On a recent reciprocity formula for Dedekind sums

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Abstract

Let $s(a,b)$ denote the classical Dedekind sum and $S(a,b) = 12s(a,b)$. Recently, Du and Zhang proved the following reciprocity formula. If a and b are odd natural numbers, $(a,b) = 1$, then

$$S(2a^*, b) + S(2b^*, a) = \frac{a^2 + b^2 + 4}{2ab} - 3,$$

where $aa^* \equiv 1 \mod b$ and $bb^* \equiv 1 \mod a$. In this paper we show that this formula is a special case of a series of similar reciprocity formulas. Whereas Du and Zhang worked with the connection of Dedekind sums and values of L-series, our main tool is the three-term relation for Dedekind sums.

1. Introduction and Result

Let a be an integer, b a natural number, and $(a,b) = 1$. The classical Dedekind sum $s(a,b)$ is defined by

$$s(a,b) = \sum_{k=1}^{b} ((k/b))((ak/b)).$$

Here

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \in \mathbb{R} \smallsetminus \mathbb{Z}; \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

(see [6, p. 1]). It is often more convenient to work with

$$S(a,b) = 12s(a,b)$$

instead. We call $S(a,b)$ a *normalized* Dedekind sum.

Probably the most important elementary result concerning Dedekind sums is reciprocity law. If a and b are coprime natural numbers, then

$$S(a,b) + S(b,a) = \frac{a^2 + b^2 + 1}{ab} - 3. \tag{1}$$

Recently, Du and Zhang have found the following hitherto unknown reciprocity law (see [2]). If a and b are coprime odd natural numbers, then

$$S(2a^*, b) + S(2b^*, a) = \frac{a^2 + b^2 + 4}{2ab} - 3, \tag{2}$$

where $aa^* \equiv 1 \mod b$ and $bb^* \equiv 1 \mod a$.

The proof given in [2] is based on the connection of Dedekind sums and values of L-series. The authors of the said paper ask for an elementary proof of their result. Here we give such an elementary proof based on the tree-term-relation of Dedekind sums. Moreover, we show that (2) is a special case of a series of similar reciprocity formulas. Indeed, we have the following.

Theorem 1 *Let a and b be coprime natural numbers and t a natural number such that $a^2+1 \equiv 0 \mod t$. Further, let $(b,t)=1$. Then*

$$S(ta^*,b)+S(tb^*,a)=\frac{a^2+b^2+t^2}{tab}-3+S(ab,t). \tag{3}$$

As to the case $t=1$, we note

$$S(a^*,b)=S(a,b) \tag{4}$$

(see [6, p. 26]) and $S(ab,1)=0$. In the case $t=2$, a and b are odd and $S(ab,2)=0$. Hence we obtain the following.

Corollary 1 *The formulas* (1) *and* (2) *are immediate consequences of Theorem 1 in the cases $t=1$ and $t=2$.*

Corollary 2 *Suppose, in the setting of Theorem 1, that $b \equiv \pm 1 \mod t$. Then*

$$S(ta^*,b)+S(tb^*,a)=\frac{a^2+b^2+t^2}{tab}-3. \tag{5}$$

Suppose, on the other hand, that $b \equiv \pm a \mod t$. Then

$$S(ta^*,b)+S(tb^*,a)=\frac{a^2+b^2+t^2}{tab}+\begin{cases}-t-2/t, & \text{if } b\equiv a \mod t;\\ t+2/t-6, & \text{if } b \equiv -a \mod t.\end{cases} \tag{6}$$

As to (5), note that $(ab)^2 \equiv -1 \mod t$, which shows that $S(ab,t)=0$ (see [6, p. 28]). In the case of (6), we use $S(1,t)=t+2/t-3$ (which is an immediate consequence of (1)) and $S(-a,b)=-S(a,b)$ (see [6, p. 26]).

Remark. The natural numbers t such that there is a natural number a with $a^2+1 \equiv 0 \mod t$ can be characterized as follows: $t=m$ or $t=2m$, where m is a natural number whose prime divisors are all $\equiv 1 \mod 4$ (this includes $m=1$).

Example. Let $t=5$, a such that $a^2+1\equiv 0 \mod 5$, and $(a,b)=(b,5)=1$. This implies $a \equiv \pm 2 \mod 5$. If $b \equiv \pm 1 \mod 5$, then

$$S(5a^*,b)+S(5b^*,a)=\frac{a^2+b^2+25}{5ab}-3.$$

In the remaining case, we have $b \equiv \pm a \mod 5$. If $b \equiv a \mod 5$, then (6) reads

$$S(5a^*,b)+S(5b^*,a)=\frac{a^2+b^2+25}{5ab}-27/5.$$

If $b \equiv -a \mod 5$, we have

$$S(5a^*,b)+S(5b^*,a)=\frac{a^2+b^2+25}{5ab}-3/5.$$

Proof of Theorem 1

Let a, b, t be natural numbers, $(a, b) = (b, t) = 1$, such that $a^2 + 1 \equiv 0 \mod t$. Put $c = b(a^2 + 1)/t$. Obviously, $(a, c) = 1$. Then [4, Th.4] says

$$S(a, c) = \frac{(b^2 - 1)a}{tb} - S(ab, t) + S(at^*, b). \tag{7}$$

where $tt^* \equiv 1 \mod b$. By the reciprocity law (1),

$$S(a, c) = -S(c, a) + \frac{a^2 + c^2 + 1}{ac} - 3.$$

However, $c \equiv bt^* \mod a$, with $tt^* \equiv 1 \mod a$. Hence $S(c, a) = S(bt^*, a)$. We replace at^* by ta^* and bt^* by tb^* in the respective normalized Dedekind sums (see (4)). Then a short calculation proves Theorem 1.

We still have to make clear that this remarkably simple proof is based on elementary results. Indeed, (7) follows from the three-term relation

$$S(a, b) = S(c, d) + \varepsilon S(r, |q|) + \frac{b^2 + d^2 + q^2}{bdq} - 3\varepsilon$$

(see [4]). Here b, d, are natural numbers, a, c integers, $(a, b) = (c, d) = 1$, $a/b \neq c/d$. Further, $q = ad - bc$ and ε is the sign of q. Finally, $r = aj - bk$, where j, k are integers such that $-cj + dk = 1$. The three-term relation, in turn, can be deduced from the composition rule of the logarithm of Dedekind's η-function (see [1, 3]). An elementary proof of this composition rule is given in [5, §4].

References

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