, s)$  is a differential operator (so that [Ma02, Prop. 3.1 & Cor. 3.2] apply) and because  $(\lambda - hsV(x))^2 \in [\lambda - \delta, \lambda + \delta] + i[ch \ln h, 0]$  for all  $\lambda$  in the zone IV and all  $x \in \mathbb{R}$  if  $s$  is small enough (so that the estimate (4.6) in [Ma02] still holds). It follows that equation (4.13) holds with  $\tilde{p}_h(\sqrt{\lambda}, s)$  instead of  $P_\theta - \rho$ <sup>11</sup>. In our setting, this reads

$$\| \chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi \| \leq Ch^{-C} \tag{42}$$

for some  $C > 0$ .

To get (39), we reproduce the argument at the end of the proof of [BoHa08, Lem. 2.4]. Choose  $f$  holomorphic satisfying the following conditions:

$$\begin{cases} |f| < 1 & \text{for } \lambda \in [a/2, 2b] + i[ch \ln h, 0], \\ |f| \geq 1 & \text{for } \lambda \in [a, b] + i[ch \ln h, 0], \\ |f| \leq h^C & \text{for } \lambda \in [a/2, 2b] \setminus [2a/3, 3b/2] + i[ch \ln h, 0] \end{cases}$$

<sup>10</sup>We show it using the sesquilinear form  $(\varphi, \psi) \mapsto \langle \langle x \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma \varphi, \psi \rangle$  first well-defined on  $\mathcal{D}(\mathcal{A}^2) \times \mathcal{D}(\mathcal{A}^2)$  because  $\langle x \rangle^{-2} \in \Psi^{0,-2}$ , and then extended to  $L^2 \times L^2$  by maximum principle.

<sup>11</sup>We can in fact insert any pseudodifferential operator here provided that hypotheses in [Ma02, §2] are verified.

where  $C > 0$  is the constant in (42). Since  $f$  is holomorphic, the function

$$g(\lambda) := \ln \|\chi \tilde{p}(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \rightarrow L^2} + \ln |f(\lambda)| + \frac{C}{ch} \Im \lambda$$

is subharmonic. We can check that  $g(\lambda) \lesssim \ln(h^{-1})$  on the boundary of  $[a/2, 2b] + i[ch \ln h, 0]$ . By the maximum principle, this estimate holds for all  $\lambda \in [a/2, 2b] + i[ch \ln h, 0]$ , whence

$$\|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \lesssim h^{-1} e^{\frac{C}{ch} |\Im \lambda|}.$$

The desired estimate (39) then follows.

## 6 Proof of Theorem 4.2

We prove in this section Theorem 4.2. The resonance expansion (30) follows from the theory of resonances as presented in [BoHa08, §3], and we can follow the proof of this paper. We only have to adapt [BoHa08, Prop. 3.1] to get an estimate for the resolvent  $\hat{R}_{\chi, \ell}(z)$ :

**Proposition 6.1.** *Let  $\ell \in \mathbb{N}$  and let  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ . There exists  $\tilde{\chi} \in C_c^\infty(\mathbb{R}, \mathbb{R})$  satisfying  $\tilde{\chi}\chi = \chi$  such that for all  $z \in \mathbb{C} \setminus \text{Res}(p_\ell)$ , the cut-off resolvent  $\chi(\hat{K}_\ell - z)^{-1}\chi$  is a bounded operator on  $\mathcal{E}_\ell$  and satisfies uniformly in  $\ell$*

$$\|\hat{R}_{\chi, \ell}(z)\|_{\mathcal{E}_\ell \rightarrow \mathcal{E}_\ell} \lesssim \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2}.$$

*Proof.* Since the norms  $\|\cdot\|_{\mathcal{E}_\ell}$  and  $\|\cdot\|_{\dot{\mathcal{E}}_\ell}$  are locally equivalent thanks to the Hardy type estimate  $\|\chi \cdot\|_{L^2} \lesssim \|P_\ell^{1/2} \cdot\|_{L^2}$  uniformly in  $\ell$  (cf. [GGH17, Lem. 9.5]), we can work on  $(\mathcal{E}_\ell, \|\cdot\|_{\mathcal{E}_\ell})$ . For  $(u_0, u_1) \in \mathcal{E}_\ell$ , we have

$$\hat{R}_{\chi, \ell}(z) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \chi p_\ell(z, s)^{-1} \chi ((z - sV)u_0 + u_1) \\ \chi(1 + (z - sV)p_\ell(z, s)^{-1}(z - sV))\chi u_0 + (z - sV)\chi p_\ell(z, s)^{-1} \chi u_1 \end{pmatrix} \quad (43)$$

and since it holds

$$\|(z - sV)\chi p_\ell(z, s)^{-1} \chi u_1\|_{L^2} \leq (1 + \|s\| \|V\|_{L^\infty}) \langle z \rangle \|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \|u_1\|_{L^2},$$

the  $\mathcal{E}_\ell$ -norm of (43) can be bounded if we show the following estimates:

$$\|P_\ell^{1/2} \chi p_\ell(z, s)^{-1} \chi (z - sV)u_0\|_{L^2} \leq C_a \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|P_\ell^{1/2} u_0\|_{L^2}, \quad (44a)$$

$$\|P_\ell^{1/2} \chi p_\ell(z, s)^{-1} \chi u_1\|_{L^2} \leq C_b \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|u_1\|_{L^2}, \quad (44b)$$

$$\|\chi(1 + (z - sV)p_\ell(z, s)^{-1}(z - sV))\chi u_0\|_{L^2} \leq C_c \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|P_\ell^{1/2} u_0\|_{L^2}. \quad (44c)$$

We use complex interpolation.

**Estimate (44a).** Let us define

$$\Lambda_a(\theta) := \langle z \rangle^{-2\theta} P_\ell^\theta \chi p_\ell(z, s)^{-1} \chi P_\ell^{-\theta}.$$

By functional calculus,  $\Lambda_a$  is analytic from  $[0, 1] + i\mathbb{R}$  to  $\mathcal{L}(L^2, L^2)$  because  $P_\ell > 0$  and  $\langle z \rangle > 0$ . We want to show that

$$\|\Lambda_a(1/2)u\|_{L^2} \leq C_a \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|u\|_{L^2} \quad \forall u \in L^2$$



for some  $C_a > 0$ . By the maximum principle, it is sufficient to bound  $\Lambda_a(\theta)$  for  $\theta \in \{0, 1\} + i\mathbb{R}$ , and since  $P_\ell$  is self-adjoint, it is sufficient by functional calculus to restrict ourselves to  $\Im\theta = 0$ . If  $\theta = 0$ , there is nothing to do. Now for  $\theta = 1$ , we put  $u = (z - sV)u_0$  and try to show that

$$\|P_\ell \chi p_\ell(z, s)^{-1} \chi u\|_{L^2 \rightarrow L^2} \leq C_a \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|P_\ell u\|_{L^2}. \quad (45)$$

Write

$$P_\ell \chi p_\ell(z, s)^{-1} \chi = \underbrace{[P_\ell, \chi] p_\ell(z, s)^{-1} \chi}_{=: A} + \underbrace{\chi P_\ell p_\ell(z, s)^{-1} \chi}_{=: B}. \quad (46)$$

We first deal with  $A$ . Pick  $z_0 \in \rho(\hat{K}_\ell) \cap \mathbb{C}^+$  so that  $p_\ell(z_0, s)^{-1}$  exists (cf. (15)). Then

$$[P_\ell, \chi] p_\ell(z, s)^{-1} = p_\ell(z_0, s)^{-1} [p_\ell(z_0, s), [P_\ell, \chi]] p_\ell(z, s)^{-1} + p_\ell(z_0, s)^{-1} [P_\ell, \chi] p_\ell(z_0, s) p_\ell(z, s)^{-1}$$

with

$$\begin{aligned} [P_\ell, \chi] &= -\chi \partial_x - \chi', \\ [p_\ell(z_0, s), [P_\ell, \chi]] &= 2\chi' \partial_x^2 + (\chi' + \chi'') \partial_x + 2z_0 s V' \chi - 2s^2 V V' \chi. \end{aligned}$$

By pseudodifferential calculus, we get:

$$p_\ell(z_0, s)^{-1} \in \Psi^{-2,0}, \quad [P_\ell, \chi] \in \Psi^{1,-\infty}, \quad [p_\ell(z_0, s), [P_\ell, \chi]] \in \Psi^{2,-\infty}.$$

On the other hand, we have

$$\begin{aligned} p_\ell(z_0, s) p_\ell(z, s)^{-1} &= (p_\ell(z, s) + (z^2 - z_0^2) - 2(z - z_0)sV) p_\ell(z, s)^{-1} \\ &= 1 + p_\ell(z, s)^{-1} (z^2 - z_0^2) - 2(z - z_0)sV p_\ell(z, s)^{-1} \\ &= p_\ell(z, s)^{-1} (P_\ell - (z_0^2 - 2zsV + s^2V^2)) - 2(z - z_0)sV p_\ell(z, s)^{-1}. \end{aligned} \quad (47)$$

Using the identity

$$\chi p_\ell(z, s)^{-1} P_\ell \chi = \chi p_\ell(z, s)^{-1} [P_\ell, \chi] + \chi p_\ell(z, s)^{-1} \chi P_\ell$$

and the uniform bound in  $\ell$

$$\|\chi' u'\|_{L^2} \lesssim \|\chi_1 u\|_{L^2} + \|\chi_2 u''\|_{L^2} \quad \chi_j \in C_c^\infty(\mathbb{R}, \mathbb{R}), \quad \text{Supp } \chi_j = \text{Supp } \chi, \quad (48)$$

we obtain from (46)

$$\|Au\|_{L^2} \leq \tilde{C}_a \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|u\|_{L^2} \quad (49)$$

where the constant  $\tilde{C}_a$  only depends on  $z_0, s, V, V', \chi, \chi', \chi'', \chi_1$  and  $\chi_2$ .

We now turn to  $B$ . Using again (47), we see that

$$\begin{aligned} \|\chi P_\ell p_\ell(z, s)^{-1} \chi u\|_{L^2} &\leq \|\chi p_\ell(z_0, s) p_\ell(z, s)^{-1} \chi u\|_{L^2} \\ &\quad + \|\chi (z_0^2 - 2z_0 sV + s^2 V^2) p_\ell(z, s)^{-1} \chi u\|_{L^2} \\ &\leq \|\chi p_\ell(z, s)^{-1} (P_\ell - (z_0^2 - 2zsV + s^2 V^2)) \chi u\|_{L^2} + 2(|z| + |z_0|) \|s\| \|V\|_{L^\infty} \|\chi p_\ell(z, s)^{-1} \chi u\|_{L^2} \\ &\quad + (|z_0|^2 + 2|z_0| \|s\| \|V\|_{L^\infty} + s^2 \|V\|_{L^\infty}^2) \|\chi p_\ell(z, s)^{-1} \chi u\|_{L^2} \\ &\leq \|\chi p_\ell(z, s)^{-1} P_\ell \chi v\|_{L^2} \\ &\quad + 2 \underbrace{\langle |z_0| + 2|s| \|V\|_{L^\infty} \rangle^2}_{=: \tilde{C}_a} \langle z \rangle \|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2} \|u\|_{L^2}. \end{aligned}$$

Commuting  $P_\ell$  with  $\chi$  and using (48), we get (45) with  $C_a = \max\{\tilde{C}_a, 1 + \tilde{C}_a\}$ .

**Estimate (44b).** Let us define

$$\Lambda_b(\theta) = \langle z \rangle^{-2\theta} P_\ell^\theta \chi p_\ell(z, s)^{-1} \chi \quad \theta \in [0, 1] + i\mathbb{R}.$$

$\Lambda_b$  is analytic from  $[0, 1] + i\mathbb{R}$  to  $\mathcal{L}(L^2, L^2)$ . As the above estimate, it is sufficient to show a bound on  $\Lambda_b(1)$ , the imaginary part of  $\theta$  playing no role and the case  $\theta = 0$  being trivial. We get (44b) if we show that

$$\|P_\ell \chi p_\ell(z, s)^{-1} \chi u\|_{L^2} \leq C_b \langle z \rangle^2 \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2} \|u\|_{L^2} \quad \forall u \in L^2 \quad (50)$$

for some  $C_b > 0$ . Using the identity (46) and the estimate (49), we obtain

$$\|P_\ell \chi p_\ell(z, s)^{-1} \chi u\|_{L^2 \rightarrow L^2} \leq \tilde{C}_a \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|u\|_{L^2} + \|\chi p_\ell(z, s)^{-1} P_\ell \chi u\|_{L^2}$$

but this time we ask for the  $L^2$  norm of  $u$ . Hence, we use that

$$p_\ell(z, s)^{-1} P_\ell = 1 + p_\ell(z, s)^{-1} (z - sV)^2$$

which yields (50) with  $C_b = \max\{\tilde{C}_a, 2\langle s \|V\|_{L^\infty} \rangle^2\}$ .

**Estimate (44c).** Let us define

$$\Lambda_c(\theta) := \langle z \rangle^{2(\theta-1)} \chi (1 + (z - (sV)^{2(1-\theta)}) p_\ell(z, s)^{-1} (z - 2^{2\theta-1} sV)) \chi P_\ell^{-\theta}.$$

Once again,  $\Lambda_c$  is analytic from  $[0, 1] + i\mathbb{R}$  to  $\mathcal{L}(L^2, L^2)$  and (dropping the imaginary part)

$$\|\Lambda_c(0)\|_{L^2 \rightarrow L^2} \leq (2 + \|s\| \|V\|_{L^\infty})^3 \|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \rightarrow L^2}.$$

We then get a bound on  $\Lambda_c(1)$ : we prove

$$\|\chi (1 + z p_\ell(z, s)^{-1} (z - 2sV)) \chi u\|_{L^2} \leq C_c \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|P_\ell u\|_{L^2} \quad \forall u \in L^2.$$

We have

$$\|\chi (1 + p_\ell(z, s)^{-1} z (z - 2sV)) \chi u\|_{L^2} \leq \|\chi (1 + p_\ell(z, s)^{-1} (z - sV)^2) \chi u\|_{L^2} + \|\chi p_\ell(z, s)^{-1} s^2 V^2 \chi u\|_{L^2}$$

and

$$\chi (1 + p_\ell(z, s)^{-1} (z - sV)^2) \chi = \chi p_\ell(z, s)^{-1} P_\ell \chi.$$

Commuting  $P_\ell$  with  $\chi$  and using (48) gives us

$$\|\chi p_\ell(z, s)^{-1} z (z - 2sV) \chi v\|_{L^2} \leq C_c \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \rightarrow L^2} \|\langle P_\ell \rangle v\|_{L^2}$$

with  $C_c = \max\{(1 + \|s\| \|V\|_{L^\infty})^3, \|\chi\|_{L^\infty}, \|\chi_1\|_{L^\infty} + \|\chi_2\|_{L^\infty}\}$ . □

The proof is now the same as in [BoHa08, §3.2]. For  $\nu > 0$  fixed and for  $\ell \in \mathbb{N}$ , we define  $L_\nu^2(\mathbb{R}, \dot{\mathcal{E}}_\ell)$  as the class of functions  $t \mapsto v(t)$  with values in  $\dot{\mathcal{E}}_\ell$  such that  $t \mapsto e^{-\nu t} v(t) \in L^2(\mathbb{R}, \dot{\mathcal{E}}_\ell)$ . For  $u \in \dot{\mathcal{E}}_\ell$ , the componentwise defined function

$$v(t) = \begin{cases} e^{-it\hat{K}_\ell} u & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

is in  $L_\nu^2(\mathbb{R}, \dot{\mathcal{E}}_\ell)$  if  $\nu$  is sufficiently large and thus

$$\tilde{v}(z) = \int_0^{+\infty} e^{izt} v(t) dt \quad (51)$$

is well-defined as soon as  $\Im z \geq \nu$ . For all  $t \geq 0$ , we have the inversion formula

$$v(t) = \frac{1}{2\pi} \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-izt} \tilde{v}(z) dz \quad (52)$$

that is

$$e^{-it\hat{K}_\ell} u = \frac{1}{2\pi i} \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-izt} (\hat{K}_\ell - z)^{-1} u dz$$

in the  $L^2_\nu(\mathbb{R}, \dot{\mathcal{E}}_\ell)$  sense. We then use the following result:

**Lemma 6.2** (Lemma 3.2 in [BoHa08]). *Let  $N \in \mathbb{N}$ ,  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$  and define for all  $j \in \mathbb{N}$  the spaces  $\dot{\mathcal{E}}_\ell^{-j} := (\hat{K}_\ell - i)^j \dot{\mathcal{E}}_\ell$ . Then for all  $k \in \{0, \dots, N\}$ , there exist bounded operators  $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-j})$  and  $B \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-N-1})$  such that*

$$\hat{R}_{\chi, \ell}(z) = \sum_{j=0}^N \frac{B_j}{(z - i(\nu + 1))^{j+1}} + \frac{B \hat{R}_{\tilde{\chi}, \ell}(z) \chi}{(z - i(\nu + 1))^{N+1}}$$

for some  $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$  satisfying  $\tilde{\chi} \chi = \chi$ .

Now define

$$\tilde{R}_{\chi, \ell}(z) := \hat{R}_{\chi, \ell}(z) - \sum_{0 \leq j \leq 1} \frac{B_j}{(z - i(\nu + 1))^{j+1}} \quad (53)$$

with  $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-j})$  as in Lemma 6.2; we thus have<sup>12</sup>

$$\|\tilde{R}_{\chi, \ell}(z)\|_{\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-2})} \lesssim \langle z \rangle^{-2} \|\hat{R}_{\chi, \ell}(z)\|_{\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell)}. \quad (54)$$

We can show that

$$\int_{-\infty+i\nu}^{+\infty+i\nu} \frac{B_j}{(z - i(\nu + 1))^{j+1}} dz = 0 \quad (55)$$

for all  $j \in \mathbb{N}$  using a contour deformation (integrate first along the square with apexes  $\pm C_0 + i\nu$ ,  $\pm C_0 - i\mu$  and let  $\mu \rightarrow -\infty$  then  $C_0 \rightarrow +\infty$ ). We thus obtain

$$\chi e^{-it\hat{K}_\ell} \chi u = \frac{1}{2\pi i} \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-izt} \tilde{R}_{\chi, \ell}(z) u dz \quad (56)$$

and the integral absolutely converges in  $\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-2})$ . We then integrate  $e^{-izt} \tilde{R}_{\chi, \ell} u$  over the (positively oriented) contour described in Figure 4 defined for  $K, \mu > 0$ . Setting

$$I_j := \frac{1}{2\pi i} \int_{\Gamma_j} e^{-izt} \tilde{R}_{\chi, \ell}(z) u dz,$$

one obtains by the residue theorem:

$$\frac{1}{2\pi i} \int_{-K+i\nu}^{K+i\nu} e^{-izt} \tilde{R}_{\chi, \ell}(z) u dz = \sum_{\substack{z_j \in \text{Res}(p_\ell) \\ \Im z_j > -\mu}} \sum_{k=0}^{m(z_j)} e^{-iz_j t} t^k \Pi_{j,k}^\chi u + \sum_{1 \leq j \leq 5} I_j. \quad (57)$$

<sup>12</sup>The purpose of Lemma 6.2 is to provide us with integrability in  $z$  at the prize of using the weaker spaces  $\dot{\mathcal{E}}_\ell^{-2}$ . The task then consists in showing that all the terms in  $\dot{\mathcal{E}}_\ell^{-2}$  vanish after deformation of contours and the remaining terms are in  $\dot{\mathcal{E}}_\ell$ .

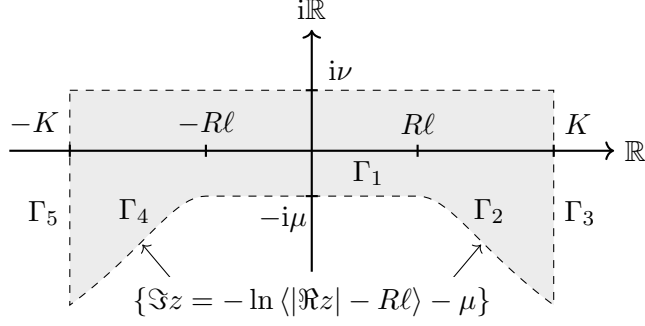


Figure 4: The contour used for the derivation of the resonance expansion.

Using the estimate (39) in the Zone IV as well as Proposition 6.1 and (54) above, we compute for  $t$  large enough:

$$\begin{aligned}
\|I_3\|_{\dot{\mathcal{E}}_\ell^{-2}} &\lesssim \int_{K-i\ln\langle K \rangle}^{K+i\nu} \|e^{-i\lambda t} \tilde{R}_{\chi,\ell}(\lambda)u\|_{\dot{\mathcal{E}}_\ell^{-2}} d\lambda \\
&\lesssim \|u\|_{\dot{\mathcal{E}}_\ell} \int_{\ln\langle K \rangle}^{\nu} \langle z \rangle^{-2} e^{\lambda t + C|\lambda|} d\lambda \\
&\lesssim \langle K \rangle^{-2} \frac{e^{\nu t}}{t} \|u\|_{\dot{\mathcal{E}}_\ell}.
\end{aligned}$$

We let  $K \rightarrow +\infty$ : the integrals  $I_3$  and  $I_5$  then vanish. We still denote by  $I_2$  and  $I_4$  the integrals over  $\Gamma_2$  and  $\Gamma_4$  which now go to infinity. As for (55), we can show that

$$\int_{\Gamma_4 \cup \Gamma_1 \cup \Gamma_2} \frac{B_j}{(z - i(\nu + 1))^{j+1}} dz = 0. \quad (58)$$

Now, using (37) for  $I_1$  and (39) for  $I_2$  and  $I_4$ , we get for  $t$  large enough:

$$\begin{aligned}
\|I_1\|_{\dot{\mathcal{E}}_\ell} &\lesssim \int_{-R\ell}^{R\ell} \|e^{-\mu t} \hat{R}_{\chi,\ell}(r - i\mu)u\|_{\dot{\mathcal{E}}_\ell} dr \\
&\lesssim e^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell} \int_{-R\ell}^{R\ell} \langle r \rangle^C dr \\
&\lesssim e^{-\mu t} \ell^{C+1} \|u\|_{\dot{\mathcal{E}}_\ell}, \\
\|I_2\|_{\dot{\mathcal{E}}_\ell} &\lesssim \int_0^{+\infty} \left\| e^{-i(R\ell + \lambda - i(\mu + \ln\langle \lambda \rangle))t} \hat{R}_{\chi,\ell}(R\ell + \lambda - i(\mu + \ln\langle \lambda \rangle))u \right\|_{\dot{\mathcal{E}}_\ell} d\lambda \\
&\lesssim e^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell} \int_0^{+\infty} e^{-\ln\langle \lambda \rangle t + C(\ln\langle \lambda \rangle + \mu)} d\lambda \\
&\lesssim e^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell}.
\end{aligned}$$

All these estimates hold in  $\dot{\mathcal{E}}_\ell$ , hence we have established part (i) of Theorem 4.2 with  $N = (C + 1)/2$ .

Let us turn to part (ii). For  $\mu < \varepsilon_0$  with  $\varepsilon_0$  as in part 2. of Theorem 3.8, we know that there is no resonance in formula 57. If  $\ell < e^{\varepsilon' t}$  for some  $\varepsilon' > 0$ , then

$$\|I_1\|_{\dot{\mathcal{E}}_\ell} \lesssim e^{((C+1)\varepsilon' - \mu)t} \|u\|_{\dot{\mathcal{E}}_\ell}.$$

Otherwise, if  $\ell \geq e^{\varepsilon' t}$ , then the exponential decay of the local energy as well as the hypotheses on  $g$  imply together:

$$\|\chi e^{-it\hat{K}_\ell} \chi u\|_{\dot{\mathcal{E}}_\ell} \lesssim 1 \lesssim \frac{g(\ell(\ell + 1))}{g(e^{2\varepsilon' t})}.$$

It remains to take  $\varepsilon'$  small enough and  $\varepsilon := \min\{2\varepsilon', \mu - (C + 1)\varepsilon'\}$  to conclude the proof.

## A Analytic extension of the coordinate $r$

In this appendix, we prove Proposition 2.1 which is analogous to [BM93, Prop IV.2]. Let  $r \in ]r_-, r_+[$ . By equation (5), we have

$$\exp\left(-\frac{\Lambda}{3A_{\pm}r_{\pm}^2}x\right) = \prod_{\alpha \in I} \left| \frac{r - r_{\alpha}}{\mathfrak{r} - r_{\alpha}} \right|^{\frac{A_{\alpha}r_{\alpha}^2}{A_{\pm}r_{\pm}^2}}.$$

Call the left-hand side  $z$  and the right-hand side  $g_{\pm}(r)$ . Observe that  $g_{\pm}(r_{\pm}) = 0$ . Since  $r \mapsto x(r)$  is increasing and analytic, we can apply the Lagrange's inversion theorem (see for example [DeB81, §2.2] and references therein) to write

$$r = r_{\pm} + \sum_{\ell=1}^{+\infty} \frac{z^{\ell}}{\ell!} \left[ \frac{d^{\ell-1}}{dr^{\ell-1}} \left( \frac{r - r_{\pm}}{g_{\pm}(r)} \right)^{\ell} \right]_{r=r_{\pm}}. \quad (59)$$

Let us introduce Kronecker's symbol

$$\delta_{\alpha, \pm} := \begin{cases} 1 & \text{if } \alpha = \pm \\ 0 & \text{otherwise} \end{cases}$$

and the notation

$$B_{\pm, \alpha} := \frac{A_{\alpha}r_{\alpha}^2}{A_{\pm}r_{\pm}^2} - \delta_{\alpha, \pm}.$$

Observe that  $B_{-, -} = B_{+, +} = 0$ . We then have

$$\frac{d^{\ell-1}}{dr^{\ell-1}} \left( \frac{r - r_{\pm}}{g_{\pm}(r)} \right)^{\ell} = \left( \prod_{\alpha \in I \setminus \{\pm\}} |\mathfrak{r} - r_{\alpha}|^{\ell B_{\pm, \alpha}} \right) \frac{d^{\ell-1}}{dr^{\ell-1}} \left( \prod_{\alpha \in I \setminus \{\pm\}} |r - r_{\alpha}|^{-\ell B_{\pm, \alpha}} \right).$$

We now fix  $\pm = +$  (the conclusion will not be changed if we choose  $\pm = -$ ). Then

$$\begin{aligned} \frac{d^{\ell-1}}{dr^{\ell-1}} \left( \prod_{\alpha \in I \setminus \{+\}} (r - r_{\alpha})^{-\ell B_{+, \alpha}} \right) &= \sum_{0 \leq k_2 \leq k_1 \leq \ell} C_{\ell, k_1, k_2} \left( \frac{d^{\ell-k_1}}{dr^{\ell-k_1}} (r - r_n)^{-\ell B_{+, n}} \right) \times \\ &\times \left( \frac{d^{k_1-k_2}}{dr^{k_1-k_2}} (r - r_c)^{-\ell B_{+, c}} \right) \left( \frac{d^{k_2}}{dr^{k_2}} (r - r_-)^{-\ell B_{+, -}} \right) \end{aligned}$$

where

$$C_{\ell, k_1, k_2} = \binom{\ell}{k_1} \binom{k_1}{k_2}.$$

Direct computation shows that

$$\frac{d^p}{dr^p} (r - r_{\alpha})^{-\ell B_{+, \alpha}} = (-1)^p (\ell B_{+, \alpha}) (\ell B_{+, \alpha} + 1) \dots (\ell B_{+, \alpha} + p - 1) (r - r_{\alpha})^{-\ell B_{+, \alpha} - p}.$$

If we let

$$K := \prod_{\alpha \in I \setminus \{+\}} (\mathfrak{r} - r_{\alpha})^{B_{+, \alpha}}, \quad B_+ := \max_{\alpha \in I \setminus \{+\}} \{ |B_{+, \alpha}| \},$$

then it follows that

$$\begin{aligned} \frac{d^{\ell-1}}{dr^{\ell-1}} \left( \frac{r - r_+}{g_+(r)} \right)^{\ell} &= K^{\ell} \sum_{0 \leq k_2 \leq k_1 \leq \ell} C_{\ell, k_1, k_2} (-1)^{\ell} \times \\ &\times (\ell B_{+, n}) (\ell B_{+, n} + 1) \dots (\ell B_{+, n} + (\ell - k_1) - 1) (r - r_n)^{-\ell B_{+, n} - (\ell - k_1)} \times \\ &\times (\ell B_{+, c}) (\ell B_{+, c} + 1) \dots (\ell B_{+, c} + (k_1 - k_2) - 1) (r - r_c)^{-\ell B_{+, c} - (k_1 - k_2)} \times \\ &\times (\ell B_{+, -}) (\ell B_{+, -} + 1) \dots (\ell B_{+, -} + k_2 - 1) (r - r_{\alpha})^{-\ell B_{+, -} - k_2} \end{aligned}$$

and thus

$$\begin{aligned}
\left| \frac{d^{\ell-1}}{dr^{\ell-1}} \left( \frac{r-r_+}{g_+(r)} \right)^\ell \right| &\leq K^\ell \ell^\ell (B_+ + 1)^\ell \left( \prod_{\alpha \in I \setminus \{+\}} (r_+ - r_\alpha)^{-B_{+,\alpha}} \right)^\ell \times \\
&\times \sum_{0 \leq k_2 \leq k_1 \leq \ell} C_{\ell, k_1, k_2} (r_+ - r_n)^{-(\ell-k_1)} (r_+ - r_c)^{-(k_1-k_2)} (r_+ - r_-)^{-k_2} \\
&= K^\ell \ell^\ell (B_+ + 1)^\ell \left( \prod_{\alpha \in I \setminus \{+\}} (r_+ - r_\alpha)^{-B_{+,\alpha}} \right)^\ell \left( \sum_{\alpha \in I \setminus \{+\}} (r_+ - r_\alpha)^{-1} \right)^\ell \\
&= \left( K(B_+ + 1) \prod_{\alpha \in I \setminus \{+\}} (r_+ - r_\alpha)^{-B_{+,\alpha}} \sum_{\alpha \in I \setminus \{+\}} (r_+ - r_\alpha)^{-1} \right)^\ell \ell^\ell \\
&=: \tilde{K}^\ell \ell^\ell.
\end{aligned}$$

Therefore, the convergence of the original series is absolute for  $z \in \mathbb{C}$  if

$$\frac{(|z| \ell \tilde{K})^\ell}{\ell!} < \ell^{-(1+\varepsilon)}$$

for any  $\varepsilon > 0$ . Using Stirling approximation  $\ell! \sim \sqrt{2\pi} \ell^{\ell+1/2}$  for large values of  $\ell$ , we see that it is sufficient to have

$$\tilde{K}|z| < \frac{e^{-(1/2+\varepsilon)\ln \ell/\ell}}{\sqrt{2\pi}^\ell} < 1.$$

This condition is fulfilled if

$$\Re x > \frac{3A_+ r_+^2}{\Lambda} \ln \tilde{K}.$$

## B Localization of high frequency resonances

We provide in this section an asymptotic approximation of resonances near the maximal energy  $W_0(0) = \max_{x \in \mathbb{R}} \{W_0(x)\}$  as  $h \rightarrow 0$ . This a generalization of the main Theorem in [SZ97] to the case  $Q \neq 0$ . More precisely, we show that the resonances associated to the meromorphic extension of  $p(z, s)^{-1}$  are close to the ones associated with the extension of  $(P - z^2)^{-1}$ , provided that  $Q$  is sufficiently small. This is a direct consequence of the fact that the extra term  $hsV$  in the semiclassical quadratic pencil is  $\mathcal{O}(hs)$ .

As in the paragraph 5.2, we set  $h := (\ell(\ell + 1))^{-1/2}$  with  $\ell > 0$  and consider  $z \in [\ell/R, R\ell] + i[-C_0, C_0]$ . We then define the spectral parameter  $\lambda := h^2 z^2$  and also  $\tilde{P}_h$  the semiclassical operator associated to  $P_\ell$ . Recall also that  $\mathfrak{r} = \frac{3M}{2} \left( 1 + \sqrt{1 - \frac{8Q^2}{9M^2}} \right)$  is the radius of the photon sphere and  $W_0(0) = F(\mathfrak{r})/\mathfrak{r}^2$  with our definition of the Regge-Wheeler coordinate  $x$  (see (5)).

**Theorem B.1.** *Let*

$$\Gamma_0(h) := \left\{ W_0(0) + h \left( 2\sqrt{W_0(0)} sV(0) + i^{-1} \sqrt{W_0''(0)/2} \left( k + \frac{1}{2} \right) \right) \mid k \in \mathbb{N} \right\}.$$

*For all  $C_0 > 0$  such that  $\partial D(W_0(0), C_0 h) \cap \Gamma_0(h) = \emptyset$ , there is a bijection  $b \equiv b(h)$  from  $\Gamma_0(h)$  onto the set of resonances of  $\tilde{P}_h$  in  $D(W_0(0), C_0 h)$  (counted with their natural multiplicity) such that*

$$b(h)(\mu) - \mu = o_{h \rightarrow 0}(h) \quad \text{uniformly for } \mu \in \Gamma_0(h).$$

*Proof.* This is a direct application of the results of Sá Barreto-Zworski [SZ97] which are based on the work of Sjöstrand [S87] (see Theorem 0.1 therein), the latter dealing with resonances generated by non-degenerate critical points when the trapping set is reduced to a single point (the difference for us is  $W_0(0) \neq 0$ ).

We recall that in the zone III the symbol of the semiclassical quadratic pencil is the function  $(x, \xi) \mapsto \xi^2 + W_0(x) + h^2 W_1(x) - (\sqrt{\lambda} - hsV(x))^2 =: p(x, \xi) - \lambda$ . We also recall the hypothesis in [S87] for the case of a Schrödinger operator of the form (0.1) in the reference:

- The trapping set is reduced to the point  $\{(0, 0)\}$  ((0.3) in [S87]),
- 0 is a non-degenerate critical point ((0.4) in [S87], which implies in the Schrödinger case the more general assumptions (0.7) and (0.9) in the reference).

Although the symbol  $p$  depends on  $\lambda$ , its principal part  $p_0$  and subprincipal part  $p_{-1}$  do not: indeed, for  $\lambda \in D(W_0(0), C_0 h)$  with  $C_0 > 0$ , we can write when  $h \ll 1$

$$p(x, \xi) = \underbrace{\xi^2 + W_0(x)}_{=p_0(x, \xi)} + h \underbrace{2\sqrt{W_0(0)}sV(x)}_{=p_{-1}(x, \xi)} + \text{lower order terms in } h.$$

This is enough to apply [S87, Thm. 0.1]: using formula (0.14) in the reference, we get the result for the set

$$\left\{ p_0(0, 0) + h \left( p_{-1}(0, 0) + i^{-1} \sqrt{W_0''(0)/2} \left( k + \frac{1}{2} \right) \right) \mid k \in \mathbb{N} \right\}$$

which is  $\Gamma_0(h)$ . □

Approximation of high frequency resonances  $\Gamma_0(h) \ni z^2 = \lambda/h^2$  is obtained as in [SZ97], by taking the square root of any element of  $\Gamma_0(h)$  and using Taylor expansion for  $0 < h \ll 1$  (corresponding to  $\ell \gg 0$ ) as well as symmetry with respect to the imaginary axis (for the choice of the sign of the square root). In our setting, we obtain the set  $\Gamma$  of Theorem 4.1.

**Remark B.2.** 1. Let  $\Gamma_{\text{DSS}}$  be the set of pseudo-poles in the De Sitter-Schwarzschild case (see the Theorem at the end of [SZ97]). Then  $\Gamma_{\text{DSS}}$  is the limit of  $\Gamma$  as  $Q \rightarrow 0$  in the sense of the sets, i.e. for all  $z \in \Gamma$ , there exists  $z_0 \in \Gamma_{\text{DSS}}$  such that  $z \rightarrow z_0$  as  $Q \rightarrow 0$ .

2. The pseudo-poles in the charged case are shifted with respect to the uncharged case. If the charges of the Klein-Gordon field and the black hole have the same sign (that is if  $qQ > 0$ ), then all the pseudo-poles go to infinity with a real part which never vanishes. However, if the charges have opposite sign ( $qQ < 0$ ), then all the pseudo-poles real part cancels precisely when  $qQ = -(k + 1/2)\sqrt{F(\mathbf{r})}$ ,  $k \in \mathbb{N} \setminus \{0\}$ , before going to infinity. Notice that no pseudo-pole goes to  $\mathbb{C}^+$  as  $|s| \rightarrow +\infty$ .
3. We can provide a physical interpretation of the set of pseudo-poles. First observe that  $\sqrt{F(\mathbf{r})}/\mathbf{r}$  is nothing but the inverse of the impact parameter  $b = |E/L|$  of trapped null geodesics with energy  $E$  and angular momentum  $L$ . Theorem 4.1 shows that resonances near the real line in the zone III are  $qQ$ -dependent multiples of this quantity: they thus correspond to impact parameters of trapped photons with high energy and angular momentum.
4. Observe that in Newtonian mechanics, the electromagnetism and gravitation do not interact with chargeless and massless photons. As a consequence, photons are not deviated and only ones with impact parameter  $|b| \leq r_-$  can “fall“ in the black hole. Hence, high frequency resonances in zone III are expected to be multiple of  $r_-^{-1}$ . As  $r_- \rightarrow 0$ , all resonances go to infinity: the trajectory are now classical straight lines as there is no obstacle anymore.

## C Abstract Semiclassical Limiting Absorption Principle for a class of Generalized Resolvents

We show in this section an abstract semiclassical limiting absorption principle for perturbed resolvents.



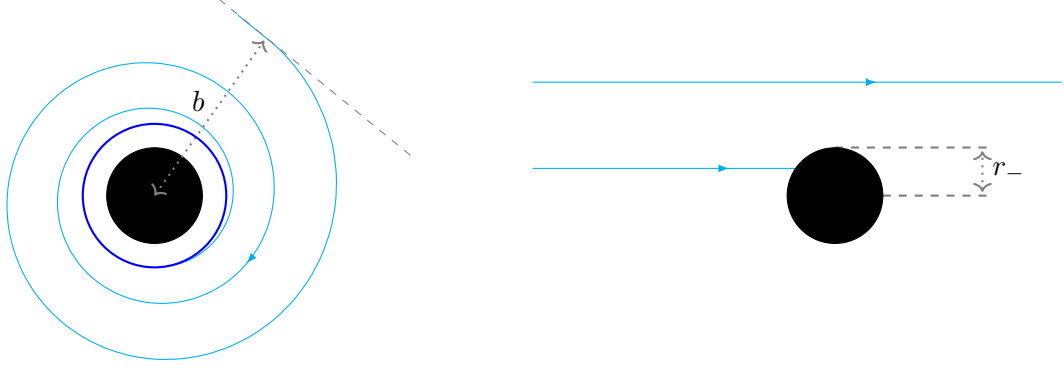


Figure 5: On the left: a relativistic trapped null geodesic. On the right: classical null geodesic trajectories.

**Abstract setting.** Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $J := [a, b] \subset \mathbb{R}$ ,  $J_\mu^+ := \{\omega \in \mathbb{C}^+ \mid \Re \omega \in J, \Im \omega < \mu\}$  for some  $\mu > 0$  fixed and  $h_0 > 0$ . The norm associated to  $\langle \cdot, \cdot \rangle$  will be denoted by  $\|\cdot\|$ . We consider families of self-adjoint operators  $P \equiv P(h)$  and  $\mathcal{A} \equiv \mathcal{A}(h)$  acting on  $\mathcal{H}$  for  $0 < h < h_0$ . We set

$$L_{\text{loc}}^\infty(P) := \{A : \mathcal{H} \rightarrow \mathcal{H} \text{ linear} \mid \forall \chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}), \forall u \in \mathcal{D}(P), \|\chi(P)Au\| < +\infty\}$$

and  $\|\cdot\|_P$  will be the operator norm on  $\mathcal{B}(\mathcal{D}(P), \mathcal{H})$ . We also define the local version of the operator  $P$ :

$$P_\tau := \tau(P)P \quad \forall \tau \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}).$$

Let then  $f : \mathbb{C} \times L_{\text{loc}}^\infty(P) \rightarrow L_{\text{loc}}^\infty(P)$  satisfying the following continuity type relation near  $0_{L_{\text{loc}}^\infty(P)}$ : there exist  $\delta_{J,\mu} : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying  $\delta_{J,\mu}(r) \rightarrow 0$  as  $r \rightarrow 0$  and  $\varepsilon_{J,\mu} : L_{\text{loc}}^\infty(P) \rightarrow L_{\text{loc}}^\infty(P)$  such that, for all  $(z, A) \in J_\mu^+ \times L_{\text{loc}}^\infty(P)$  with  $\|A\|_P$  small,

$$f(z, A) = z + \delta_{J,\mu}(\|A\|_P) \varepsilon_{J,\mu}(A). \quad (\text{C})$$

We make the following assumptions:

$$(P - f(z, hA))^{-1} \text{ exists for all } z \in J_\mu^+ \text{ and } A \in L_{\text{loc}}^\infty(P) \text{ if } h \leq h_0 \quad (\text{I})$$

$$P \in \mathcal{C}^2(\mathcal{A}) \quad (\text{P})$$

$$\mathbb{1}_I(P)[P, i\mathcal{A}]\mathbb{1}_I(P) \geq ch\mathbb{1}_I(P) \quad \text{for some } c > 0 \text{ and } J \Subset I := ]\alpha, \beta[ \subset \mathbb{R} \quad (\text{M})$$

$$\text{ad}_{\chi(P)}^k(\varepsilon_{J,\mu}(A)) \in h^k \mathcal{B}(\mathcal{D}(\mathcal{A})) \quad \text{for all } k \in \{0, 1\}, \chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}) \text{ and } \|A\|_P < c' \text{ for } c' > 0. \quad (\text{A})$$

Recall that  $P \in \mathcal{C}^2(\mathcal{A})$  means for all  $z \in \mathbb{C} \setminus \sigma(P)$  that the map

$$\mathbb{R} \ni t \mapsto e^{itA}(P - z)^{-1}e^{-itA}$$

is  $\mathcal{C}^2$  for the strong topology of  $L^2$ . Recall also that for all linear operators  $L_1, L_2$  acting on  $\mathcal{H}$ ,  $\text{ad}_{L_1}^0(L_2) := L_2$  and  $\text{ad}_{L_1}^{k+1}(L_2) := [L_1, \text{ad}_{L_1}^k(L_2)]$ . Our goal is to show the following result:

**Theorem C.1.** *Assume hypotheses (C), (I), (P), (M) and (A). Then for all  $\sigma > 1/2$ ,*

$$\sup_{z \in J_\mu^+} \|\langle \mathcal{A} \rangle^{-\sigma} (P - f(z, hB))^{-1} \langle \mathcal{A} \rangle^{-\sigma}\| \lesssim h^{-1}. \quad (60)$$

In the sequel, we will write  $R(z, hB) := (P - f(z, hB))^{-1}$  and call it the generalized resolvent (of  $P$ ). Also, since  $J$  and  $\mu$  are now fixed, we will simply write  $J$ ,  $\delta$  and  $\varepsilon$  instead of  $J_\mu$ ,  $\delta_{J,\mu}$  and  $\varepsilon_{J,\mu}$ .

**Preliminary results.** The purpose of this paragraph is to show preliminary results used to prove Theorem C.1. We first prove an adapted version of [Ge08, Lem. 2.1] to our situation.

**Lemma C.2.** *Let  $0 \leq \sigma \leq 1$ ,  $z \in J^+$  and let  $\chi \in C_c^\infty(\mathbb{R}, \mathbb{R})$ . If  $h$  is small enough, then  $R(z, hB)$  and  $\chi(P)$  are bounded on  $\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ .*

*Proof.* The result is true for  $(P-z)^{-1}$  and  $\chi(P)$  by [Ge08, Lem. 2.1]. Let us show that  $R(z, hB)\mathcal{D}(\langle \mathcal{A} \rangle^\sigma) \subset \mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ :

$$\begin{aligned} \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\| &\leq \|\langle \mathcal{A} \rangle^\sigma (P-z)^{-1}\langle \mathcal{A} \rangle^{-\sigma}\| + \|\langle \mathcal{A} \rangle^\sigma (R(z, hB) - (P-z)^{-1})\langle \mathcal{A} \rangle^{-\sigma}\| \\ &\lesssim 1 + \|\langle \mathcal{A} \rangle^\sigma R(z, hB)(z - f(z, hB))(P-z)^{-1}\langle \mathcal{A} \rangle^{-\sigma}\| \end{aligned}$$

and (using that  $\varepsilon(hB) \in \mathcal{B}(\mathcal{D}(\mathcal{A}))$  by Assumption (A) for  $k=0$ )

$$\begin{aligned} &\|\langle \mathcal{A} \rangle^\sigma R(z, hB)(z - f(z, hB))(P-z)^{-1}\langle \mathcal{A} \rangle^{-\sigma}\| \\ &\leq \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma (z - f(z, hB))\langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma (P-z)^{-1}\langle \mathcal{A} \rangle^{-\sigma}\| \\ &\lesssim \delta(h\|B\|_p) \|\langle \mathcal{A} \rangle^\sigma \varepsilon(hB)\langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\|. \end{aligned}$$

We then use the uniformity in assumption (A) for  $k=0$  to write for  $h$  very small

$$\delta(h\|B\|_p) \|\langle \mathcal{A} \rangle^\sigma \varepsilon(hB)\langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\| < \frac{1}{2} \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\|.$$

The proof is complete.  $\square$

**Corollary C.3.** *Let  $0 \leq \sigma \leq 1$ ,  $z \in J^+$  and  $\tau, \chi \in C_c^\infty(\mathbb{R}, [0, 1])$  such that  $\chi \equiv 1$  on  $I$  and  $\tau\chi = \chi$ . If  $h$  is small enough, then  $(P_\tau - f(z, hB))\chi(P)$ ,  $(P_\tau - f(z, hB))\chi(P)(P+i)^{-1}$  and  $(P - f(z, hB))(P+i)^{-1}$  preserve  $\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ .*

*Proof.* We have

$$\begin{aligned} \langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)\langle \mathcal{A} \rangle^{-\sigma} &= \langle \mathcal{A} \rangle^\sigma (P_\tau - z)\chi(P)\langle \mathcal{A} \rangle^{-\sigma} \\ &\quad + \langle \mathcal{A} \rangle^\sigma (z - f(z, hB))\langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma \chi(P)\langle \mathcal{A} \rangle^{-\sigma} \end{aligned}$$

which is bounded by assumption (A) for  $k=0$ , Lemma C.2 and the fact that  $P_\tau\chi(P) = \varphi(P)$  with  $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$  by functional calculus. Next, [Ge08, Lem. 2.1] implies that  $(P+i)^{-1}$  preserves  $\mathcal{D}(\mathcal{A})$ , so we can write

$$\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)(P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} = \langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)\langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma (P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma}$$

which is clearly bounded thanks to the above computation. Finally,

$$\begin{aligned} \langle \mathcal{A} \rangle^\sigma (P - f(z, hB))(P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} &= \langle \mathcal{A} \rangle^\sigma (P+i-i-z+z-f(z, hB))(P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} \\ &= \text{Id} - (i+z)\langle \mathcal{A} \rangle^\sigma (P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} \\ &\quad + \langle \mathcal{A} \rangle^\sigma (z - f(z, hB))\langle \mathcal{A} \rangle^{-\sigma} \langle \mathcal{A} \rangle^\sigma (P+i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} \end{aligned}$$

and we again use [Ge08, Lem. 2.1] and assumption (A) for  $k=0$ .  $\square$

The next result is an adaptation of [Ge08, Lem. 3.1] to our setting.

**Lemma C.4.** *Let  $0 < \sigma \leq 1$  and let  $\tau, \chi \in C_c^\infty(\mathbb{R}, [0, 1])$  such that  $\chi \equiv 1$  on  $I$  and  $\tau\chi = \chi$ . Consider the following three statements:*

$$(i) \quad \sup_{z \in J^+} \|\langle \mathcal{A} \rangle^{-\sigma} R(z, hB)\langle \mathcal{A} \rangle^{-\sigma}\| \lesssim h^{-1};$$

(ii) *For all  $z \in J^+$  and all  $u \in (P+i)^{-1}\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ ,*

$$\|\langle \mathcal{A} \rangle^{-\sigma} u\| \lesssim h^{-1} \|(P - f(z, hB))u\| + h^{-1} \|\langle \mathcal{A} \rangle^\sigma (P - f(z, hB))\chi(P)u\|;$$

(iii) *For all  $z \in J^+$  and all  $u \in \mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ ,*

$$\|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| \lesssim h^{-1} \|\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)u\|.$$

*If  $h$  is sufficiently small, then (iii) implies (ii) and (ii) implies (i).*

*Proof.* First of all, observe that (i) makes sense by Lemma (C.2), and (ii), (iii) make sense by Corollary C.3 and because  $P\chi(P) = P_\tau\chi(P)$ .

- We show that (ii) implies (i). Let  $u \in \mathcal{H}$  and let  $v := R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u$ . Then

$$w := u - \langle \mathcal{A} \rangle^\sigma (f(z, hB) - i)R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u \in \mathcal{H}.$$

This makes sense if  $h$  is small enough because  $R(z, hB)$  preserves  $\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$  by Lemma C.2 and because

$$\langle \mathcal{A} \rangle^\sigma (f(z, hB) - i)\langle \mathcal{A} \rangle^{-\sigma} = \langle \mathcal{A} \rangle^\sigma (f(z, hB) - z)\langle \mathcal{A} \rangle^{-\sigma} + (z - i)$$

is bounded by assumption (A) for  $k = 0$ . Next, using the resolvent identity  $(P + i)^{-1} - R(z, hB) = (P + i)^{-1}(f(z, hB) - i)R(z, hB)$ , we see that

$$\begin{aligned} (P + i)^{-1}\langle \mathcal{A} \rangle^{-\sigma} w &= ((P + i)^{-1} - (P + i)^{-1}(f(z, hB) - i)R(z, hB))\langle \mathcal{A} \rangle^{-\sigma} u \\ &= R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u \\ &= v \end{aligned}$$

so that  $v \in (P + i)^{-1}\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ . Hence, applying (ii) to  $v$  yields

$$\begin{aligned} \|\langle \mathcal{A} \rangle^{-\sigma} R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| &= \|\langle \mathcal{A} \rangle^{-\sigma} v\| \\ &\lesssim h^{-1}\|\langle \mathcal{A} \rangle^{-\sigma} u\| + h^{-1}\|\langle \mathcal{A} \rangle^\sigma (P - f(z, hB))\chi(P)R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| \\ &\lesssim h^{-1}\|\langle \mathcal{A} \rangle^{-\sigma} u\| + h^{-1}\|\langle \mathcal{A} \rangle^\sigma [P - f(z, hB), \chi(P)]R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| \\ &\quad + h^{-1}\|\langle \mathcal{A} \rangle^\sigma \chi(P)\langle \mathcal{A} \rangle^{-\sigma} u\|. \end{aligned}$$

By assumption (A) for  $k = 1$  and Lemma C.2, we have

$$\begin{aligned} &\|\langle \mathcal{A} \rangle^\sigma [P - f(z, hB), \chi(P)]R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| \\ &= \|\langle \mathcal{A} \rangle^\sigma [z - f(z, hB), \chi(P)]R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| \\ &\leq \delta(h\|B\|_p)\|\langle \mathcal{A} \rangle^\sigma [\varepsilon(hB), \chi(P)]\langle \mathcal{A} \rangle^{-\sigma}\| \|\langle \mathcal{A} \rangle^\sigma R(z, hB)\langle \mathcal{A} \rangle^{-\sigma} u\| \\ &\lesssim h\delta(h\|B\|_p). \end{aligned}$$

Therefore, (i) follows from (ii) if  $h$  is small enough.

- We show that (iii) implies (ii). Let  $\tilde{\chi} := 1 - \chi$  and let  $u \in (P + i)^{-1}\mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ . We write

$$\|\langle \mathcal{A} \rangle^{-\sigma} u\| \leq \|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| + \|\langle \mathcal{A} \rangle^{-\sigma} \tilde{\chi}(P)u\| \quad (61)$$

and (iii) implies that

$$\|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| \lesssim h^{-1}\|\langle \mathcal{A} \rangle^\sigma (P - f(z, hB))\chi(P)u\|$$

because  $\tau \equiv 1$  on  $\text{Supp } \chi$ . In order to control the term involving  $\tilde{\chi}(P)$  in (61), we write  $\tilde{\chi} = \psi_- + \psi_+$  with  $\psi_\pm \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$  such that  $\text{Supp } \psi_- \subset ]-\infty, \alpha]$  and  $\text{Supp } \psi_+ \subset [\beta, +\infty[$ . We also pick  $\rho \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$  such that  $\rho\psi_- = \psi_-$ . Since  $B \in L_{loc}^\infty(P)$ , we have for any  $v \in \mathcal{D}(P)$

$$\begin{aligned} &\Re\langle \psi_-(P)^2(f(z, hB) - P)v, v \rangle \\ &= \Re\langle \psi_-(P)^2 z v, v \rangle + \Re\langle \psi_-(P)^2 \delta(h\|B\|_p)\varepsilon(hB)v, v \rangle - \Re\langle \psi_-(P)^2 P v, v \rangle \\ &\geq a\|\psi_-(P)v\|^2 - \delta(h\|B\|_p)\|\rho(P)\varepsilon(hB)\|_p\|\psi_-(P)v\|^2 - \alpha\|\psi_-(P)^2 v\|^2 \\ &\geq c_-\|\psi_-(P)v\|^2 \end{aligned} \quad (62)$$

where  $c_- > 0$  if  $h$  is sufficiently small. Using Cauchy-Schwarz inequality, we get  $\|\psi_-(P)(P - f(z, hB))v\| \geq c_-\|\psi_-(P)v\|$  and thus  $\|\psi_-(P)R(z, hB)v\| \lesssim \|\psi_-(P)v\|$ . Similarly, one can show  $\|\psi_+(P)R(z, hB)v\| \lesssim \|\psi_+(P)v\|$ . These inequalities and  $\tilde{\chi}^2 = (\psi_- + \psi_+)^2 = \psi_-^2 + \psi_+^2$  then imply

$$\|\tilde{\chi}(P)R(z, hB)v\| \lesssim \|\tilde{\chi}(P)v\|$$

which in turn implies for  $u \in \mathcal{D}(P)$

$$\begin{aligned} \|\langle \mathcal{A} \rangle^{-\sigma} \tilde{\chi}(P)u\| &\lesssim \|\tilde{\chi}(P)u\| \\ &= \|\tilde{\chi}(P)R(z, hB)(P - f(z, hB))u\| \\ &\lesssim \|\tilde{\chi}(P)(P - f(z, hB))u\| \\ &\lesssim \|(P - f(z, hB))u\|. \end{aligned}$$

□

*Proof of Theorem C.1.* We show that the regularity (P) and the Mourre estimate (M) are enough to establish (60). As pointed out at the beginning of [Ge08], the key point is the following energy estimate: for any self-adjoint operators  $H$  acting on  $\mathcal{H}$ ,  $u \in \mathcal{D}(H)$ ,  $\tau \in C_c^\infty(\mathbb{R}, [0, 1])$  and  $P_\tau := \tau(P)P$ , we have

$$2\Im\langle Hu, (P_\tau - f(z, hB))u \rangle = \langle u, [P_\tau, iH]u \rangle - 2\Im\langle u, f(z, hB)Hu \rangle \quad (63)$$

where the commutator must be understood as a quadratic form on  $\mathcal{D}(H)$ .

We follow the proof of [Ge08, Thm. 1]. Let  $\tau, \chi \in C_c^\infty(\mathbb{R}, [0, 1])$  such that  $\chi \equiv 1$  on  $I$  and  $\tau\chi = \chi$  and let

$$F(\xi) := - \int_\xi^{+\infty} g(\zeta)^2 d\zeta$$

with  $g \in C^\infty(\mathbb{R}, [0, 1])$  satisfying  $g(\xi) = 0$  for  $\xi \geq 2$  and  $g(\xi) = 1$  for  $\xi \leq 1$ . By Lemma C.4, it is sufficient to prove the following estimate: for any  $z \in J^+$  and  $u \in \mathcal{D}(\langle \mathcal{A} \rangle^\sigma)$ ,

$$\|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| \lesssim h^{-1} \|\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)u\|.$$

As  $P \in \mathcal{C}^2(\mathcal{A})$ ,  $P$  and  $\mathcal{A}$  are self-adjoint and satisfy the Mourre estimate (M) on  $I$ , we can apply the estimate (3.30) in the proof of [Ge08, Thm. 1]:

$$\chi(P)[P_\tau, iF(\mathcal{A})]\chi(P) \gtrsim h\chi(P)\langle \mathcal{A} \rangle^{-2\sigma}\chi(P). \quad (64)$$

Now we apply the identity (63) with  $H = F(\mathcal{A})$ : for all  $u \in \mathcal{D}(\mathcal{A})$ ,

$$2\Im\langle F(\mathcal{A})u, (P_\tau - f(z, hB))u \rangle = \langle u, [P_\tau, iF(\mathcal{A})]u \rangle + 2\Im\langle f(z, hB)u, F(\mathcal{A})u \rangle.$$

Since  $F < 0$  is bounded and  $\Im z > 0$ , we can write for all  $h$  sufficiently small

$$\begin{aligned} &2\Im\langle F(\mathcal{A})u, (P_\tau - f(z, hB))u \rangle \\ &= \langle u, [P_\tau, iF(\mathcal{A})]u \rangle - 2(\Im z)\langle u, F(\mathcal{A})u \rangle - 2\delta(h\|B\|_P)\Im\langle u, \varepsilon(hB)F(\mathcal{A})u \rangle \\ &> \langle u, [P_\tau, iF(\mathcal{A})]u \rangle - 2\delta(h\|B\|_P)\|\varepsilon(hB)u\|\|F(\mathcal{A})u\| \end{aligned}$$

where we used that that  $\varepsilon(hB) \in \mathcal{B}(\mathcal{D}(\mathcal{A}))$  by Assumption (A). It thus follows

$$2\Im\langle F(\mathcal{A})u, (P_\tau - f(z, hB))u \rangle \geq \langle u, [P_\tau, iF(\mathcal{A})]u \rangle. \quad (65)$$

Plugging the estimate (64) into inequality (65) and putting  $\chi(P)u$  instead of  $u$  yield

$$\begin{aligned} \|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\|^2 &= \langle u, \chi(P)\langle \mathcal{A} \rangle^{-2\sigma}\chi(P)u \rangle \\ &\lesssim h^{-1}\langle u, \chi(P)[P_\tau, iF(\mathcal{A})]\chi(P)u \rangle \\ &\leq h^{-1}|\langle F(\mathcal{A})\chi(P)u, (P_\tau - f(z, hB))\chi(P)u \rangle|. \end{aligned}$$

Using again the boundedness of  $F$ , we get

$$\|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\|^2 \lesssim h^{-1} \|\langle \mathcal{A} \rangle^{-\sigma} \chi(P)u\| \|\langle \mathcal{A} \rangle^\sigma (P_\tau - f(z, hB))\chi(P)u\|$$

which establishes the point (iii) and thus the point (i) in Lemma C.4. □

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