Periods of generalized Fermat curves

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Abstract

Let $k, n \geq 2$ be integers. A generalized Fermat curve of type (k, n) is a compact Riemann surface S that admits a subgroup of conformal automorphisms $H \leq \mathrm{Aut}(S)$ isomorphic to $\mathbb{Z}_k^n$, such that the quotient surface S/H is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$ and has $n+1$ branch points, each one of order k. There exists a good algebraic model for these objects, which makes them easier to study. Using tools from algebraic topology and integration theory on Riemann surfaces, we find a set of generators for the first homology group of a generalized Fermat curve. Finally, with this information, we find a set of generators for the period lattice of the associated Jacobian variety.

Keywords: Complex Geometry, Riemann Surfaces, Jacobian Variety, Generalized Fermat Curve

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1. Introduction

The Jacobian variety JS of a compact Riemann surface S of genus g is isomorphic to a complex torus of dimension g, i.e., a quotient $\mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is the period lattice ($\Lambda \cong \mathbb{Z}^{2g}$) of S that depends on the analytical and algebraic-topological structure of S. The importance of JS is due to Torelli's theorem, which states that the principally polarized abelian variety JS determines the Riemann surface S up to biholomorphism.

Thus, if the Jacobian variety is in the form $\mathbb{C}^g/\Lambda$, the period lattice Λ with the corresponding polarization determines S. However, to find an explicit form for the period lattice of a particular compact Riemann surface is a difficult task and there is no standard method to do it.

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We restrict attention to an interesting family of compact Riemann surfaces called generalized Fermat curves of type (k, n), where $k, n \geq 2$ are integers. In [2] it was noticed that such a Riemann surface can be described as a suitable fiber product of $(n-1)$ classical Fermat curves of degree k. In this paper we find a generating set for the period lattice of a generalized Fermat curve, based on the work of Rohrlich [1] who found a generating set for the period lattice of the classical Fermat curve of degree $k \geq 4$.

2. Preliminaries

2.1. The Jacobian variety

Let S be a compact Riemann surface of genus $g \geq 0$. Its first homology group $H_1(S, \mathbb{Z})$ is a free Abelian group of rank $2g$, and the complex vector space $H^{1,0}(S)$ of its holomorphic 1-forms has dimension g. There is a natural $\mathbb{Z}$-linear injective map

$$\tau : H_1(S, \mathbb{Z}) \hookrightarrow (H^{1,0}(S))^*$$
$$\gamma \mapsto \tau(\gamma)(\cdot) := \int_\gamma \cdot,$$

where $(H^{1,0}(S))^*$ is the dual space of $H^{1,0}(S)$. The image $\tau(H_1(S, \mathbb{Z}))$ is a lattice in $(H^{1,0}(S))^*$, and the quotient g-dimensional torus

$$JS := (H^{1,0}(S))^* / \tau(H_1(S, \mathbb{Z}))$$

is called the *Jacobian variety* of S. It is a fact that JS admits a principal polarization defined by the Hermitian form on $H^{1,0}(S)$ given by

$$(\omega_1, \omega_2) \to \int \omega_1 \wedge \overline{\omega_2}.$$

If $\{\omega_1, \dots, \omega_g\}$ is a basis for $H^{1,0}(S)$, then we have the isomorphism $(H^{1,0}(S))^* \cong \mathbb{C}^g$, and if $\{\gamma_1, ..., \gamma_m\}$ is a finite generating set for $H_1(S, \mathbb{Z})$ (need not be a basis), then we can see $\tau(H_1(S, \mathbb{Z}))$ as the lattice Λ in $\mathbb{C}^g$ generated by the collection

$$C_i = \left(\int_{\gamma_i} \omega_1, \int_{\gamma_i} \omega_2, \dots, \int_{\gamma_i} \omega_g \right), \;\; 1 \leq i \leq m.$$

The lattice generated by the C_i‘s is called the *period lattice* of S, and in the case where m is the rank $2g$ of $H_1(S, \mathbb{Z})$, we can find the Riemann matrix of S, which allows us to study JS as a polarized Abelian variety.

2.2. Generalized Fermat curves

Let $k, n \geq 2$ be integers. A compact Riemann surface S is called a *generalized Fermat curve* of type (k, n) if it admits a subgroup of conformal automorphisms $H \leq \mathrm{Aut}(S)$ that is isomorphic to $\mathbb{Z}_k^n$ (where $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$), such that the quotient surface S/H is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$ and has $n+1$ branch points, each one of order k. In this case the subgroup H is called a *generalized Fermat group* of type (k, n), and the pair (S, H) is called a *generalized Fermat pair of type* (k, n). As a consequence of the Riemann-Hurwitz formula given in Corollary 1.2 of [6] or Proposition 1.2 of [7], the genus $g_{k,n}$ of a generalized Fermat curve of type (k, n) is

$$g_{k,n} = \frac{2 + k^{n-1}((n-1)(k-1) - 2)}{2}.$$

We say that two generalized Fermat pairs (S_1, H_1) and (S_2, H_2) are *holomorphically equivalent* if there exists a biholomorphism $f : S_1 \to S_2$ such that $fH_1f^{-1} = H_2$.

Remark 1. *Note that generalized Fermat curves of type* $(k, 1)$ *are just cyclic covers of degree* k *of* $\hat{\mathbb{C}}$ *with two branch points, which are all of genus* 0*. From [5] we know that the Fermat curve of degree* $k \geq 2$ *given by*

$$\{[x_0, x_1, x_2] \in \mathbb{P}^2\mathbb{C} : x_0^k + x_1^k + x_2^k = 0\}$$

has a subgroup of conformal automorphisms isomorphic to $\mathbb{Z}_k^2$*, where the quotient surface is biholomorphic to the Riemann sphere with three branch points* $\infty, 0, 1$*. Thus the classical Fermat curves are generalized Fermat curves of type* $(k, 2)$*.*

Remark 2. *The non-hyperbolic case, i.e., when* $g_{k,n} \leq 1$*, are given by* $(k, n) \in \{(2, 2), (2, 3), (3, 2)\}$*, or* $k = 1$*. See [2] for explicit examples.*

Let (S, H) be a generalized Fermat pair of type (k, n) and, up to a Moebius transformation, let $\{\infty, 0, 1, \lambda_1, \lambda_2, ..., \lambda_{n-2}\}$ be the branch points of the quotient S/H. Let us consider the following fiber product of $n - 1$ classical

Fermat curves:

$$C_k(\lambda_1, ..., \lambda_{n-2}) := \left\{ \begin{array}{ccc} x_0^k + x_1^k + x_2^k & = & 0 \\ \lambda_1 x_0^k + x_1^k + x_3^k & = & 0 \\ \lambda_2 x_0^k + x_1^k + x_4^k & = & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{n-2} x_0^k + x_1^k + x_n^k & = & 0 \end{array} \right\} \subset \mathbb{P}^n\mathbb{C}. \qquad (1)$$

Since the values λ_i are pairwise different and each one is different from 0 and 1, the algebraic curve $C_k(\lambda_1, ..., \lambda_{n-2})$ is a non-singular projective algebraic curve, hence a compact Riemann surface.

On $C_k(\lambda_1, ..., \lambda_{n-2})$ we have the abelian group $H_0 \cong \mathbb{Z}_k^n$ of conformal automorphisms generated by the maps

$$a_i([x_0, \cdots, x_n]) = [x_0, \cdots, x_{i-1}, \zeta_k x_i, x_{i+1}, \cdots, x_n], \quad i = 0, ..., n,$$

where $\zeta_k = e^{2\pi i/k}$. Let us consider the holomorphic map of degree k^n

$$\begin{array}{cccc} \pi : & C_k(\lambda_1, ..., \lambda_{n-2}) & \to & \hat{\mathbb{C}} \\ & [x_0, \cdots, x_n] & \mapsto & -\left(\frac{x_1}{x_0}\right)^k, \end{array}$$

with the property $\pi \circ a_i = \pi$ for each $i = 1, \dots, n$. So π induces a biholomorphism

$$\begin{array}{cccc} \hat{\pi} : & C_k(\lambda_1, ..., \lambda_{n-2})/H_0 & \to & \hat{\mathbb{C}} \\ & H_0 p & \mapsto & \pi(p). \end{array}$$

Furthermore, the map π has $n+1$ branch points given by

$$\{\infty, 0, 1, \lambda_1, \lambda_2, ..., \lambda_{n-2}\}.$$

It follows that $C_k(\lambda_1, ..., \lambda_{n-2})$ is a generalized Fermat curve of type (k, n) with generalized Fermat group H_0, whose standard generators are $a_1, \dots, a_n$ and $a_0 = (a_1 a_2 \dots a_n)^{-1}$. Using the above notation, the following result was proved in [2].

Theorem 1. *The generalized Fermat pairs (S, H) and $(C_k(\lambda_1, ..., \lambda_{n-2}), H_0)$ are holomorphically equivalent.*

On $C_k(\lambda_1, ..., \lambda_{n-2})$ we have the following meromorphic maps

$$y_j = \frac{x_j}{x_0} : C_k(\lambda_1, ..., \lambda_{n-2}) \to \hat{\mathbb{C}}, \ \ j = 1, ..., n.$$

We consider the set $I_{k,n}$ of tuples $(\alpha_1, \dots, \alpha_n)$ such that

$$\alpha_i \in \mathbb{Z}, \quad 0 \leq \alpha_2, \dots, \alpha_n \leq k-1, \quad 0 \leq \alpha_1 \leq \sum_{i=2}^{n} \alpha_i - 2,$$

and define the meromorphic form

$$\theta_{\alpha_1,...,\alpha_n} := \frac{y_1^{\alpha_1} dy_1}{y_2^{\alpha_2} \dots y_n^{\alpha_n}},$$

for each $(\alpha_1, \dots, \alpha_n) \in I_{k,n}$.The paper [3] proved the following.

Theorem 2. *With the above notation, the following holds:*

1. $\theta_{\alpha_1,...,\alpha_n}$ *is holomorphic for every* $(\alpha_1, \dots, \alpha_n) \in I_{k,n}$.
2. $\#I_{k,n} = g_{k,n}$.
3. *The collection*

$$\{\theta_{\alpha_1,...,\alpha_n}\}_{(\alpha_1,...,\alpha_n) \in I_{k,n}}$$

is a basis for the space $H^{1,0}(C_k(\lambda_1, ..., \lambda_{n-2}))$ *of holomorphic* 1*-forms.*

For simplicity, in the rest of this paper we write $C_{k,n}$ instead of $C_k(\lambda_1, ..., \lambda_{n-2})$.

2.3. The logarithm symbol on the punctured plane

Let $R \subset \mathbb{C}$ a finite subset with $0 \in R$ and $|R| \geq 2$. The elements of R are denoted by r_i, with $1 \leq i \leq |R|$. Then we consider a universal covering of $\mathbb{C} - R$ given by

$$p : U \to \mathbb{C} - R.$$

Since p is holomorphic, we have the family of holomorphic functions $p_i = p - r_i$ with $1 \leq i \leq |R|$. The function p_i does not vanish on U, so there exists a determination of $\log p_i$ on U such that

$$\exp(\log p_i) = p_i.$$

Let $\mathrm{Deck}(p)$ be the group of covering transformations of p. Then for every $\phi \in \mathrm{Deck}(p)$ the function

$$u \to \frac{1}{2\pi i}(\log p_i(\phi(u)) - \log p_i(u))$$

on U is identically an integer. This integer is independent of the choice of $\log p_i$, and we denote it by $L(p_i, \phi)$. It is not difficult to see that the symbol $L(p_i, \cdot)$ satisfies

$$L(p_i, \phi \circ \psi) = L(p_i, \phi) + L(p_i, \psi), \tag{2}$$

for every $\phi, \psi \in \text{Deck}(p)$. Furthermore, we observe the following.

Lemma 1. *Let $\hat{x}_i : U \to \mathbb{C}$ be the kth root of p_i defined by*

$$\hat{x}_i = \exp\left(\frac{1}{k}\log p_i\right).$$

Then for every $\phi \in \text{Deck}(p)$ we have

$$\hat{x}_i \circ \phi = \zeta_k^{L(p_i,\phi)} \hat{x}_i.$$

Recall that $\text{Deck}(p)$ is isomorphic to the fundamental group $\pi_1(\mathbb{C} - R)$, which is a free group generated by $|R|$ elements, each one homotopic to a circle with center r_i and index one. Then we consider the generators $\phi_1, ..., \phi_{|R|} \in \text{Deck}(p)$ associated with each generator of $\pi_1(\mathbb{C} - R)$, and for any $u \in U$ we have the equality

$$L(p_i, \phi_j) = \frac{1}{2\pi i}\int_u^{\phi_j(u)} d\log p_i = \delta_{ij},$$

where δ_{ij} is the usual Kronecker delta.

For general aspects of the logarithm symbol on Riemann surfaces, see [4].

3. Generating set for the period lattice of $C_{k,n}$

Consider the generalized Fermat curve of type (k, n) given by Equation (1) and the set of $n + 1$ branch points $R \cup \{\infty\}$, where

$$R = \{r_1 = 0, r_2 = \lambda_0 = 1, r_3 = \lambda_1, \ldots, r_n = \lambda_{n-2}\}.$$

3.1. A finite generating set for $H_1(C_{k,n}, \mathbb{Z})$

Associated with R, we have the universal covering $p : U \to \mathbb{C} - R$. We have the set of functions

$$\begin{cases} p_1 &= -p \\ p_i &= p - r_i \quad \text{for } 2 \leq i \leq n. \end{cases}$$

There exists a kth root $\hat{x}_i$ of p_i, which by Lemma 1 satisfies

$$\hat{x}_i \circ \phi = \zeta_k^{L(p_i,\phi)} \hat{x}_i$$

for each $\phi \in \mathrm{Deck}(p)$. From Equation (2) we have the surjective homomorphism

$$\Psi : \mathrm{Deck}(p) \to \mathbb{Z}_k^n, \quad \Psi(\phi) = (L(p_1, \phi), \dots, L(p_n, \phi)) \mod k,$$

so it is not difficult to deduce the following fact.

Lemma 2. *The subgroup of* $\mathrm{Deck}(p)$ *which leaves each* $\hat{x}_i$ *invariant is*

$$\mathrm{Deck}(p)_k := Ker(\Psi).$$

Now we consider the punctured Riemann surface $C'_{k.n} = C_{k,n} - \pi^{-1}(R \cup \{\infty\})$. We now prove the following result.

Lemma 3. *The map*

$$q : U \to C'_{k,n}, \quad u \mapsto q(u) = [1, \hat{x}_1(u), \cdots, \hat{x}_n(u)]$$

is a universal covering of $C'_{k,n}$*, with* $\mathrm{Deck}(q) = \mathrm{Deck}(p)_k$.

Proof. Since $r_i + \hat{x}_1^k + \hat{x}_i^k = \lambda_{i-2} - p + (p - r_i) = 0$ for $i \geq 2$, we have $q(U) \subset C'_{k,n}$. Let $a \in C'_{k,n}$ and assume $a = [1, a_1, ..., a_n]$, with $a_i \neq 0$ for every i. If $q(u) = a$, then $\hat{x}_i(u) = a_i$ for each i. In particular $\hat{x}_1(u)^k = -p(u) = a_1^k$. Since p is surjective, there exists $u \in U$ such that $p(u) = -a_1^k$, and hence $\hat{x}_1(u) = \zeta_k^{j_1} a_1$ for some integer j_1. Since $a_i^k = -a_1^k - r_i$ for $1 < i \leq n$ and

$$\hat{x}_i^k(u) = p(u) - \lambda_{i-2} = -a_1^k - r_i = a_i^k, \quad 1 < i \leq n,$$

we have $\hat{x}_i(u) = \zeta_k^{j_i} a_i$ with an integer j_i for each i. Now we choose $\phi \in \mathrm{Deck}(p)$ such that

$$L(p_i, \phi) = -j_i$$

for every i, we get $q(\phi(u)) = a$, and therefore q is surjective. Now q is a covering map because every $a \in C'_{k,n}$ has an evenly covered neighborhood since p is a covering map. Finally, if $u, v \in q^{-1}(a)$ with $a \in C'_{k,n}$, then $v = \phi(u)$ for some $\phi \in \operatorname{Deck}(p)$. Now, as

$$q(u) = [1, \hat{x}_1(u), ..., \hat{x}_n(u)] = [1, \zeta_k^{L(p_1,\phi)} \hat{x}_1(u), ..., \zeta_k^{L(p_n,\phi)} \hat{x}_n(u)] = q(v),$$

we have $\phi \in \operatorname{Deck}(p)_k$. □

From the previous two lemmas, we have

Lemma 4. *The map Ψ gives an isomorphism*

$$\operatorname{Deck}(p)/\operatorname{Deck}(q) \cong \mathbb{Z}_k^n.$$

We denote by $\phi_1, \ldots, \phi_n$ the n generators of $\operatorname{Deck}(p)$ with

$$L(p_i, \phi_j) = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Lemma 5. $\operatorname{Deck}(q)$ *is generated by*

$$\phi_i^k \textit{ for each } 1 \leq i \leq n \quad \textit{and} \quad [\operatorname{Deck}(p), \operatorname{Deck}(p)],$$

where $[\operatorname{Deck}(p), \operatorname{Deck}(p)]$ *is the commutator subgroup of* $\operatorname{Deck}(p)$.

Proof. Since $\operatorname{Deck}(p)/\operatorname{Deck}(q)$ is Abelian, we have that $[\operatorname{Deck}(p), \operatorname{Deck}(p)] \trianglelefteq \operatorname{Deck}(q)$. We also know that the free generators $\phi_1, ..., \phi_n$ of $\operatorname{Deck}(p)$ correspond to the canonical basis of $\mathbb{Z}_k^n$ by Ψ, hence $\Psi(\phi_i^k) = 0$ for each i. If $K \leq \operatorname{Deck}(p)$ is the subgroup generated by each ϕ_i^k and $[\operatorname{Deck}(p), \operatorname{Deck}(p)]$, then we have

$$\operatorname{Deck}(p)/K \cong \mathbb{Z}_k^n.$$

So we must have $\operatorname{Deck}(q) = K$. □

Recall that $\pi_1(C'_{k,n}) = \operatorname{Deck}(q)$ by Lemma 3, so we have

Theorem 3. *The first homology group of $C'_{k,n}$, namely*

$$H_1(C'_{k,n}, \mathbb{Z}) \cong \frac{\operatorname{Deck}(q)}{[\operatorname{Deck}(q), \operatorname{Deck}(q)]},$$

is generated by the classes of the elements

$$\phi_i^k, \quad 1 \leq i \leq n$$

and

$$\left(\prod_{d=1}^{n} \phi_d^{g_d}\right) [\phi_j, \phi_l] \left(\prod_{d=1}^{n} \phi_d^{g_d}\right)^{-1},$$

with integers $1 \leq j < l \leq n$ *and* $0 \leq g_d \leq k-1$.

Proof. Since $\mathrm{Deck}(q)$ is generated by ϕ_i^k and $[\mathrm{Deck}(p), \mathrm{Deck}(p)]$, it is generated by

$$\phi_i^k, \quad 1 \leq i \leq n,$$

and

$$\gamma[\phi_j, \phi_l]\gamma^{-1},$$

with $\gamma \in \mathrm{Deck}(p)$ and $1 \leq j < l \leq n$. We have $\mathrm{Deck}(p)/\mathrm{Deck}(q) = \mathbb{Z}_k^n$, so $\{\prod_{d=1}^n \phi_d^{g_d}\}_{0 \leq g_d \leq k-1}$ is a set of representatives such that every $\gamma \in \mathrm{Deck}(p)$ lies in $\mathrm{Deck}(q)\rho$ for precisely one ρ from this set.
Choosing the representative $\rho \in \{\prod_{d=1}^n \phi_d^{g_d}\}_{0 \leq g_d \leq k-1}$, we have $\gamma = \sigma\rho$ with $\sigma \in \mathrm{Deck}(q)$, and

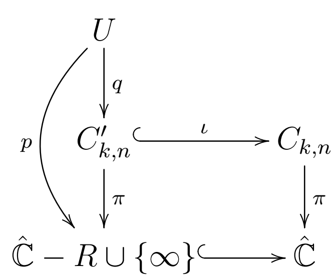
$$\gamma[\phi_j, \phi_l]\gamma^{-1} = \sigma(\rho[\phi_j, \phi_l]\rho^{-1})\sigma^{-1}$$

as a product of elements in $\mathrm{Deck}(q)$. Quotienting by $[\mathrm{Deck}(q), \mathrm{Deck}(q)]$ the product commutes, and the σ's cancel. $\square$

Since the inclusion map $\iota : C'_{k,n} \hookrightarrow C_{k,n}$ induces a surjective homomorphism between the homology groups, we have

Corollary 1. *The images of the generating set of* $H_1(C'_{k,n}, \mathbb{Z})$ *under the homomorphism induced by the inclusion* $\iota : C'_{k,n} \hookrightarrow C_{k,n}$ *forms a generating set for* $H_1(C_{k,n}, \mathbb{Z})$.

We summarize the maps used in the following diagram.



3.2. Computing periods

Let $\phi \in \mathrm{Deck}(p)$ and fix $u \in U$. We denote by l_ϕ a curve from u to $\phi(u)$ on U. So a generating set for $H_1(C_{k,n}, \mathbb{Z})$ are the homology classes of the curves $\iota \circ q(l_\phi)$ for each ϕ of the form

$$\phi_i^k, \quad \left(\prod_{d=1}^n \phi_d^{g_d}\right)[\phi_j, \phi_l]\left(\prod_{d=1}^n \phi_d^{g_d}\right)^{-1}$$

for $1 \leq i \leq n$, $1 \leq j < l \leq n$, and $0 \leq g_d \leq k-1$. Thus, to find an explicit generating set for the period lattice of $C_{k,n}$ we need to calculate

$$\int_{\iota \circ q \circ l_\phi} \theta_{\alpha_1,\dots,\alpha_n} = \int_{l_\phi} q^*\theta_{\alpha_1,\dots,\alpha_n} = \int_u^{\phi(u)} q^*\theta_{\alpha_1,\dots,\alpha_n}.$$

Lemma 6. *We have the following relations between the induced pullbacks of the generators of* $H^{1,0}(C_{k,n})$*:*

$$q^*\theta_{\alpha_1,\dots,\alpha_n} = \frac{\hat{x}_1^{\alpha_1} d\hat{x}_1}{\hat{x}_2^{\alpha_2}\cdots\hat{x}_n^{\alpha_n}}, \tag{3}$$

$$\phi^* q^*\theta_{\alpha_1,\dots,\alpha_n} = \zeta_k^{(\alpha_1+1)L(p_1,\phi)-\sum_{d=2}^n \alpha_d L(p_d,\phi)\alpha_d} q^*\theta_{\alpha_1,\dots,\alpha_n}, \tag{4}$$

for each $\phi \in \mathrm{Deck}(p)$*. In particular,* $q^*\theta_{\alpha_1,\dots,\alpha_n}$ *is an eigenvector for each* $\phi^* \in \mathrm{Deck}(p)$*.*

Proof. The first result follows from the observation that $\hat{x}_i = y_i \circ q$ for each i. The second follows from Lemma 1 in Section 2.3. □

We denote by M_i the value $(\alpha_1+1)L(p_1,\phi_i) - \sum_{d=2}^n \alpha_d L(p_d,\phi_i)$. We observe that

$$M_i = \begin{cases} \alpha_1 + 1 & i = 1 \\ -\alpha_{i+1} & 2 \leq i \leq n. \end{cases}$$

Moreover, since U is a simply connected domain, we have for $\phi, \psi \in \mathrm{Deck}(p)$ and $\omega \in H^{1,0}(U)$ the relation

$$\int_u^{\phi\circ\psi(u)} \omega = \int_u^{\phi(u)} \omega + \int_u^{\psi(u)} \phi^*\omega.$$

Lemma 7. *For each ϕ_i^k with $i \in \{1, ..., n\}$ we have*

$$\int_{\iota \circ q \circ l_{\phi_i^k}} \theta_{\alpha_1,\dots,\alpha_n} = 0.$$

Proof. If M_i is non zero, then from Equation (4) of Lemma 6 we obtain

$$\begin{aligned}\int_{\iota \circ q \circ l_{\phi_i^k}} \theta_{\alpha_1,\dots,\alpha_n} &= \int_u^{\phi_i^k u} q^*\theta_{\alpha_1,\dots,\alpha_n} \\ &= \sum_{m=1}^k \int_u^{\phi_i u} (\phi_i^{m-1})^* q^*\theta_{\alpha_1,\dots,\alpha_n} \\ &= \int_u^{\phi_i u} q^*\theta_{\alpha_1,\dots,\alpha_n} \sum_{m=1}^k \zeta_k^{(m-1)M_i} \\ &= 0.\end{aligned}$$

In the case where $M_i = 0$, the differential form $\theta_{\alpha_1,\dots,\alpha_n}$ is holomorphic on the interior of the loop $\iota \circ q \circ l_{\phi_i^k}$, so that the integral vanishes. □

We conclude that the homology class of each ϕ_i^k is null in $H(C_{k,n}, \mathbb{Z})$, which reduces the problem to computing the integrals over

$$\left(\prod_{d=1}^n \phi_d^{g_d}\right) [\phi_j, \phi_l] \left(\prod_{d=1}^n \phi_d^{g_d}\right)^{-1}.$$

Lemma 8. *For each $\sigma = \rho[\phi_j, \phi_l]\rho^{-1}$ with $\rho = \prod_{d=1}^n \phi_d^{g_d}$ we have*

$$\int_{\iota \circ q \circ l_\sigma} \theta_{\alpha_1,\dots,\alpha_n} = \zeta_k^{\sum_{d=1}^n g_d M_d} \int_{\iota \circ q \circ l_{[\phi_j,\phi_l]}} \theta_{\alpha_1,\dots,\alpha_n}.$$

Proof. From Lemma 6 and the observation that $[\phi_j, \phi_l] \in \mathrm{Deck}(q)$ leaves

$q^*\theta_{\alpha_1,\dots,\alpha_n}$ invariant, it follows that

$$\begin{aligned}\int_{\iota\circ q\circ l_\sigma}\theta_{\alpha_1,\dots,\alpha_n} &= \int_u^{\rho[\phi_j,\phi_l]\rho^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \int_{\rho\rho^{-1}u}^{\rho[\phi_j,\phi_l]\rho^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \int_{\rho^{-1}u}^{[\phi_j,\phi_l]\rho^{-1}u} \rho^*q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \zeta_k^{\sum_{d=1}^n g_dM_d}\int_{\rho^{-1}u}^{[\phi_j,\phi_l]\rho^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \zeta_k^{\sum_{d=1}^n g_dM_d}\left(\int_{\rho^{-1}u}^{[\phi_j,\phi_l]u}+\int_{[\phi_j,\phi_l]u}^{[\phi_j,\phi_l]\rho^{-1}u}\right) q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \zeta_k^{\sum_{d=1}^n g_dM_d}\int_u^{[\phi_j,\phi_l]u} q^*\theta_{\alpha_1,\dots,\alpha_n}.\end{aligned}$$

□

Lemma 9. *For each $j,l\in\{1,...,n\}$ we have*

$$\begin{aligned}\int_{\iota\circ q\circ l_{[\phi_j,\phi_l]}}\theta_{\alpha_1,\dots,\alpha_n} &= (1-\zeta_k^{M_l})\int_u^{\phi_j u} q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &\quad -(1-\zeta_k^{M_j})\int_u^{\phi_l u} q^*\theta_{\alpha_1,\dots,\alpha_n}.\end{aligned}$$

Proof. Again from Lemma 6 we have the relations

$$\begin{aligned}\int_u^{[\phi_j,\phi_l]u} q^*\theta_{\alpha_1,\dots,\alpha_n} &= \int_u^{\phi_j u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \int_{\phi_j u}^{\phi_j\phi_l\phi_j^{-1}\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \int_u^{\phi_j u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \int_u^{\phi_l\phi_j^{-1}\phi_l^{-1}u} \phi_j^*q^*\theta_{\alpha_1,\dots,\alpha_n}\\ &= \int_u^{\phi_j u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \zeta_k^{M_j}\int_u^{\phi_l\phi_j^{-1}\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}\end{aligned}$$

and

$$\begin{aligned}\int_u^{\phi_l\phi_j^{-1}\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n} &= \int_u^{\phi_l u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \int_{\phi_l u}^{\phi_l\phi_j^{-1}\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n} \\ &= \int_u^{\phi_l u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \int_u^{\phi_j^{-1}\phi_l^{-1}u} \phi_l^* q^*\theta_{\alpha_1,\dots,\alpha_n} \\ &= \int_u^{\phi_l u} q^*\theta_{\alpha_1,\dots,\alpha_n} + \zeta_k^{M_l} \int_u^{\phi_j^{-1}\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n}.\end{aligned}$$

Doing the same for the integral from u to $\phi_j^{-1}\phi_l^{-1}u$ and observing that

$$\int_u^{\phi_l^{-1}u} (\phi_j^{-1})^* q^*\theta_{\alpha_1,\dots,\alpha_n} = \zeta_k^{-M_j} \int_u^{\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n},$$

$$\int_u^{\phi_j^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n} = -\zeta_k^{M_j} \int_u^{\phi_j u} q^*\theta_{\alpha_1,\dots,\alpha_n}$$

and

$$\int_u^{\phi_l^{-1}u} q^*\theta_{\alpha_1,\dots,\alpha_n} = -\zeta_k^{M_l} \int_u^{\phi_l u} q^*\theta_{\alpha_1,\dots,\alpha_n},$$

the result follows.

□

Finally we reduced the problem to computing

$$\int_u^{\phi_i u} q^*\theta_{\alpha_1,\dots,\alpha_n} \qquad \text{for } 1 \le i \le n.$$

Lemma 10. *Fix $u \in U$ and let $z_0 = p(u) \in \mathbb{C} \setminus R$ with*

$$R = \{r_1 = 0, r_2 = 1, r_3 = \lambda_1, \dots, r_n = \lambda_{n-2}\}.$$

Then for each $i \in \{1, ..., n\}$ we have

$$\int_u^{\phi_i u} q^*\theta_{\alpha_1,\dots,\alpha_n} = -\frac{1}{k}(1 - \zeta_k^{M_i}) \int_{z_0}^{r_i} (-w)^{\frac{\alpha_1+1}{k}-1} \prod_{t=2}^{n} (w - r_t)^{-\alpha_t/k} dw,$$

where the choice of the branch is determined by the preimage u of z_0.

Proof. Since $-\hat{x}_1^k = p$, making a change of variable $w = -\hat{x}_1^k$ we obtain

$$\int_u^{\phi_i u} q^*\theta_{\alpha_1,\dots,\alpha_n} = -\frac{1}{k}\int_{\gamma_i}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw,$$

where γ_i is the projection by p of the curve from u to $\phi_i(u)$, i.e., γ_i is an element of $\pi_1(\mathbb{C}-R, z_0)$ that surrounds r_i with index 1 . Choose $s_0 \in [0,1)$ such that $z_0 = |z_0|e^{2\pi i s_0}$, and consider the circle with center 0 and radius $\epsilon > 0$ given by $\beta_\epsilon(s) = \epsilon e^{2\pi i(s+s_0)}$. Let β be the line from z_0 to $z_\epsilon \in \beta_\epsilon \cap \overline{z_0, 0}$. Thus γ_1 is homotopic to $\beta + \beta_\epsilon - e^{2\pi i}\beta$, where the factor $e^{2\pi i}$ is due to the continuation of the argument through the critic line $(-\infty, 0]$, as we see in Figure 1.

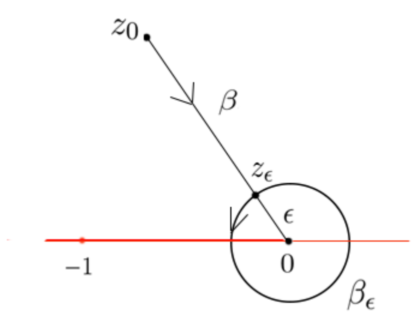


Figure 1

Then

$$-k\int_u^{\phi_1 u} q^*\theta_{\alpha_1,\dots,\alpha_n} = (1-\zeta_k^{\alpha_1+1})\int_{z_0}^{z_\epsilon}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw$$
$$+\int_{\beta_\epsilon}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw.$$

For small ϵ the maps $(\epsilon e^{2\pi i(s+s_0)} - \lambda_t)^{-\alpha_t/k}$ are continuous on $[0,1]$, thus bounded. So there exists a positive constant C independent of ϵ such that

$$\begin{aligned}
&\left|\int_{\beta_\epsilon}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw\right| \\
&=\left|2\pi(-\epsilon)^{\frac{\alpha_1+1}{k}}\int_0^1\frac{e^{\frac{2\pi i(\alpha_1+1)(s+s_0)}{k}}}{\prod_{t=2}^{n}(\epsilon e^{2\pi i(s+s_0)}-\lambda_t)^{\alpha_t/k}}ds\right| \\
&\leq 2\pi\epsilon^{\frac{\alpha_1+1}{k}}C.
\end{aligned}$$

Since $\frac{\alpha_1+1}{k}, -\alpha_i/k$ are larger than -1 for each $i\geq 2$, in the limit $\epsilon\to 0$ we obtain

$$\int_u^{\phi_1 u}q^*\theta_{\alpha_1,\dots,\alpha_n}=-\frac{(1-\zeta_k^{\alpha_1+1})}{k}\int_{z_0}^{0}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw.$$

For γ_i with $i\geq 2$ we apply an analogous argument. □

Remark 3. *For the integral*

$$\int_{z_0}^{r_i}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw,$$

the convergence is given by the fact that $\frac{\alpha_1+1}{k}, -\alpha_j/k$ *are larger than* -1 *for each* j*, so the maps* $|w-r_j|^{M_j/k}$ *with* $i\neq j$ *are well defined, continuous and bounded when* z_0 *is in a neighborhood of* r_i*.*

Theorem 4. *Let*

$$R=\{r_1=0,r_2=1,r_3=\lambda_1,\dots,r_n=\lambda_{n-2}\}$$

be the set of branch points of $(C_{k,n},H_0)$ *distinct of* ∞*. If we denote*

$$W(R,\vec{\alpha})(w):=(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}$$

for each $\vec{\alpha}=(\alpha_1,...,\alpha_n)\in I_{k,n}$*, then the period lattice* $\Lambda\cong\tau(H_1(C_{k,n},\mathbb{Z}))$ *is generated by the period vectors*

$$\left(\zeta_k^{\sum_{d=1}^{n}g_dM_d}\frac{(1-\zeta_k^{M_j})(1-\zeta_k^{M_l})}{k}\int_{r_j}^{r_l}W(R,\vec{\alpha})dw\right)_{\vec{\alpha}\in I_{k,n}}$$

for each generator $\rho[\phi_j,\phi_l]\rho^{-1}\in H_1(C_{k,n},\mathbb{Z})$ *with* $\rho=\prod_{d=1}^{n}\phi_d^{g_d}$ *and* $0\leq g_d\leq k-1$*.*

Proof. From Lemmas 9 and 10 we obtain

$$\int_u^{[\phi_j,\phi_l]u} q^*\theta_{\alpha_1,\dots,\alpha_n} = -\frac{(1-\zeta_k^{M_j})(1-\zeta_k^{M_l})}{k}\int_{r_j}^{r_l}(-w)^{\frac{\alpha_1+1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw.$$

Thus by Lemma 8 for each generator $\sigma = \rho[\phi_j,\phi_l]\rho^{-1} \in H_1(C_{k,n},\mathbb{Z})$ with $\rho = \prod_{d=1}^{n}\phi_d^{g_d}$ we have

$$\int_{\iota\circ q\circ l_\sigma}\theta_{\alpha_1,\dots,\alpha_n} = -\zeta_k^{\sum_{d=1}^{n} g_d M_d}\frac{(1-\zeta_k^{M_j})(1-\zeta_k^{M_l})}{k}\int_{r_j}^{r_l}(-w)^{\frac{\alpha_1-1}{k}-1}\prod_{t=2}^{n}(w-r_t)^{-\alpha_t/k}dw$$

with $1 \leq j < l \leq n$ and $0 \leq g_d \leq k-1$. □

Remark 4. *In the case of the classical Fermat curves $C_{k,2}$ with $R = \{r_1 = 0, r_2 = 1\}$, the integrals to compute are*

$$\int_0^1 \frac{(-w)^{\frac{\alpha_1+1}{k}-1}dw}{(w-1)^{\alpha_2/k}} = -\eta^{\alpha_1-\alpha_2+1}\int_0^1 w^{\frac{\alpha_1+1}{k}-1}(1-w)^{-\alpha_2/k}dw,$$

where $\eta = (-1)^{1/k}$. If we consider the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}, \quad Re(x), Re(y) > 0,$$

then

$$\int_0^1 \frac{(-w)^{\frac{\alpha_1+1}{k}-1}dw}{(w-1)^{\alpha_1/k}} = -\eta^{\alpha_1-\alpha_2+1}B\left(\frac{\alpha_1+1}{k}, 1-\frac{\alpha_2}{k}\right),$$

which yields a result similar to that of Rohrlich in [1] for the standard Fermat curve $X^k + Y^k = Z^k$. In the case of the generalized Fermat curve we need to compute

$$\int_{\lambda_j}^{\lambda_l} \frac{(-w)^{\frac{\alpha_1+1}{k}-1}dw}{(w-1)^{\alpha_2/k}(w-\lambda_1)^{\alpha_3/k}\cdots(w-\lambda_{n-2})^{\alpha_n/k}},$$

which we can view as a natural generalization of the Beta function.

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References

[1] Gross. B (appendix by Rohrlich. D), *On the periods of Abelian Integrals and a Formula of Chowla and Selberg*, Inventiones mathematicae **45**, 1978, 193-211.

[2] Gonzalez-Diez. G, Hidalgo. R, Leyton. M, *Generalized Fermat curves*, Journal of Algebra **321**, 2009, 1643-1660.

[3] Hidalgo. R, *Holomorphic differentials of Generalized Fermat curves*, arXiv:1710.01349, 2017.

[4] Lang. S, *Introduction to Algebraic and Abelian Functions*, Graduate Texts in Mathematics, Vol. **89**. Springer-Verlag, New York-Heidelberg, 1982.

[5] Tzermias. P, *The Group of Automorphism of the Fermat Curve*, Journal of Number Theory **53**, 1995, 173-178.

[6] Kopeliovich. Y, Zemel. S, *On Spaces Associated with Invariant Divisors on Galois Covers of Riemann Surfaces and Their Applications*, Israel Journal of Mathematics **234**, 2019, 393-450.

[7] Kopeliovich. Y, Zemel. S, *Thomae formula for Abelian covers of* $\mathbb{CP}^1$, Transactions of the American Mathematical Society **372**, 2019, 7025-7069.