ALGORITHMIC ASPECTS OF IMMERSIBILITY AND EMBEDDABILITY

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Abstract. We analyze an algorithmic question about immersion theory: for which m, n, and CAT = **Diff** or **PL** is the question of whether an m-dimensional CAT-manifold is immersible in $\mathbb{R}^n$ decidable? As a corollary, we show that the smooth embeddability of an m-manifold with boundary in $\mathbb{R}^n$ is undecidable when $n-m$ is even and $11m \geq 10n+1$.

1. Introduction

The problem of classifying immersions of one smooth manifold M in another N was, in a sense, solved by Smale and Hirsch [Sma59] [Hir59], who reduced the question to one in homotopy theory. This is now viewed as an important example of the philosophy of h-principles [Gro86] [EM02]. While embedding seems to be much harder, many relevant questions have likewise been reduced to algebraic topology at least in principle, with, in our view, the signal achievements due to Whitney, Haefliger [Hae95, e.g.], and Goodwillie–Klein–Weiss [GKW03, e.g.].

Analogous work has been done, to a less complete degree, in the PL category, with an analogue of the Smale–Hirsch theorem given by Haefliger and Poénaru [HP64].

In this paper we discuss, mainly in the case $N = \mathbb{R}^n$ (or, equivalently, S^n), whether these classifications can actually be performed algorithmically given some finite data representing the pair of manifolds. This has consequences not only for computational topology but also for geometry.

In several papers, Gromov emphasized that topological existence results do not directly enable us to understand the geometric object that is supposed to exist. Indeed, the eversion of the sphere took quite a while to make explicit (but can now be observed in several nice animations). A basic question is: how complicated are embeddings or immersions, when they exist?

In [CDMW18], an analogous problem was studied in the case of cobordism, which was reduced to homotopy theory in a similar way by the work of Thom. In that case, we showed that if a nullcobordism exists, its complexity can be made at worst slightly superlinear with respect to a natural measure of the complexity of the manifold.

In contrast, in the setting of immersion and embedding, there is sometimes no computable upper bound to the complexity of solutions. Consider smooth m-manifolds M with sectional curvature $|K| \leq 1$ and injectivity radius ≥ 1; we say such manifolds have *bounded local geometry.* By results of Cheeger and Gromov, these bounds guarantee that the manifold doesn't have "too much topology" per unit volume, and volume can therefore be used as a measure of complexity. In particular, for every V, there are finitely many diffeomorphism types of manifolds of bounded local geometry and volume at most V. By taking the maximum over a finite set of manifolds-up-to-small-deformation, we get a function $F_{m,c}(V)$ such that any such smooth manifold M, if immersible in $\mathbb{R}^{m+c}$, has an immersion whose bilipschitz constant and norm of the second derivative are bounded by $F_{m,c}(\operatorname{vol} M)$.

Our undecidability result Theorem 3.1 then implies:

Corollary 1.1. *If $c \leq m/4$ and is even, $F_{m,c}(V)$ is not bounded by any computable function.*

The proof follows an outline originating in the work of Nabutovsky [Nab95]. Suppose a computable bound existed. Then one could solve the decision problem of whether M is immersible by a brute force search *all* candidate functions (up to a C^2-small deformation) whose geometry is below the bound. If an immersion is not found in this search, then the manifold is not immersible.

In this way logical complexity of decision problems is reflected in lower complexity bounds for solutions of related variational problems. The difference between this case and the situation in [CDMW18] is that in the case of cobordism, the relevant algebraic topology is stable homotopy theory, which is algorithmically tractable, while for immersions the relevant problems are unstable.

1.1. **Immersion vs. embedding.** Some prior work has been done on the decidability of various questions involving embeddings. In a pair of papers from the 1990's, Nabutovsky and the second author [NW99] [NW96] considered the problem of recognizing embeddings, that is, deciding whether two embeddings of a manifold M in a manifold N are isotopic. When M and N are both simply connected, this is decidable as long as the codimension is not 2; in codimension 2, even equivalence of knots (embeddings of S^{n-2} in $\mathbb{R}^n$) for $n \geq 5$ is not decidable.

A related result asserted in [NW99] says that for closed (even simply-connected) manifolds the problem of embedding is in general undecidable, as in our paper, for reasons related to Hilbert's tenth problem. Here, we study the special case where the target is a sphere, and do not know what to expect for the case of closed manifolds embedding in the sphere.

More recent work has considered the problem of embedding simplicial complexes in $\mathbb{R}^n$. In [MTW11] it is shown that this problem is undecidable in codimensions zero and one, when $n \geq 5$; in [ČKV17], that it is decidable in the so-called *metastable range*, when the dimension of the complex is at most roughly $\frac{2}{3}n$.

Between codimension 3 and the metastable range, embedding theory is best described via the *calculus of embeddings*, due to Goodwillie, Klein and Weiss. This describes smooth embeddings of manifolds via a rather complicated homotopical construction which nevertheless can be arbitrarily closely approximated via finite descriptions (see e.g. [GKW03]); unlike in the metastable range, where work of Haefliger shows that immersion theory is essentially irrelevant, immersions form the "bottom layer" of this construction. Thus, understanding immersion theory from a computational point of view seems to be a good first step towards solving this set of problems. As we show, it also directly leads to some results regarding embeddings.

While a similar construction for PL embeddings of simplicial complexes is not currently in the literature, it seems plausible that such a construction can be developed and will be quite similar to the smooth version. We believe that many variations of the embeddability question can eventually be shown to be undecidable using this correspondence.

1.2. **Summary of old and new results.** The properties of embedding and immersion questions, including their decidability, depend heavily on the ratio between the dimensions m and n of the two objects considered. Our main result concerns the decidability of immersibility in $\mathbb{R}^n$.

Theorem 3.1. The results break up into the following ranges.

- **The *stable range*, $m < \frac{1}{2}n + 1$:** The Whitney immersion theorem states that every manifold in this range has an immersion in $\mathbb{R}^n$.
- **The *metastable range*, $\frac{1}{2}n + 1 \leq m < \frac{2}{3}n$:** For manifolds in this range, both smooth and PL immersibility are always decidable.
- **$\frac{2}{3}n \leq m < \frac{4}{5}n$:** In this range, PL immersibility of manifolds in $\mathbb{R}^n$ is decidable, as is smooth immersibility as long as $n - m$ is odd. We do not know whether smooth immersibility in even codimension is decidable.
- **$\frac{4}{5}n \leq m \leq n - 3$:** In this range, PL immersibility of manifolds is decidable, whereas smooth immersibility is decidable if and only if $n - m$ is odd.
- **$m = n - 2$:** In codimension 2, there are two notions of PL immersion: in a *locally flat* immersion, links of vertices are always unknotted in the ambient space; but one may also

study PL immersions which are not necessarily locally flat. Here, smooth immersibility is undecidable at least when $n \geq 10$, as is PL locally flat immersibility, which is equivalent; PL not necessarily locally flat immersibility is decidable.

$m = n - 1$**:** In codimension 1, immersibility is decidable.

This parallels the overall picture for embedding theory, about which we still know much less. Note that the stable and metastable ranges are slightly different from the immersion case.

The stable range, $m \leq n/2$**:** The Whitney embedding theorem states that every manifold in this range has an embedding in $\mathbb{R}^n$. For simplicial complexes, one needs $m < n/2$; for $n = 2m$, embeddability is obstructed by the Van Kampen obstruction.

The metastable range, $m \leq \frac{2}{3}n - 1$**:** Here the embeddability of simplicial complexes is decidable; this is a theorem of Čadek, Krčál and Vokřínek [ČKV17]. Moreover, PL embeddings in this range are smoothable, so smooth embeddability is decidable as well.

$\frac{2}{3}n \leq m \leq \frac{10}{11}n$**:** In this range, nothing is known about whether embeddability is decidable; however, see [MTW11] for some lower bounds on computational complexity. Moreover, ongoing work of Filakovský, Wagner and Zhechev on the embedding extension problem (is it possible to extend an embedding of a subcomplex to an embedding of the whole space?) suggests that the more general problem of *classifying* embeddings of simplicial complexes up to isotopy is undecidable in the vast majority of this range.

$\frac{10}{11}n < m \leq n - 2$**:** The state of the art on embeddability of simplicial complexes is much the same here as in the previous range. However, our results on immersions are enough to show:

Theorem 4.2. When $\frac{10}{11}n < m \leq n - 2$ and $n - m$ is even, the embeddability of a smooth m-manifold with boundary in $\mathbb{R}^n$ is undecidable.

However, the reduction (from Hilbert's tenth problem via the immersion problem) which we use to show undecidability creates examples that are always PL embeddable; the construction relies on the smooth structure of the manifold. Moreover, we do not know whether embeddability is decidable when restricted to closed manifolds; as discussed in §4.1, the method of Theorem 4.2 cannot work in that case.

$n = m - 1$**:** Here, PL embeddability is undecidable, as shown in [MTW11].

1.3. **Methods.** Questions of immersibility and embeddability are classically handled by reducing them first to pure homotopy theory and then reducing the homotopy theory to algebra. To resolve any particular instance, then, one has to do the corresponding algebraic computation. To decide whether the answers can be obtained algorithmically, one has to (1) find an algorithm to perform the reduction and (2) determine whether the resulting algebra problem is decidable.

The homotopy-theoretic side of these questions is fairly well-studied. Novikov showed in 1955 that it is undecidable whether a given finite presentation yields the trivial group; in particular, this means that whether a given simplicial complex is simply connected is undecidable. This was extended by Adian to show that many other properties of groups are likewise undecidable. Soon after, Brown [Bro57] showed, by way of contrast, that the higher homotopy groups of a simply connected space are computable.

Much more recently, Čadek, Krčál, Matoušek, Vokřínek and Wagner [ČKM$^+$14a] showed that the set of homotopy classes $[X, Y]$ is in general uncomputable, even when Y is a simply connected space. This is because the problem of determining which rational invariants can be attained is tantamount to resolving a system of diophantine equations; this is the famously undecidable Hilbert's tenth problem.

It seems as if fundamental group issues and Hilbert's tenth problem are the only obstructions to computability in homotopy theory. The same group of authors, along with Filakovský, Franek, and Zhechev, have authored a number of papers [ČKM$^+$14b, FV20, Vok17, ČKV17, FFWZ18] describing algorithms for various problems in homotopy theory that do not encounter these. While some open questions do remain, all of the homotopy-theoretic problems encountered in this paper can easily be reduced to ones covered by their results.

The main issue, then, is that of the reduction. The h-principles of Hirsch–Smale [Hir59] and Haefliger–Poénaru [HP64], respectively, show that immersions of codimension k in the smooth and PL categories are classified via lifts of the stable tangent bundle to the classifying spaces BO_k and BPL_k. While BO_k can easily be approximated by a Grassmannian of k-planes in a high-dimensional Euclidean space, and therefore classifying maps are also not difficult to compute, BPL_k is more recalcitrant. While it is known to be of finite type, that is, homotopy equivalent to a complex with finite skeleta, this equivalence is inexplicit and it is not clear how to algorithmically reduce the tangential data of a PL manifold to a finite amount of data. In this paper, we employ various workarounds; the question of understanding BPL more directly remains open and is also relevant to the quantitative topology of PL manifolds.

1.4. **Complexity.** Our algorithms do not give any information about the complexity of the computations. In many cases, we perform a construction by iterating through all objects of a given form until we find the needed one; this uses the fact that its existence is known and that it is algorithmically recognizable. However, often such an object only exists when the input is a manifold; this means that the algorithm will not terminate if presented with an invalid input (for example, a simplicial complex all of whose links are homology spheres, but not spheres.)

We believe that this issue can be circumvented and that these algorithms can be made much more efficient, but this is beyond the scope of this paper.

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2. Effective representation

In this section, we discuss algorithms and data structures which we borrow from previous work, as well as new ones to represent certain objects which have not been worked out in detail previously.

2.1. **Computational homotopy theory.** There has been a fair amount of work on computational homotopy theory: taking a finite simplicial complex and algorithmically describing its homotopy groups, Postnikov tower, and so forth. Here we summarize some of this past work and cite algorithms for various operations which we will use as building blocks.

Proposition 2.1. *(a) The isomorphism type of homotopy groups of spheres is computable* [Bro57].
(b) Moreover, there is an algorithm that, given a finite simply connected simplicial complex X, computes a generating set and relations for $\pi_k(X)$ and an explicit simplicial representative for each generator [FFWZ18]. *From this, one can compute a simplicial representative for any linear combination of the generators.*
(c) Given a map $Y \to B$ between simply connected finite simplicial complexes or simplicial sets, its relative (or Moore–)Postnikov tower may be computed to any finite stage. The output is given in a form so that the (co)homology of every Postnikov stage P_n may be computed, as well as the maps

$$Y \to P_n \to P_{n-1} \to B$$

and the maps on (co)homology induced by them [ČKV17, Theorem 3.3].

(d) *Given a diagram*

$$\begin{array}{ccc} A & \longrightarrow & P_n \\ \big\downarrow & \nearrow & \big\downarrow \\ X & \longrightarrow & P_{n-1}, \end{array}$$

where (X, A) is a finite simplicial pair and $P_n \to P_{n-1}$ is a (relative) Postnikov stage, the obstruction in $H^n(X, A; \pi)$ to filling in the dotted arrow can be computed. Moreover, if the obstruction is zero, a lifting-extension may be constructed [ČKV17, Proposition 3.7]*. Finally, given a lifting-extension $f : X \to P_n$ and a cochain $w \in C^{n-1}(X, A; \pi)$, where $P_n \to P_{n-1}$ is a $K(\pi, n-1)$-fibration, one can construct another lifting-extension $g : X \to P_n$ such that the obstruction to homotoping f and g is $[w] \in H^{n-1}(X, A; \pi)$.*

(e) *Given two maps $X \to S^n$, for any simplicial complex X, there is an algorithm to determine whether they are homotopic; more generally, one can replace S^n with any simply connected finite complex Y* [FV20]*. In particular, given an explicit map $S^k \to S^n$, one can (by iterating over the combinations) determine its homotopy class as a combination of the generators computed in* [FFWZ18].

(f) *Given a map $X \to Y$ known to be nullhomotopic, we can compute an explicit nullhomotopy.*

Proof. We prove only the parts which are not given a citation in the statement.

Part (d)**.** The last part is not explicitly done in [ČKV17], but the representation of Postnikov stages given there makes it easy to modify a map $f : X \to P_n$ by a cochain in $C^{n-1}(X; \pi)$.

Part (f)**.** This can be done through an exhaustive search for maps from increasingly fine subdivisions of the cone on X. □

2.2. **Smooth manifolds.** There are several possible ways of representing smooth manifolds computationally; as far as we know, this topic has not been thoroughly explored. According to the Nash–Tognoli theorem, every smooth n-manifold embedded in $\mathbb{R}^{2n+1}$ is closely approximated by a smooth real algebraic variety cut out by rational polynomials. This is one way of specifying smooth manifolds, however it is not clear whether it can be computed from other possible representations.

2.2.1. *Our model.* For our purposes, compact smooth n-manifolds will be specified via C^1 triangulations. (Classically, the categories of C^1 and C^∞ manifolds are equivalent, and C^1 immersions or embeddings can be approximated by C^∞ ones [Mun66, §4]. Therefore we treat "smooth" and "C^1" as synonymous.) That is, we take a simplicial complex and specify a polynomial map with rational coefficients of each top-dimensional simplex to $\mathbb{R}^N$, for some N, such that the derivatives are nonsingular and coincide where simplices meet. (To be precise, given adjacent simplices $\delta_1, \delta_2 : \Delta^n \to \mathbb{R}^N$, and viewing Δ^n as a subset of $\mathbb{R}^n$, there is an affine transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ such that δ_1 and the map $\delta_2 \circ L : L^{-1}(\Delta^n) \to \mathbb{R}^N$ patch together to form a C^1 immersion of the connected set $\Delta^n \cup L^{-1}(\Delta^n)$ in $\mathbb{R}^N$.)

Remark 2.2. To represent every diffeomorphism type of smooth n-manifold, it suffices to use polynomials of bounded degree, with the bound depending on n. This is because, for any smoothly embedded manifold, there is a piecewise polynomial approximation of the embedding using splines on a triangulation: start by fixing the tangent spaces at vertices, then interpolate via cubic functions on each edge, and so on. If the triangulation is sufficiently fine, so that the embedding is close to linear on each simplex, then the approximation will also be an embedding.

Now suppose that we have a simplicial complex M and map $f : M \to \mathbb{R}^N$ satisfying these properties. Under what conditions does this actually define a closed manifold smoothly immersed in $\mathbb{R}^N$? Since there is a well-defined tangent space at every point, it suffices to show that for every vertex v, the "derivative" $Df_v : \mathrm{st}(v) \to T_vM$ (that is, the map whose restriction to each simplex of the star is the derivative at v of the restriction of f to that simplex) is a bijection to a neighborhood of the origin; then these derivatives at every vertex give an atlas of C^1 charts for M. The derivative is easily computed from the polynomial maps defining f. To show that it is injective, it suffices to show that the intersection between the images of any two simplices is the image of their intersection, which is a matter of showing that a set of linear inequalities is unsatisfiable. To show that it is surjective, it then suffices to show that $H_{n-1}(\mathrm{lk}(v))$ is nontrivial, since any proper subset of S^{n-1} has trivial $(n-1)$st homology. Thus whether a given piece of data represents an immersed manifold can be decided algorithmically.

Since every n-manifold embeds in $\mathbb{R}^{2n+1}$ and every embedding can be approximated by a C^1 piecewise polynomial map with rational coefficients on a sufficiently fine triangulation, this gives a way of enumerating all smooth closed n-manifolds: one iterates through all pure n-dimensional simplicial complexes and piecewise polynomial maps and checks the conditions to decide whether the data defines a closed manifold.

We can enumerate compact smooth manifolds with boundary using a similar strategy. To check whether a set of data encodes a smooth n-manifold M with boundary, we need to check whether the boundary (defined combinatorially) is a smooth $(n-1)$-manifold; whether M smoothly embeds at interior vertices; and, for each boundary vertex v, that Df_v injects into the tangent half-space and that the relative homology group $H_{n-1}(\mathrm{lk}_M(v), \mathrm{lk}_{\partial M}(v))$ is nontrivial.

2.2.2. *Grassmannians and Pontryagin classes.* Determining explicit triangulations of real Grassmannians is an interesting and apparently open problem in combinatorial algebraic geometry. However, it is well-established that some triangulation can be computed algorithmically. The oriented Grassmannian $\mathrm{Gr}_n(\mathbb{R}^N)$ can be explicitly expressed as an algebraic variety, for example in $\Lambda^n\mathbb{R}^N$ via the (double cover of the) Plücker embedding which sends an oriented n-plane to $v_1 \wedge \ldots \wedge v_n$, for any ordered orthonormal basis $v_1, \ldots, v_n$. And in fact a triangulation can be computed algorithmically for any semi-algebraic set [BPR06, Remark 11.19(b)].

The Plücker embedding also has the advantage that, given the derivative of a map from an n-manifold M to $\mathbb{R}^N$, the corresponding point in the Grassmannian can be readily computed as the normalized wedge product of its columns. In particular we can compute the value of the classifying map $\varphi : M \to \mathrm{Gr}_n(\mathbb{R}^N)$ of the tangent bundle at any rational point of any simplex. Likewise:

Lemma 2.3. *One can compute an upper bound for the Lipschitz constant of φ with respect to the induced Riemannian metrics on M and $\mathrm{Gr}_n(\mathbb{R}^N)$.*

Proof. The global Lipschitz constant is the maximum of the Lipschitz constants of each polynomial piece. On each polynomial piece, an upper bound for $\mathrm{Lip}\,\varphi$ is given by

$$\frac{\max\{\|D\Phi_x(v)\| : x \in \Delta^n, v \in \mathbb{R}^n, \|Df_x(v)\| = 1\}}{\min\{\|\Phi(x)\| : x \in \Delta^n\}},$$

where $\Phi(x) = (Df_x)_1 \wedge \cdots \wedge (Df_x)_n$ is the pre-normalization version of φ. The numerator and denominator are instances of optimization of a polynomial function over a semialgebraic set, which is computable via [BPR06, Algorithm 14.9]. □

Lemma 2.4. *This data suffices to produce a simplicial approximation of φ.*

Proof. Given a triangulation of the Grassmannian, thought of as a map $\tau : K \to \mathrm{Gr}_n(\mathbb{R}^N)$ where K is a simplicial complex equipped with the standard simplexwise linear length metric, let

$$s = \frac{\text{inradius of a standard } \dim(\mathrm{Gr}_n(\mathbb{R}^N))\text{-simplex}}{\mathrm{Lip}(\tau^{-1})\,\mathrm{Lip}\,\varphi}.$$

Note that the inradius of an r-simplex is given by $\frac{1}{\sqrt{2r(r+1)}}$. Moreover, $\mathrm{Lip}(\tau^{-1})$ can be computed algorithmically. For every simplex, one must minimize the derivative of a polynomial function over that simplex, which can again be done via [BPR06, Algorithm 14.9]. Therefore we can compute a lower bound for s.

The constant s is defined so that for any ball B of radius s in M, $\varphi(B)$ is contained in the star of some vertex of τ. Suppose we have a subdivision τ'' of our triangulation τ' of M in which the diameter of every simplex is at most $s/2$. By the usual proof of the simplicial approximation theorem, see e.g. [Hat02, Theorem 2C.1], the map from the 0-skeleton of τ'' taking each vertex v to the vertex of τ nearest to $\varphi(v)$ extends linearly to a simplicial map homotopic to φ.

To compute such a subdivision, we first compute the Lipschitz constant C of the triangulation τ' of M (this boils down to maximizing derivatives of polynomial functions with linear constraints). Then we compute the k-fold edgewise subdivision of τ' in the sense of [EG00], for some $k > 4C/s$ (that is, each edge of τ' is divided into k edges). This subdivision has rational vertices and the simplices of the k-fold edgewise subdivision of the standard simplex have diameter at most $2/k$. Therefore the diameters of our new simplices are at most $s/2$. □

We would like to use this to compute the rational Pontryagin classes of M, by pulling back simplicial cochains on $\mathrm{Gr}_n(\mathbb{R}^N)$. Since $\mathrm{Gr}_n(\mathbb{R}^N)$ is a finite simplicial complex, we can compute its rational cohomology algorithmically as an abelian group finitely generated by explicit cochains, whose simplicial cup products we can also take.

To determine the generators corresponding to the Pontryagin classes $p_1, \ldots, p_{n/4}$, we can use the Pontryagin numbers of products of complex projective spaces. If n is a multiple of 4, for every partition $n/4 = i_1 + \cdots + i_r$, the Pontryagin numbers of $\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r}$ are known explicitly, and by a result of Thom [MS74, Theorem 16.8], the matrix of Pontryagin numbers of all such n-dimensional products is nonsingular. If (by induction) we have computed cochain representatives of $p_1, \ldots, p_{n/4-1}$ in $H^*(\mathrm{Gr}_n(\mathbb{R}^N); \mathbb{Q})$, and we have constructed classifying maps for the various $\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r}$, then we can solve for $p_{n/4} \in H^n(\mathrm{Gr}_n(\mathbb{R}^N); \mathbb{Q})$.

We can compute explicit embeddings of such products in $\mathbb{R}^N$ for N sufficiently large, using an explicit embedding of $\mathbb{C}P^{2i_j}$ as an affine algebraic variety. Since Pontryagin classes are stable, we can therefore compute the corresponding cohomology classes in $\mathrm{Gr}_{n+k}(\mathbb{R}^{N+k}; \mathbb{Q})$ for any k. If we need to reduce N, we can pull the classes back along a simplicial approximation to the inclusion $\mathrm{Gr}_n(\mathbb{R}^{N_0}) \to \mathrm{Gr}_n(\mathbb{R}^N)$. This proves:

Proposition 2.5. *There is an algorithm to construct a triangulation of* $\mathrm{Gr}_n(\mathbb{R}^N)$ *and compute cochain representatives of the Pontryagin classes* $p_1, \ldots, p_{\lfloor n/4 \rfloor} \in H^*(\mathrm{Gr}_n(\mathbb{R}^N); \mathbb{Q})$.

Our algorithms will not require the computation of $p_{\lfloor n/4 \rfloor + 1}, \ldots, p_{\lfloor n/2 \rfloor}$.

2.2.3. *Other possible models.* We note that there are a number of other ways of specifying a smooth manifold via a combinatorial structure. We list some of these here; the extent to which they can be transformed into each other requires further study.

A more general way of specifying smooth (that is, C^1) n-dimensional submanifolds of $\mathbb{R}^N$ is by patching them together from smooth real semialgebraic sets, with a consistent derivative along the boundaries of the patches. This includes the case of a single smooth variety; a triangulated smooth manifold with semialgebraic simplices; and a handle decomposition with semialgebraic handles.

One can dispense with the explicit embedding by taking a triangulated manifold and assigning an element of $\mathrm{Gr}_n(\mathbb{R}^N)$ to each vertex. If the triangulation is sufficiently fine, we can send adjacent vertices close enough to each other (at most some constant distance depending on n and N) that one can interpolate linearly over the simplices, uniquely determining a smooth structure on the manifold. One must ensure, of course, that this structure is compatible with the PL structure.

Finally, one can specify a manifold via an atlas of coordinate patches and transition functions; for example one may require the patches to be real semialgebraic and the transition functions to be rational functions (in one direction).

2.3. **Classifying spaces for spherical fibrations.** Manifold topology makes use of a variety of classifying spaces, the most familiar of which are the classifying spaces BO_n and BSO_n for unoriented and oriented vector bundles. These have relatively straightforward models as Grassmannians of n-planes in $\mathbb{R}^\infty$. The classifying spaces BPL_n for PL structures are much more complicated to model combinatorially; while some work in this direction has been done by Mnëv [Mnë07], in this paper we do not attempt to make BPL_n and classifying maps to it concrete enough to manipulate algorithmically. Instead, we use some well-known computations to avoid talking about BPL_n at all and focus instead on BG_n, the classifying space for the much weaker structure of S^{n-1}-fibrations.

Transition functions between fibers in an oriented S^{n-1}-fibration are chosen from the topological monoid G_n of homotopy automorphisms of S^{n-1} homotopic to the identity; this fits into a fiber sequence

$$\Omega^{n-1}_{\pm\,\mathrm{id}} S^{n-1} \to G_n \to S^{n-1},$$

where the fiber consists of degree ± 1 maps $S^{n-1} \to S^{n-1}$ in the iterated loop space $\Omega^{n-1}S^{n-1}$ and the fibration is induced by evaluation at the basepoint. This monoid has a classifying space BG_n; as $n \to \infty$, this converges to a stable object BG. In order to compute with BG_n and BG, we need to construct finite models for their skeleta as well as the tautological bundles over them.

Lemma 2.6. *(i) There is an algorithm which, given natural numbers m and n, constructs a finite simplicial set $B_{m,n}$ which has an m-connected map to BG_n, together with any stage of the relative Postnikov tower of the pullback to $B_{m,n}$ of the tautological bundle over BG_n.*

(ii) There is an algorithm that constructs the map $B_{m,n} \to B_{m,m+1}$ induced by iterated suspension. Since the map $BG_{m+1} \to BG$ is m-connected, this map models the stabilization $BG_n \to BG$.

(iii) There is an algorithm which, given a stable range PL embedding $M^m \to \mathbb{R}^{2m+k}$, $k \geq 1$, constructs the classifying map $M \to B_{m,m+k}$ of the normal bundle.

We note that the model we compute is extremely inexplicit in how it classifies fibrations.

Proof. We start by outlining the algorithm for (i). We will use the algorithms outlined in Proposition 2.1 mostly without comment as building blocks.

We first show that we can compute $\pi_k(G_n)$, given n and k. From the fiber sequence

$$\Omega^{n-1}_{\pm\,\mathrm{id}} S^{n-1} \xrightarrow{i} G_n \xrightarrow{j} S^{n-1},$$

we obtain the homotopy exact sequence

$$\pi_{k+1}(S^{n-1}) \xrightarrow{\phi_{k+1}} \pi_{n+k-1}(S^{n-1}) \xrightarrow{i_*} \pi_k(G_n) \xrightarrow{j_*} \pi_k(S^{n-1}) \xrightarrow{\phi_k} \pi_{n+k-2}(S^{n-1}).$$

Thus it is enough to perform the following steps:

(1) Compute representatives of generators for $\pi_k(S^{n-1})$ and $\pi_{k+1}(S^{n-1})$.

(2) Compute the obstruction theoretic map $\phi_k : \pi_k(S^{n-1}) \to \pi_{n+k-2}(S^{n-1})$. Given a map $f : S^k \to S^{n-1}$, $\phi_k(f)$ is the obstruction to lifting f to a map $\tilde{f} : S^k \to \Omega^{n-1}_{\mathrm{id}} S^{n-1}$ which sends the base point to the identity map. In other words, it is the obstruction to extending the map

$$f \vee \mathrm{id} : S^k \vee S^{n-1} \to S^{n-1}$$

to $S^k \times S^{n-1}$. This obstruction is the *Whitehead product* $[f, \mathrm{id}_{S^{n-1}}]$: the composition

$$S^{n+k-2} \to S^k \vee S^{n-1} \xrightarrow{f \vee \mathrm{id}} S^{n-1}$$

where the first map is homotopic to the attaching map of the top cell of $S^k \times S^{n-1}$. From this map, we compute its homotopy class as an element of $\pi_{n+k-2}(S^{n-1})$; doing this for a representative of each generator gives a finite description of the map ϕ_k.

(3) Now $\pi_k(G_n)$ is generated by lifts of $\ker \phi_k$ and the image of $\pi_{n+k-1}(S^{n-1})$ under i_*. We compute, via exhaustive search, a homotopy lift of each generator of $\ker \phi_k$ to a map $S^k \times S^{n-1} \to S^{n-1}$; the generators of $\pi_{n+k-1}(S^{n-1})$ determine maps $S^k \times S^{n-1} \to S^{n-1}$ by precomposing with the map collapsing $S^k \vee S^{n-1}$. We then determine the isomorphism type of $\pi_k(G_n)$ by computing all relations.

When both groups are finite, this is a finite computation. This leaves the following cases:

$k = n-2$, n odd: Then

$$\pi_{n+k-1}(S^{n-1}) = \pi_{2n-3}(S^{n-1}) \cong \mathbb{Z} \oplus A_{n-1},$$

where A_{n-1} is a finite group, and the restriction of the map

$$\phi_{k+1} : \pi_{k+1}(S^{n-1}) \to \pi_{n+k-1}(S^{n-1})$$

to the $\mathbb{Z}$ factor is either surjective or has cokernel $\mathbb{Z}/2\mathbb{Z}$, depending on the resolution of the Hopf invariant one problem in that dimension, since $[\mathrm{id}, \mathrm{id}]$ always has Hopf invariant two. Thus $\operatorname{im} i_*$ is finite; this can be hardcoded into the computation.

$k = n-1$, n even: In this case $\ker \phi_k \cong \mathbb{Z}$, whereas $\pi_{n+k-1}(S^{n-1})$ is finite. Therefore

$$\pi_k(G_n) \cong \mathbb{Z} \oplus \operatorname{im}(i_*),$$

and we only need to compute the relations within the image of i_*.

$k = 2n-3$, n odd: In this case $\ker \phi_k$ contains a $\mathbb{Z}$ factor and a finite factor A, and $\pi_{k+n-1}(S^{n-1})$ is again finite. This makes this case similar to the previous one:

$$\pi_k(G_n) \cong \mathbb{Z} \oplus B,$$

where B is an extension of A by π_{k+n-1}, a finite group all of whose relations can be computed.

In addition, the case $k = 0$ must be coded separately: G_n always has two components.

Now we use the fact that $\pi_k(G_n) = \pi_{k+1}(BG_n)$ to build successive approximations $B_{i,n}$ of BG_n, together with the pullbacks $p_i : E_{i,n} \to B_{i,n}$ of the tautological S^{n-1}-fibration.

Given a map $f : S^k \times S^{n-1} \to S^{n-1}$ representing an element $\alpha \in \pi_k(G_n)$ (in particular with $f|_{* \times S^{n-1}} = \mathrm{id}$) we define the space

$$E_f = D^{k+1} \times S^{n-1} / \{(x, y) \sim (*, f(x, y)) : x \in \partial D^{k+1}, y \in S^{n-1}\};$$

then the projection $E_f \to S^{k+1}$ onto the first factor has homotopy fiber S^{n-1}, and α is the obstruction to constructing a fiberwise homotopy equivalence $S^{n-1} \times S^{k+1} \to E_f$. We use the E_f for representatives of a generating set for $\pi_k(G^n)$ as building blocks for our construction.

We set $p_1 : E_{1,n} \to B_{1,n}$ to be the map

$$\bigvee_{[f] \text{ generating } \pi_0(G_n)} E_f \to \bigvee S^1.$$

Now suppose we have constructed $p_i : E_{i,n} \to B_{i,n}$ which is the homotopy pullback of the tautological bundle over BG_n along an i-connected map. Then we construct the spaces $E_{i+1,n}$ and $B_{i+1,n}$ and the map p_{i+1} using the following algorithm. Here the CW structure can be given via simplicial maps from subdivided simplices corresponding to each cell.

(1) First, we compute the kernel of the map $\pi_i(B_{i,n}) \to \pi_i(BG_n)$; since $E_{i,n}$ is the pullback of the tautological bundle, this means determining which elements of $\pi_i(B_{i,n})$ pull $E_{i,n}$ back to a trivial fibration over S^i.

To do this, we first evaluate, for each generator of $\pi_i(B_{i,n})$, the obstruction in $\pi_{i-1}(S^{n-1})$ to lifting it to $E_{i,n}$. This allows us to compute the kernel K of this obstruction, as well as a generating set for this kernel and explicit lifts $h_j : S^{n-1} \to E_{i,n}$ of the generators of K.

The obstruction to extending h_j to a fiberwise homotopy equivalence $S^i \times S^{n-1} \to E_{i,n}$ depends on the choice of lift; as above, it is the Whitehead product $[h_j, \iota]$ where $\iota : S^{n-1} \to E_{i,n}$ is the inclusion of a fiber. For each h_j and each $g : S^i \to S^{n-1}$ representing a generator of $\pi_i(S^{n-1})$, we compute the obstruction in $\pi_{n+i-2}(S^{n-1})$ to completing the diagram

$$\begin{array}{ccc} S^i \vee S^{n-1} & \xrightarrow{(h_j+\iota g)\vee\iota} & E_{i,n} \\ \big\downarrow & \nearrow & \big\downarrow p_i \\ S^i \times S^{n-1} & \xrightarrow{\overline{h_j}} & B_{i,n}, \end{array}$$

where $\overline{h_j}(x, y) = p_i \circ h_j(y)$. These obstructions generate a unique homomorphism

$$K \times \pi_i(S^{n-1}) \to \pi_{n+i-2}(S^{n-1})$$

which describes the obstruction to extending any lift of any element of K. This allows us to compute the kernel of this obstruction as a subgroup of $K \times \pi_i(S^{n-1})$. Its projection to K is the subgroup of $\pi_i(B_{i,n})$ we desire.

Note that both of these obstruction-theoretic calculations only depend on one stage of the Postnikov tower of $E_{i,n} \to B_{i,n}$ and therefore fit into the computational framework of [ČKV17] described in Proposition 2.1(d).

(2) Now given a generating set for this kernel, we glue in an $(i + 1)$-cell for each generator to $B_{i,n}$ and a corresponding copy of $D^{i+1} \times S^{n-1}$ to $E_{i,n}$. This ensures that the map $\pi_i(B_{i+1,n}) \to \pi_i(BG_n)$ is an isomorphism.

(3) Finally, we wedge on $E_f \to S^{i+1}$ for a set of functions f which generate $\pi_i(G_n)$. This ensures that $B_{i+1,n} \to BG_n$ is an $(i + 1)$-connected map.

Finally, we construct the relative Postnikov tower of the map $p_m : E_{m,n} \to B_{m,n}$. This completes the proof of (i).

For both (ii) and (iii), we will need a subroutine which, given a map $f : E \to B$ whose homotopy fiber is S^{n-1} and such that B is m-dimensional, computes the classifying map to $B_{m,m+1}$. We note that since the homotopy groups of BG are finite, so are the homotopy groups of $B_{m,m+1}$ through dimension m. Therefore there are a finite number of homotopy classes of maps $B \to B_{m,m+1}$, which can be enumerated via obstruction theory; we choose the one for which f is the homotopy pullback of p_m, which can be verified by induction on the relative Postnikov tower.

For (ii), we can construct the map f by repeatedly taking the double mapping cone of p_m.

For (iii), given a PL embedding of M, we need to compute the Spivak normal S^{m+k-1}-fibration $E \to M$ before we compute its classifying map. It is enough to find the following data:

- a compact PL $(2m+k)$-manifold with boundary $N(M)$ embedded in $\mathbb{R}^{2m+k}$ which contains a subdivision of M in its interior;
- a strong deformation retraction of $N(M)$ to M.

Then the induced map $\partial N \to M$ is the Spivak normal fibration. Since these properties are checkable and we can iterate through all subdivisions of M, simplicial complexes in $\mathbb{R}^{2m+k}$ with rational vertices, and simplicial maps from a subdivision of $N(M) \times I$ to M, we can find this data via exhaustive search. □

3. Immersibility

Theorem 3.1. *Let $n \geq 4$ and $m < n$ be natural numbers.*

(i) Whenever $n - m$ is odd or $3m \leq 2n - 1$, the immersibility of a smooth m-manifold with boundary (given as a semialgebraic set in some $\mathbb{R}^N$) in $\mathbb{R}^n$ is algorithmically decidable.

(ii) Whenever $n - m$ is even and $5m \geq 4n$, the immersibility of a smooth m-manifold in $\mathbb{R}^n$ is undecidable (including if only closed manifolds are considered.)
(iii) Whenever $n - m \neq 2$, the immersibility of a PL m-manifold with boundary in $\mathbb{R}^n$ is decidable.
(iv) When $n - m = 2$, it is undecidable (at least for $n \geq 10$) whether a PL m-manifold has a locally flat immersion in $\mathbb{R}^n$, but there is an algorithm to decide whether it has a not necessarily locally flat immersion.

Moreover, over the cases for which an algorithm exists, it can be made uniform with respect to m and n.

Note that for certain pairs with $n - m$ even, we have not determined whether immersibility is decidable. We suspect that it is in fact undecidable in those cases, since the corresponding homotopy-theoretic problem is undecidable.

Proof. We assume at first that the manifold is oriented, to avoid fundamental group issues.

In each case, the problem of immersibility can be reduced to a homotopy lifting problem: these are the h-principles of Smale–Hirsch [Hir59] in the smooth case and Haefliger–Poenaru [HP64] in the PL case. Both of these results state that the space of immersions $M \to N$ in the appropriate category is homotopy equivalent to the space of tangent bundle monomorphisms $TM \to TN$, or simply $TM \to \mathbb{R}^n$ when $N = \mathbb{R}^n$. These in turn can be thought of as lifts of the classifying map of the tangent bundle to the Grassmannian of m-planes in $\mathbb{R}^n$.

The smooth case. The previous paragraph reduces immersibility to the homotopy lifting problem

$$\begin{array}{ccc} & & \mathrm{Gr}_m(\mathbb{R}^n) \\ & \nearrow & \downarrow \\ M & \xrightarrow{\kappa} & BSO_m \end{array}$$

where κ is the classifying map of the tangent bundle of M. Moreover, this lifting property is *stable*: such a lift exists if and only if the corresponding lift

$$\begin{array}{ccccc} & & \mathrm{Gr}_m(\mathbb{R}^n) & \longrightarrow & \mathrm{Gr}_{m+1}(\mathbb{R}^{n+1}) \\ & & \downarrow & \nearrow & \downarrow \\ M & \xrightarrow[\kappa]{} & BSO_m & \longrightarrow & BSO_{m+1} \end{array}$$

exists. This is because the map between the corresponding homotopy fibers

$$V_m(\mathbb{R}^n) \to V_{m+1}(\mathbb{R}^{n+1})$$

is $(n-1)$-connected. Therefore, when $m < n$, it suffices to resolve the lifting problem

$$\begin{array}{ccc} & & BSO_{n-m} \\ & \nearrow & \downarrow \\ M & \xrightarrow{\kappa} & BSO \end{array} \tag{3.2}$$

which arises from the limit of this sequence.

The spaces BSO_{n-m} and BSO each have the rational homotopy type of a product of Eilenberg–MacLane spaces [FHT12, Prop. 15.15]. In particular, the rational homotopy groups are dual to a subset of the rational cohomology algebra. Moreover, a map to either of these spaces is determined up to a finite set by the pullbacks of cohomology generators. Specifically, $H^*(BSO)$ is a free algebra generated by the Pontryagin classes in degree $4k$ for every k. When $n - m$ is odd, $H^*(BSO_{n-m})$ is generated by Pontryagin classes in degree $4k$ where $2k < n - m$; when $n - m$ is even, it is generated by these plus an Euler class in degree $n - m$ whose square is the top Pontryagin class.

The smooth case in odd codimension. In the case when $n-m$ is odd, whether a lift exists in (3.2), and therefore the immersibility of M in $\mathbb{R}^n$, can be determined via the following algorithm. Let M be given to us by a C^1 triangulation immersed in $\mathbb{R}^N$; without loss of generality we can take $N \geq 2m+1$. We can construct the classifying map $f : M \to \mathrm{Gr}_m(\mathbb{R}^N)$ as described in §2.2.2.

Now we determine whether there is a lift of f, using the special properties of the relative Postnikov tower of the map $p : BSO_{n-m} \to BSO$. After rationalization, p becomes the inclusion

$$\prod_{k=1}^{2(n-m)} K(\mathbb{Q}, 4k) \to \prod_{k=1}^{\infty} K(\mathbb{Q}, 4k).$$

This means that a lift of f exists if and only if the kth Pontryagin classes of M are zero, for $2(n-m) < 4k \leq m$, and some finite obstructions can be resolved. Moreover, this lift is rationally unique.

We start by computing the Pontryagin classes of M. If there is a nonzero class above dimension $2(n-m)$, then a lift does not exist. Otherwise, we must compute representatives of the finitely many homotopy classes of maps $g : M \to BSO_{n-m}$ which have the same Pontryagin classes. Then, for each of these representatives, we must test whether $p \circ g$ is homotopic to the classifying map of the tangent bundle of M.

As approximations to BSO_{n-m} and BSO, we use $\mathrm{Gr}_{m+N'}(\mathbb{R}^{n+N'})$ and $\mathrm{Gr}_{m+N'}(\mathbb{R}^{N+N'})$, for N' sufficiently large. As explained in §2.2.2, we can construct the stabilization maps and therefore have a map $M \to \mathrm{Gr}_{m+N'}(\mathbb{R}^{N+N'})$, which by abuse of notation we also call f. The map p can be constructed similarly.

It remains to explain in detail how to construct the finite set of representatives. We first compute the Postnikov tower of the space $\mathrm{Gr}_{m+N'}(\mathbb{R}^{n+N'})$ up to dimension m, obtaining Postnikov stages and maps $P_k \xrightarrow{p_k} P_{k-1}$. Note that a map $M \to P_{4k}$ contains information about Pontryagin classes up to the kth. At the end of the induction, we will have maps $g : M \to P_m$; homotopy classes of such maps are in bijection with $[M, BSO_{n-m}]$.

Suppose now that we have computed the set Σ_{k-1} of candidate maps $M \to P_{k-1}$. For each map in Σ_{k-1}, we first decide whether it lifts to P_k and if it does, compute a lift $f_k : M \to P_k$. Now if k is not a multiple of 4 or $k > 2(n-m)$, then the obstruction group $H^k(M; \pi_k(\mathrm{Gr}_{m+N'}(\mathbb{R}^{N+N'})))$ which determines the set of lifts is finite, and we can compute a representative of each homotopy class of lifts.

If k is a multiple of 4 and $k \leq 2(n-m)$, then there are an infinite number of lifts, and we must restrict to those which have the desired $(k/4)$th Pontryagin class. Since BSO_{n-m} is rationally a product,

$$H^k(P_k; \mathbb{Q}) \cong H^k(\mathrm{Gr}_{m+N'}(\mathbb{R}^{N+N'})),$$

and by computing the induced map on cohomology we can find the Pontryagin class $\eta \in H^k(P_k; \mathbb{Q})$. Now we state a lemma:

Lemma 3.3. *There is a homomorphism $\varphi : H^k(M; \pi) \to H^k(M; \mathbb{Q})$ such that if $f_k, f'_k : M \to P_k$ are lifts of a map $f_{k-1} : M \to P_{k-1}$ such that the obstruction to homotoping them over f_{k-1} is $\omega \in H^k(M; \pi)$, then*

$$(f'_k)^*\eta = f_k^*\eta + \varphi(\omega).$$

Assuming the lemma, we can compute the values of the homomorphism φ on the generators of $H^k(M; \pi)$ by computing $f_k^*\eta$ as well as the pullback along those maps f'_k for which the obstruction to homotoping f_k to f'_k is one of the generators of $H^k(M; \pi)$. From this, we can now compute the finite set of elements of $H^k(M; \pi)$ such that the corresponding lifts of f_{k-1} have the desired Pontryagin class, and the corresponding finite set of lifts. This completes the inductive step.

Proof of the lemma. The homomorphism φ is induced by a map $\pi \to \mathbb{Q}$, which in turn is induced by the composition

$$K(\pi, n) \xrightarrow{i} P_k \xrightarrow{\eta} K(\mathbb{Q}, n),$$

where i is the inclusion of the fiber of the fibration $P_k \to P_{k-1}$ and η is the classifying map of the Pontryagin class. From this the lemma can be proved by explicit computation. Suppose we have a triangulation of M and two lifts f_k and f_k' of f_{k-1} which coincide on M^{k-1}; this can be achieved by a homotopy without loss of generality. Then one has that

$$(f_k')^*\eta = f_k^*\eta + [(\eta \circ i)_* w]$$

where $w \in C^k(M; \pi)$ is the obstruction cochain. $\square$

The smooth case in the metastable range. When $2n \geq 3m + 1$, all Pontryagin classes in the relevant range are zero. However, when $n - m$ is even, there may be a nonzero Euler class in degree $n - m$, whose square is always zero. This is the only infinite-order homotopy group of the fiber of the map $BSO_{n-m} \to BSO$ below dimension n. Moreover, this map is $(n - m)$-connected. To show that the resulting lifting problem is decidable, we can use the results of Vokřínek [Vok17], who shows that a lifting problem through a k-connected fiber is decidable if the only infinite-dimensional homotopy groups of this fiber are of dimensions $< 2k$. As in the odd-dimensional case, we can approximate the map $BSO_{n-m} \to BSO$ by maps between finite-dimensional Grassmannians.

The smooth case in even codimension. Now suppose that $n - m$ is even; write $c = n - m$. We will show that immersibility is undecidable in this situation if $m \geq 4c$. In order to do this, we first show that the lifting problem

$$\begin{array}{ccc} & & BSO_c \\ & \nearrow & \downarrow \\ X & \xrightarrow{f} & BSO \end{array}$$

is undecidable for general $2c$-complexes X and maps $f : X \to BSO$. We prove this by a method used in [ČKM$^+$14a], reducing an algebraic problem which is undecidable by [ČKM$^+$14a, Lemma 2.1] to a question about lifts. The undecidable problem in question is a special case of Hilbert's 10th problem: determining the existence of an integer solution to a system of equations each of the form

$$\sum_{1 \leq i < j \leq r} a_{ij}^{(k)} x_i x_j = b_k, \tag{3.4}$$

where $x_1, \ldots, x_r$ are variables and b_k and $a_{ij}^{(k)}$ are coefficients.

The idea is to build a CW complex X with a map $f : X \to BSO$ determined by the b_k, such that any lift of f to BSO_c determines an assignment of the variables x_i. This is done by having the b_k determine the $2c$-dimensional Pontryagin class of f; since in BSO_c, this Pontryagin class is the square of a c-dimensional Euler class, this forces us to find a corresponding c-dimensional class in X which is the pullback of this Euler class under the lift. The relationships between the pairings of these classes with c- and $2c$-dimensional homology classes will be given by (3.4). The process is complicated by some finite-order phenomena; we now give the construction in detail.

Let $\alpha \in \pi_{2c}(BSO) \cong \mathbb{Z}$ be a generator; we have

$$\alpha^* p([S^{2c}]) = n_1$$

where $p \in H^{2c}(BSO)$ is the Pontryagin class in degree $2c$ and n_1 is some integer. Denote the pullback of this Pontryagin class by $\tilde{p} \in H^{2c}(BSO_c)$. Similarly, let $\beta \in \pi_c(BSO_c)$ be an element which (if c is a multiple of 4) pairs trivially with the Pontryagin class in degree c and (subject to this restriction) has the smallest possible nontrivial pairing with the Euler class $\eta \in H^c(BSO_c)$. Let n_2 be an integer such that

$$\beta^*\eta([S^c]) = n_2.$$

Note that rationally, η^2 is a multiple of $\tilde{p}$, so there are integers $n_3, n_4 \neq 0$ such that

$$n_3\eta^2 = n_4\tilde{p}.$$

Finally, let $n_5 = |\pi_{2n-1}(BSO_c)|$, since this homotopy group is of finite order.

Given a system of s equations of the form (3.4), we form a CW complex X as follows. We take the wedge of r copies of S^c, which we label $S^c_1, \ldots, S^c_r$, and attach s $2c$-cells, the kth cell e_k via an attaching map whose homotopy class is a linear combination of Whitehead products

$$\sum_{1\leq i<j\leq r} n_1n_4n_5a^{(k)}_{ij}[\mathrm{id}_i, \mathrm{id}_j],$$

where id_i is the inclusion map of S^c_i. We fix a map $f : X \to BSO$ by taking the c-cells to the basepoint and the kth $2c$-cell to a representative of $2n_2^2n_3n_5b_k\alpha$.

Then for any homotopy lift $\tilde{f} : X \to BSO_c$ of f, we have

$$n_4\tilde{f}^*\tilde{p}(e_k) = n_3\tilde{f}^*\eta^2(e_k) = n_1n_3n_4n_5 \sum_{1\leq i<j\leq r} 2a^{(k)}_{ij}\tilde{f}^*\eta([S^c_i])\tilde{f}^*\eta([S^c_j]), \tag{3.5}$$

where the last equation can be obtained by analyzing, for each i and j, the maps $X \to S^c_i \times S^c_j$ which send all the other spheres to a point. Moreover, if c is a multiple of 4, the pullback of the degree c Pontryagin class along $\tilde{f}$ is zero. Therefore, the numbers $x_i = n_2^{-1}\tilde{f}^*\eta([S^c_i])$ are integers, and (3.5) reduces to

$$2n_1n_2^2n_3n_4n_5b_k = 2n_1n_2^2n_3n_4n_5 \sum_{1\leq i<j\leq r} a^{(k)}_{ij}x_ix_j,$$

showing that (3.4) is satisfied.

Conversely, if we choose $x_1, \ldots, x_r$ satisfying (3.4), then the map $\bigvee_{i=1}^r S^c \to BSO_c$ which maps S^c_i via a representative of $x_i\beta$ extends to a map $\tilde{f} : X \to BSO_c$ because the attaching map of every $2c$-cell is divisible by n_5 and therefore is zero in $\pi_{2c-1}(BSO_c)$. The projection of $\tilde{f}$ to BSO is then nullhomotopic on each c-cell and maps each $2c$-cell e_k to BSO via a representative of $2n_2^2n_3n_5b_k\alpha$.

It remains to show that one can construct a manifold whose homotopy type and Pontryagin classes determine any such system. This can be done, at the cost of some increase in dimension; our examples are of dimension at least $4c$, which is probably not optimal.

Such manifolds exist by an argument of Wall [Wal66, §5], who shows that for any $2c$-complex X and map $f : X \to BSO$, and any $q \geq 4c$, there is a corresponding $(q+1)$-dimensional *thickening* of X, i.e. a manifold with boundary M homotopy equivalent to X such that the classifying map of its tangent bundle is homotopic to f. Moreover, the pair $(M, \partial M)$ is $(q-2c)$-connected and any extra topology of ∂M is sent to zero by the classifying map. Thus ∂M is a closed q-manifold which immerses in $\mathbb{R}^{q+c}$ if and only if the system of equations above has a solution.

Now suppose there were an algorithm to decide smooth immersibility of q-manifolds in $\mathbb{R}^{q+c}$ for some fixed even c and $q \geq 4c$. Then given a system of equations, we could iterate over smooth closed q-manifolds M and bases for $H^c(M)$ until we find one with the right cohomology algebra and classifying map. This search terminates since Wall guarantees the existence of such a manifold. Then we could decide whether the system has a solution using our solution to the immersibility problem. Thus immersibility cannot be decidable.

The PL case in codimension ≥ 3. Denote the universal cover of a space X by $\widetilde{X}$. In the PL case, similarly to the smooth case, the unstable lifting problem reduces to the stable problem

$$\begin{array}{ccc} & & \widetilde{BPL}_{n-m} \\ & \nearrow & \downarrow \\ M & \xrightarrow{\kappa} & \widetilde{BPL}. \end{array}$$

Moreover, when $n-m \geq 3$, the diagram

$$\begin{array}{ccc} BPL_{n-m} & \longrightarrow & BPL \\ \downarrow & & \downarrow \\ BG_{n-m} & \longrightarrow & BG, \end{array}$$

where BG is the classifying space of spherical fibrations, is a homotopy pullback square [Wal99, p. 123]. Thus, equivalently, we must solve the lifting problem

$$\begin{array}{ccc} & & \widetilde{BG}_{n-m} \\ & \nearrow & \downarrow \\ M & \xrightarrow{\kappa} & \widetilde{BG}. \end{array}$$

The argument in §2 shows that the only infinite homotopy group of G_{n-m} is

$$\begin{cases} \pi_{n-m-1} & \text{when } n-m \text{ is even} \\ \pi_{2(n-m)-3} & \text{when } n-m \text{ is odd,} \end{cases}$$

and therefore BG_{n-m} only has a single infinite homotopy group in dimension $n-m$ or $2(n-m)-2$, depending on parity. Moreover, in both cases the map $\widetilde{BG}_{n-m} \to \widetilde{BG}$ is $(n-m-2)$-connected, by the stability of homotopy groups of spheres.

Thus to decide immersibility we can use the following algorithm. Suppose M is given to us as a simplicial complex. By abuse of notation we refer to $\widetilde{BG}_n$ when we really mean the approximations constructed in §2.3.

(1) We embed the simplicial complex linearly in $\mathbb{R}^N$, for some large N.
(2) This gives us a map $M \to \widetilde{BG}_N$ which can be computed by Lemma 2.6(iii).
(3) Decide whether the map lifts to a map $M \to \widetilde{BG}_{n-m}$. In the even case, this can be done by the aforementioned work of Vokřínek [Vok17], since the only infinite obstruction is below twice the connectivity of the map $\widetilde{BG}_{n-m} \to \widetilde{BG}$. In the odd-dimensional case, we can split the work into two steps:
    - Compute all possible lifts to the $(2(n-m)-3)$rd stage of the relative Postnikov tower of $\widetilde{BG}_{n-m} \to \widetilde{BG}$. This can be done since all the obstructions are finite.
    - For each lift computed, use the algorithm of Vokřínek to decide whether it can be extended to $\widetilde{BG}_{n-m}$.

PL immersions in codimension 2. In codimension 2, there are two somewhat different things we may mean by PL immersion: locally flat immersion, in which the link of every vertex is unknotted, and immersion which is not necessarily locally flat.

A PL manifold M has a locally flat immersion in codimension 2 if and only if it has a smoothing which immerses smoothly in codimension 2. This is because by the fundamental theorem of smoothing theory [HM74, Part II], M is smoothable if and only if the classifying map $M \to \widetilde{BPL}$

of the stable tangent bundle lifts to BSO; but immersibility is equivalent to the existence of a further lift

$$\begin{array}{ccc} & & BSO_2 \cong \widetilde{BPL}_2 \\ & \nearrow & \downarrow \\ & & BSO \\ & & \downarrow \\ M & \longrightarrow & \widetilde{BPL}. \end{array}$$

Moreover, the homotopy fiber PL/O has finite homotopy groups, so the rational obstructions discussed above are the same in the PL case as in the smooth case. Therefore, this problem is undecidable for $\dim M \geq 8$ by the same argument as above: the examples we produced are PL immersible if and only if they are smoothly immersible.

The case of immersions which are not necessarily locally flat was studied by Cappell and Shaneson [CS76, CS73]. Such immersions are classified by maps to a space $BSRN_2$. Unlike in the higher codimension case, the diagram

$$\begin{array}{ccc} BSRN_2 & \longrightarrow & \widetilde{BG_2} \\ \downarrow & & \downarrow \\ \widetilde{BPL} & \longrightarrow & \widetilde{BG} \end{array}$$

is not a homotopy pullback square, but the map from $BSRN_2$ to the pullback splits up to homotopy. Therefore it is again sufficient to solve the lifting problem from $\widetilde{BG}$ to $\widetilde{BG_2}$.

Codimension 1. In codimension one, the lifting problem above, and therefore the question of smooth immersibility, boils down to whether the suspension of the tangent bundle is trivial, that is, whether the composition

$$M \to BSO_m \to BSO_{m+1}$$

is nullhomotopic. Once this composition is given as an explicit map, whether it is nullhomotopic is a decidable question by Proposition 2.1(e) (due to [FV20]).

The oriented PL case is formally identical: one needs to determine whether the map $M \to \widetilde{BPL}_{m+1}$ induced by the tangent bundle, or equivalently the map $M \to \widetilde{BPL}$, is trivial. However, up until now we have gotten away with only studying maps to $\widetilde{BG}$, and we have neither an explicit finite-type model for $\widetilde{BPL}$ nor a way of constructing the map. One way of getting around this would be to first determine whether the map to $\widetilde{BG}$ is trivial; if it is, then there is an induced map to G/PL which must also be trivial. To determine its triviality, we would need to compute the Pontryagin and Kervaire classes of M from its combinatorial structure. While it is known in principle that local combinatorial formulas can be used to compute these classes [LR78], no explicitly computable such formulas are known, except for the first Pontryagin class computed by Gaifullin; see the survey article [Gai05]. For example, the well-known construction of rational Pontryagin classes by Gelfand and MacPherson [GM92] uses either a smooth structure or an additional piece of data replacing it.

The path we take uses smoothing theory. As in the codimension 2 case, if the classifying map $M \to \widetilde{BPL}$ is trivial, M admits a smoothing (that is, a smooth structure which is compatible with the PL structure) which immerses smoothly in $\mathbb{R}^{m+1}$. Thus it is enough to construct all possible smoothings of M (finitely many, and perhaps none); then we can use the smooth algorithm to determine whether one of them immerses. This construction is given in §5.

Non-orientable manifolds. In this case constructing an immersion is equivalent to constructing a $\mathbb{Z}/2\mathbb{Z}$-invariant immersion of the oriented double cover. In other words, we must do what we did above but in a way that respects the natural free $\mathbb{Z}/2\mathbb{Z}$-action on each of the classifying spaces. This action is easy to encode computationally; moreover, as pointed out by Vokřínek [Vok17, §5] and elaborated in [ČKV17], the relevant homotopy theory can be done as easily as in the non-equivariant case. □

4. Applications to embeddings

4.1. **Immersions which extend to embeddings.** The following is a well-known fact, noted for example in [Mas59].

Lemma 4.1. *The normal bundle to an embedded smooth closed oriented submanifold $M^m \subseteq \mathbb{R}^n$ always has vanishing Euler class.*

Proof. Consider the diagram

$$\begin{array}{ccccc} H^{n-m}(\mathbb{R}^n, \mathbb{R}^n \setminus M) & \longrightarrow & H^{n-m}(\mathbb{R}^n) & & \\ \downarrow & & \downarrow & & \\ H^{n-m}(\nu_M, \nu_M \setminus M) & \xrightarrow{(*)} & H^{n-m}(\nu_M) & \longrightarrow & H^{n-m}(M). \end{array}$$

The Euler class is the image of the generator of $H^{n-m}(\nu_M, \nu_M \setminus M)$ along the bottom row. The left vertical arrow is an isomorphism by excision. Since $H^{n-m}(\mathbb{R}^n) = 0$, the arrow labeled $(*)$ is zero. □

This means that if M is closed and oriented, an immersion of M can only be regularly homotopic to an embedding if it has zero Euler class. Unlike the existence of an immersion in general, the existence of such an immersion is decidable via the same algorithm as in odd codimension: test whether all Pontryagin classes in degrees $2(n-m) \leq 4i \leq 2m$ are zero, and then resolve the remaining finite-order questions.

In other words, while it may well be that the embeddability of closed smooth manifolds in $\mathbb{R}^n$ is undecidable outside the metastable range, this cannot be a result of immersion theory.

4.2. **Embeddability is undecidable.**

Theorem 4.2. *Whenever $n - m$ is even and $11m \geq 10n + 1$, the embeddability of a smooth m-manifold with boundary in $\mathbb{R}^n$ is undecidable.*

We note that the method used here depends both on using the smooth category and on allowing the manifold to have boundary.

Proof. We reduce this statement to Theorem 3.1(ii). We note first that by the stability property discussed above, when $n \geq m+2$, an m-manifold M immerses smoothly in $\mathbb{R}^n$ if and only if $M \times D^k$ immerses smoothly in $\mathbb{R}^{n+k}$.

In general position, the self-intersection of an immersion $f : M \to N$ is a $(2m-n)$-dimensional CW complex. If we stabilize by crossing with $\mathbb{R}^k$ for $k \geq 4m - 2n + 1$, then this complex always has an embedding in $\mathbb{R}^k$; therefore the immersion

$$f \times \mathrm{id} : M \times D^k \to \mathbb{R}^{n+k}$$

can be deformed to an embedding, by pushing a neighborhood of the self-intersection off itself in the $\mathbb{R}^k$ direction. Conversely, if M does not immerse in $\mathbb{R}^n$, then $M \times D^k$ does not embed in $\mathbb{R}^{n+k}$.

If $m = 4c$ and $n = 5c$, then we can choose $k = 6c + 1$. In other words, it is undecidable whether a $(10c+1)$-manifold embeds into $\mathbb{R}^{11c+1}$ when c is even. □

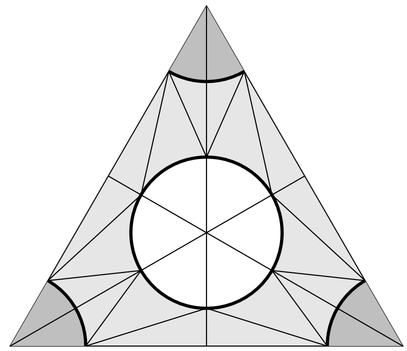

FIGURE 1. A subdivision of Δ^2 satisfying the required conditions. The sets U_0 and U_1 are highlighted in shades of gray. We have drawn the simplices curved to suggest how U_0 and U_1 will look as subsets of a smooth manifold with boundary.

5. Computing all smoothings of a PL manifold

In this section we sketch an algorithm which, given a triangulation of a PL manifold M^m, computes a set of C^1 manifolds which contains at least one (but usually many) representatives of each diffeomorphism type of smoothing of M. The manifolds are given in the form of a subdivision of the original triangulation equipped with a simplexwise polynomial immersion to some $\mathbb{R}^N$, as described in §2.2.1. Of course, if M is not smoothable, the algorithm yields the empty set.

The algorithm naturally splits into two pieces: some computations related to the groups Θ_k of exotic spheres in dimensions $k \le m$, and an inductive procedure which relies on those computations.

The inductive procedure. We start by fixing a subdivision of the m-simplex such that for each $0 \le k \le m$ there is a pure m-dimensional subcomplex $U_k \subset \Delta^m$ such that:

- U_k deformation retracts to the k-skeleton of Δ^m.
- U_k is invariant under permutations of the vertices of Δ^m.
- U_k is contained in the interior of U_{k+1}.

An example of such a subdivision of Δ^2 is illustrated in Figure 1. We write

$$V_k = \bigcup_{\sigma \in M} U_k.$$

We then construct all possible smoothings of M by induction on k: first we construct all possible smoothings of V_0, then extend them to V_1 in all possible ways, and so on. At each step each representative will be encoded via a smooth map from a further subdivision to $\mathbb{R}^N$, for some fixed $N \ge 2m+1$. Once we have extended to $V_m = M$, we have generated representatives of all possible smoothings. The base case is clear: since there is a unique smooth structure on a compact PL disk, we can choose an arbitrary smooth map $V_0 \to \mathbb{R}^N$.

Now suppose we have defined a smoothing $f_{k-1} : V_{k-1} \to \mathbb{R}^N$. Then the kth step of the induction proceeds as follows, for every k-simplex σ of M:

1. Determine whether the map on $\partial V_{k-1} \cap \sigma$ is diffeomorphic to the standard $(k-1)$-sphere. If it isn't, then the smoothing does not extend to σ.
2. If the smoothing extends, we extend it over σ by iterating over all possible piecewise polynomial smooth maps until we find one that works. The result is a smoothing of $V_{k-1} \cup M^k$: that is, it is both a C^1 embedding of an m-manifold with boundary when restricted to V_{k-1} and a C^1 embedding of a k-manifold when restricted to the interior of each k-simplex.
3. For every exotic k-sphere, we surger in a D^k which modifies the smoothing on $\sigma \setminus V_{k-1}$ by that k-sphere. That is, we cut out a (subdivided) simplex of $\sigma \setminus V_{k-1}$ and glue in an exotic

sphere missing a simplex, with a cylindrical "neck" making a C^1 connection between them. Thinking of this as a map from σ entails a further subdivision.

(4) For every smoothing of $V_{k-1} \cup M^k$ thus generated, the stability of smoothing theory guarantees that there is a unique smooth structure extending it over V_k, which deformation retracts to $V_{k-1} \cup M^k$. We construct such an extension by exhaustive search.

Algorithms for exotic spheres. To give detailed instructions for steps (1) and (3), we must describe algorithms for constructing and classifying exotic spheres. In every dimension k, the exotic spheres are classified by a finite abelian group Θ_k whose group operation is connected sum. To perform steps (1) and (3), it would be enough to have an algorithm which, given a smooth manifold PL homeomorphic to the sphere, computes the corresponding element of Θ_k. For step (1), we must simply test whether the element of Θ_k induced by $f_{k-1}|\partial V_{k-1} \cap \sigma$ is zero. For step (3), we can generate all the exotic spheres by iterating over all possible piecewise polynomial smooth maps from subdivisions of $\partial\Delta^{k+1}$ to $\mathbb{R}^N$ until we find representatives for every element of Θ_k. Then we can get the desired disks by cutting out a k-simplex from each of these.

In fact, we find something slightly weaker. To analyze the group Θ_k, we look at the original paper of Kervaire and Milnor [KM63] where it is defined, as well as an expository paper of Levine [Lev85] which fills in certain details developed later. It turns out that Θ_k naturally fits into an exact sequence, whose terms we will define later:

$$0 \to bP_{k+1} \to \Theta_k \xrightarrow{\psi} \operatorname{coker}\big(\pi_k(SO_{k+1}) \xrightarrow{J_k} \pi_{2k+1}(S^{k+1})\big) \xrightarrow{\phi} P_k.$$

We sketch algorithms which, given a smooth manifold PL homeomorphic to the sphere,

(∗) compute the corresponding element of Θ_k/bP_{k+1};

(†) if this element is zero, compute the corresponding element of bP_{k+1}.

This is clearly enough for step (1); for step (3), if we generate representatives of all elements of Θ_k/bP_{k+1} and all elements of bP_{k+1}, we can generate representatives of all elements of Θ_k by taking connect sums.

We now discuss the terms of the exact sequence above:

- The group $P_k = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & k \equiv 2 \mod 4 \\ \mathbb{Z} & k \equiv 0 \mod 4. \end{cases}$
- The map ϕ sends a smooth map $f : S^{2k+1} \to S^{k+1}$ to the Kervaire invariant (if $k \equiv 2 \mod 4$) or 1/8 times the signature (if $k \equiv 0 \mod 4$) of the preimage of a regular point.
- The map J_k is the usual J-homomorphism, defined as follows. An element of $\pi_r(SO_q)$ can be interpreted as a map $S^r \times S^{q-1} \to S^{q-1}$. This in turn induces a map from the join $S^r * S^{q-1} \cong S^{r+q}$ to the suspension of S^{q-1}, that is, S^q.
- The group bP_{k+1} is a certain finite quotient of P_{k+1}. In the nontrivial case $k+1 = 2r$, this has order which divides
$$2^{2r-1} \cdot (2^{2r-1} - 1) \cdot \operatorname{numerator}(B_r/r),$$
where $2r = k+1$ and B_r is the rth Bernoulli number.
- The map ψ is constructed as follows. Every smooth homotopy sphere Σ is stably parallelizable. This means that given an embedding $\Sigma \hookrightarrow S^{2k+1}$, one can construct a trivialization of the normal bundle and use the Pontryagin–Thom construction to give a map $S^{2k+1} \to S^{k+1}$. This depends on the choice of trivialization, and the indeterminacy is exactly the image of the J-homomorphism.
- Finally, an isomorphism between $\ker\psi$ and bP_{k+1} is given as follows. If $\Sigma \in \ker\psi$, then Σ is framed nullcobordant. Then the corresponding element in P_{k+1} is given by the Kervaire invariant (when $k+1$ is odd) or 1/8 signature (rel boundary, when $k+1$ is even) of a

nullcobordism with parallelizable normal bundle; this has an indeterminacy which induces the quotient map b.

It remains to show that all of these elements can be computed.

The signature and Kervaire invariant are cohomological notions and so are unproblematic to compute from a triangulation.

A generator for $\pi_k(SO_{k+1})$ can be constructed explicitly as a simplicial map by the main theorem of [FFWZ18]. Then a corresponding simplicial map $S^k \times S^k \to S^k$ can be constructed by induction on skeleta of S^r. Finally, the Hopf construction of a map from the join to the suspension is clearly algorithmic. This gives an algorithm for determining the image of the J-homomorphism.

By results of [ČKM$^+$14b], $\pi_{2k+1}(S^{k+1})$ is fully effective: that is, we can compute a set of generators and find the combination of generators which represents the homotopy class of a given simplicial map. In particular, this allows us to compute the cokernel of the J-homomorphism.

The main remaining obstacle is implementing the map ψ. We can do this by explicitly constructing, given a smooth homotopy k-sphere Σ, a map $S^{2k+1} \to S^{k+1}$ realizing the Pontryagin–Thom construction.

If Σ is specified by a C^1 piecewise polynomial embedding $f : X \to \mathbb{R}^N$, where X is a simplicial complex homeomorphic to S^k, then a smooth structure on $\Sigma \times D^{k+1}$ (with each product of simplices triangulated in a standard way) is given by

$$f \times i : \Sigma \times D^{k+1} \to \mathbb{R}^{N+k+1}.$$

We then iterate through simplexwise polynomial maps with rational coefficients from subdivisions of $\Sigma \times D^{k+1}$ to $[0,1]^{2k+1}$ until we find a map $g : \Sigma \times D^{k+1} \to [0,1]^{2k+1}$ that is C^1 and injective. Both these conditions can be checked. One checks C^1 by checking that derivatives match between neighboring simplices. To check injectivity, one shows that every simplex is injective, the images of any two non-adjacent simplices are disjoint, and the intersection of the images of adjacent simplices is the image of their intersection. Each of these can be expressed as a sentence in the language of the reals and can be decided by the Tarski–Seidenberg theorem; see e.g. [BPR06, Ch. 11] for relatively practical algorithms.

Now, let $p : \Sigma \times D^{k+1} \to S^{k+1}$ be the map which projects to the D^{k+1} factor and then collapses the boundary. This map is easily made simplicial. The desired map $S^{2k+1} \to S^{k+1}$ is obtained from a map $u : [0,1]^{2k+1} \to S^{k+1}$ which maps the image of g to S^{k+1} via $p \circ g^{-1}$ and everything outside the image of g to the base point. Although g^{-1} is not piecewise polynomial, we can approximate its values to arbitrary precision and bound its Lipschitz constant. By Lemma 2.4, this suffices to construct a simplicial approximation to u from a sufficiently fine subdivision of $[0,1]^{2k+1}$. From this we can compute its homotopy class in $\pi_{2k+1}(S^{k+1})$, and therefore the image of Σ under ψ.

Moreover, given a map in $\ker \psi$, we can find a framed nullcobordism by iterating over all candidate smooth manifolds Υ whose boundary is Σ and smooth embeddings $\Upsilon \times D^{k+1} \to D^{2k+2}$ extending g. Thus, given a homotopy sphere, we can assign it either to a nonzero element of Θ_k/bP_{k+1} or an element of bP_{k+1}. By iterating over all smooth triangulations corresponding to barycentric subdivisions of $\partial\Delta^{k+1}$, we eventually generate representatives for all the elements of both the subgroup and the quotient group.

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