SCATTERING FOR 3D CUBIC FOCUSING NLS ON THE DOMAIN OUTSIDE A CONVEX OBSTACLE REVISITED

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Abstract. In this article, we consider the focusing cubic nonlinear Schrödinger equation(NLS) in the exterior domain outside of a convex obstacle in $\mathbb{R}^3$ with Dirichlet boundary conditions. We revisit the scattering result below ground state in Killip-Visan-Zhang [16] by utilizing the method of Dodson and Murphy [4,5] and the dispersive estimate in Ivanovici and Lebeau [9], which avoids using the concentration compactness. We conquer the difficulty of the boundary in the focusing case by establishing a local smoothing effect of the boundary. Based on this effect and the interaction Morawetz estimates, we prove the solution decays at a large time interval, which meets the scattering criterions.

Key Words: Schrödinger equation; exterior domain; global well-posedness; scattering criterions.
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1. Introduction

Consider the Cauchy problem of the nonlinear Schrödinger equation with Dirichlet boundary condition

$$\begin{cases} i\partial_t u + \Delta u = -|u|^2 u =: F(u), & (t,x) \in \mathbb{R} \times \Omega \\ u(0,x) = \phi(x), \\ u(t,x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the exterior of a smooth, compact, strictly convex obstacle $\Omega^c \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, and Δ is the Dirichlet Laplacian operator. It is easy to find that the solution u to equation (1.1) with sufficient smooth conditions posses the energy conservation

$$E_\Omega(u(t)) := \int_\Omega \left[\frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{4}|u(t,x)|^4\right] dx = E_\Omega(u_0) \tag{1.2}$$

and mass conservation

$$M_\Omega(u(t)) := \int_\Omega |u(t,x)|^2 dx = M_\Omega(u_0). \tag{1.3}$$

When $\Omega = \mathbb{R}^3$, the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.4}$$

is scale invariant. More precisely, the class of solutions to (1.4) is left invariant by the scaling

$$u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.5}$$

Moreover, one can also check that the only homogeneous L_x^2-based Sobolev space that is left invariant under (1.5) is $\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^3)$. Hence, we say that the Cauchy problem (1.1) is $\dot{H}^{\frac{1}{2}}$-critical. We will consider the well-posedness and long time behavior of the Cauchy problem (1.1) with initial data in the energy spaces. To do it, we first recall the classical Sobolev spaces on the domain Ω.

Definition 1.1. For integer $k \geq 1$ and $1 \leq p \leq \infty$, we denote $H_0^{k,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{H_0^{k,p}(\Omega)} := \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^p(\Omega)}.$$

If $p = 2$, we also write $H_0^k(\Omega) = H_0^{k,2}(\Omega)$ for simplicity.

In fact, $-\Delta$ is an unbounded and positive semi-define symmetric operator on $C_c^\infty(\Omega)$. We define the corresponding quadratic form by for $u, v \in C_c^\infty(\Omega)$

$$Q(u,v) = \int_\Omega \nabla u(x)\nabla \bar{v}(x)dx.$$

The extension of form Q is unique and defined on $H_0^1(\Omega)$. Then the Friedrichs extension of $-\Delta$ gives the Dirichlet Laplacian on Ω, $-\Delta_\Omega$, which is a self-adjoint operator and with form domain $Q(-\Delta_\Omega) = D(\sqrt{-\Delta_\Omega})$. By the spectral theorem, we are able to denote the spectral measure $E(\lambda)$ and the operators by

$$\varphi(\sqrt{-\Delta_\Omega}) = \int_{[0,\infty)} \varphi(\lambda)dE(\lambda).$$

Thus, the linear operator $e^{it\Delta_\Omega}$ associated to the free Schrödinger equation on Ω is well defined and unitary on $L^2(\Omega)$. And we can define the Sobolev spaces based on the operator Δ_Ω.

Definition 1.2. For $s \geq 0$ and $1 < p < \infty$, let $\dot{H}_D^{s,p}(\Omega)$ and $H_D^{s,p}(\Omega)$ denote the completions of $C_c^\infty(\Omega)$ under the norms

$$\|f\|_{\dot{H}_D^{s,p}(\Omega)} := \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p(\Omega)} \quad \text{and} \quad \|f\|_{H_D^{s,p}(\Omega)} := \left\|(1-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p(\Omega)}.$$

When $p = 2$ we also write $\dot{H}_D^s(\Omega)$ and $H_D^s(\Omega)$ for $\dot{H}_D^{s,2}(\Omega)$ and $H_D^{s,2}(\Omega)$, respectively.

These two definitions are equivalent under certain conditions, see Proposition 2.1 below.

For the Euclidean space $\mathbb{R}^d$, the linear operator $e^{it\Delta}$ obeys the dispersive estimates and the Strichartz estimates. Owing to this, the local well-posedness theory of the solutions to equation (1.4) with the general power type nonlinearities $F(u) = |u|^{p-1}u$ is standard. For the defocusing energy subcritical($F(u) = -|u|^{p-1}u,\ 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$) cases, the solutions with initial datum in $H^1(\mathbb{R}^d)$ are global well-posed and scatter, see [2] [15] and references therein.

In general domains, we do not have the dispersive estimate and the Strichartz estimates for $e^{it\Delta_\Omega}$. For the case of exterior domain of a convex obstacle, Ivanovici [8] proved the Strichartz estimates except endpoint case by using the Melrose and Taylor parametrix and she also proved the scattering theory energy subcritical NLS for exterior domain of smooth convex obstacle in $3D$. Ivanovici and Lebeau [9] proved the dispersive estimates holds only in the $3D$ case. For more scattering results of defocusing subcritical NLS in the general exterior domains, we refer to Planchon-Vega [18], Ivanovici-Planchon [10], and Blair-Smith-Sogge [1].

In this paper, we consider scattering theory of the solutions to focusing equation (1.1), which is mass supercritical and energy subcritical. In fact, the nonlinear elliptic equation

$$-\Delta\varphi+\varphi=|\varphi|^2\varphi, \tag{1.6}$$

has infinite number of solutions in $H^1(\mathbb{R}^3)$. Then for any solution $\varphi\in H^1(\mathbb{R}^3)$ to (1.6), $e^{it}\varphi$ is a global and non-scattering solution to the Cauchy problem (1.4). Furthermore, there exists a minimal mass solution and we often denote it as Q and call it the ground state, which is positive, radial, exponentially decaying, see Cazenave [2] and Tao [21]. Holmer-Roudenko [7] proved the global well-posedness and scattering theory for radial solutions to equation (1.4) such the following conditions in $\mathbb{R}^3$:

$$E_{\mathbb{R}^3}(u_0)M_{\mathbb{R}^3}(u_0)<E_{\mathbb{R}^3}(Q)M_{\mathbb{R}^3}(Q), \tag{A}$$

$$\|\nabla u_0\|_{L^2(\mathbb{R}^3)}\|u_0\|_{L^2(\mathbb{R}^3)}<\|\nabla Q\|_{L^2(\mathbb{R}^3)}\|Q\|_{L^2(\mathbb{R}^3)}. \tag{B}$$

Duyckaerts-Holmer-Roudenko [6] removed the radial assumption. Killip-Visan-Zhang [16] proved the results for exterior domains of convex obstacles in $\mathbb{R}^3$:

Theorem 1.1. *Let Ω is exterior of a convex obstacle in $\mathbb{R}^3$. If the initial data $u_0\in H^1_D(\Omega)$ satisfies*

$$E_\Omega(u_0)M_\Omega(u_0)<E_{\mathbb{R}^3}(Q)M_{\mathbb{R}^3}(Q), \tag{1.7}$$

$$\|\nabla u_0\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)}<\|\nabla Q\|_{L^2(\mathbb{R}^3)}\|Q\|_{L^2(\mathbb{R}^3)}, \tag{1.8}$$

then, the corresponding solution to the Cauchy problem (1.1) *with initial u_0 is globally well-posed and scatters.*

The proofs of [6] and [16] utilized the concentration-compactness arguments basing on the profile decomposition introduced by Kenig-Merle [12, 13], which have become powerful and effective methods for many dispersive equations and many other equations.

In this article, we revisit Theorem 1.1, by employing an idea of Dodson-Murphy [4], [5], which provide new proofs in the Euclidean case avoiding uses of concentration and compactness.

Outline of proof: By the Strichartz estimates and the equivalence of various Sobolev norm definitions, we have the local well-posedness of (1.1) in $H^1_D(\Omega)$. From the coercivity property(Lemma 2.10 below) under the ground state, we know the solution u is globally well-posed and of bounded $H^1_D(\Omega)$ norm. Utilizing the dispersive estimates, we prove that the scattering criterion given by [5] also holds in our case, that is: if for any large time window, there exists a large subinterval such that a space-time norm of of u is small in it, then u must scatter.

To end the proof, the main difficulties are how to overcome the effect from boundary $\partial\Omega$ and the lack of the Galilean invariance. Combining with the concavity of $\partial\Omega$ and the coercivity property, the Morawetz estimates yields a weaker local smoothing effect on the boundary. On the other hand, as in [6], for the Euclidean case, by the Galilean invariance, one can assume the critical solution u_c has zero conserved momentum, which yields the spatial translation parameter $x(t)=o(t)$ (as $t\to\infty$). This fact is essential to the preclusion of the critical solution by making use of the Morawetz estimates centered at origin. For our case, the momentum is obvious bounded since $u \in L_t^\infty H_x^1$. Based on this fact, one could just expect $|x(t)| \lesssim |t|$. However, the interaction Morawetz identity is defined as an average of the Morawetz action that is centered any point in $\mathbb{R}^3$. Fortunately, since $u \in L_t^\infty H_x^1$, we are able to prove the smallness $L_{t,x}^3$-norm in a large subinterval of any large time interval without employing the Galilean transformation.

Finally, this and a standard continuity argument imply the solution such that the conditions of the scattering criterion.

Remark 1.2. Our proof is based on the the dispersive estimates of [9], which does not hold true in higher dimensions. Nevertheless, in these cases, it is hopeful that one may prove the corresponding results via establishing weaker dispersive estimates(see for example [23]).

Remark 1.3. We remark that the interaction Morawetz estimates also reflect that the solution decays in big ball around any point. In fact, for any fixed $R > 0$, we have

$$\liminf_{t\to\infty} \sup_{x(t)\in\mathbb{R}^3} \|u(t, \cdot - x(t))\|_{L_x^2(\Omega\cap B_R)} \;=\; 0,$$

where B_r is the ball center at origin with radius r. This suffices the scattering criterion for non-radial NLS (1.1) in [19] when $\Omega = \mathbb{R}^3$.

Remark 1.4. In fact, as in [20] and [5], one can check that our proof would imply

$$\|u\|_{L_{t,x}^5(\mathbb{R}\times\Omega)} \lesssim \exp\{\exp A(E(u_0), M(u_0))\},$$

where A is a rational polynomial of $E(u_0), M(u_0)$ and $E(Q), M(Q)$. The double-exponential growth derives from the local smoothing effect of boundary and the interaction Morawetz estimates.

Remark 1.5. Our arguments can be used to prove the similar results for general focusing energy subcritical cases($F(u) = -|u|^{p-1}u$, $\frac{7}{3} < p < 5$), which has been considered in [22].

This article is organized as follows: in Section 2, we recall some basics facts on the domain. Section 3 is devoted to prove the scattering under the assumption of smallness of $L_{t,x}^5$ norm of the solution. In Section 4, we verify the scattering criterion.

We conclude the introduction by giving some notations which will be used throughout this paper. We always use $X \lesssim Y$ to denote $X \le CY$ for some constant $C > 0$. $X \sim Y$ stands for $X \lesssim Y$ and $Y \lesssim X$. Similarly, $X \lesssim_u Y$ indicates there exists a constant $C := C(u)$ depending on u such that $X \le C(u)Y$. The symbol ∇ refers to the spatial derivation. For $M = \mathbb{R}^3$ or a domain in $\mathbb{R}^3$, we use $L^r(M)$ to denote the Banach space of functions $f : M \to \mathbb{C}$ whose norm

$$\|f\|_{L^r(M)} = \Big(\int_M |f(x)|^r dx\Big)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$. For a time slab I, we use $L_t^q L_x^r(I \times M)$ to denote the space-time norm

$$\|f\|_{L_t^q L_x^r(I\times M)} = \Big(\int_I \|f(t,x)\|_{L_x^r(M)}^q dt\Big)^{\frac{1}{q}}$$

with the usual modifications when q or r is infinite.

2. Basic tools and the local theory

In this section we give some basic harmonic tools and the local well-posedness theory for the Cauchy problem (1.1). In this section, we assume that Ω is the complement of a compact convex body $\Omega^c \subset \mathbb{R}^3$ with smooth boundary.

First, we recall the following proposition.

Proposition 2.1 (Equivalence of the Sobolev norms, [17])**.** *Let* $1 < p < \infty$*. If* $0 \le s < \min\{1 + \frac{1}{p}, \frac{3}{p}\}$*, then*

$$\left\|(-\Delta_{\mathbb{R}^3})^{\frac{s}{2}} f\right\|_{L^p(\mathbb{R}^3)} \sim_{p,s} \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^p(\Omega)} \tag{2.1}$$

for all $f \in C_c^\infty(\Omega)$.

Using this proposition, we have

Corollary 2.2 (Fractional product rule, [17])**.** *For all* $f, g \in C_c^\infty(\Omega)$*, we have*

$$\left\|(-\Delta_\Omega)^{\frac{s}{2}}(fg)\right\|_{L^p(\Omega)} \lesssim \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_2}(\Omega)} + \|f\|_{L^{q_1}(\Omega)} \left\|(-\Delta_\Omega)^{\frac{s}{2}} g\right\|_{L^{q_2}(\Omega)}$$

with the exponents satisfying $1 < p, p_1, q_2 \le \infty$, $1 < p_2, q_1 \le \infty$,

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}, \quad \text{and} \ \ 0 < s < \min\left\{1 + \frac{1}{p_1}, 1 + \frac{1}{q_2}, \frac{3}{p_1}, \frac{3}{q_2}\right\}.$$

Corollary 2.3 (Fractional chain rule, [17])**.** *Suppose* $G \in C^1(\mathbb{C})$*,* $s \in (0,1]$*, and* $1 < p, p_1, p_2 < \infty$ *are such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $0 < s < \min\left\{1 + \frac{1}{p_2}, \frac{3}{p_2}\right\}$*. Then*

$$\left\|(-\Delta_\Omega)^{\frac{s}{2}} G(f)\right\|_{L^p(\Omega)} \lesssim_{s,p,p_1} \|G'(f)\|_{L^{p_1}(\Omega)} \left\|(-\Delta_\Omega)^{\frac{s}{2}} f\right\|_{L^{p_2}}.$$

We need the chain rule for fractional derivatives on $\mathbb{R}^d$, which will be useful for the local theory.

Proposition 2.4 (Chain rule for fractional derivatives, [14])**.** *If* $F \in C^2$*, with* $F(0) = 0, F'(0) = 0$*, and* $|F''(a+b)| \le C\{|F''(a)| + |F''(b)|\}$*, and* $|F'(a+b)| \le C\{|F'(a)| + |F'(b)|\}$*, we have, for* $0 < \alpha < 1$,

$$\|\Lambda^\alpha F(u)\|_{L_x^p(\mathbb{R}^d)} \le C \|F'(u)\|_{L^{p_1}(\mathbb{R}^d)} \|\Lambda^\alpha u\|_{L^{p_2}(\mathbb{R}^d)}, \ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

and

$$\begin{aligned} &\|\Lambda^\alpha [F(u) - F(v)]\|_{L_x^p(\mathbb{R}^d)} \\ \le\ & C[\|F'(u)\|_{L^{p_1}(\mathbb{R}^d)} + \|F'(v)\|_{L^{p_1}(\mathbb{R}^d)}] \|\Lambda^\alpha (u-v)\|_{L^{p_2}(\mathbb{R}^d)} \\ &+ C \||F''(u)| + |F''(v)|\|_{L^{r_1}(\mathbb{R}^d)} \left(\|\Lambda^\alpha u\|_{L^{r_2}(\mathbb{R}^d)} + \|\Lambda^\alpha v\|_{L^{r_2}(\mathbb{R}^d)}\right) \|(u-v)\|_{L^{r_3}(\mathbb{R}^d)}, \end{aligned}$$

where $\Lambda = (-\Delta_{\mathbb{R}^3})^{\frac{1}{2}}$.

Next, we recall the dispersive estimates.

Lemma 2.5 (Dispersive estimate, [9])**.**

$$\left\|e^{it\Delta_\Omega} f\right\|_{L_x^\infty(\Omega)} \lesssim t^{-\frac{3}{2}} \|f\|_{L_x^1(\Omega)}. \tag{2.2}$$

Combining this with the endpoint Strichartz estimate of Keel-Tao, we have the following Strichartz estimates:

Proposition 2.6 (Strichartz estimates [8] [11]). *Let $q, \tilde{q} \geq 2$, and $2 \leq r, \tilde{r} \leq \infty$ satisfying $\frac{3}{2} = \frac{2}{q} + \frac{3}{r} = \frac{2}{\tilde{q}} + \frac{3}{\tilde{r}}$. Then, the solution u to $(i\partial_t + \Delta)u = F$ on an interval $I \ni 0$ satisfies*

$$\|u\|_{L_t^q L_x^r(I\times\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\Omega)}. \tag{2.3}$$

We define the $S(I)$ and $W(I)$ norm for a interval I by

$$\|u\|_{S(I)} = \|u\|_{L^5_{t,x}(I\times\Omega)} \quad \text{and} \quad \|u\|_{W(I)} = \|u\|_{L_t^5 L_x^{\frac{30}{11}}(I\times\Omega)}. \tag{2.4}$$

Note that $1 < \min\{1 + \frac{11}{30}, \frac{11}{10}\}$. Thus, by Strichartz, Corollary 2.3, Proposition 2.4, we have:

Theorem 2.7 (Local well-posedness, [3] [12]). *Assume that $u_0 \in \dot{H}^1_D(\Omega), 0 \in I$, and $\|u_0\|_{\dot{H}^1_D(\Omega)} \leq A$. Then there exists $\delta = \delta(A)$ such that if $\left\|e^{it\Delta_\Omega} u_0\right\|_{S(I)} \leq \delta$, there exists a unique solution u to (1.1) in $I \times \Omega$, with $u \in C(I; \dot{H}^1_D(\Omega))$ such that*

$$\left\|(-\Delta_\Omega)^{\frac{1}{2}} u\right\|_{W(I)} + \sup_{t\in I} \|u(t)\|_{\dot{H}^1_D(\Omega)} \leq CA, \quad \|u\|_{S(I)} \leq 2\delta. \tag{2.5}$$

Moreover, if $u_{0,k} \to u_0$ in $\dot{H}^1_D(\Omega)$, we obtain the corresponding solutions $u_k \to u$ in $C(I; \dot{H}^1_D(\Omega))$.

Remark 2.8. From standard arguments, we have if u is a global solution and such that

$$\|u\|_{S(\mathbb{R})} < \infty,$$

then u scatters both directions.

We need the following refined Gagliardo-Nirenberg inequality, which follows from the sharp Gagliardo-Nirenberg inequality and the Pohozaev identities of the ground state.

Lemma 2.9 (Refined Gagliardo-Nirenberg inequality, [5]). *For $f \in H^1(\mathbb{R}^3)$ and any $\xi \in \mathbb{R}^3$,*

$$\|f\|^4_{L^4(\mathbb{R}^3)} \leq \frac{4}{3}\Big(\frac{\|f\|_{L^2(\mathbb{R}^3)}\|f\|_{\dot{H}^1_x(\mathbb{R}^3)}}{\|Q\|_{L^2(\mathbb{R}^3)}\|Q\|_{\dot{H}^1_x(\mathbb{R}^3)}}\Big) \inf_{\xi\in\mathbb{R}^3} \left\|e^{ix\xi} f\right\|^2_{\dot{H}^1_x(\mathbb{R}^3)}. \tag{2.6}$$

Before the end of this section, we recall the coercivity property for functions under the ground state Q(i.e., satisfying the conditions (A) and (B)). We denote $M_{\mathbb{R}^3}$ and $E_{\mathbb{R}^3}$ as the Mass and energy on $\mathbb{R}^3$ respectively.

Lemma 2.10 (Coercivity). *Let $u_0 \in H^1_D(\Omega)$ satisfy the conditions (1.7). If $\|u_0\|_{L^2(\Omega)} \|u_0\|_{\dot{H}^1_D(\Omega)} \leq \|Q\|_{L^2_x(\mathbb{R}^3)} \|Q\|_{\dot{H}^1_x(\mathbb{R}^3)}$, then there exists $\delta' = \delta'(\delta) > 0$ so that*

$$\|u(t)\|_{L^2_x(\Omega)} \|u(t)\|_{\dot{H}^1_x(\Omega)} \leq (1-\delta') \|Q\|_{L^2_x(\mathbb{R}^3)} \|Q\|_{\dot{H}^1_x(\mathbb{R}^3)} \tag{2.7}$$

holds for all $t \in I$, where $u : I \times \Omega \to \mathbb{C}$ is the maximal lifespan solution to (1.1). In particular, $I = R$ and u is uniformly bounded in $H^1(\Omega)$.

Moreover, for any function $f \in H^1_D(\Omega)$ such that (2.9), there exists $\rho = \rho(\delta') > 0$ such that

$$\|f\|^2_{\dot{H}^1_x(\Omega)} - \frac{3}{4}\|f\|^4_{L^4_x(\Omega)} \geq \rho(\|f\|^2_{\dot{H}^1_x(\Omega)} + \|f\|^4_{L^4_x(\Omega)}). \tag{2.8}$$

Proof. The proof follows from Proposition 2.1 above, Lemma 2.3 and Lemma 2.4 in [4].

□

Remark 2.11. Suppose $u_0 \in H^1_D(\Omega)$ satisfies (1.7) and (1.8). Then by the above lemma, the maximal-lifespan solution u to (1.1) with initial data u_0 obeys

$$\|u(t)\|_{L^2_x(\Omega)} \|u(t)\|_{\dot H^1_x(\Omega)} \le (1-\delta') \|Q\|_{L^2_x(\mathbb{R}^3)} \|Q\|_{\dot H^1_x(\mathbb{R}^3)} \tag{2.9}$$

for all t in the lifespan of u. In particular, u remains bounded in $H^1_D(\Omega)$ and hence is global.

3. Scattering criterion

In this section, we prove a scattering criterion for solutions of the Cauchy problem (1.1).

Proposition 3.1. *Suppose that u is a global solution to* (1.1)*, satisfying*

$$\|u\|_{L_t^\infty H^1_D(\mathbb{R}\times\Omega)} \le E. \tag{3.1}$$

There exist $\epsilon = \epsilon(E,\Omega) > 0$ and $T_0 = T_0(\epsilon, E, \Omega) > 0$ satisfying that if for any $a \in \mathbb{R}$ there exists $T \in \mathbb{R}$ such that $[T-\epsilon^{-5}, T] \subset (a, a+T_0)$ and

$$\|u\|_{L^5_{t,x}([T-\epsilon^{-5},T)\times\Omega)} \ \le \ \epsilon, \tag{3.2}$$

then u scatters forward in time.

Proof. By the Strichartz estimates and continuity method, there exists $\varepsilon = \varepsilon(E,\Omega)$ such that if for any $T > 0$,

$$\left\|e^{i(t-T)\Delta_\Omega} u(T)\right\|_{L^5_{t,x}([T,\infty)\times\Omega)} \le \varepsilon, \tag{3.3}$$

then the u scattering forward.

By the Duhamel formula, we have

$$e^{i(t-T)\Delta_\Omega} u(T) = e^{it\Delta_\Omega} u_0 + i\int_0^T e^{i(t-s)\Delta_\Omega}(|u|^2 u)(s)ds. \tag{3.4}$$

First, by the Strichartz estimates, there exists $T_1 > 0$ such that, if $T > T_1$

$$\left\|e^{it\Delta_\Omega} u_0\right\|_{L^5_{t,x}([T,\infty)\times\Omega)} < \frac{1}{2}\varepsilon. \tag{3.5}$$

Take $a = T_1$, $\epsilon = \varepsilon^2$, T as in the assumption (3.2) and make a decomposition

$$[0,T] = [0, T-\epsilon^{-5}] \cup [T-\epsilon^{-5}, T] := I_1 \cup I_2.$$

Then by (3.2), the Strichartz estimates, and the continuity method, we have

$$\|u\|_{L^5_t H_D^{\frac{30}{11},3}([T-\epsilon^{-5},T]\times\Omega)} \lesssim 1.$$

Thus, we have

$$\left\|\int_{I_2} e^{i(t-s)\Delta_\Omega}(|u|^2u)(s)ds\right\|_{L^5_{t,x}([T,\infty)\times\Omega)} \lesssim \ \|u\|^2_{L^5_{t,x}(I_2\times\Omega)} \|u\|_{L^5_t H_D^{\frac{30}{11},3}(I_2\times\Omega)} \lesssim \epsilon^2. \tag{3.6}$$

Next, we consider the corresponding contribution of I_1. By the Duhamel formula and the Strichartz estimates, we have

$$\begin{aligned} &\left\|\int_{I_1} e^{i(t-s)\Delta_\Omega}(|u|^2u)(s)ds\right\|_{L^5_t L_x^{\frac{30}{11}}([T,\infty)\times\Omega)} \\ =&\left\|e^{i(t-(T-T_0^{\frac13}))\Delta_\Omega} u(T-T_0^{\frac13}) - e^{it\Delta_\Omega}u_0\right\|_{L^5_t L_x^{\frac{30}{11}}([T,\infty)\times\Omega)} \lesssim 1. \end{aligned}$$

On the other hand, employing the dispersive estimates and the Sobolev embedding, we have

$$\begin{aligned}&\Big\| \int_{I_1} e^{i(t-s)\Delta_\Omega}(|u|^2u)(s)ds\Big\|_{L_t^5 L_x^\infty([T,\infty)\times\Omega)}\\ \lesssim &\ \Big\| \int_{I_1} \tfrac{1}{(t-s)^{\frac{3}{2}}}ds\Big\|_{L^5([T,\infty))} \|u\|^3_{L_t^\infty H_D^1(\mathbb{R}\times\Omega)} \lesssim \epsilon^{\frac{3}{2}}.\end{aligned}$$

Thus, by interpolation, we have

$$\Big\| \int_{I_1} e^{i(t-s)\Delta_\Omega}(|u|^2u)(s)ds\Big\|_{L_t^5 L_x^5([T,\infty)\times\Omega)} \lesssim \epsilon^{\frac{15}{22}},$$

which together with (3.5) and (3.6) implies (3.3). Therefore, we complete the proof. □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First we prove a local smoothing effect property on the Boundary $\partial\Omega$ by utilizing a Morawetz-type estimate. Then we prove the interaction Morawetz estimates for the solution in the Theorem 1.1. Finally, we prove Theorem 1.1 by showing the solution such that the conditions of the scattering criterion in previous section.

Let $\chi_R(x)$ be a smooth function on $\mathbb{R}^3$ and such that $\chi_R(x) = 1$ when $|x| \leq \frac{R}{4}$ and $\chi_R(x) = 0$ when $|x| \geq \frac{R}{2}$. We need the following coercivity property, which follows similar proof of Lemma 3.2 in [4].

Lemma 4.1 (Coercivity on balls)**.** *There exists $R = R(\delta, M(u), Q) > 0$ sufficiently large such that for any point $z \in \mathbb{R}^3$,*

$$\sup_{t\in\mathbb{R}} \|\chi_R(\cdot - z)u(t)\|_{L_x^2(\Omega)} \|\chi_R(\cdot - z)u(t)\|_{\dot H_x^1(\Omega)} < (1-\delta)\|Q\|_{L_x^2(\mathbb{R}^3)} \|Q\|_{\dot H_x^1(\mathbb{R}^3)} \tag{4.1}$$

In particular, by Lemma 2.10, there exists $\delta' = \delta'(\delta) > 0$ so that

$$\|\chi_R(\cdot - z)u(t)\|^2_{\dot H_x^1(\Omega)} - \frac{3}{4}\|\chi_R(\cdot - z)u(t)\|^4_{L_x^4(\Omega)} \geq \delta' \|\chi_R(\cdot - z)u(t)\|^2_{\dot H_x^1(\Omega)} \tag{4.2}$$

uniformly for $t \in \mathbb{R}$.

Next, we make some preparation for the Morawetz estimates. Let $n(x)$ be the outer normal vector at $x \in \partial\Omega$ and define the outer derivative by $\partial_n f = \nabla f \cdot n$. Denote dS be the induced measure on $\partial\Omega$.

Let $\eta > 0$ small, $\chi(x) = 1$ for $|x| \leq 1-\eta$ and $\chi = 0$ for $x \geq 1$. Let $R > 1$ large, and define

$$\phi(x) = \frac{1}{\omega_3 R^3}\int_{\mathbb{R}^3} \chi^2(\tfrac{x-s}{R})\chi^2(\tfrac{s}{R})ds,$$

and

$$\phi_1(x) = \frac{1}{\omega_3 R^3}\int_{\mathbb{R}^3} \chi^2(\tfrac{x-s}{R})\chi^4(\tfrac{s}{R})ds,$$

where ω_3 is the volume of unit ball in $\mathbb{R}^3$. Then we have

$$|\phi - \phi_1| \lesssim \eta.$$

Let

$$\psi(x) = \frac{1}{|x|}\int_0^{|x|} \phi(r)dr,$$

which satisfies

$$|\psi(x)| \le \min\left\{1, \frac{R}{|x|}\right\} \quad \text{and} \quad \partial_k \psi(x) = \frac{x_k}{|x|^2}[\psi(x) - \phi(x)].$$

One can also deduce that

$$\partial_k[\psi(x)x_k] = 3\phi(x) + 2(\psi - \phi)(x), \tag{4.3}$$

where the repeated indices are summed.

4.1. **Local smoothing effect.** We define the Morawetz action by

$$M(t) = 2 \text{ Im } \int_\Omega \psi(x)x[\bar{u}\nabla u]dx. \tag{4.4}$$

Then, $|M(t)| \lesssim R$.

Proposition 4.2. *For large $T_0 > 1$ and any time interval $I = [a, a+T_0] \subset \mathbb{R}$, we have*

$$\frac{1}{T_0}\int_I \int_{\partial\Omega} |\partial_n u|^2(t,x)dS(x)dt \lesssim \frac{1}{(\log T_0)^{\frac{1}{2}}}. \tag{4.5}$$

Proof. From the identity

$$2\partial_t \text{ Im } (\bar{u}u_k) \ = \ \partial_k|u|^4 + \partial_k\Delta|u|^2 - 4\partial_j \text{ Re } (\bar{u}_j u_k), \tag{4.6}$$

(4.3), and integration by parts, we have

$$\begin{aligned} &\partial_t M(t) \\ &= 4\int_\Omega \Big[\phi(x)|\nabla u|^2(t,x) - \frac{3}{4}\phi_1(x)|u|^4(t,x)\Big]dx \qquad (4.7)\\ &\quad - 2\int_{\partial\Omega}\psi(x)x\cdot n(x)|\partial_n u|^2 dS(x) + 4\int_\Omega(\psi-\phi)(x)|\not\nabla u|^2(t,x)dx \qquad (4.8)\\ &\quad - \int_\Omega [3(\phi-\phi_1)+2(\psi-\phi)](x)|u|^4(t,x)dx + \int_\Omega \nabla[3\phi+2(\psi-\phi)](x)\cdot\nabla|u|^2(t,x)dx, \qquad (4.9)\end{aligned}$$

where $\not\nabla$ is the angular derivation centered at the origin.

By the definition of χ, (4.7) equals

$$\frac{4}{R^3}\int_{\mathbb{R}^3}\int_\Omega |\nabla\big(\chi(\tfrac{x-s}{R})u\big)|^2 - \frac{3}{4}|\chi(\tfrac{x-s}{R})u|^4 dx\chi^2(\tfrac{s}{R})ds + O(\tfrac{1}{\eta^2R^2}). \tag{4.10}$$

By the Coercivity property Lemma 4.1, there exists $R_1 > 0$, such that the first term of (4.10) is nonnegative for $R > R_1$. And the nonnegativity for second term of (4.8) follows from the fact $\psi - \phi \ge 0$. From the facts $\phi - \phi_1 \lesssim \eta$ and

$$|\psi-\phi|+|\nabla\phi|+|\nabla\psi| \lesssim |\psi-\phi|(1+\frac{1}{|x|})+|\nabla\phi| \le \frac{1}{\eta R}+\min\left\{\frac{|x|}{\eta R}, \frac{R}{|x|}\right\}+\min\left\{\frac{1}{\eta R}, \frac{R}{|x|^2}\right\}, \tag{4.11}$$

we have

$$\frac{1}{J}\int_{R_0}^{e^J R_0} |\phi-\phi_1| + |\psi-\phi| + |\nabla\phi| + |\nabla\psi|\frac{dR}{R} \lesssim \eta + \frac{1}{J\eta} + \frac{1}{R_0\eta J}. \tag{4.12}$$

Thus, we can deduce that

$$-\frac{1}{T_0}\int_I \frac{1}{J}\int_{R_0}^{e^J R_0}\int_{\partial\Omega}\psi(x)x\cdot n(x)|\partial_n u|^2(t,x)dx\frac{dR}{R}dt \lesssim \eta + \frac{1}{J\eta} + \frac{1}{R_0\eta J} + \frac{R_0 e^J}{T_0 J} + \frac{1}{\eta^2 J R_0^2}. \tag{4.13}$$

Since the boundary $\partial\Omega$ of Ω is concave and compact, we have $-\psi(x)x\cdot n(x) = x\cdot n(x)\gtrsim 1$ for $x\in\partial\Omega$, which yields

$$\frac{1}{T_0}\int_I\int_{\partial\Omega}|\partial_n u|^2(t,x)dxdt \lesssim \frac{1}{J\eta R_0}+\frac{1}{J\eta}+\eta+\frac{R_0e^J}{T_0J}+\frac{1}{\eta^2JR_0^2}. \tag{4.14}$$

Then the conclusion follows by taking $\eta = R_0^{-1} = J^{-\frac12} = (\log T_0)^{-\frac12}$.

□

4.2. **Interaction Morawetz estimates.** We define the interaction Morawetz quantity

$$M_R(t) = 2\iint_{\Omega\times\Omega}|u|^2(t,y)\psi(x-y)(x-y)\ \mathrm{Im}\ [\bar u\nabla u](t,x)dxdy, \tag{4.15}$$

which reflects the information of u on whole Ω. One can easily find that for any $R>0$ and $t\in\mathbb{R}$,

$$|M_R(t)|\lesssim RE_0^2.$$

Theorem 4.3 (Interaction Morawetz estimates). *For arbitrary small $\varepsilon>0$, there exists $T_0, R_0>0$ large and $\eta>0$ small enough satisfying that: for any interval $I=[a,a+T_0]$, there exists $\xi=\xi(s,t,R)\in\mathbb{R}^3$ such that*

$$\frac{1}{JT_0}\int_{R_0}^{R_0e^J}\int_I\frac{1}{R^3}\int_{\mathbb{R}^3}\iint_{\Omega\times\Omega}\left|\chi\left(\tfrac{\cdot-s}{R}\right)u\right|^2(t,y)\left|\nabla(\chi\left(\tfrac{\cdot-s}{R}\right)u^\xi)\right|^2(t,x)dxdydsdt\frac{dR}{R}\lesssim\varepsilon. \tag{4.16}$$

Proof. By the identities (4.6) and

$$\partial_t|u|^2 \ = \ -2\partial_k\ \mathrm{Im}\ (\bar u u_k), \tag{4.17}$$

we have

$$\partial_t M_R(t) = \int\int_{\Omega\times\Omega}|u|^2(t,y)\psi(x-y)(x-y)\nabla|u|^4(t,x)dxdy \tag{4.18}$$

$$+\int\int_{\Omega\times\Omega}|u|^2(t,y)\psi(x-y)(x-y)\nabla\Delta|u|^2(t,x)dxdy \tag{4.19}$$

$$-4\int\int_{\Omega\times\Omega}|u|^2(t,y)\psi(x-y)(x_k-y_k)\ \mathrm{Re}\ (\partial_j(\bar u_j u_k)(t,x)dxdy \tag{4.20}$$

$$-4\int\int_{\Omega\times\Omega}\partial_j\ \mathrm{Im}\ (\bar u u_j)(t,y)\psi(x-y)(x-y)_k\ \mathrm{Im}\ (\bar u u_k)(t,x)dxdy. \tag{4.21}$$

By integration by parts and the Dirichlet boundary condition of u, we have

$$(4.18) = -\iint_{\Omega\times\Omega}|u|^2(t,y)[3\phi(x-y)+2(\psi-\phi)(x-y)]|u|^4(t,x)dxdy$$

$$= -3\iint_{\Omega\times\Omega}|u|^2(t,y)\phi_1(x-y)|u|^4(t,x)dxdy \tag{4.22}$$

$$-2\iint_{\Omega\times\Omega}|u|^2(t,y)(\psi-\phi)(x-y)|u|^4(t,x)dxdy \tag{4.23}$$

$$-3\iint_{\Omega\times\Omega}|u|^2(t,y)(\psi-\phi_1)(x-y)|u|^4(t,x)dxdy. \tag{4.24}$$

Here, we view (4.23) and (4.24) as error terms from the definitions the cutoff functions.

$$(4.19) = \iint_{\Omega\times\Omega} |u|^2(t,y)\nabla_x\left[3\phi(x-y)+2(\psi-\phi)(x-y)\right]\nabla_x\left[|u|^2(t,x)\right]dxdy \tag{4.25}$$

$$+2\int_\Omega\int_{\partial\Omega}|u|^2(t,y)\psi(x-y)(x-y)\vec{n}_x|\partial_n u|^2(t,x)dS(x)dy. \tag{4.26}$$

As above, we also regard (4.25) as an error term. We will apply the local smoothing effect to the estimation of (4.26)

$$(4.20) \ = 4\iint_{\Omega\times\Omega}|u|^2(t,y)\phi(x-y)|\nabla u|^2(t,x)dxdy \tag{4.27}$$

$$+4\iint_{\Omega\times\Omega}|u|^2(t,y)P_{ij}(x-y)(\psi-\phi)(x-y)\ \mathrm{Re}\ [\bar{u}_j u_k]dxdy \tag{4.28}$$

$$-4\int_\Omega\int_{\partial\Omega}|u|^2(t,y)\psi(x-y)(x-y)_k\ \mathrm{Re}\ (\partial_n\bar{u}u_k)(t,x)dS(x)dy. \tag{4.29}$$

$$(4.21) \ = \ -4\int\int_{\Omega\times\Omega}\partial_{y_j}\ \mathrm{Im}\ (\bar{u}u_j)(t,y)\psi(x-y)(x-y)_k\ \mathrm{Im}\ (\bar{u}u_k)(t,x)dxdy$$

$$= -4\iint_{\Omega\times\Omega}\phi(x-y)\ \mathrm{Im}\ (\bar{u}\nabla u)(t,y)\ \mathrm{Im}\ (\bar{u}\nabla u)(t,x)dxdy \tag{4.30}$$

$$-4\iint_{\Omega\times\Omega}\ \mathrm{Im}\ (\bar{u}\nabla u_j)(t,y)P_{jk}(x-y)[\psi(x-y)-\phi(x-y)]\ \mathrm{Im}\ (\bar{u}\nabla u_k)(t,x)dxdy, \tag{4.31}$$

where $P_{ij}(x)=\delta_{ij}-\frac{x_ix_j}{|x|^2}$.

From the fact that $\psi-\phi\geq 0$ and Cauchy-Schwarz, we have

$$\begin{aligned}&(4.28)+(4.31)\\ =&4\iint_{\Omega\times\Omega}|u|^2(t,y)|\ \not\nabla_y u(t,x)|^2[(\psi-\phi)(x-y)]dxdy\\ &-4\iint_{\Omega\times\Omega}\ \mathrm{Im}\ [\bar{u}\ \not\nabla_x u](t,y)\ \mathrm{Im}\ [\bar{u}\ \not\nabla_y u](t,x)[(\psi-\phi)(x-y)]dxdy\geq 0,\end{aligned} \tag{4.32}$$

where $\not\nabla_z$ is the angular derivation centered at $z\in\mathbb{R}^3$. By the compactness and convexity of $\partial\Omega$, we have

$$\begin{aligned}&|(4.26)+(4.29)|\\ =&\left|2\int_\Omega\int_{\partial\Omega}|u|^2(t,y)\psi(x-y)(x-y)n(x)|\partial_n u|^2(t,x)dxdy\right|\\ \lesssim& R\int_\Omega\int_{\partial\Omega}|u|^2(t,y)\ |\partial_n u|^2(t,x)dxdy.\end{aligned} \tag{4.33}$$

By a direct computation, one has

$$\begin{aligned}
&\frac{\omega_3 R^3}{4}[(4.27)+(4.30)]\\
=&\int_{\mathbb{R}^3}\iint_{\Omega\times\Omega}\chi^2\big(\tfrac{x-s}{R}\big)\chi^2\big(\tfrac{y-s}{R}\big)\left[|u|^2(t,y)|\nabla u|^2(t,x)-\text{ Im }(\bar{u}\nabla u)(t,y)\text{ Im }(\bar{u}\nabla u)(t,x)\right]dxdyds\\
=&\int_{\mathbb{R}^3}\iint_{\Omega\times\Omega}\chi^2\big(\tfrac{x-s}{R}\big)\chi^2\big(\tfrac{y-s}{R}\big)|u|^2(t,y)|\nabla u^\xi|^2(t,x)dxdyds,
\end{aligned}$$

for $u^\xi(t,x)=e^{ix\xi}u(t,x)$ and

$$\xi(t,s,R)=-\frac{\int_\Omega \chi^2\big(\frac{x-s}{R}\big)\text{ Im }(\bar{u}\nabla u)(t,x)dx}{\int_\Omega \chi^2\big(\frac{x-s}{R}\big)|u|^2(t,x)dx}$$

or $\xi=0$ if $\int_\Omega\chi^2\big(\frac{x-s}{R}\big)|u|^2(t,x)dx=0$.

Combining these estimates above, we have

$$\begin{aligned}
&\frac{1}{R^3}\int_{\mathbb{R}^3}\iint_{\Omega\times\Omega}|\chi(\tfrac{\cdot-s}{R})u|^2(t,y)\Big[|\nabla\big(\chi(\tfrac{\cdot-s}{R})u^\xi\big)|^2(t,x)-\frac{3}{4}|\chi(\tfrac{\cdot-s}{R})u|^4(t,x)\Big]dxdyds\\
\lesssim&\frac{1}{\eta^2R^2}+\partial_t M_R(t)+\int_\Omega\int_{\partial\Omega}|u|^2(t,y)|x\cdot n(x)|\;|\partial_n u|^2(t,x)dxdy\\
&+\iint_{\Omega\times\Omega}|u|^2(t,y)|u|^4[(\psi-\phi)(x-y)+(\phi-\phi_1)(x-y)]dxdy\\
&+\iint_{\Omega\times\Omega}|u|^2(t,y)|u\nabla u|(t,x)\,|\nabla(\psi+\phi)(x-y)|\,dxdy.
\end{aligned}$$

By the Lemma 4.1 and (4.11), for sufficiently large $R>0$, we have

$$\begin{aligned}
&\frac{1}{JT_0}\int_{R_0}^{R_0e^J}\int_I\frac{1}{R^3}\int_{\mathbb{R}^3}\iint_{\Omega\times\Omega}|\chi(\tfrac{\cdot-s}{R})u|^2(t,y)|\nabla(\chi\big(\tfrac{\cdot-s}{R}\big)u^\xi)|^2(t,x)dxdydsdt\frac{dR}{R}\\
\lesssim&\frac{1}{J\eta^2R_0^2}+\frac{R_0e^J}{(\log T_0)^{\frac12}J}+\frac{1}{R_0\eta J}+\frac{1}{J\eta}+\eta,
\end{aligned}$$

which implies the conclusion (4.16) by taking $\eta=J^{-\frac12}=R_0^{-1}=\varepsilon$ and $\log T_0=e^{\varepsilon^{-2}}$.

□

4.3. **Proof of Theorem 1.1.** By the interaction Morawetz estimates and the Sobolev embedding, there exists $T_0>0$ and $R\in[R_0,e^JR_0]$ such that for any interval $I=[a,a+T_0]$

$$\frac{1}{T_0}\int_I\frac{1}{R^3}\int_{\mathbb{R}^3}\Big\|\chi\big(\tfrac{\cdot-s}{R}\big)u(t)\Big\|_{L^2(\Omega)}^2\Big\|\nabla\Big(\chi\big(\tfrac{\cdot-s}{R}\big)u^\xi(t)\Big)\Big\|_{L^2(\Omega)}^2dsdt\lesssim\varepsilon.$$

Thus there exists $\theta\in[0,1]^3$ such that

$$\frac{1}{T_0}\int_I\sum_{z\in\mathbb{Z}^3}\Big\|\chi\big(\tfrac{\cdot-\frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^2(\Omega)}^2\Big\|\nabla\Big(\chi\big(\tfrac{\cdot-\frac{R}{4}(z+\theta)}{R}\big)u^\xi(t)\Big)\Big\|_{L^2(\Omega)}^2dt\lesssim\varepsilon.$$

Therefore, there exists a subinterval $I_0=[b-\varepsilon^{-\frac14},b]\subset I$ such that

$$\int_{I_0}\sum_{z\in\mathbb{Z}^3}\Big\|\chi\big(\tfrac{\cdot-\frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^2(\Omega)}^2\,\Big\|\nabla\Big(\chi\big(\tfrac{\cdot-\frac{R}{4}(z+\theta)}{R}\big)u^\xi(t)\Big)\Big\|_{L^2(\Omega)}^2\,dt\lesssim\varepsilon^{\frac34}.$$

This together with the Gagliardo-Nirenberg inequality

$$\|f\|_{L^3}^4 \lesssim \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2$$

implies that

$$\int_{I_0} \sum_{z\in\mathbb{Z}^3} \Big\|\chi\big(\tfrac{\cdot - \frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^3(\Omega)}^4 dt \lesssim \varepsilon^{\frac{3}{4}}. \tag{4.34}$$

On the other hand, by Hölder's inequality and Sobolev embedding, we have

$$\sum_{z\in\mathbb{Z}^3} \Big\|\chi\big(\tfrac{\cdot - \frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^2(\Omega)} \Big\|\chi\big(\tfrac{\cdot - \frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^6(\Omega)} \lesssim 1,$$

which yields

$$\sum_{z\in\mathbb{Z}^3} \Big\|\chi\big(\tfrac{\cdot - \frac{R}{4}(z+\theta)}{R}\big)u(t)\Big\|_{L^3(\Omega)}^2 \lesssim 1, \tag{4.35}$$

Now, we have, by (4.34) and(4.35),

$$\begin{aligned}
&\|u\|_{L^3(I_0\times\Omega)}^3 \\
\le & \int_{I_0} \sum_{z\in\mathbb{Z}^3} \Big\|\chi\Big(\frac{\cdot - \frac{R}{4}(z+\theta)}{R}\Big)u(t)\Big\|_{L^3(\Omega)}^3 dt \\
\le & \int_{I_0} \left(\sum_{z\in\mathbb{Z}^3} \Big\|\chi\Big(\frac{\cdot - \frac{R}{4}(z+\theta)}{R}\Big)u(t)\Big\|_{L^3(\Omega)}^4\right)^{\frac{1}{2}} \left(\sum_{z\in\mathbb{Z}^3} \Big\|\chi\Big(\frac{\cdot - \frac{R}{4}(z+\theta)}{R}\Big)u(t)\Big\|_{L^3(\Omega)}^2\right)^{\frac{1}{2}} dt \\
\le & \left(\int_{I_0} \sum_{z\in\mathbb{Z}^3} \Big\|\chi\Big(\frac{\cdot - \frac{R}{4}(z+\theta)}{R}\Big)u(t)\Big\|_{L^3(\Omega)}^4 dt\right)^{\frac{1}{2}} \left(\int_{I_0} \sum_{z\in\mathbb{Z}^3} \Big\|\chi\Big(\frac{\cdot - \frac{R}{4}(z+\theta)}{R}\Big)u(t)\Big\|_{L^3(\Omega)}^2 dt\right)^{\frac{1}{2}} \\
\le & \varepsilon^{\frac{1}{4}}.
\end{aligned} \tag{4.36}$$

By interpolation, we have

$$\|u\|_{L^5_{t,x}(I_0\times\Omega)} \le \|u\|_{L^3_{t,x}(I_0\times\Omega)}^{\frac{3}{7}} \|u\|_{L^{10}_{t,x}(I_0\times\Omega)}^{\frac{4}{7}} \lesssim \varepsilon^{\frac{1}{28} - \frac{1}{4}\frac{1}{10}\frac{4}{7}} \lesssim \varepsilon^{\frac{3}{140}}, \tag{4.37}$$

where we have used the fact that

$$\|u\|_{L^{10}_{t,x}(I\times\Omega)} \lesssim \langle |I| \rangle^{\frac{1}{10}},$$

which is a direct consequence of the Strichartz estimates, the Sobolev inequality and a standard continuity argument. Then, by the scattering criterion in Proposition 3.1, the conclusion follows.

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