On a class of stochastic differential equations with random and Hölder continuous coefficients arising in biological modeling

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Abstract

Inspired by the paper Greenhalgh et al. [5] we investigate a class of two dimensional stochastic differential equations related to susceptible-infected-susceptible epidemic models with demographic stochasticity. While preserving the key features of the model considered in [5], where an *ad hoc* approach has been utilized to prove existence, uniqueness and non explosivity of the solution, we consider an encompassing family of models described by a stochastic differential equation with random and Hölder continuous coefficients. We prove the existence of a unique strong solution by means of a Cauchy-Euler-Peano approximation scheme which is shown to converge in the proper topologies to the unique solution.

Key words and phrases: two dimensional susceptible-infected-susceptible epidemic model, Brownian motion, stochastic differential equation

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1 Introduction

Susceptible-infected-susceptible (SIS) epidemic model is one of the most popular models for how diseases spread in a population. In such a model an individual starts off being susceptible to a disease and at some point of time gets infected and then recovers after some time becoming susceptible again. The literature of such mathematical models is very rich: for probabilistic/stochastic models one may look for instance at Allen [2], Allen and Burgin [3], A. Gray et al. [4], Hethcote and van den Driessche [6], Kryscio and Lefvre [8], McCormack and Allen [10] and Nasell [12]. We also refer the reader to the detailed account presented in Greenhalgh et al. [5] for an overview on both deterministic

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and stochastic models.

The focus of the present paper is on the model presented in [5]. One of its distinguishing features is the nature of the births and deaths that are regarded as stochastic processes with per capita disease contact rate depending on the population size. Contrary to many other previously proposed models, this stochasticity produces a variable population size which turns out to be a reasonable assumption for slowly spreading diseases.

From a mathematical point of view, the SIS model proposed in [5] amounts at the following two dimensional stochastic differential equation for the vector (S_t, I_t) where S_t and I_t stand for the number of susceptible and infected individuals at time t, respectively:

$$\begin{cases} dS = \left[-\frac{\lambda(N)SI}{N} + (\mu + \gamma)I\right] dt + \sqrt{\frac{\lambda(N)SI}{N} + (\mu + \gamma)I + 2\mu S} dW_3 \\ dI = \left[\frac{\lambda(N)SI}{N} - (\mu + \gamma)I\right] dt + \sqrt{\frac{\lambda(N)SI}{N} + (\mu + \gamma)I} dW_4. \end{cases}$$
(1.1)

Here, N := S + I denotes the total population size while μ , γ and $\lambda : [0, +\infty[\rightarrow [0, +\infty[$ are suitably chosen parameters. The system (1.1) is driven by the two dimensional correlated Brownian motion (W_3, W_4) resulting from a certain application of the martingale representation theorem (see Section 2.1 below for technical details). The system (1.1) is then shown to be equivalent to the triangular system

$$\begin{cases} dI = \left[\frac{\lambda(N)}{N}(N-I)I - (\mu+\gamma)I\right]dt + \sqrt{\frac{\lambda(N)}{N}(N-I)I + (\mu+\gamma)I}dW_4 \\ dN = \sqrt{2\mu N}dW_5 \end{cases}$$
(1.2)

where now the second equation, the so-called square root process (see for instance the book by Mao [9] for the properties of this process), is independent of the first one. To prove the existence of a solution to the first equation in (1.2) the authors resort to Theorem 2.2 in Chapter IV of Ikeda and Watanabe [7] while for the uniqueness they need to construct a localized version of Theorem 3.2, Chapter IV in [7]. The equation for I in (1.2) exhibits random (for the dependence on the process N) and Hölder continuous (for the presence of the square root in the diffusion term) coefficients resulting in a stochastic differential equation for which the issue of the existence of a unique solution has not been addressed in the literature yet.

Our aim in the present paper is to propose a more general approach allowing for the investigation of a richer family of models characterized by the same distinguishing features of the model analyzed in [5].

The paper is articulated as follows: In Section 2 we present a general review using the exposition in the book by Allen (see [1]) of a two-state dynamics leading to a Fokker-Planck partial differential equation and its associated stochastic system. This is followed by Section 2.1 where we consider the more specific situation of a bio-demographic model like the one presented in [5]. Our idea is to embed the rather special system of SDE's of the model in a slightly more encompassing class, like the one in (3.9) below, in order to establish a general proof of strong existence and uniqueness. Our technique relies on the construction of an explicit approximating sequence of stochastic processes (inspired by the work of Zubchenko [13]) in such a way that all the relevant features of the solution appear to be directly constructed from scratch. In Section 3 we give a detailed proof of existence and uniqueness of the SDE (3.9). We would like to point out that systems of SDE's with non-Lipschitz or Hölder coefficients exhibit non-standard difficulties as far as general results for existence and uniqueness are concerned. This model conforms

to the aforementioned difficulties and that is what has motivated us in approaching the problem. Our idea has been to how we could encase the model proposed in [5] within a more general framework, thus bypassing some of the computations done there, and hopefully allowing for larger class of models to be treated.

2 A general two-state system

In this section we review the construction of a general two-state system presented in the book by Allen ([1]). The model will then be made concrete through the assumptions contained in the paper by Greenhalgh et al. ([5]) and this will lead to the class of stochastic differential equations investigated in the present manuscript.

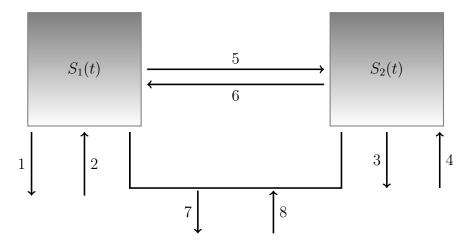


Figure 1: A two-state dynamical process

We begin by considering a representative two-state dynamical process which is illustrated in Figure 1. Let $S_1(t)$ and $S_2(t)$ represent the values of the two states of the system at time t. It is assumed that in a small time interval Δt , state S_1 can change by $-\lambda_1$, 0 or λ_1 and state S_2 can change by $-\lambda_2$, 0 or λ_2 , where $\lambda_1, \lambda_2 \geq 0$. Let $\Delta S := [\Delta S_1, \Delta S_2]^T$ be the change in a small time interval Δt . As illustrated in Figure 1, there are eight possible changes for the two states in the time interval Δt not including the case where there is no change in the time interval. The possible changes and the probabilities of these changes are given in Table 1. It is assumed that the probabilities are given to $O((\Delta t)^2)$. For example, change 1 represents a loss of λ_1 in S_1 with probability $d_1\Delta t$, change 5 represents a transfer of λ_1 out of state S_1 with a corresponding transfer of λ_2 into state S_2 with probability $m_{12}\Delta t$ and change 7 represents a simultaneous reduction in both states S_1 and S_2 . As indicated in the table, all probabilities may depend on $S_1(t), S_2(t)$ and the time t. Also notice that it is assumed that the probabilities for the changes are proportional to the time interval Δt .

It is useful to calculate the mean vector and covariance matrix for the change $\Delta S = [\Delta S_1, \Delta S_2]^T$ fixing the value of S at time t. Using the table below,

$$E[\Delta S] = \sum_{j=1}^{9} p_j \Delta S^{(j)} = \begin{bmatrix} (-d_1 + b_1 - m_{12} + m_{21} + m_{22} - m_{11})\lambda_1 \\ (-d_2 + b_2 + m_{12} - m_{21} + m_{22} - m_{11})\lambda_2 \end{bmatrix} \Delta t$$

Change	Probability
$\Delta \mathbf{S}^{(1)} = [-1, 0]^T$	$p_1 = d_1(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(2)} = \begin{bmatrix} 1, 0 \end{bmatrix}^T$	$p_2 = b_1(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(3)} = [0, -1]^T$	$p_3 = d_2(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(4)} = [0, 1]^T$	$p_4 = b_2(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(5)} = [-1, 1]^T$	$p_5 = m_{12}(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(6)} = [1, -1]^T$	$p_6 = m_{21}(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(7)} = [-1, -1]^T$	$p_7 = m_{11}(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(8)} = [1, 1]^T$	$p_8 = m_{22}(t, S_1, S_2)\Delta t$
$\Delta \mathbf{S}^{(9)} = [0, 0]^T$	$p_9 = 1 - \sum_{j=1}^8 p_j$

 Table 1: Possible changes in the representative two-state system with the corresponding probabilities

$$E[\Delta S(\Delta S)^{T}] = \sum_{j=1}^{9} p_{j}(\Delta S^{(j)})(\Delta S^{(j)})^{T}$$

=
$$\begin{bmatrix} (d_{1} + b_{1} + m_{a})\lambda_{1}^{2} & (-m_{12} - m_{21} + m_{22} + m_{11})\lambda_{1}\lambda_{2} \\ (-m_{12} - m_{21} + m_{22} + m_{11})\lambda_{1}\lambda_{2} & (d_{2} + b_{2} + m_{a})\lambda_{2}^{2} \end{bmatrix} \Delta t$$

where we set $m_a := m_{12} + m_{21} + m_{11} + m_{22}$. Notice that the covariance matrix is set equal to $E(\Delta S(\Delta S)^T)/\Delta t$ because $E(\Delta S)(E(\Delta S))^T = O((\Delta t)^2)$. We now define

$$\mu(t, S_1, S_2) = E[\Delta S] / \Delta t \quad \text{and} \quad V(t, S_1, S_2) = E[\Delta S (\Delta S)^T] / \Delta t \tag{2.1}$$

and we denote by $B(t, S_1, S_2)$ the symmetric square root matrix of V. A forward Kolmogorov equation can be determined for the probability distribution at time $t + \Delta t$ in terms of the distribution at time t. If we write $p(t, x_1, x_2)$ for the probability that $S_1(t) = x_1$ and $S_2(t) = x_2$, then referring to Table 1 we get

$$p(t + \Delta t, x_1, x_2) = p(t, x_1, x_2) + \Delta t \sum_{i=1}^{10} T_i$$
(2.2)

where

$$\begin{array}{rcl} T_{1} &=& p(t,x_{1},x_{2})(-d_{1}(t,x_{1},x_{2})-b_{1}(t,x_{1},x_{2})-d_{2}(t,x_{1},x_{2})-b_{2}(t,x_{1},x_{2}))\\ T_{2} &=& p(t,x_{1},x_{2})(-m_{a}(t,x_{1},x_{2}))\\ T_{3} &=& p(t,x_{1},x_{2})(-m_{a}(t,x_{1},x_{2}))\\ T_{4} &=& p(t,x_{1}+\lambda_{1},x_{2})d_{1}(t,x_{1}+\lambda_{1},x_{2})\\ T_{5} &=& p(t,x_{1},x_{2}-\lambda_{2})b_{2}(t,x_{1},x_{2}-\lambda_{2})\\ T_{6} &=& p(t,x_{1},x_{2}+\lambda_{2})d_{2}(t,x_{1},x_{2}+\lambda_{2})\\ T_{7} &=& p(t,x_{1}+\lambda_{1},x_{2}-\lambda_{2})m_{12}(t,x_{1}+\lambda_{1},x_{2}-\lambda_{2})\\ T_{8} &=& p(t,x_{1}-\lambda_{1},x_{2}+\lambda_{2})m_{21}(t,x_{1}-\lambda_{1},x_{2}+\lambda_{2})\\ T_{9} &=& p(t,x_{1}+\lambda_{1},x_{2}-\lambda_{2})m_{11}(t,x_{1}+\lambda_{1},x_{2}+\lambda_{2})\\ T_{10} &=& p(t,x_{1}-\lambda_{1},x_{2}-\lambda_{2})m_{22}(t,x_{1}-\lambda_{1},x_{2}-\lambda_{2}). \end{array}$$

Now, expanding out the terms T_3 through T_{10} in second order Taylor polynomials around the point (t, x_1, x_2) , it follows that

$$\begin{split} T_{3} &\approx pd_{1} + \partial_{x_{1}}(pd_{1})\lambda_{1} + 1/2\partial_{x_{1}x_{2}}^{2}(pd_{1})\lambda_{1}^{2} \\ T_{4} &\approx pb_{1} - \frac{\partial(pb_{1})}{\partial x_{1}}\lambda_{1} + \frac{1}{2}\frac{\partial^{2}(pb_{1})}{\partial x_{1}^{2}}\lambda_{1}^{2} \\ T_{5} &\approx pb_{2} - \frac{\partial(pb_{2})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\frac{\partial^{2}(pb_{2})}{\partial x_{2}^{2}}\lambda_{2}^{2} \\ T_{6} &\approx pd_{2} - \frac{\partial(pd_{2})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\frac{\partial^{2}(pd_{2})}{\partial x_{2}^{2}}\lambda_{2}^{2} \\ T_{7} &\approx pm_{12} + \frac{\partial(pm_{12})}{\partial x_{1}}\lambda_{1} - \frac{\partial(pm_{12})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\frac{\partial^{2}(pm_{12})}{\partial x_{i}\partial x_{j}}\lambda_{i}\lambda_{j} \\ T_{8} &\approx pm_{21} - \frac{\partial(pm_{21})}{\partial x_{1}}\lambda_{1} + \frac{\partial(pm_{21})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\frac{\partial^{2}(pm_{21})}{\partial x_{i}\partial x_{j}}\lambda_{i}\lambda_{j} \\ T_{9} &\approx pm_{11} + \frac{\partial(pm_{11})}{\partial x_{1}}\lambda_{1} + \frac{\partial(pm_{11})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\frac{\partial^{2}(pm_{11})}{\partial x_{i}\partial x_{j}}\lambda_{i}\lambda_{j} \\ T_{10} &\approx pm_{22} - \frac{\partial(pm_{22})}{\partial x_{1}}\lambda_{1} - \frac{\partial(pm_{22})}{\partial x_{2}}\lambda_{2} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\frac{\partial^{2}(pm_{22})}{\partial x_{i}\partial x_{j}}\lambda_{i}\lambda_{j} \end{split}$$

Substituting these expressions into (2.2) and assuming that Δt , λ_1 and λ_2 are small, then it is seen that $p(t, x_1, x_2)$ approximately solves the Fokker-Planck equation

$$\frac{\partial p(t, x_1, x_2)}{\partial t} = -\sum_{i=1}^2 \frac{\partial}{\partial x_1} \left[\mu_i(t, x_1, x_2) p(t, x_1, x_2) \right] \\
+ \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial}{\partial x_i \partial x_j} \left[\sum_{k=1}^2 b_{ik}(t, x_1, x_2) b_{jk}(t, x_1, x_2) p(t, x_1, x_2) \right] (2.3)$$

where $\mu = (\mu_1, \mu_2)$ and $B = \{b_{ij}\}_{1 \le i,j \le 2}$. On the other hand, it is well known that the probability distribution $p(t, x_1, x_2)$ that solves equation (2.3) coincides with the distribution of the solution at time t to the following system of stochastic differential equations

$$dS = \mu(t, S)dt + B(t, S)dW(t), \quad S(0) = S_0$$
(2.4)

where W is a two-dimensional standard Brownian motion and S_0 is a given deterministic initial condition. The stochastic differential equation (2.4) describes the random evolution of the two-state system S related to the changes described in Table 1.

2.1 The Greenhalgh et al. [5] model

We now specialize the general model introduced in the previous section to the case investigated in Greenhalgh et al. [5] (where the process (S_1, S_2) is denoted as (S, I)). The values of the parameters in Table 1 are chosen as follows:

Change	Probability
$\Delta \mathbf{S}^{(1)} = [-1, 0]^T$	$\mu S_1 \Delta t$
$\Delta \mathbf{S}^{(2)} = [1, 0]^T$	$\mu N \Delta t$
$\Delta \mathbf{S}^{(3)} = [0, -1]^T$	$\mu S_2 \Delta t$
$\Delta \mathbf{S}^{(4)} = [0, 1]^T$	0
$\Delta \mathbf{S}^{(5)} = [-1, 1]^T$	$\frac{\lambda(N)S_1S_2}{N}\Delta t$
$\Delta \mathbf{S}^{(6)} = [1, -1]^T$	$\gamma S_2 \Delta t$
$\Delta \mathbf{S}^{(7)} = [-1, -1]^T$	0
$\Delta \mathbf{S}^{(8)} = [1, 1]^T$	0
$\Delta \mathbf{S}^{(9)} = [0, 0]^T$	$1 - \sum_{j=1}^{8} p_j$

Table 2: Probabilities in *Greenhalgh et al.*'s paper

where $N := S_1 + S_2$, $\lambda : [0, +\infty[\rightarrow [0, +\infty[$ is a continuous monotone increasing function and μ and γ are positive constants. We refer to the paper [5] for the biological interpretation of these quantities. Now, according to Table 2 the vector μ and matrix V in (2.1) read

$$\mu(t, S_1, S_2) = \begin{bmatrix} -\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 \\ \frac{\lambda(N)S_1S_2}{N} - (\mu + \gamma)S_2 \end{bmatrix}$$

and

$$V(t, S_1, S_2) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where to ease the notation we set

$$a := \frac{\lambda(N)S_{1}S_{2}}{N} + (\mu + \gamma)S_{2} + 2\mu S_{1}$$

$$b := -\frac{\lambda(N)S_{1}S_{2}}{N} - \gamma S_{2}$$

$$c := \frac{\lambda(N)S_{1}S_{2}}{N} + (\mu + \gamma)S_{2}.$$

Therefore,

$$B(t, S_1, S_2) = V(t, S_1, S_2)^{\frac{1}{2}} = \frac{1}{d} \begin{bmatrix} a+w & b \\ b & c+w \end{bmatrix}$$

with

$$w := \sqrt{ac - b^2}$$
 and $d := \sqrt{a + c + 2w}$.

We are then lead to study the following two dimensional system of stochastic differential equations

$$\begin{cases} dS_1 = \left[-\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 \right] dt + \frac{a+w}{d} dW_1 + \frac{b}{d} dW_2 \\ dS_2 = \left[\frac{\lambda(N)S_1S_2}{N} - (\mu + \gamma)S_2 \right] dt + \frac{b}{d} dW_1 + \frac{c+w}{d} dW_2 \end{cases}$$
(2.5)

where $W = (W_1, W_2)$ is a standard two dimensional Brownian motion. We observe that by construction

$$\left(\frac{a+w}{d}\right)^2 + \left(\frac{b}{d}\right)^2 = a$$

Therefore, by the martingale representation theorem (see for instance Theorem 3.9 Chapter V in [11]) there exists a Brownian motion W_3 such that the first equation in (2.5) can be rewritten as

$$dS_1 = \left[-\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 \right] dt + \sqrt{\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 + 2\mu S_1} dW_3$$

Similarly, since

$$\left(\frac{b}{d}\right)^2 + \left(\frac{c+w}{d}\right)^2 = c$$

by the martingale representation theorem there exists a Brownian motion W_4 such that the second equation in (2.5) can be rewritten as

$$dS_2 = \left[\frac{\lambda(N)S_1S_2}{N} - (\mu + \gamma)S_2\right]dt + \sqrt{\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2}dW_4.$$

This implies that the system (2.5) is equivalent to

$$\begin{cases} dS_1 = \left[-\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 \right] dt + \sqrt{\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2 + 2\mu S_1} dW_3 \\ dS_2 = \left[\frac{\lambda(N)S_1S_2}{N} - (\mu + \gamma)S_2 \right] dt + \sqrt{\frac{\lambda(N)S_1S_2}{N} + (\mu + \gamma)S_2} dW_4. \end{cases}$$
(2.6)

We remark that by construction the Brownian motions W_3 and W_4 are now correlated. Moreover, if we notice that the drift of the first equation in (2.5) is the opposite of the one in the second equation in (2.5), recalling that $N = S_1 + S_2$ we may write

$$dN = \frac{a+b+w}{d}dW_1 + \frac{b+c+w}{d}dW_2$$

and, exploiting the definitions of a, b, c, d and w, we conclude as before that there exists a Brownian motion W_5 such that

$$dN = \sqrt{2\mu N} dW_5. \tag{2.7}$$

Hence, instead of studying the system (2.5), the authors in [5] study the equivalent system

$$\begin{cases} dS_2 = \left[\frac{\lambda(N)}{N}(N-S_2)S_2 - (\mu+\gamma)S_2\right]dt + \sqrt{\frac{\lambda(N)}{N}(N-S_2)S_2 + (\mu+\gamma)S_2}dW_4 \\ dN = \sqrt{2\mu N}dW_5 \end{cases} (2.8)$$

where the Brownian motions W_4 and W_5 are correlated. In the system (2.8) the equation for N does not depend on S_2 and it belongs to the family of the square root processes ([9]). Once the equation for N is solved, the equation for S_2 contains random (for the presence of N) Hölder continuous coefficients. Moreover, due to the presence of the square root in the diffusion coefficient of S_2 , the authors of [5] consider a modified version of the first equation in (2.8) to make the coefficients defined on the whole real line. They consider

$$dS_2(t) = \bar{a}(t, N(t), S_2(t))dt + \bar{g}(t, N(t), S_2(t))dW_4(t)$$
(2.9)

where

$$\bar{a}(t,y,x) = \begin{cases} 0 & \text{for } x < 0\\ \frac{\lambda(y)x}{y}(y-x) - (\mu+\gamma)x & \text{for } 0 \le x \le y \left(1 + \frac{\mu+\gamma}{\lambda(y)}\right)\\ \bar{a}\left(t,y,y\left(1 + \frac{\mu+\gamma}{\lambda(y)}\right)\right) & \text{for } x > y \left(1 + \frac{\mu+\gamma}{\lambda(y)}\right) \end{cases}$$

and

$$\bar{g}(t,y,x) = \begin{cases} 0 & \text{for } x < 0\\ \sqrt{\frac{\lambda(y)x}{y}(y-x) + (\mu+\gamma)x} & \text{for } 0 \le x \le y \left(1 + \frac{\mu+\gamma}{\lambda(y)}\right)\\ 0 & \text{for } x > y \left(1 + \frac{\mu+\gamma}{\lambda(y)}\right) \end{cases}$$

The existence of a unique non explosive strong solution to equation (2.9) is obtained through a localization argument in terms of stopping times and comparison inequalities to control the non explosivity of the solution. In the next section we will consider a class of stochastic differential equations, which includes equation (2.9), allowing for more general models where the existence of a unique non explosive strong solution is proved via a standard Caychy-Euler-Peano approximation method.

3 Main theorem

Motivated by the discussion in the previous sections, we are now ready to state and prove the main result of our manuscript. We begin by specifying the class of coefficients involved in the stochastic differential equations under investigation. Let $g: [0, +\infty] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of the form

$$g(t, y, x) = \sqrt{-x^2 + \alpha(t, y)x + \beta(t, y)}$$

$$(3.1)$$

where $\alpha, \beta : [0, +\infty] \times \mathbb{R} \to \mathbb{R}$ are measurable functions satisfying the condition

$$\alpha(t,y)^2 + 4\beta(t,y) \ge 0 \quad \text{for all } (t,y) \in [0,+\infty[\times\mathbb{R}.$$
(3.2)

We observe that condition (3.2) implies that

$$-x^{2} + \alpha(t, y)x + \beta(t, y) \ge 0 \quad \text{if and only if} \quad r_{1}(t, y) \le x \le r_{2}(t, y)$$

where we set

$$r_1(t,y) := \frac{\alpha(t,y) - \sqrt{\alpha(t,y)^2 + 4\beta(t,y)}}{2}$$

and

$$r_2(t,y) := \frac{\alpha(t,y) + \sqrt{\alpha(t,y)^2 + 4\beta(t,y)}}{2}$$

Now, we define

$$\bar{g}(t,y,x) := \begin{cases} 0 & \text{if} \quad x < r_1(t,y) \\ g(t,y,x) & \text{if} \quad r_1(t,y) \le x \le r_2(t,y) \\ 0 & \text{if} \quad x > r_2(t,y) \end{cases}$$
(3.3)

The function \bar{g} will be the diffusion coefficient of our stochastic differential equation.

Assumption 3.1 There exist a positive constant M such that

$$|\alpha(t,y)| \le M(1+|y|)$$
 and $|\beta(t,y)| \le M(1+|y|)$ (3.4)

for all $(t, y) \in [0, \infty[\times \mathbb{R}]$. Moreover, there exists a positive constant H such that

$$|\bar{g}(t, y_1, x_1) - \bar{g}(t, y_2, x_2)| \le H(\sqrt{|y_1 - y_2|} + \sqrt{|x_1 - x_2|})$$
(3.5)

for all $t \in [0, \infty[$ and $y_1, y_2, x_1, x_2 \in \mathbb{R}$.

We observe that assumption (3.4) implies the bound

$$\begin{split} |\bar{g}(t,y,x)| &\leq \max_{x\in\mathbb{R}} |\bar{g}(t,y,x)| \\ &= \sqrt{\frac{\alpha(t,y)^2}{4} + \beta(t,y)} \\ &\leq M(1+|y|) \end{split}$$

for all $t \in [0, \infty[$ and $y \in \mathbb{R}$. Here the constant M may differ from the one appearing in (3.4); we will adopt this convention for the rest of the paper. We also remark that by construction inequality (3.5) for $y_1 = y_2$ is satisfied with a constant $H = \sqrt{|\alpha(t, y_1)|}$.

We now introduce the drift coefficient of our SDE. We start with a measurable function $a: [0, +\infty[\times\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ with the following property.}]$

Assumption 3.2 There exists a positive constant M such that

$$|a(t, y, x)| \le M(1 + |y| + |x|) \tag{3.6}$$

for all $t \in [0, \infty[$ and $x, y \in \mathbb{R}$. Moreover, there exists a positive constant L such that

$$|a(t, y_1, x_1) - a(t, y_2, x_2)| \le L(|y_1 - y_2| + |x_1 - x_2|)$$
(3.7)

for all $t \in [0, \infty[$ and $y_1, y_2, x_1, x_2 \in \mathbb{R}$.

Then, we set

$$\bar{a}(t,y,x) := \begin{cases} a(t,y,r_1(t,y)) & \text{if} \quad x < r_1(t,y) \\ a(t,y,x) & \text{if} \quad r_1(t,y) \le x \le r_2(t,y) \\ a(t,y,r_2(t,y)) & \text{if} \quad x > r_2(t,y) \end{cases}$$
(3.8)

Observe that by construction also the function \bar{a} satisfies Assumption 3.2.

We now consider the following one dimensional stochastic differential equation

$$dX_t = \bar{a}(t, Y_t, X_t)dt + \bar{g}(t, Y_t, X_t)dW_t^2, \quad X_0 = x \in \mathbb{R}$$
(3.9)

where $\{Y_t\}_{t\geq 0}$ is the unique strong solution of the stochastic differential equation

$$dY_t = m(t, Y_t)dt + \sigma(t, Y_t)dW_t^1, \quad Y_0 = y \in \mathbb{R}.$$
(3.10)

Here $\{(W_t^1, W_t^2)\}_{t\geq 0}$ is a two dimensional correlated Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the process $\{(W_t^1, W_t^2)\}_{t\geq 0}$. Strong solutions are meant to be $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted.

Regarding equation (3.10), the coefficients m and σ are assumed to entail existence and uniqueness of a strong solution $\{Y_t\}_{t>0}$ such that

$$E\left[\sup_{t\in[0,T]}|Y_t|^2\right]$$
 is finite for all $T>0$.

Equations (3.9) and (3.10) describe a class of equations which includes equations (2.9) and (2.7) as a particular case.

Remark 3.3 If $r_1(t, y) = r_2(t, y)$ for all $(t, y) \in [0, \infty[\times\mathbb{R}, which is equivalent to say that <math>\alpha(t, y)^2 + 4\beta(t, y) = 0$, then the diffusion coefficient \overline{g} is identically zero and the drift coefficient becomes $\overline{a}(t, y, x) = a(t, y, \alpha(t, y)/2)$. Therefore, in this particular case the SDE (3.9) takes the form

$$dX_t = a(t, Y_t, \alpha(t, Y_t)/2)dt, \quad X_0 = x$$

whose solution is explicitly given by the formula

$$X_t = x + \int_0^t a(s, Y_s, \alpha(s, Y_s)/2) ds.$$

Theorem 3.4 (Strong existence and uniqueness) Let Assumption 3.1 and Assumption 3.2 be fulfilled. Then, the stochastic differential equation (3.9) possesses a unique strong solution $\{X_t\}_{t>0}$.

PROOF. To ease the notation we consider the time-homogeneous case and hence we drop the explicit dependence on t from all the coefficients.

We fix an arbitrary T > 0 and prove existence and uniqueness of a solution for the SDE

$$X_t = x + \int_0^t \bar{a}(Y_s, X_s) ds + \int_0^t \bar{g}(Y_s, X_s) dW_s^2, \quad X_0 = x.$$
(3.11)

on the time interval $t \in [0, T]$. The proof for the existence is rather long and proceeds as follows: using a Cauchy-Euler-Peano approximate solutions technique we define, associated to a partition Δ_n of [0, T] a stochastic process X^n . We will, at the beginning, prove a convergence result for X^n in the space $L^1([0, T] \times \Omega)$, then we will prove a convergence result for X^n in the space $\mathcal{C}[0, T]$ with the norm of the uniform convergence and this will eventually yield the result. **Existence**: We consider a sequence of partitions $\{\Delta_n\}_{n\geq 1}$ of the interval [0,T] with $\Delta_n \subseteq \Delta_{n+1}$. Each partition Δ_n will consist of a set of $N_n + 1$ points $\{t_0^n, t_1^n, ..., t_{N_n}^n\}$ satisfying

$$0 = t_0^n < t_1^n < \dots < t_{N_n}^n = T.$$

We denote by $\|\Delta_n\| := \max_{0 \le k \le N_n - 1} |t_{k+1}^n - t_k^n|$, the mesh of the partition Δ_n , and assume that $\lim_{n\to\infty} \|\Delta_n\| = 0$. In the sequel, we will write t_k instead of t_k^n when the membership to the partition Δ_n will be clear from the context.

For a given partition Δ_n we construct a continuous and $\{\mathcal{F}_t\}_{t\geq 0}$ -adapted stochastic process $\{X_t^n\}_{t\in[0,T]}$ as follows: for t=0 we set $X_t^n = x$ while for $t\in]t_k, t_{k+1}]$ we define

$$X_t^n := X_{t_k}^n + \bar{a}(Y_{t_k}, X_{t_k}^n)(t - t_k) + g(Y_{t_k}, X_{t_k}^n)(W_t - W_{t_k}).$$
(3.12)

It is useful to observe that, denoting $\eta_n(t) = t_k$ when $t \in]t_k, t_{k+1}]$, we may represent X_t^n in the compact form:

$$X_t^n = x + \int_0^t \bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) ds + \int_0^t \bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) dW_s^2.$$
(3.13)

Step one: $\mathbb{E}|X_{\eta_n(t)}^n|$ is uniformly bounded with respect to n and t

We begin with equation (3.12). Using the triangle inequality and upper bounds for \bar{a} and \bar{g} we get

$$\begin{split} \mathbb{E}[|X_{t_{k+1}}^{n}|] &\leq \mathbb{E}[|X_{t_{k}}^{n}|] + \mathbb{E}[|\bar{a}(Y_{t_{k}}, X_{t_{k}}^{n})(t_{k+1} - t_{k})|] \\ &+ \mathbb{E}[|\bar{g}(Y_{t_{k}}, X_{t_{k}}^{n})(W_{t_{k+1}} - W_{t_{k}})|] \\ &\leq \mathbb{E}[|X_{t_{k}}^{n}|] + M|t_{k+1} - t_{k}|\mathbb{E}[1 + |Y_{t_{k}}|] + M|t_{k+1} - t_{k}|\mathbb{E}\left[|X_{t_{k}}^{n}|\right] \\ &+ M\mathbb{E}\left[(1 + |Y_{t_{k}}|)|W_{t_{k+1}} - W_{t_{k}}|\right] \\ &\leq (1 + M||\Delta_{n}||)\mathbb{E}[|X_{t_{k}}^{n}|] + M|t_{k+1} - t_{k}|\mathbb{E}\left[1 + |Y_{t_{k}}|\right] \\ &+ \frac{M}{2}\left(\mathbb{E}\left[(1 + |Y_{t_{k}}|)^{2}\right] + \mathbb{E}\left[|W_{t_{k+1}} - W_{t_{k}}|^{2}\right]\right) \\ &\leq (1 + M||\Delta_{n}||)\mathbb{E}[|X_{t_{k}}^{n}|] + M||\Delta_{n}||\sup_{t\in[0,T]}\mathbb{E}\left[1 + |Y_{t}|\right] \\ &+ \frac{M}{2}\sup_{t\in[0,T]}\mathbb{E}\left[(1 + |Y_{t}|)^{2}\right] + \frac{M}{2}|t_{k-1} - t_{k}| \\ &\leq (1 + M||\Delta_{n}||)\mathbb{E}[|X_{t_{k}}^{n}|] + M||\Delta_{n}||\sup_{t\in[0,T]}\mathbb{E}\left[1 + |Y_{t}|\right] \\ &+ \frac{M}{2}\sup_{t\in[0,T]}\mathbb{E}\left[(1 + |Y_{t}|)^{2}\right] + \frac{M}{2}||\Delta_{n}|| \\ &\leq (1 + M||\Delta_{n}||)\mathbb{E}[|X_{t_{k}}^{n}|] + \frac{M}{2}\sup_{t\in[0,T]}\mathbb{E}\left[(1 + |Y_{t}|)^{2}\right] + \varepsilon. \end{split}$$

Here we used the fact that $\|\Delta_n\|$ tends to zero as n tends to infinity and that $\sup_{t \in [0,T]} \mathbb{E} [1 + |Y_t|]$ is finite: we can therefore choose n big enough to make

$$M \|\Delta_n\| \sup_{t \in [0,T]} \mathbb{E} \left[1 + |Y_t|\right] + \frac{M}{2} \|\Delta_n\|$$

smaller than a given positive ε . Comparing the first and last terms of the previous chain of inequalities we get for all $k \in \{0, ..., N_n - 1\}$

$$\mathbb{E}[|X_{t_{k+1}}^{n}|] \le (1+M\|\Delta_{n}\|)\mathbb{E}[|X_{t_{k}}^{n}|] + \frac{M}{2}\sup_{t\in[0,T]}\mathbb{E}\left[(1+|Y_{t}|)^{2}\right] + \varepsilon$$

which by recursion implies

$$\mathbb{E}[|X_{t_k}^n|] \leq \gamma_1^k |x| + \frac{\gamma_1^k - 1}{\gamma_1 - 1} \gamma_2$$

$$\leq \gamma_1^{N_n} |x| + \frac{\gamma_1^{N_n} - 1}{\gamma_1 - 1} \gamma_2$$

where for notational convenience we set

$$\gamma_1 := 1 + M \|\Delta_n\|$$
 and $\gamma_2 := \frac{M}{2} \sup_{t \in [0,T]} \mathbb{E} \left[(1 + |Y_{t_k}|)^2 \right] + \varepsilon.$

Since $\eta_n(t)$ is a step function in [0, T] with values $\{t_0, t_1, ..., t_{N_n}\}$, the previous estimate for $k \in \{0, ..., N_n - 1\}$ entails the boundedness of the function $[0, T] \ni t \to E[|X_{\eta_n(t)}^n|]$. We now obtain an estimate for $E[|X_{\eta_n(t)}^n|]$ which is also uniform with respect to n. Using the triangle inequality in (3.13) we can write

$$\mathbb{E}[|X_{\eta_n(t)}^n|] \leq |x| + \mathbb{E}\left[\left|\int_0^{\eta_n(t)} \bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n)ds\right|\right] \\
+ \mathbb{E}\left[\left|\int_0^{\eta_n(t)} \bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n)dW_s^2\right|\right].$$
(3.14)

For the first expected value on the right hand side above we employ the assumptions on \bar{a} :

$$\begin{split} \mathbb{E}\left[\left|\int_{0}^{\eta_{n}(t)} \bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})ds\right|\right] &\leq \mathbb{E}\left[\int_{0}^{t} |\bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})|ds\right] \\ &\leq M\mathbb{E}\left[\int_{0}^{t} (1 + |X_{\eta_{n}(s)}^{n}| + |Y_{\eta_{n}(s)}|)ds\right] \\ &= M\int_{0}^{t} \mathbb{E}[|X_{\eta_{n}(s)}^{n}|]ds + M\int_{0}^{t} \mathbb{E}\left[1 + |Y_{\eta_{n}(s)}|\right]ds \\ &\leq M\int_{0}^{t} \mathbb{E}[|X_{\eta_{n}(s)}^{n}|]ds + MT\sup_{t \in [0,T]} \mathbb{E}\left[1 + |Y_{t}|\right]. \end{split}$$

Using the Itô isometry and the assumptions on \bar{g} we can treat the second expected value as follows:

$$\mathbb{E}\left[\left|\int_{0}^{\eta_{n}(t)} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) dW_{s}^{2}\right|\right] \leq \left(\mathbb{E}\left[\left|\int_{0}^{\eta_{n}(t)} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) dW_{s}^{2}\right|^{2}\right]\right)^{\frac{1}{2}} \leq \left(\int_{0}^{t} \mathbb{E}[|\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})|^{2}] ds\right)^{\frac{1}{2}}$$

$$\leq M \left(\int_0^t \mathbb{E}[(1+|Y_{\eta_n(s)}|)^2] ds \right)^{\frac{1}{2}} \\ \leq M \sqrt{T \sup_{t \in [0,T]} \mathbb{E}[(1+|Y_t|)^2]}.$$

Plugging the last two estimates in (3.14) gives

$$\begin{split} \mathbb{E}[|X_{\eta_n(t)}^n|] &\leq |x| + \mathbb{E}\left[\left|\int_0^{\eta_n(t)} \bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n)ds\right|\right] \\ &+ \mathbb{E}\left[\left|\int_0^{\eta_n(t)} \bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n)dW_s^2\right|\right] \\ &\leq |x| + M \int_0^t \mathbb{E}[|X_{\eta_n(s)}^n|]ds + MT \sup_{t \in [0,T]} \mathbb{E}\left[1 + |Y_t|\right] \\ &+ M \sqrt{T \sup_{t \in [0,T]} \mathbb{E}\left[(1 + |Y_t|)^2\right]} \\ &= G + M \int_0^t \mathbb{E}[|X_{\eta_n(s)}^n|]ds \end{split}$$

where

$$G := |x| + MT \sup_{t \in [0,T]} \mathbb{E} \left[1 + |Y_t| \right] + M \sqrt{T \sup_{t \in [0,T]} \mathbb{E} \left[(1 + |Y_t|)^2 \right]}.$$

By the Gronwall inequality (we proved before that $t \to \mathbb{E}[|X_{\eta_n(t)}^n|]$ is a non negative, bounded and measurable function) we conclude that

$$\mathbb{E}[|X_{\eta_n(t)}^n|] \le Ge^{Mt} \le Ge^{MT} \tag{3.15}$$

which provides the desired uniform bound (with respect to n and t) for $\mathbb{E}[|X_{\eta_n(t)}^n|]$.

Step two: $\mathbb{E}[|X_t^n - X_{\eta_n(t)}^n|]$ tends to zero as n tends to infinity, uniformly with respect to $t \in [0,T]$

We proceed as in step one. Recalling the identity (3.13) we can write

$$\begin{split} \mathbb{E}[|X_{t}^{n} - X_{\eta_{n}(t)}^{n}|] &= \mathbb{E}\left[\left|\int_{\eta_{n}(t)}^{t} \bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})ds + \int_{\eta_{n}(t)}^{t} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})dW_{s}^{2}\right|\right] \\ &\leq \int_{\eta_{n}(t)}^{t} \mathbb{E}[|\bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})|]ds + \mathbb{E}\left[\left|\int_{\eta_{n}(t)}^{t} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})dW_{s}^{2}\right|\right] \\ &\leq M \int_{\eta_{n}(t)}^{t} \mathbb{E}[(1 + |X_{\eta_{n}(s)}^{n}| + |Y_{\eta_{n}(s)}|)]ds \\ &+ \left(\mathbb{E}\left[\left|\int_{\eta_{n}(t)}^{t} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n})dW_{s}^{2}\right|^{2}\right]\right)^{\frac{1}{2}} \\ &\leq M(t - \eta_{n}(t)) \left(Ge^{MT} + \sup_{t \in [0,T]} \mathbb{E}[1 + |Y_{t}|]\right) \end{split}$$

$$\begin{aligned} &+ \left(\int_{\eta_n(t)}^t \mathbb{E}[|\bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n)|^2] ds \right)^{\frac{1}{2}} \\ &\leq M(t - \eta_n(t)) \left(Ge^{MT} + \sup_{t \in [0,T]} \mathbb{E}[1 + |Y_t|] \right) \\ &+ M\sqrt{t - \eta_n(t)} \sqrt{\sup_{t \in [0,T]} \mathbb{E}[(1 + |Y_t|)^2]} \\ &\leq M\sqrt{\|\Delta_n\|} \left(Ge^{MT} + \sup_{t \in [0,T]} \mathbb{E}[1 + |Y_t|] + \sqrt{\sup_{t \in [0,T]} \mathbb{E}[(1 + |Y_t|)^2]} \right). \end{aligned}$$

Here, in the third equality, we utilized the uniform upper bound (3.15). We have therefore proved that

$$\mathbb{E}[|X_t^n - X_{\eta_n(t)}^n|] \leq M\sqrt{\|\Delta_n\|} \left(Ge^{MT} + \sup_{t \in [0,T]} \mathbb{E}[1 + |Y_t|] + \sqrt{\sup_{t \in [0,T]} \mathbb{E}[(1 + |Y_t|)^2]} \right) \\ =: M_1\sqrt{\|\Delta_n\|}$$

This in turn implies that $\mathbb{E}[|X_t^n - X_{\eta_n(t)}^n|]$ tends to zero as *n* tends to infinity, uniformly with respect to $t \in [0, T]$.

Step three: $\{X^n\}_{n\geq 1}$ is a Cauchy sequence in $L^1([0,T]\times \Omega)$.

We need to prove that for any $\varepsilon > 0$ there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$\mathbb{E}\left[\int_0^T |X_t^n - X_t^m| dt\right] < \varepsilon \quad \text{ for all } n, m \ge n_{\varepsilon}.$$

We have:

$$X_{t}^{n} - X_{t}^{m} = \int_{0}^{t} \left[\bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{a}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m}) \right] ds + \int_{0}^{t} \left[\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m}) \right] dW_{s}^{2}$$

We now aim to apply the Itô formula to the stochastic process $\{X_t^n - X_t^m\}_{t \in [0,T]}$ for a suitable smooth function that we now describe.

Consider the decreasing sequence of real numbers $\{a_h\}_{h\geq 0}$ defined by induction as follows:

$$a_0 = 1$$
 and for $h \ge 1$ $\int_{a_{h-1}}^{a_h} \frac{1}{u} du = h.$

It is easy to see that $a_h = e^{-\frac{h(h+1)}{2}}$ and therefore that $\lim_{h\to+\infty} a_h = 0$. Define the function $\Phi_h(u)$ for $u \in [0,\infty)$ such that $\Phi_h(0) = 0$, $\Phi_h(u) \in \mathcal{C}^2([0,\infty[))$ and

$$\Phi_h''(u) = \begin{cases} 0, & 0 \le u \le a_h \\ \text{a value between 0 and } \frac{2}{hu}, & a_h < u < a_{h-1} \\ 0, & u \ge a_{h-1} \end{cases}$$
(3.16)

with

$$\int_{a_h}^{a_{h-1}} \Phi_h''(u) du = 1.$$

Integrating Φ_h'' we get

$$\Phi'_{h}(u) = \begin{cases} 0, & 0 \le u \le a_{h} \\ \text{a value between 0 and 1,} & a_{h} < u < a_{h-1} \\ 1, & u \ge a_{h-1} \end{cases}$$
(3.17)

Finally we choose $\theta_h(u) = \Phi_h(|u|)$. Then, we have:

$$\begin{aligned} \theta_h(X_t^n - X_t^m) &= \int_0^t \theta'_h(X_s^n - X_s^m) \left[\bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{a}(Y_{\eta_m(s)}, X_{\eta_m(s)}^m) \right] ds \\ &+ \int_0^t \theta'_h(X_s^n - X_s^m) \left[\bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{g}(Y_{\eta_m(s)}, X_{\eta_m(s)}^m) \right] dW_s^2 \\ &+ \frac{1}{2} \int_0^t \theta''(X_s^n - X_s^m) \left[\bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{g}(Y_{\eta_m(s)}, X_{\eta_m(s)}^m) \right]^2 ds \\ &=: I_1(\theta_h) + I_2(\theta_h) + I_3(\theta_h) \end{aligned}$$

Since for any $h \ge 0$ and $u \in \mathbb{R}$ we have by construction that $|u| - a_{h-1} \le \theta_h(u)$, we can write

$$\mathbb{E}[|X_t^n - X_t^m|] \leq a_{h-1} + \mathbb{E}[\theta_h(X_t^n - X_t^m)]
= a_{h-1} + \mathbb{E}[I_1(\theta_h) + I_2(\theta_h) + I_3(\theta_h)]
= a_{h-1} + \mathbb{E}[I_1(\theta_h)] + \mathbb{E}[I_3(\theta_h)].$$
(3.18)

Let us now estimate $\mathbb{E}[|I_1(\theta_h)|]$:

$$\begin{split} \mathbb{E}[|I_{1}(\theta_{h})|] &= \mathbb{E}\left[\left|\int_{0}^{t} \theta_{h}'(X_{s}^{n}-X_{s}^{m})\left[\bar{a}(Y_{\eta_{n}(s)}.X_{\eta_{n}(s)}^{n})-\bar{a}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m})\right]ds\right|\right] \\ &\leq \mathbb{E}\left[\int_{0}^{t} |\theta_{h}'(X_{s}^{n}-X_{s}^{m})|\cdot|\bar{a}(Y_{\eta_{n}(s)}.X_{\eta_{n}(s)}^{n})-\bar{a}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m})|ds\right] \\ &\leq \mathbb{E}\left[\int_{0}^{t} |\bar{a}(Y_{\eta_{n}(s)}.X_{\eta_{n}(s)}^{n})-\bar{a}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m})|ds\right] \\ &\leq L\int_{0}^{t} \mathbb{E}[|X_{\eta_{n}(s)}^{n}-X_{\eta_{m}(s)}^{m}|]ds+L\int_{0}^{t} \mathbb{E}[|Y_{\eta_{n}(s)}-Y_{\eta_{m}(s)}|]ds \end{split}$$

In the second inequality we utilized the bound $|\theta'_h(u)| \leq 1$ which is valid for all $h \geq 0$ and $u \in \mathbb{R}$. By means of the estimate obtained in step two we can write

$$\mathbb{E}[|X_{\eta_n(s)}^n - X_{\eta_m(s)}^m|] \leq \mathbb{E}[|X_{\eta_n(s)}^n - X_s^n|] + \mathbb{E}[|X_s^n - X_s^m|] + \mathbb{E}[|X_s^m - X_{\eta_m(s)}^m|]$$

$$\leq M_1(\sqrt{\|\Delta_n\|} + \sqrt{\|\Delta_m\|}) + \mathbb{E}[|X_s^n - X_s^m|].$$

Similarly we get

$$\mathbb{E}[|Y_{\eta_n(s)} - Y_{\eta_m(s)}|] \leq \mathbb{E}[|Y_{\eta_n(s)} - Y_s|] + \mathbb{E}[Y_s - Y_{\eta_m(s)}|]$$

$$\leq C(\sqrt{\|\Delta_n\|} + \sqrt{\|\Delta_m\|})$$

where the last inequality is due to well known estimates for strong solutions of stochastic differential equations. Combining the last two bounds we conclude that

$$\mathbb{E}[|I_{1}(\theta_{h})|] \leq L \int_{0}^{t} \mathbb{E}[|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}|]ds + L \int_{0}^{t} \mathbb{E}[|Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}|]ds \\
\leq TL(M_{1} + C)(\sqrt{\|\Delta_{n}\|} + \sqrt{\|\Delta_{n}\|}) + L \int_{0}^{t} \mathbb{E}[|X_{s}^{n} - X_{s}^{m}|]ds. \quad (3.19)$$

We now treat $\mathbb{E}[I_3(\theta_h)]$; by the assumption (3.5) and properties of θ_h we get:

$$\mathbb{E}[I_{3}(\theta_{h})] = \frac{1}{2}\mathbb{E}\left[\int_{0}^{t} \theta_{h}''(X_{s}^{n} - X_{s}^{m})(\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m}))^{2}ds\right] \\
\leq \frac{H^{2}}{2}\mathbb{E}\left[\int_{0}^{t} \theta_{h}''(X_{s}^{n} - X_{s}^{m})\left(\sqrt{|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}|} + \sqrt{|Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}|}\right)^{2}ds\right] \\
\leq H^{2}\mathbb{E}\left[\int_{0}^{t} \theta_{h}''(X_{s}^{n} - X_{s}^{m})\left(|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}| + |Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}|\right)ds\right] \\
\leq H^{2}\mathbb{E}\left[\int_{0}^{t} \frac{2}{h|X_{s}^{n} - X_{s}^{m}|}|X_{s}^{n} - X_{s}^{m}|ds\right] \\
+ H^{2}\|\theta_{h}''\|\mathbb{E}\left[\int_{0}^{t}(|X_{\eta_{n}(s)}^{n} - X_{s}^{n}| + |X_{\eta_{m}(s)}^{m} - X_{s}^{m}|)ds\right] \\
+ H^{2}\|\theta_{h}''\|\mathbb{E}\left[\int_{0}^{t}(|Y_{\eta_{n}(s)} - Y_{s}| + |Y_{s} - Y_{\eta_{m}(s)}|)ds\right] \\
\leq \frac{2H^{2}T}{h} + \|\theta_{h}''\|TH^{2}(M_{1} + C)(\sqrt{\|\Delta_{n}\|} + \sqrt{\|\Delta_{n}\|}).$$
(3.20)

Here $\|\theta_h''\|$ denotes the supremum norm of θ_h'' while in the last inequality we used the same bound to obtain inequality (3.19). Now, let us fix $\varepsilon > 0$. For this ε let h be such that $0 < a_{h-1} < \varepsilon$ and $\frac{2H^2T}{h} < \varepsilon$. With this h being so chosen and fixed, $\|\theta_h''\|$ is bounded. Then, there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$(M_1 + C)(T + \|\theta_h''\|TH^2)(\sqrt{\|\Delta_n\|} + \sqrt{\|\Delta_n\|}) < \varepsilon$$

for all $n, m \ge n_{\varepsilon}$. We can now insert estimates (3.19) and (3.20) in (3.18) to obtain

$$\begin{split} \mathbb{E}[|X_{t}^{n} - X_{t}^{m}|] &\leq a_{h-1} + \mathbb{E}[I_{1}(\theta_{h})] + \mathbb{E}[I_{3}(\theta_{h})] \\ &\leq a_{h-1} + TL(M_{1} + C)(\sqrt{\|\Delta_{n}\|} + \sqrt{\|\Delta_{n}\|}) + L\int_{0}^{t} \mathbb{E}[|X_{s}^{n} - X_{s}^{m}|]ds \\ &\quad + \frac{2H^{2}T}{h} + \|\theta_{h}^{\prime\prime}\|TH^{2}(M_{1} + C)(\sqrt{\|\Delta_{n}\|} + \sqrt{\|\Delta_{n}\|}) \\ &\leq 3\varepsilon + L\int_{0}^{t} \mathbb{E}[|X_{s}^{n} - X_{s}^{m}|]ds. \end{split}$$

By Gronwall's inequality we conclude then that

$$\mathbb{E}[|X_t^n - X_t^m|] \le 3e^{Lt}\varepsilon \le 3e^{LT}\varepsilon,$$

for all $n, m \ge n_{\varepsilon}$ and all $t \in [0, T]$. Hence,

$$\begin{split} \mathbb{E}\left[\int_{0}^{T}|X_{t}^{n}-X_{t}^{m}|dt\right] &= \int_{0}^{T}\mathbb{E}[|X_{t}^{n}-X_{t}^{m}|]dt\\ &\leq T\sup_{t\in[0,T]}\mathbb{E}[|X_{t}^{n}-X_{t}^{m}|]\\ &\leq 3Te^{LT}\varepsilon. \end{split}$$

The claim of step three is proved.

Step four: $\{X^n\}_{n\geq 1}$ is a Cauchy sequence in $L^1(\Omega; C([0,T]))$.

We know that $\{X^n\}_{n\geq 1}$ is a Cauchy sequence in $L^1([0,T]\times\Omega)$ which is a complete space. We can therefore conclude that there exists a stochastic process $X \in L^1([0,T]\times\Omega)$ such that

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^T |X_t^n - X_t| dt\right] = 0.$$

From Step two we can also deduce that

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^T |X_{\eta_n(t)}^n - X_t| dt\right] = 0.$$

Hence, there exists a subsequence (we keep the same indexes though for easy notations) such that

$$\lim_{n \to \infty} X_t^n(\omega) = \lim_{n \to \infty} X_{\eta_n(t)}^n(\omega) = X_t(\omega) \quad dt \times d\mathbb{P}\text{-almost surely}.$$

Since the process $\{X_t^n\}_{t\in[0,T]}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$ -adapted for any $n \in \mathbb{N}$ and almost sure convergence preserves measurability, we deduce that $\{X_t\}_{t\in[0,T]}$ is also $\{\mathcal{F}_t\}_{t\in[0,T]}$ -adapted. To prove the continuity of $\{X_t\}_{t\in[0,T]}$ we need to check the convergence in the uniform topology, i.e. we need to estimate $\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^n - X_t^m|\right]$. As before we employ the representation (3.13):

$$\begin{split} \mathbb{E}\left[\sup_{t\in[0,T]}|X_{t}^{n}-X_{t}^{m}|\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]}\int_{0}^{t}|\bar{a}(Y_{\eta_{n}(s)},X_{\eta_{n}(s)}^{n})-\bar{a}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m})|ds\right] \\ &+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{0}^{t}(\bar{g}(Y_{\eta_{n}(s)},X_{\eta_{n}(s)}^{n})-\bar{g}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m}))dW_{s}^{2}\right|\right] \\ &\leq \int_{0}^{T}\mathbb{E}[|\bar{a}(Y_{\eta_{n}(s)},X_{\eta_{n}(s)}^{n})-\bar{a}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m})|]ds \\ &+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{0}^{t}(\bar{g}(Y_{\eta_{n}(s)},X_{\eta_{n}(s)}^{n})-\bar{g}(Y_{\eta_{m}(s)},X_{\eta_{m}(s)}^{m}))dW_{s}^{2}\right|^{2}\right]^{\frac{1}{2}} \\ &=: J_{1}+J_{2} \end{split}$$

To treat J_1 we proceed as before; using inequality (3.19) we obtain

$$J_1 = \int_0^T \mathbb{E}[|\bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{a}(Y_{\eta_m(s)}, X_{\eta_m(s)}^m)|] ds$$

$$\leq L \int_{0}^{T} \mathbb{E}[|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}|]ds + \int_{0}^{T} \mathbb{E}[|Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}|]ds \qquad (3.21)$$

$$\leq TL(M_{1} + C)(\sqrt{\|\Delta_{n}\|} + \sqrt{\|\Delta_{n}\|}) + L \int_{0}^{T} \mathbb{E}[|X_{s}^{n} - X_{s}^{m}|]ds.$$

Since we proved in Step three that $\{X^n\}_{n\geq 1}$ is a Cauchy sequence in $L^1([0, T[\times \Omega)]$ and by assumption $\|\Delta_n\|$ tends to zero as n tends to infinity, we can find n and m big enough to make the last row of the previous chain of inequalities smaller than any positive ε . We now evaluate J_2 . Invoking the Doob maximal inequality and Itô isometry we can write

$$J_{2} = \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{0}^{t} (\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m}))dW_{s}^{2}\right|^{2}\right]^{\frac{1}{2}}$$

$$\leq 2\mathbb{E}\left[\left|\int_{0}^{T} (\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m}))dW_{s}^{2}\right|^{2}\right]^{\frac{1}{2}}$$

$$= 2\mathbb{E}\left[\int_{0}^{T} |\bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{\eta_{m}(s)}, X_{\eta_{m}(s)}^{m})|^{2}ds\right]^{\frac{1}{2}}$$

$$\leq 2H\mathbb{E}\left[\int_{0}^{T} \left(\sqrt{|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}|} + \sqrt{|Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}|}\right)^{2}ds\right]^{\frac{1}{2}}$$

$$\leq 2\sqrt{2}H\mathbb{E}\left[\int_{0}^{T} |X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}| + |Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}||ds\right]^{\frac{1}{2}}$$

$$= 2\sqrt{2}H\left(\int_{0}^{T} \mathbb{E}[|X_{\eta_{n}(s)}^{n} - X_{\eta_{m}(s)}^{m}|] + \mathbb{E}[|Y_{\eta_{n}(s)} - Y_{\eta_{m}(s)}||ds\right]^{\frac{1}{2}}.$$

If we now observe that the last member above is equivalent to (3.21), we can proceed as before and conclude that for any $\varepsilon > 0$ there exists $n_{\varepsilon} \in \mathbb{N}$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^n-X_t^m|\right]<\varepsilon\quad\text{for all }n,m\geq n_{\varepsilon}.$$

This proves that $\{X_n\}_{n\geq 1}$ is a Cauchy sequence in $L^1(\Omega; C([0,T]))$ and thus

$$\lim_{n \to \infty} \mathbb{E} \left[\sup_{t \in [0,T]} |X_t^n - X_t| \right] = 0$$

where $\{X_t\}_{t \in [0,T]}$ is the stochastic process obtained in Step three. Moreover, we can find a subsequence (we keep the same indexes though for easy notations) such that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} |X_t^n(\omega) - X_t(\omega)| = 0 \quad d\mathbb{P}\text{-almost surely.}$$

Since the processes $\{X_t^n\}_{t\in[0,T]}$ are continuous by construction for each $n \in \mathbb{N}$, we deduce that the process $\{X_t\}_{t\in[0,T]}$ is also continuous being a uniform limit of continuous functions.

Step five: The stochastic process $\{X_t\}_{t \in [0,T]}$ solves equation (3.9).

Finally we show that

$$\mathbb{P}\left(X(t) = x + \int_0^t \bar{a}(Y_s, X_s)ds + \int_0^t \bar{g}(Y_s, X_s)dW_s^2 \quad \text{for all } t \in [0, T]\right) = 1.$$

This in turn will be proven by showing that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t - x - \int_0^t \bar{a}(Y_s, X_s)ds - \int_0^t \bar{g}(Y_s, X_s)dW_s^2\right|\right] = 0$$

In fact, the equality

$$X_{t} - x - \int_{0}^{t} \bar{a}(Y_{s}, X_{s}) ds - \int_{0}^{t} \bar{g}(Y_{s}, X_{s}) dW_{s}^{2}$$

= $X_{t} - X_{\eta_{n}(t)}^{n} + \int_{0}^{t} \bar{a}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{a}(Y_{s}, X_{s}) ds$
+ $\int_{0}^{t} \bar{g}(Y_{\eta_{n}(s)}, X_{\eta_{n}(s)}^{n}) - \bar{g}(Y_{s}, X_{s}) dW_{s}^{2}$

implies

$$\sup_{t \in [0,T]} \left| X_t - x - \int_0^t \bar{a}(Y_s, X_s) ds - \int_0^t \bar{g}(Y_s, X_s) dW_s^2 \right|$$

$$\leq \sup_{t \in [0,T]} \left| X_t - X_{\eta_n(t)}^n \right| + \int_0^T \left| \bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{a}(Y_s, X_s) \right| ds$$

$$+ \sup_{t \in [0,T]} \left| \int_0^t \bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{g}(Y_s, X_s) dW_s^2 \right|.$$

If we take the expectation and use the technique utilized in Step four to bound the terms in the right hand side of the previous inequality we get

$$\begin{split} & \mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t - x - \int_0^t \bar{a}(Y_s, X_s)ds - \int_0^t \bar{g}(Y_s, X_s)dW_s^2\right|\right] \\ &= \lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t - x - \int_0^t \bar{a}(Y_s, X_s)ds - \int_0^t \bar{g}(Y_s, X_s)dW_s^2\right|\right] \\ &\leq \lim_{n\to\infty}\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left|X_t - X_{\eta_n(t)}^n\right|\right] + \mathbb{E}\left[\int_0^T\left|\bar{a}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{a}(Y_s, X_s)\right|ds\right]\right) \\ &+ \lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \bar{g}(Y_{\eta_n(s)}, X_{\eta_n(s)}^n) - \bar{g}(Y_s, X_s)dW_s^2\right|\right] \\ &= 0. \end{split}$$

Uniqueness: We use a standard approach. Let $\{X_t\}_{t \in [0,T]}$ and $\{Z_t\}_{t \in [0,T]}$ be two strong solutions of equation (3.9). Setting,

$$\delta_t := X_t - Z_t = \int_0^t [\bar{a}(Y_s, X_s) - \bar{a}(Y_s, Z_s)] ds + \int_0^t [\bar{g}(Y_s, X_s) - \bar{g}(Y_s, Z_s)] dW_s^2 \quad (3.22)$$

we get by the Itô formula

$$\theta_h(\delta_t) = \int_0^t \theta'_h(\delta_s) [\bar{a}(Y_s, X_s) - \bar{a}(Y_s, Z_s)] ds$$

+
$$\int_0^t \theta'_h(\delta_s) [\bar{g}(Y_s, X_s) - \bar{g}(Y_s, Z_s)] dW_s^2$$

+
$$\frac{1}{2} \int_0^t \theta''_h(\delta_s) [\bar{g}(Y_s, X_s) - \bar{g}(Y_s, Z_s)]^2 ds$$

where $\{\theta_h\}_{h\geq 0}$ is the collection of functions defined in Step three. Using the assumptions on \bar{a} and \bar{g} and the bounds $|\theta'_h(u)| \leq 1$ and $|\theta''_h(u)| \leq \frac{2}{hu}$ we get

$$\begin{split} \mathbb{E}[\theta_h(\delta_t)] &\leq \mathbb{E}\left[\int_0^t \theta'_h(\delta_s)[\bar{a}(Y_s, X_s) - \bar{a}(Y_s, Z_s)]ds\right] + \frac{tH^2}{h} \\ &\leq L \int_0^t \mathbb{E}[|\delta_s|]ds + \frac{tH^2}{h} \end{split}$$

If we let $h \to \infty$, the function θ_h approaches the absolute value function; hence, Gronwall's inequality and sample path continuity imply that $\{X_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$ are indistinguishable.

References

- [1] E. Allen, *Modelling with It Stochastic Differential Equations*, Springer-Verlag, London, 2007.
- [2] L.J.S. Allen, An introduction to stochastic epidemic models in mathematical epidemiology, in: F. Brauer, P. van den Driessche, J. Wu (Eds.), Lecture Notes in Biomathematics, Mathematical Biosciences Subseries, vol. 1945, Springer-Verlag, Berlin (2008) 81130.
- [3] L.J.S. Allen, A.M. Burgin, Comparison of deterministic and stochastic SIS and SIR models in discrete time, *Math. Biosci.* 163 (2000) 133.
- [4] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math. 71 (3) (2011) 876902.
- [5] D. Greenhalgh, Y.Liang, X.Mao, SDE SIS epidemic model with demographic stochasticity and varying population size, *Applied Mathematics and Computation*, 276 (2016) 218-238
- [6] H.W. Hethcote, P. van den Driessche, An SIS epidemic model with variable population size and a delay, J. Math. Biol. 34 (1995) 177194.
- [7] N. Ikeda and S.Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, New York, Oxford, Kodansha, 1981.
- [8] R.J. Kryscio, C. Lefvre, On the extinction of the SIS stochastic logistic epidemic, J. Appl. Probab. 26 (4) (1989) 685694.

- [9] X. Mao, Stochastic Differential Equations and Applications, Second edition, Horwood, Chichester, UK, 2008.
- [10] R.K. McCormack, L.J.S. Allen, Stochastic SIS and SIR multihost epidemic models, in: R.P. Agarwal, K. Perera (Eds.), Proceedings of the Conference on Differential and Difference Equations and Applications, Hindawi, New York (2006) 775786.
- [11] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, Third Edition, Springer-Verlag, Berlin, 1999.
- [12] I. Nasell, The quasi-stationary distribution of the closed endemic SIS model, Adv. Appl. Probab. 28 (1996) 895932.
- [13] V.P. Zubchenko, Properties of solutions of stochastic differential equations with random coefficients, non-Lipschitzian coefficients and Poisson measure, *Theor. Probability and Math. Statist.* 82 (2011) 11-26