

RAIL KNOTOIDS

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Dedicated to Louis H. Kauffman for his 70th birthday

ABSTRACT. We work on the notions of rail arcs and rail isotopy in \mathbb{R}^3 , and we introduce the notions of rail knotoid diagrams and their equivalence. Our main result is that two rail arcs in \mathbb{R}^3 are rail isotopic if and only if their knotoid diagram projections to the plane of two lines which we call rails, are equivalent. We also make a connection between the rail isotopy in \mathbb{R}^3 and the knot theory of the handlebody of genus 2.

INTRODUCTION

We study isotopies in \mathbb{R}^3 between arcs that have their endpoints on two fixed parallel lines (we call them rails), that allow the endpoints to move freely on the rails but do not allow any other point of the arcs to touch them. We call such arcs as rail arcs and such isotopies among them as rail isotopies. As we remarked in [15], it turns out that rail isotopies are connected to the knot theory of the handlebody of genus 2. Rail arcs and rail isotopies are convenient renamings of what in [9] are called open arcs and line isotopies respectively. It was proved in [9] that rail isotopies can be studied diagrammatically with the notion of knotoid diagram which is a kind of generalization of knot diagram. A planar knotoid diagram is what one gets by projecting a rail arc onto a plane perpendicular to the rails (keeping track of over/under data at crossings).

Here we develop a new diagrammatic setting, which we call rail knotoid diagram, by projecting rail arcs onto the plane of the rails, and we prove that rail isotopy corresponds to (gives rise and comes from) an appropriately defined equivalence between such diagrams. Although we do not make use of any previous result on knotoids, the current article belongs in the general theory about them, and familiarizing with knotoids helps in putting the current work into context. Thus in §3 we recall some basic facts about knotoids. In §1 we introduce the basic notions of rail arcs and rail isotopy between such arcs and we develop the notion of triangle move for studying such isotopies. In §2 we remark on the connection of the study of rail isotopy to the study of knot theory in the handlebody of genus 2. In §4 we introduce and study rail knotoid diagrams. We define a notion of equivalence between such diagrams and we prove that two rail arcs are rail isotopic if and only if their rail knotoid diagrams are equivalent.

We will be working in the piecewise linear (p.l.) category, thus all curves will be p.l. curves, all maps will be p.l. maps etc. Due to the usual p.l. approximation theorems for the analogous smooth objects, our results hold in the smooth category as well.

1. RAIL ISOTOPY

Henceforth \mathbb{R}^3 will be considered equipped with two given parallel lines ℓ_1, ℓ_2 (in this order) which we will call as *rails*. We define:

Definition 1. A *rail arc* is any oriented, connected, embedded arc c in \mathbb{R}^3 with its interior in $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$, its first endpoint on ℓ_1 and the last on ℓ_2 . We call two rail arcs c_1, c_2 as *rail isotopic*,

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if there exists an isotopy of \mathbb{R}^3 taking one onto the other (thus c_1, c_2 are connected by a homotopy of embeddings in \mathbb{R}^3), so that each rail maps onto itself (not necessarily pointwise) throughout the isotopy. In particular, this implies that at each time throughout the isotopy, the image of the arc is a rail arc, and each endpoint remains on the same rail, but with the freedom to move up and down on it. We call such an isotopy as a *rail isotopy* in \mathbb{R}^3 .

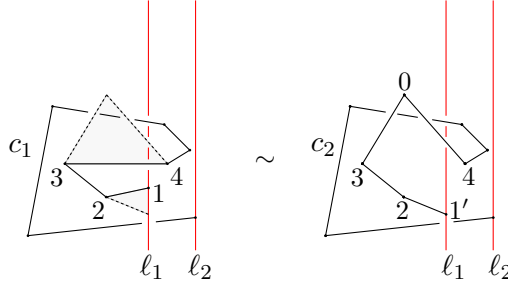


FIGURE 1. Two triangle moves in \mathbb{R}^3 which are compatible for the rail arc c_1 and transform it to the rail arc c_2 .

Similarly to the case of isotopy between p.l. knots in \mathbb{R}^3 , rail isotopy between p.l. rail arcs can be effected via a finite sequence of *triangle moves* or *elementary moves*: a rail arc c is modified so that it either replaces its edge AB by two new edges AC, CB , or vice versa, where the triangle $\Delta = ABC$ does not intersect c or the rails at any other point (see Figure 1) or else it replaces edge AB with CB , where A, C lie on the same rail and the triangle $\Delta = ABC$ has no other common points with the arc c , and no other common point with the rails other than segment AC . When it will be necessary to distinguish the second kind of move from the first one, we will be calling it as a *space slide move*. Each triangle move, which from now on will be denoted Δ as the triangle itself, is actually the restriction on an edge (or on two consecutive edges) of the result of an isotopy h_Δ in \mathbb{R}^3 which fixes all points in the complement of the interior of a closed 3-ball neighborhood of Δ (with the vertices of Δ on the boundary of the ball if we wish). This can readily be made to keep the rails fixed (although not necessarily pointwise) which means it is a rail isotopy. So instead of rail isotopies between arcs, we can indistinguishably use, if we wish, triangle moves. We talk in general about triangle moves in \mathbb{R}^3 without any reference to rail arcs, as long as the triangles ABC do not intersect the rails as explained above. For a triangle move Δ replacing edge AB by edges BC, CA , its *inverse* move Δ^{-1} is the triangle move replacing edges AC, CB by edge AB ; the inverse of Δ^{-1} is Δ .

If $\Delta_1, \dots, \Delta_k$ are the triangles for a sequence of triangle moves in \mathbb{R}^3 and $h_{\Delta_1}, \dots, h_{\Delta_k}$ corresponding isotopies in space, instead of writing $h_{\Delta_k} \circ \dots \circ h_{\Delta_1}$ we can write $\Delta_k \circ \dots \circ \Delta_1$, and say that $\Delta_k \circ \dots \circ \Delta_1$ is the *composite move* of the triangle moves $\Delta_1, \dots, \Delta_k$ (in this order). We call the last ones as *submoves* of the former. If $\Delta_1, \dots, \Delta_k$ are the triangles for a sequence of triangle moves for getting c_2 from c_1 , we say $\Delta = \Delta_k \circ \dots \circ \Delta_1$ is a *composite move* for c_1 and write $c_1 \xrightarrow{\Delta} c_2$. Let us note that a triangle move in \mathbb{R}^3 performed in an edge of a rail arc is not necessary a triangle move of the arc itself as the triangle of the move may interfere with the rest of the arc. Similarly, whenever we get a rail arc c_2 from another rail arc c_1 via a sequence of triangle moves $\Delta_1, \dots, \Delta_k$, it is not always true that we can replace any Δ_i by other moves δ_j in \mathbb{R}^3 that compose to it and still claim that we get c_2 from c_1 , since for example the δ_j 's may interfere with the rest of $(\Delta_{i-1} \circ \dots \circ \Delta_1)(c_1)$. But clearly, we can do so in case the triangles of the δ_j 's are just part of Δ_i . Also, two moves on distinct edges of an arc c whose triangles have no common points other than possibly a common vertex of c , can be performed in either order and we can think of them

as being performed simultaneously. We call two such moves as *compatible for c* and extend the definition to any finite number of moves when each is performed on its own edge or pair of edges of c and any two of their triangles have nothing else in common other than possibly a vertex of c . Such moves can be performed in any order; rigorously, this means that their composition is defined in any order, and the result is always the same.

Finally, applying to an arc a sequence of moves, and then performing in reverse order their inverses, we return to the original arc. Thus:

Lemma 1. *For triangle moves Δ_i and rail arcs c_1, c_2 so that $c_1 \stackrel{\Delta_k \circ \dots \circ \Delta_1}{\sim} c_2$, the following are true:*

- [For some $i, \Delta_i = \delta_i \circ \dots \circ \delta_1$, and $\delta_j \subseteq \Delta_i, \forall j \Rightarrow c_1 \stackrel{\Delta_k \circ \dots \circ (\delta_i \circ \dots \circ \delta_1) \circ \dots \circ \Delta_1}{\sim} c_2$
- [$\Delta_1, \dots, \Delta_k$ compatible for c_1 and τ is a permutation of $\{1, \dots, k\} \Rightarrow c_1 \stackrel{\Delta_{\tau(1)} \circ \dots \circ \Delta_{\tau(k)}}{\sim} c_2$
- $c_2 \stackrel{\Delta_1^{-1} \circ \dots \circ \Delta_k^{-1}}{\sim} c_1$

Any rail arc without its endpoints lies in $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$ and we call it as *open rail arc*. The boundary of an open rail arc as a subset of \mathbb{R}^3 consists of two points, one on ℓ_1 and the other on ℓ_2 . We call an isotopy of $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$ as *rail isotopy*, if it moves one open rail arc onto another, keeping at all times the image of the open rail arc as an open rail arc (the boundary points of the images of the rail arc considered as subsets of \mathbb{R}^3 , are onto the rails).

Clearly, the definitions imply that each rail arc corresponds to a unique open rail arc and vice versa (the first lies inside \mathbb{R}^3 , and the second inside $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$), and also that a rail isotopy of rail arcs gives rise to a rail isotopy of open rail arcs and vice versa (the first takes place inside \mathbb{R}^3 , and the second inside $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$).

2. RAIL ISOTOPY AND HANDLEBODIES

The space $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$ is homeomorphic to the interior $(\mathcal{H}_2)^o$ of a handlebody $\mathcal{H}_2 = A \times [0, 1]$ of genus 2, where A is an annular thickening of a figure eight plane curve, with boundary three circles, say c_0, c_1, c_2 . Let $h : \mathbb{R}^3 - (\ell_1 \cup \ell_2) \rightarrow (\mathcal{H}_2)^o$ be a homeomorphism of the two manifolds, sending points close to ℓ_i to points close to $c_i \times [0, 1]$. Then it is meaningful to repeat the definitions for rails, rail arcs, open rail arcs and rail isotopies, replacing (i) \mathbb{R}^3 by $\mathcal{H}_2 - ((c_0 \times [0, 1]) \cup (A \times 0) \cup (A \times 1))$, (ii) $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$ by $(\mathcal{H}_2)^o$ and (iii) ℓ_i by $c_i \times [0, 1]$ for $i = 1, 2$.

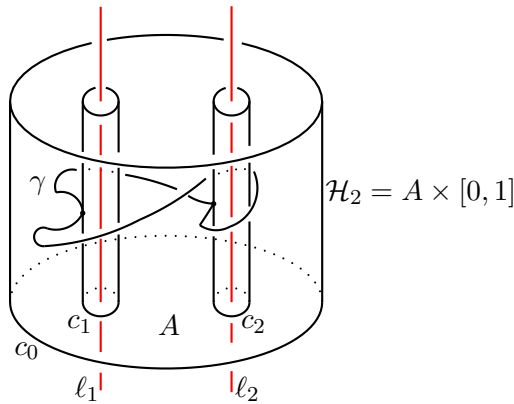


FIGURE 2. The handlebody \mathcal{H}_2 of genus 2 standardly embedded in \mathbb{R}^3 , a knot in \mathcal{H}_2 with two points on its boundary, and two ‘rail’ lines ℓ_1, ℓ_2 in \mathbb{R}^3 through the two ‘holes’ of the handlebody.

With these definitions, rail isotopy in $\mathcal{H}_2 - ((c_0 \times [0, 1]) \cup (A \times 0) \cup (A \times 1))$ corresponds by h^{-1} to rail isotopy in $\mathbb{R}^3 - (\ell_1 \cup \ell_2)$. But the first rail isotopy, clearly corresponds (gives rise in the natural way, and vice versa) to rail isotopy in \mathcal{H}_2 , where now we allow the isotopies to touch the part $(c_0 \times [0, 1]) \cup (A \times 0) \cup (A \times 1)$ of the boundary of the handlebody of genus two, retaining the rest of the above mentioned properties about keeping endpoints on the cylinders $c_i \times [0, 1]$. And as already said, the second rail isotopy above, corresponds (gives rise in a natural way, and vice versa) to the rail isotopy of \mathbb{R}^3 . Summarizing:

Proposition 1. *Rail isotopies in \mathcal{H}_2 are in one to one correspondence with rail isotopies in \mathbb{R}^3 .*

Let us now notice that any knot in \mathcal{H}_2 is isotopic to some knot k with a unique point on $c_1 \times [0, 1]$ and a unique point on $c_2 \times [0, 1]$. Let us call k as *rail knot*, its points on $c_i \times [0, 1]$, $i = 1, 2$ as rail points and the two arcs of k with endpoints the rail point as its corresponding rail arcs. For two rail knots k_1, k_2 , we can modify any isotopy of one onto the other so as at each stage its image is a rail knot. We get then a rail isotopy of the two rail arcs of k_1 to the two rail arcs of k_2 . Throughout this isotopy, the images of the two arcs are disjoint except for the two rail points of the first arc which have the same images as the two rail points of the second. Let us in general, call two rail isotopies of two rail arcs with the same endpoints, as *matching* whenever these two properties hold. Then the converse of the first observation about knots holds, and summarizing we get:

Proposition 2. *Knot isotopies in \mathcal{H}_2 are in one to one correspondence with matching rail isotopies of two rail arcs with the same endpoints.*

3. A REMINDER ON KNOTOIDS

In this section we recall some facts about knotoids. A *knotoid diagram* K in an oriented surface Σ is an immersion of the unit interval $[0, 1]$ in Σ with a finite number of double points each of which is a transversal self-intersection endowed with over/under data. These are the crossings of K . The images of 0 and 1 are two distinct points called the *endpoints* of K and are specifically called *leg* and *head*, respectively, so that K is naturally oriented from its leg to its head. The trivial knotoid diagram is assumed to be an immersed arc without any self-intersections. See Figure 3 for some examples.

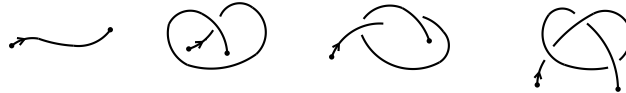


FIGURE 3. Examples of knotoid diagrams.

On the set of knotoid diagrams in Σ the usual local moves are allowed away from the endpoints. Namely the three Reidemeister moves together with planar isotopy, which includes also the *swing moves* for the endpoints, whereby an endpoint can be pulled within its region, without crossing any other arc of the diagram. See Figure 4. All these moves generate an equivalence relation in the set of knotoid diagrams in Σ and the equivalence classes are called *knotoids*.



FIGURE 4. Moves for knotoid diagrams.

The moves consisting of pulling the arc adjacent to an endpoint over or under another arc, as shown in Figure 5, are the *forbidden moves* in the theory. Notice that, if both forbidden moves

were allowed, any knotoid diagram in any surface could be clearly turned into the trivial knotoid diagram.



FIGURE 5. Forbidden knotoid moves.

The theory of knotoids was introduced by Turaev [17] in 2010. The theory of knotoids in the 2-sphere (spherical knotoids) extends classical knot theory and also proposes a new diagrammatic approach to classical knot theory [17]. This approach promises reducing of the computational complexity of knot invariants, see [17, 4]. In [17] basic properties of knotoids were studied, including the introduction of several invariants of knotoids in the 2-sphere, such as the complexity (or height [9]) and the Jones/bracket polynomial. Knotoids in S^2 were classified by Bartholomew in [2] up to 5 crossings by using Turaev's generalization of the bracket polynomial for knotoids. There is also a recent classification table for prime knotoids of positive complexity with up to 5 crossings [16], obtained by using the correspondence between knotoids in S^2 and knots in the thickened torus. New invariants for knotoids were introduced in [8, 9] in analogy with invariants from virtual knot theory.

Planar knotoids surject to spherical knotoids, but do not inject [17]. For example, the first two illustrations of Figure 3 are equivalent as spherical knotoids but distinct as planar ones. This means that planar knotoids provide a much richer combinatorial structure. This fact has interesting implications in the study of proteins. Indeed, recently knotoids have been studied in the field of biochemistry as they suggest new topological models for open protein chains [5, 6, 7]. Such studies are enabled due to the following lifting of knotoids to open space curves (what we call rail arcs), proposed by Ggmc and Kauffman in [9]. Namely, an open space curve in \mathbb{R}^3 projects to a planar knotoid diagram when projected along the two lines passing through its endpoints (what we call rails) and are perpendicular to a chosen projection plane, and this curve can be viewed as a lifting of the diagram. Figure 6 illustrates two such projections. The method in [5, 6, 7] is to project an open protein chain to several planes and to consider all possible knotoid types obtained this way, choosing the dominant one for representing the protein. Then, the invariants introduced in [17, 8, 9] are used for determining its topological type.

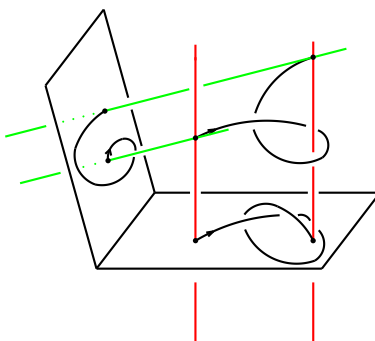


FIGURE 6. Projections of a rail arc as knotoid diagrams.

A line isotopy in [9] between two open space curves is what we call rail isotopy, and Ggmc and Kauffman have proved (in our new terminology) the following:

Theorem. *Two rail arcs are rail isotopic if and only if their knotoid diagram projections in a plane perpendicular to the rails are equivalent.*

In [15] we observed that this lifting of knotoids in 3-space is related to the knot theory of the handlebody of genus 2. Some other recent works on knotoids include: the theory of braidoids [8, 11, 12], a study of biquandle coloring invariants [13], the study of knots that are knotoid equivalent [1] and the construction of double branched covers of knotoids [3]. For a survey on the subject the interested reader may consult [10].

4. RAIL ISOTOPY AND RAIL KNOTOIDS

We are now going to investigate rail isotopy in \mathbb{R}^3 in a new diagrammatic setting by projecting the rail arcs to the plane π defined by the rails ℓ_1, ℓ_2 , which we can call as *rail plane*. Figure 7 conveys a feeling of the difference between the new setting (rightmost figure) and that of the usual planar knotoids (leftmost figure).

Let us note that projections of rail arcs onto planes can be as bad as projections of knots to planes, but clearly any rail arc is rail isotopic to one with a generic projection to any given plane, meaning it has only a finite number of intersection points with itself and the rails, all of them double points. Thus from now on we can restrict attention to such projections. Keeping track of the over/under crossings at the double points of the onto the plane π of the rails ℓ_1, ℓ_2 , we get a generic immersion of the unit interval in the plane with its endpoints on the two rails. This projection is actually just a planar knotoid diagram c_{pr} on π , whose endpoints are on the rails (the leg on ℓ_1 and the head on ℓ_2).

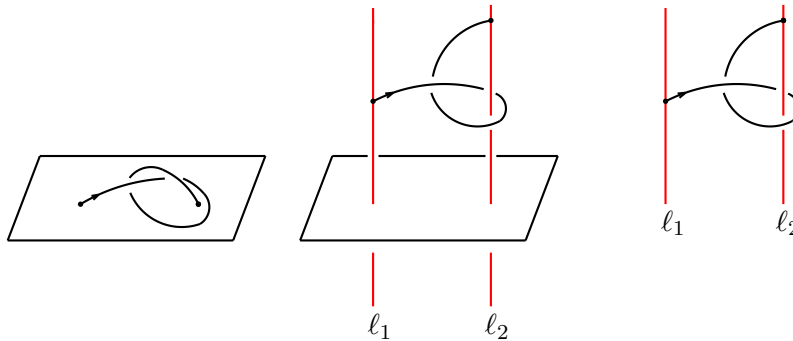


FIGURE 7. In the middle we see a rail arc in space and a plane perpendicular to the rails, whereas on the left and right we see the knotoid diagram projections of this rail arc on the perpendicular plane, and on the plane of the rails respectively.

So first we define:

Definition 2. A *planar rail knotoid diagram* or just *rail knotoid diagram* is an immersion of the unit interval in the rail plane π of the rails with only a finite number of transversal intersection points with itself and the rails. All intersection points, except for the endpoints, are double points with additional over/under data. The endpoints, the first one on ℓ_1 and the second one on ℓ_2 , are trivalent.

Two rail knotoid diagrams on π are *rail equivalent* whenever one can be obtained from the other via a finite sequence of the *rail knotoid equivalence moves* defined locally in Figure 8, which include the usual Reidemester moves $\Omega_1, \Omega_2, \Omega_3$ and their versions where parts of the rails are involved, along with slide *slide moves* which involve the rails and the endpoints of the rail knotoids, and finally some planar rail isotopy moves or just *planar isotopies* of π .

Clearly, equivalence between rail knotoid diagrams as defined is indeed an equivalence relation in the set of all planar rail knotoid diagrams. We call the equivalence classes as *planar rail knotoids* or simply as rail knotoids.

Although rather obvious, it is no harm to call upon some more terminology: if a move δ_1 is applied to a diagram c_1 resulting to a diagram c_2 , and δ_2 is applied to c_2 resulting to c_3 , we call δ_2 as successive to δ_1 , we say that they compose and write $\delta_2 \circ \delta_1$, and we also say that their composition is applied to c_1 resulting to c_3 . Similarly we define the notion of a sequence of successive moves (or just a sequence of moves), and of their composition.

We prefer the planar isotopies of Figure 8 instead of plane isotopies in π , so that the above equivalence relation which we'll try to investigate can get an entirely diagrammatic setting.

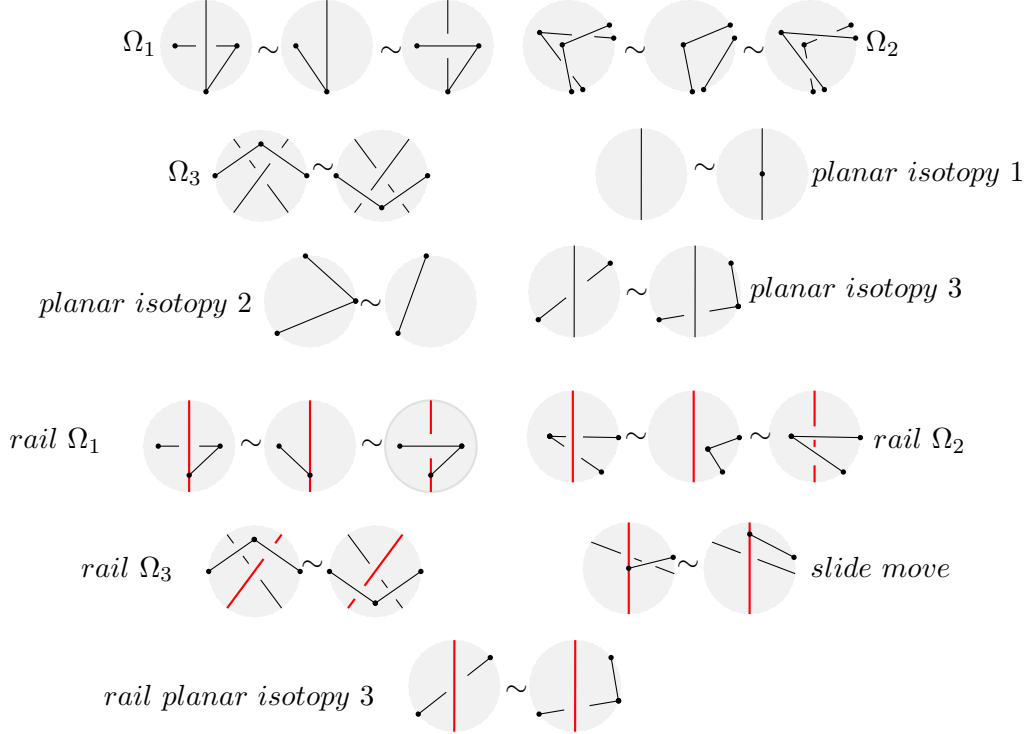


FIGURE 8. The set of rail knotoid equivalence moves, which generate the equivalence in the set of planar rail knotoid diagrams. The moves are split in two parts: those which do not involve the rails, and those which do involve them. Dots denote vertices, and rails get a red colour. The moving part in Ω_3 -moves can be on top, bottom or in the middle, and especially in the rail Ω_3 moves, the rail part can be on top or bottom of the other non-moving part. The non-moving part in planar isotopies of type 3 can be either on top or bottom, whereas the non-moving arc in slide moves can only be on top or bottom of all other parts.

For a triangle move Δ between two rail arcs c_1, c_2 in \mathbb{R}^3 , let us denote Δ_{pr} the projection on π of the edges of the triangle Δ , keeping track of over/under data with respect to the rails and to the projections c_{1pr}, c_{2pr} of c_1, c_2 . Let us call Δ as *nice* whenever Δ_{pr} is a finite composition of Reidemeister, slide or planar isotopy moves between c_{1pr}, c_{2pr} . And let us call $\Delta = ABC$ which exchanges edge AB with edges AC, CB as *good*, whenever the following entering-exiting condition is satisfied: for Δ_{pr} the entering and exiting edges (if any) MA_{pr}, BN_{pr} of c_{1pr} (and c_{2pr}) do not intersect the interior of ABC_{pr} . The following is true:

Lemma 2. *Any triangle move Δ between two rail arcs c_1, c_2 is a composition of other triangle moves Δ_i , each being a nice or good one, with triangles satisfying $\Delta_i \subseteq \Delta$ for all i .*

Proof. Let $\Delta = ABC$ be a move which, say for definiteness, replaces edge AB of c_1 by edges AC, CB of c_2 . If on π it happens for example that the entering edge, say MA_{pr} , intersects the interior of ABC_{pr} , then by small triangle moves we put a new vertex on c_1 at a point B' nearby A on edge AB . We take care that the triangle of each one of these moves is a tiny triangle part of ABC which projects its vertices away from edges of c_{pr} other than AB and whose projection on π contains no vertices of c_{pr} other than A, B . Let δ be the composition of these moves. Let C' be a point close to A on the segment AC , and let Δ_1 be the triangle move replacing AB' by $AC', C'B'$. Let Δ_2 be the triangle move replacing $C'B', B'B$ by $C'B$. Let Δ_3 be the triangle move replacing $C'B$ by $C'C, CB$. Then $\Delta = \Delta_3 \circ \Delta_2 \circ \Delta_1 \circ \delta$ (no question here about if these moves compose).

Note now that each submove of δ projects on π to a planar isotopy of type 2 or 3, thus δ is a nice move. Also, choosing B', C' appropriately close to A , the projection Δ_{1pr} is an Ω_1 move, thus Δ_1 is a nice move. Finally $\Delta_{2pr}, \Delta_{3pr}$ are good moves with respect to their entering edge. Since the triangle of δ and all of Δ_i 's is a subset of ABC we would have finished, if only Δ_2, Δ_3 were good with respect to their exiting edge as well.

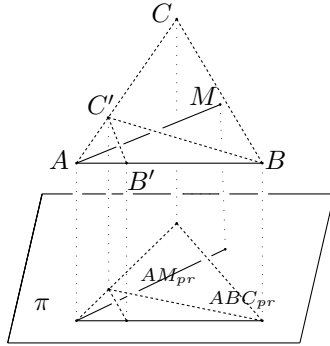


FIGURE 9. The only original vertices of c_1 are M, A, B . The small triangles (sub-triangles of ABC) of the moves inserting vertex B' to c_1 are not shown.

If still any one of Δ_2, Δ_3 is not a good move, then working similarly for it, but this time for the exiting edge, we write it as a composition of submoves which are nice or satisfy the exiting edge condition. Since for the latter moves the entering edge condition is automatically satisfied, they are good, and by Lemma 1 we are done.

If Δ is a space slide move, observe that one of the entering or exiting conditions is automatically satisfied as there exists no such edge for the move. Also, let AB be the edge that become AC performing the move, and BC be part of a rail. The arguments developed above work for the vertex A with the slight modification that $C'B$ is not replaced by the two edges $C'C, CB$, but rather just by the single edge $C'C$. \square

Lemma 3. *Any good triangle move Δ between two rail arcs c_1, c_2 is a finite composition of nice moves Δ_i with triangles satisfying $\Delta_i \subseteq \Delta, \forall i$.*

Proof. Let us call c_{pr} the projection of what is left in common after removing the non-common points of c_1, c_2 on Δ . We consider two cases:

Case (I). The move $\Delta = ABC$ exchanges the edge AB with the edges AC, CB between c_1 and c_2 (it replaces AB by AC, CB or vice versa). Then no side of the triangle Δ can be part of a rail, and the move Δ is not a slide move.

Since Δ is good, the entering and exiting edges (if any) on π , say MA_{pr} and BN_{pr} , do not intersect the interior of Δ_{pr} . But the rails, as well as the projection c_{pr} might do. As always our projections keep track of over/under data.

Since no rail and no arc of c_1, c_2 pierces triangle Δ , the parts of $S = \ell_1 \cup \ell_2 \cup c_{pr}$ that intersect Δ_{pr} are equipped with data rendering them either entirely over or entirely under Δ_{pr} . The vertices and crossing points of S in Δ_{pr} are finite and have to appear in the interior of Δ_{pr} . Thus in the interior of Δ_{pr} we can consider small enough, disjoint triangles around each one of these vertices and crossings, and we can extend these triangles to a finite triangulation of the whole triangle Δ_{pr} , taking care of putting no vertex or side of the triangulation on S . Then each triangle δ_i of the triangulation contains a part of S falling to one of four types: (i) δ_i contains a single crossing point of S with branches through it that intersect two sides of δ_i at interior points, (ii) δ_i contains a single vertex of S and parts of the two edges with endpoint this vertex, that intersect one or two sides of δ at interior points, (iii) δ_i contains only a part of an edge that intersect two sides of δ at interior points, (iv) δ_i contains no no point of S . Constructing the triangulation, it is convenient to consider triangles of types (i), (ii) that look as in Figure 10.

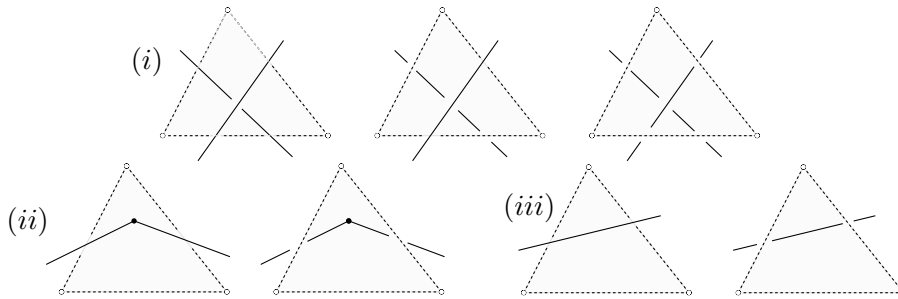


FIGURE 10. A triangle move Δ in space, decomposes to smaller moves that project on π to triangles that contain either no parts of the projected arcs and the rails, or else they do so in the three ways shown here.

Let us notice that the triangulation by the δ_i 's of Δ_{pr} , implies a triangulation of Δ by triangles Δ_i 's that project onto the δ_i 's on π . Let us also notice that by Lemma 1, it is legitimate to perform consecutive moves in each one of the Δ_i 's. And let for definiteness, the move Δ replaces AB by AC, CB . Then since the triangles of the Δ_i 's have Δ as their union, we can arrive at the same result of replacing AB by AC, CB performing moves through the Δ_i 's starting from triangles with sides on AB , and ending up with those with sides on AC, CB .

The Δ_i moves, which are moves between arcs in space, project on π to corresponding δ_i moves between the projected arcs on the plane. But such a move for a triangle δ_i of type (i) is either an Ω_3 move composed with a planar isotopy 1 move, or it decomposes to an Ω_2 move, an Ω_3 move, and some planar isotopies (Figure 11). Similarly, for δ_i of type (ii) the corresponding move is either an Ω_2 move or it decomposes to some planar isotopies; for δ_i of type (iii) the corresponding move is an isotopy 1 move, an isotopy 3 move, or decomposes to an Ω_2 and some isotopies; whereas for δ_i of type (iv) the corresponding move is a planar isotopy. Thus each Δ_i is a nice move and we are done.

Case (II). $\Delta = ABC$ is a space slide move. Then it does not exchange an edge by two others or vice versa as above, but it instead replaces in space one position of the initial or final edge of c_1, c_2 by another position, say from AB to CB with A, C on a rail ℓ .

We spot all crossing points of AC with c_{pr} . If any, we call these points in increasing distance from A as X_1, X_2, \dots, X_k . Let Y_i be an interior point in the segment $X_i X_{i+1}$ for $i = 1, \dots, k-1$. We decompose move Δ to a sequence of moves through the triangles $BAY_1, BY_1 Y_2, \dots, BY_{k-2} Y_{k-1}, BY_{k-1} C$; we are allowed to, by Lemma 1. We'll prove that each one of these moves is as required, and then by Lemma 1 we'll get that Δ is also as required and we will be done. It is enough to prove that

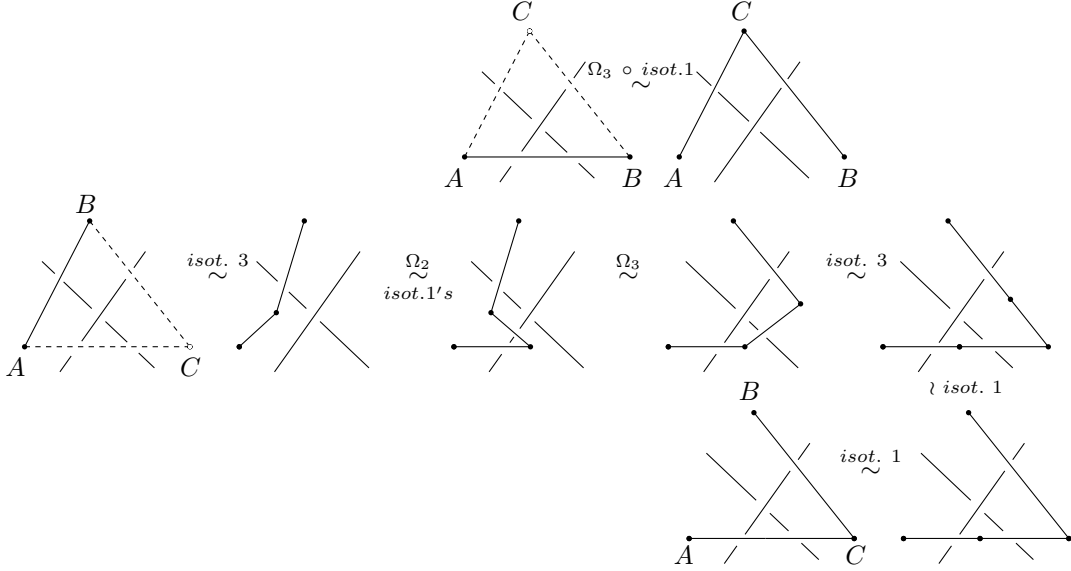


FIGURE 11. Two cases of decomposing a move through a triangle δ_i of type (i) on π to a sequence of Reidemeister, slide or planar isotopies moves.

the move δ through BAY_1 satisfies our Lemma, since the reasoning will apply to all other triangles in this sequence as well. To this end:

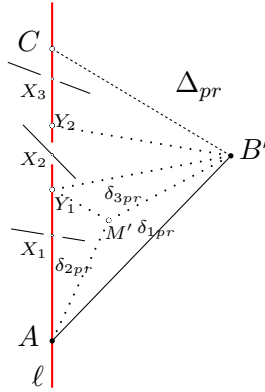


FIGURE 12. Decomposing a space slide move $\Delta = ABC$ to smaller moves. $ABC = AY_1B \cup Y_1Y_2B \cup \dots$, and $AY_1B = \delta_1 \cup \delta_2 \cup \delta_3$ are depicted here as projected on π . Solid black lines are part of the projected arc. Triangle moves via the δ_i 's are good.

A and Y_1 project on π onto themselves. Let us call the projection of B as B' . We triangulate $B'AY_1$ so that one of the triangles is AY_1M' , where M' is the projection of a point M on AX_1B chosen so close to the rail ℓ so that AY_1M' contains no crossings of arcs of pr_c . Let us consider the following be moves in space: δ_1 in triangle AMB that replaces AB by AM, MB , δ_2 in triangle MY_1B that replaces AM by Y_1M (a slide move), and δ_3 in triangle CMB that replaces Y_1M, MB by Y_1B . Then $\delta = \delta_3 \circ \delta_2, \delta_1$. But by Case (I), we know that δ_1, δ_3 are as required by our Lemma, whereas δ_2 projects to a slide move thus it is also as required by our Lemma. Since the triangles of these moves are part of the triangle ABC of move Δ , Lemma 1 implies that δ is as required and we are done. \square

So if Δ is a triangle move between two rail arcs c_1, c_2 and $\Delta = \Delta_k \circ \dots \circ \Delta_i$ a decomposition to nice and good moves as assured in Lemma 2, then Lemma 1 allows us to replace in the decomposition of Δ , any good move Δ_i with a product of nice moves as in Lemma 3. Thus we proved:

Lemma 4. *Any triangle move Δ between two rail arcs c_1, c_2 is a finite composition $\Delta_k \circ \dots \circ \Delta_i$ of nice moves with triangles satisfying $\Delta_i \subseteq \Delta, \forall i$.*

So if the two rail arcs c_1, c_2 are rail isotopic, thus related by a finite sequence of triangle moves, we can replace by Lemma 1 any one of these moves by a sequence of moves as assured in Lemma 3, and we get:

Corollary 1. *If the two rail arcs c_1, c_2 are rail isotopic, then they are related by a finite sequence of nice moves.*

The following is a special case of (one part of) the main Theorem that follows.

Lemma 5. *If c_1, c_2 are two rail arcs with exactly the same (pointwise) projection $c_{1pr} = c_{2pr}$ on the plane π of the rails, then there exists a rail isotopy between the arcs.*

Proof. If necessary, we subdivide the two arcs so that they get the same number of vertices and every vertex of each one of them lies on the same vertical line (with respect to π) with a vertex of the other. Since this subdivision can be performed via triangle moves, the resulting arcs c'_1, c'_2 are related to the original ones via isotopies. Pointwise the subdivided arcs have not changed.

We perform a further subdivision of c'_1, c'_2 as thus: if e_1, f_1 are two edges of c'_1 whose projections have a crossing point A on π , and e_2, f_2 the corresponding edges of c'_2 with the same projections on π , then we chose on $e_{1pr} = e_{2pr}$ two nearby points on each of the two sides of A and lift them vertically to two points on each of e_1, e_2 as new vertices; and similarly for f_1, f_2 . We do so for every crossing point on the projections on π , and remaining very close to each crossing point, the segments on the old edges between the new vertices, remain disjoint, even if they happen to appear on the same old edge. Since the insertion of all these new vertices can be performed via triangle moves, the new arcs c''_1, c''_2 are related to c'_1, c'_2 , and then to the original, ones via isotopies. Pointwise the subdivided arcs have not changed.

So actually we need to show that there exists an isotopy between c''_1, c''_2 . This is not that hard to see, but it is rather technical:

c''_1, c''_2 have the same number of vertices, paired so that any one of the first, along with its pair from the second lie on the same vertical line. Let's call such vertices as corresponding. Let us also call two edges, one from each arc, as corresponding whenever their endpoints are corresponding. We cannot just slide each vertex of c''_1 on its vertical (w.r.t. to π) line to make it take the place of the corresponding vertex of c''_2 because this causes a sliding of the issuing edges from these vertices, and these edges may be obstructed by other edges below them. Even if we try to slide all vertices at once in order to prevent such obstructions, it is not clear at all that such a simultaneous slide will have the desired result. So we deal first with the possible obstructions. Throughout below, we think of c''_2 as the fixed ideal position to push c''_1 to. And we perform isotopies on space changing the position of c''_1 , but the ideal position of c''_2 remains fixed.

Obstructions occur whenever edges of c_1 project on π forming crossing points. We'll deal first with the crossing points of two projected edges, ignoring any crossing points of a projected edge with the rails.

If A is a crossing point of the common projection of c''_1, c''_2 on π (Figure 13), then on the first arc there exist consecutive edges e_1, e_2, e_3 on a line segment and also exist consecutive edges f_1, f_2, f_3 on another line segment, with the projections of e_2, f_2 containing A . Let E_1, E_2, E_3 on a line segment, and F_1, F_2, F_3 on another line segment be the corresponding edges on the second arc of the edges e_1, e_2, e_3 and f_1, f_2, f_3 respectively. Let π_1 be the vertical plane to π containing the edges e_i, E_i ,

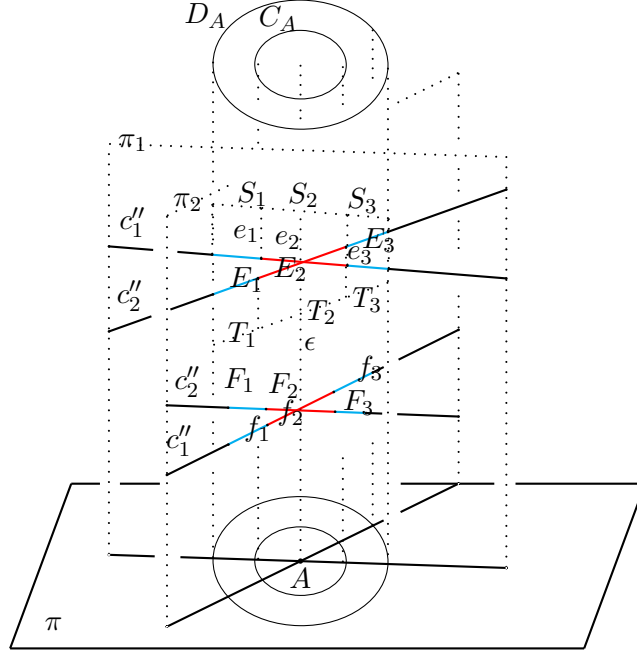


FIGURE 13. The parts of c''_1, c''_2 over a neighborhood of each crossing A of their projections are isolated via two cylinders C_A, D_A , the first inside the second. Red edges are inside C_A , blue are between C_A, D_A . We move c''_1 to a new position, by a p.l. isotopy fixing everything outside D_A and pushing vertically everything inside D_A , so that the e_2 red arc assumes the position E_2 . The blue arcs e_1, e_3 remain segments and retain their endpoints on the surface of D_A , but have moved their endpoints on the surface of C_A vertically. The over/under relation of the red arcs e_2, f_2 has not changed, thus $(e_2 \equiv E_2, f_2)$, is isotopic to (E_2, F_2) . We perform this isotopy vertically, in a similar way to the previous one, and now c''_1 is made to coincide with c''_2 over the neighborhood of the crossing A .

and π_2 the vertical plane containing the corresponding edges f_i, F_i . Call S_1, S_2, S_3 the zones on π_1 that lie on and between the vertical lines passing from the endpoints of e_i, E_i for $i = 1, 2, 3$. And call T_1, T_2, T_3 the zones on π_2 that lie on and between the vertical lines passing from the endpoints of f_i, F_i for $i = 1, 2, 3$. Because of our construction of the second subdivision above, a small solid infinite cylinder C_A (surface union its interior) contains $S_2 \cup T_2$ and nothing more of c''_1, c''_2 . Similarly, an infinite solid cylinder D_A contains $(\cup_i S_i) \cup (\cup_i T_i)$ and nothing more of c''_1, c''_2 . Now, e_2, f_2, E_2, F_2 are segments intersecting the vertical line ϵ through A at points 1, 2, 3, 4 respectively. Make ϵ an axis and compare relative positions of the points 1, 2, 3, 4 using their coordinates. For definiteness, let 1 be above 2 (i.e. have greater coordinate). Since c''_{1pr}, c''_{2pr} carry the same over/under data, 3 should be above 4 too. In $S_2 \cup T_2$ we perform a vertical (w.r.t. π) push in S_2, T_2 up until e_2 is made to coincide with E_2 . Then 1 coincides with 3, and the points 2, 4 of ϵ both lie below $1 \equiv 3$ ((a) of Figure 14 cannot happen). Then the edges f_2, F_2 define in the zone T_2 on their plane π_2 , a quadrangle q either as edges or as diagonals. In both cases, we can push the edge f_2 vertically inside q (Figure 14 (b)), until it coincides with F_2 without disturbing edge e_2 that lies above.

We take care to perform the above vertical pushes as p.l. isotopies in space fixing everything on the surface and outside cylinder D_A . As a result, in their new positions e_2, f_2 coincide with E_2, F_2 , and $e_1, e_3, f_1, f_3, E_1, E_3, F_1, F_3$ change their carrier lines, all projections on π remain as before the

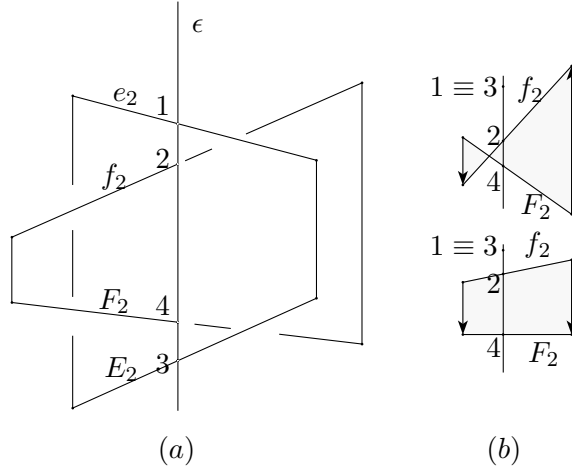


FIGURE 14. The situation in (a) cannot happen because c_1'', c_2'' carry the same over/under data. After pushing e_2 to assume position E_2 , we make f_2 to assume position F_2 by moving it on π_2 as shown in (b).

isotopy. Performing for each crossing point such isotopies consecutively, and even all at once, we get a new c_1'' , say c_1''' . Some of its vertices still may not coincide with those of c'' . So at the exterior of the union of the C_A 's we perform a simultaneous vertical pushing of all vertices of c_1''' that do not already coincide with their corresponding vertices of c_2'' until they do coincide. This push is not obstructed by edges crossing the moving ones, but it can in principal be obstructed by the rails. Nevertheless, for corresponding edges, say e, f of c_1''', c_2'' , the common projection to π is for both an underpassing or an overpassing for the crossing rail. This means that both pairs of corresponding endpoints-vertices, form vertical segments that do not intersect the rails, and as parallel segments, they form a quadrilateral q not containing the rails. So the push of such vertices takes place in these quadrilaterals and is not obstructed by the rails. As above we perform this push as a p.l. isotopy which does not disturb those vertices which were previously made to coincide with their corresponding of c_2'' , and we are done. \square

We can now prove the following:

Theorem 3. *Two rail arcs in \mathbb{R}^3 are rail isotopic iff their rail knotoid diagram projections on the plane π of the rails are rail equivalent. In other words, rail isotopy in \mathbb{R}^3 corresponds to rail equivalence on π (rail arcs are isotopic iff they correspond to the same rail knotoid).*

Proof. (\Rightarrow) If c_1, c_2 are rail arcs which are rail isotopic in \mathbb{R}^3 , then by Corollary 1 there exists a rail isotopy between them expressed by a finite sequence of nice triangle moves. By definition, the projection of any such move is a Reidemeister or slide move or a planar isotopy move. Thus c_{pr1}, c_{pr2} are equivalent rail knotoid diagram projections as wanted.

(\Leftarrow) Let for two arcs c_1, c_2 in \mathbb{R}^3 , their corresponding rail knotoid diagram projections c_{1pr}, c_{2pr} on π differ by a single Reidemeister move, slide move or planar isotopy move δ . In each case one can readily check that there exist a few obvious triangles Δ_1, \dots in space so that for their projected moves on π it is $(\dots \circ \Delta_1)_{pr} = \delta$ whereas in space $(\dots \circ \Delta_1)(c_1)$ coincides with c_2 , thus c_1, c_2 are rail isotopic as wanted. Let us notice that no matter what kind of move δ is, the Δ_i 's indeed provide us with triangle moves in space: when the rails are not involved in δ the result is immediate, and whenever a part of a rail is involved in δ , the Δ_i 's either do not have common points with the rails,

or they have a whole side on a rail, thus by the definition of a triangle move, we get the required result.

In the general case of two rail arcs c_1, c_2 in \mathbb{R}^3 with equivalent rail knotoid diagram projections c_{1pr}, c_{2pr} on π , we note that c_{2pr} comes from c_{1pr} via a finite sequence of Reidemeister moves, slide moves or planar isotopies. Let on π be $d_0 = c_{1pr}, d_1, \dots, d_k = c_{2pr}$ the diagrams obtained consecutively by such a sequence of moves, and let in space, $f_0 = c_1, f_1, \dots, f_k$ be arcs with projections the d_i 's respectively. Then by what we proved just above, there exist a space isotopy between any f_i and the next one f_{i+1} , thus there exists such an isotopy between $c_1 = f_0$ and f_k . But f_k and c_2 share the same projection c_{2pr} on π . Thus Lemma 5 assures the existence of an isotopy between f_k and c_2 , and we finally get a desired isotopy between c_1 and c_2 . \square

5. RAIL KNOTOIDS AND THETA-CURVES

Closing this article, it is worth mentioning that a rail arc together with the two rails is a kind of a trivalent graph embedded in the 3-space \mathbb{R}^3 , containing the end points of the rail arc as its only two vertices, the rail arc as an edge, and the four half-lines of the rails emanating from the two vertices, as infinitely extended edges. Clearly there is a direct connection of these graphs with the θ -curves, where a theta-curve is a graph with the form of the Greek letter θ , embedded in the 3-sphere or the 3-space as a trivalent graph with only two vertices and exactly three edges (upper, middle, lower) each one of which joins the two vertices [18], [14] [19]. From our infinitely extended trivalent graph, one gets a theta-curve as in Figure 15, where the two horizontal line segments joining the two rails are chosen far away from the rail arc (say outside a 3-disk containing the rail arc), one segment on either side (on each rail, the corresponding endpoint of the rail arc lies between the endpoints of the two horizontal segments). Conversely, from a θ -curve we can get a rail knotoid as follows: from each one of the upper and lower edges of the θ -curve, we first delete an arc from its interior leaving two small disjoint arcs touching one vertex each, then we make the two arcs around each vertex into a line segment, and finally make the two segments parallel extending them indefinitely to form the parallel lines of the rails; the middle edge of the theta-curve becomes a rail arc. These remarks suggest the possibility of exchanging information between the theory of rail knotoids and the theory of theta-curves. Theta-curves exhibit a richness of properties and recently, Turaev has connected them to the spherical knotoids as well [17].

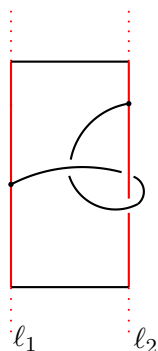


FIGURE 15. The solid red segments on the two rails, together with the two horizontal line segments joining the two rails and also together with the pictured rail arc, form a theta-curve with the endpoints of the rail arc as its vertices, and the rail knotoid as its middle edge.

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