Nonlinear Robust Filtering of Sampled-Data Dynamical Systems

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Abstract

This work is concerned with robust filtering of nonlinear sampled-data systems with and without exact discrete-time models. A linear matrix inequality (LMI) based approach is proposed for the design of robust H_∞ observers for a class of Lipschitz nonlinear systems. Two type of systems are considered, Lipschitz nonlinear discrete-time systems and Lipschitz nonlinear sampled-data systems with Euler approximate discrete-time models. Observer convergence when the exact discrete-time model of the system is available is shown. Then, practical convergence of the proposed observer is proved using the Euler approximate discrete-time model. As an additional feature, maximizing the admissible Lipschitz constant, the solution of the proposed LMI optimization problem guaranties robustness against some nonlinear uncertainty. The robust H_∞ observer synthesis problem is solved for both cases. The maximum disturbance attenuation level is achieved through LMI optimization. At the end, a path to extending the results to higher-order approximate discretizations is provided.

1 Introduction

Design of discrete-time nonlinear observers has been the subject of significant attention in recent years. [1], [2], [3], [4]. The study of the nonlinear discrete-time observers is important at least for two reasons. First, most continuous-time control system designs are implemented digitally. Given that in most practical cases it is impossible to measure every state variable in real time, these controllers require the reconstruction of the states

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of the discrete-time model of the true continuous-time plant. Second, there are systems which are inherently discrete-time and do not originate from discretization of a continuous-time plant. Of those, discrete-time observers of continuous-time systems are particularly challenging. The reason is that exact discretization of a continuous-time nonlinear model is usually not possible to obtain. Approximate discrete-time models, on the other hand, are affected by the consequent approximation error. In this paper, we address both problems. First, we consider a class of nonlinear discrete-time systems with exact model. A nonlinear H_∞ observer design algorithm is proposed for these systems based on an LMI approach. Then, the nonlinear sampled data system with Euler approximate model is considered. The Euler approximation is important because not only it is easy to derive but also maintains the structure of the original nonlinear model. We will show that by appropriate selection of one of the parameters in our proposed LMIs (actually the only design parameter in our algorithm), the practical convergence of the observer via Euler approximation is guaranteed as well as the robust H_∞ cost. Our approach is based on the recent results of [5]. See [6] and [7] for other approaches. We emphasize that while the algorithms in [6] and [7] are specifically designed for Euler discretization, our proposed algorithm can be applied either to the nominal exact discrete-time model or its Euler approximation.

There is a large body of literature for control and estimation of nonlinear systems satisfying a Lipschitz continuity condition. See for example [8, 9, 10, 11, 12, 13, 14, 15, 6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and the references therein, for details of the approach and applications to control and filtering of different classes of nonlinear systems. The significance of this condition is that it guarantees the existence and uniqueness of the solution of the nonlinear systems. Also, it provides a mathematically tractable framework to apply Lyapunov stability theory and establish stability and performance conditions in the form of Riccati equations or LMIs.

The LMI based observer design for uncertain discrete-time systems has been addressed in several works e.g. [12], [16] and [13]. In all these studies, the proposed LMIs are nonlinear in the Lipschitz constant and thus it can not be considered as one of the LMI variables. In the algorithm proposed here, first the problem is addressed in the general case, then, having a bound on the Lipschitz constant, the LMIs become linear in the Lipschitz constant and we can take advantage of this feature to solve an optimization problem over it. Providing that the optimal solution is larger than the actual Lipschitz constant of the system in hand, we show that the redundancy achieved can guarantee robustness against some nonlinear uncertainty in the original continuous-time model for both exact and Euler approximate discretizations.

The rest of the paper is organized as follows: Section II briefly describes the filtering framework. In Section III, an observer design method for a class of nonlinear discrete-time systems is introduced. In Section IV the practical convergence of the proposed observer via the Euler approximate models is shown. In section V, the results of the two previous sections will extend into the H_∞ context followed by an illustrative example showing satisfactory performance of our algorithm.

2 Filtering Framework

Figure 1 shows a classification of state estimators in terms of their functionality, and their computational framework [29]. The *filtering* problem deals with state estimation under noise/disturbance; the *robust observation* problem addresses state estimation under model uncertainty, while *robust filtering* combines the two. *Multi-objective robust filtering* provides tools to tune the trade-offs between robustness bounds, disturbance attenuation level and convergence rate [26, 22, 9]. While general matrix inequalities, including bilinear matrix inequalities (BMIs), are not numerically tractable, semidefinite programming problems (SDP) and LMIs can be solved using efficient interior-point methods. *Strict LMI*s are referred to those LMIs in which all inequalities are strictly positive or negative definite and no semidefinite matrices or equality constraints are allowed. Strict LMI solvers are often more efficient than SDP solvers. The solutions provided in this work are *robust filters* whose gains are computed using *strict LMI*s. Following an $\mathcal{L}_2$ filtering framework, we assume that the noise is energy-bounded and the model uncertainties are norm-bounded.

3 Observer Design For Nonlinear Discrete-Time Systems

We consider the following system

$$x(k+1) = A_d x(k) + F(x(k), u(k)) \tag{1}$$

$$y(k) = C_d x(k) \tag{2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and $F(x(k), u(k))$ contains nonlinearities of second order or higher. The above system can be either an inherently discrete-time system or the exact discretization of a continuous-time system.

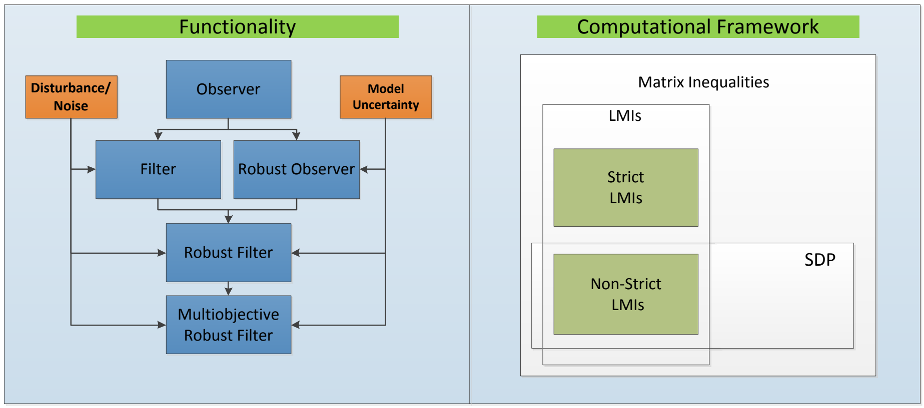

Figure 1: State estimation functionality and computational framework

We assume that $F(x(k), u(k))$ is locally Lipschitz with respect to x in a region $\mathcal{D}$, uniformly in u, i.e. $\forall\, x_1(k), x_2(k) \in \mathcal{D}$:

$$\|F(x_1, u^*) - F(x_2, u^*)\| \leq \gamma_d \|x_1 - x_2\| \tag{3}$$

where $\|.\|$ is the induced 2-norm, u^* is any admissible control sequence and $\gamma_d > 0$ is called the Lipschitz constant. If the nonlinear function F satisfies the Lipschitz continuity condition globally in $\mathbb{R}^n$, then all the results in this and the ensuing sections will be valid globally. All matrices and vectors have appropriate dimensions unless otherwise mentioned. The proposed observer is in the following form:

$$\hat{x}(k+1) = A_d \hat{x}(k) + F(\hat{x}(k), u(k)) + L(y(k) - C_d \hat{x}(k)) \tag{4}$$

the observer error is thus:

$$\begin{aligned} e(k+1) \triangleq x(k+1) - \hat{x}(k+1) = (A_d - LC_d)e(k) \\ + F(x(k), u(k)) - F(\hat{x}(k), u(k)). \end{aligned} \tag{5}$$

Our goal is two-fold: (i) In the first place, we want to find an observer gain, L, such that the observer error dynamics is asymptotically stable. (ii) We want to maximize γ_d, the allowable Lipschitz constant of the nonlinear system.

Theorem 1. *Consider the system (1)-(2) with given Lipschitz constant γ_d. The observer error dynamics (5) is (globally) asymptotically stable if* *there exist scalar $\varepsilon > 0$, fixed matrix $Q > 0$ and matrices $P > 0$ and G such that the following set of LMIs has a solution*

$$\begin{bmatrix} P - Q - \varepsilon I & A_d^T P - C_d^T G^T \\ PA_d - GC_d & P \end{bmatrix} > 0 \tag{6}$$

$$\begin{bmatrix} \Psi_1 I & P \\ P & \Psi_1 I \end{bmatrix} > 0 \tag{7}$$

where

$$\Psi_1 = \frac{-\lambda_{\max}(Q) + \sqrt{\lambda_{\max}^2(Q) + \frac{1}{\gamma_d^2}\lambda_{\min}^2(Q)}}{\gamma_d + 2}. \tag{8}$$

P, G, and ϵ are the LMI variables and Q is a design parameter to be chosen. Once the problem is solved:

$$L = P^{-1}G \tag{9}$$

Proof: Consider the Lyapunov function candidate as follows:

$$V_k = e_k^T P e_k \tag{10}$$

then:

$$\begin{aligned} \Delta V = V_{k+1} - V_k = & e_k^T (A_d - LC_d)^T P (A_d - LC_d) e_k \\ & + 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) \\ & + (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k) - e_k^T P e_k \end{aligned} \tag{11}$$

where for simplicity:

$$F_k \triangleq F(x(k), u(k)), \; \hat{F}_k \triangleq F(\hat{x}(k), u(k)). \tag{12}$$

Suppose $\exists\, P, Q > 0$ such that the following discrete-time Lyapunov equation has a solution:

$$(A_d - LC_d)^T P (A_d - LC_d) - P = -Q \tag{13}$$

then (11) becomes:

$$\begin{aligned} \Delta V = & - e_k^T Q e_k + 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) \\ & + (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k) \end{aligned} \tag{14}$$

using Rayleigh and Schwartz inequalities, we have:

$$\|e_k^T Q e_k\| \geq \lambda_{\min}(Q)\|e_k\|^2 \tag{15}$$

$$\begin{aligned} &\|2e_k^T(A_d - LC_d)^T P(F_k - \hat{F}_k)\| \leq \|2e_k^T P(F_k - \hat{F}_k)\| \cdots \\ &\quad \cdots \|A_d - LC_d\| \leq 2\gamma_d \lambda_{\max}(P)\|e_k\|^2 \|A_d - LC_d\| \\ &\quad = 2\gamma_d \lambda_{\max}(P)\|e_k\|^2 \bar{\sigma}(A_d - LC_d) \end{aligned} \tag{16}$$

$$\begin{aligned} &\|(F_k - \hat{F}_k)^T P(F_k - \hat{F}_k)\| \leq \lambda_{\max}(P)\|(F_k - \hat{F}_k)\|^2 \\ &\quad \leq \gamma_d^2 \lambda_{\max}(P)\|e_k\|^2 \end{aligned} \tag{17}$$

so for $\Delta V < 0$ it is sufficient to have:

$$-\lambda_{\min}(Q) + \lambda_{\max}(P)[2\gamma_d \bar{\sigma}(A_d - LC_d) + \gamma_d^2] < 0. \tag{18}$$

Condition (18) along with (13) are sufficient conditions for asymptotic stability. We now endeavor to convert these nonlinear inequalities into LMIs. There exists a solution for (13) if

$$\begin{aligned} &\exists\, \varepsilon > 0,\ (A_d - LC_d)^T P(A_d - LC_d) - P < -Q - \varepsilon I \\ &\qquad \Rightarrow (P - Q - \varepsilon I) - (A_d - LC_d)^T P P^{-1} P(A_d - LC_d) > 0 \end{aligned} \tag{19}$$

using Schur's complement lemma, defining $G = PL$ and knowing that $P^T = P$, the first LMI in Theorem 1 is obtained. The Lyapunov equation in (13) can be rewritten as

$$P = (A_d - LC_d)^T P(A_d - LC_d) + Q \tag{20}$$

and taking into account that:

$$|\bar{\sigma}(A) - \bar{\sigma}(B)| \leq \bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \tag{21}$$

we have that

$$\begin{aligned} &|\bar{\sigma}\left[(A_d - LC_d)^T P(A_d - LC_d)\right] - \bar{\sigma}(Q)| \leq \bar{\sigma}(P) \\ &\Rightarrow \bar{\sigma}\left[(A_d - LC_d)^T P(A_d - LC_d)\right] \leq \bar{\sigma}(Q) + \bar{\sigma}(P) \end{aligned} \tag{22}$$

using Schwartz inequality:

$$\bar{\sigma}\left[(A_d - LC_d)^T P(A_d - LC_d)\right] \leq \bar{\sigma}^2(A_d - LC_d)\bar{\sigma}(P) \tag{23}$$

comparing (22) and (23), a sufficient condition for (22) is

$$\begin{aligned} &\bar{\sigma}^2(A_d - LC_d)\bar{\sigma}(P) \leq \bar{\sigma}(P) + \bar{\sigma}(Q) \\ &\qquad \Rightarrow \bar{\sigma}(A_d - LC_d) \leq \sqrt{1 + \frac{\bar{\sigma}(Q)}{\bar{\sigma}(P)}} \end{aligned} \tag{24}$$

note that since P and Q are positive definite their eigenvalues and singular values are the same. Now, we want to find a sufficient condition for (18). Using (24):

$$\begin{aligned}\bar{\sigma}(A_d - LC_d)\lambda_{\max}(P) &\leq \sqrt{1 + \frac{\bar{\sigma}(Q)}{\bar{\sigma}(P)}}\bar{\sigma}(P) \\ &= \sqrt{\bar{\sigma}^2(P) + \bar{\sigma}(Q)\bar{\sigma}(P)}.\end{aligned} \tag{25}$$

Suppose Q is given, define,

$$g(\bar{\sigma}(P)) \triangleq \sqrt{\bar{\sigma}^2(P) + \bar{\sigma}(Q)\bar{\sigma}(P)} \tag{26}$$

then $g(\bar{\sigma}(P))$ is strictly increasing so there is no constant upper limit for this function but we can still bound this nonlinear function with a linear one.

$$\begin{aligned}g(\bar{\sigma}(P)) &< \sqrt{\bar{\sigma}^2(P) + \bar{\sigma}(Q)\bar{\sigma}(P) + \left[\frac{\bar{\sigma}(Q)}{2}\right]^2} \\ &= \sigma(P) + \frac{\bar{\sigma}(Q)}{2}\end{aligned} \tag{27}$$

$$\Rightarrow \bar{\sigma}(A_d - LC_d)\lambda_{\max}(P) < \bar{\sigma}(P) + \frac{\bar{\sigma}(Q)}{2} \tag{28}$$

which is a sufficient condition for (22). Substituting the above into (18), a sufficient condition for (18) is

$$\gamma_d \lambda_{\max}(P)\left[2\sqrt{1 + \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}} + \gamma_d\right] < \lambda_{\min}(Q). \tag{29}$$

For any $a, b > 0$, $a^2 < b^2$ implies $a < b$, thus, by squaring the two sides of the above inequality, substituting from (27) and after some algebra, to have (29) it suffices to

$$\begin{aligned}(\gamma_d + 2)\lambda_{\max}^2(P) + 2\lambda_{\max}(Q)\lambda_{\max}(P) &< \frac{\lambda_{\min}^2(Q)}{\gamma_d^2(\gamma_d + 2)} \\ &\Rightarrow \lambda_{\max}(P) < \Psi_1\end{aligned} \tag{30}$$

or equivalently,

$$\Psi_1^2 I - PP^T > 0 \tag{31}$$

which is by means of Schur's complement lemma, equivalent to the second LMI in Theorem 1 where Ψ_1 is as in (8). This ends the proof. $\triangle$

In continuance, consider the case where the Lipschitz constant of system, γ_d, is less than 1. This is not restrictive since the Lipschitz constant can reduced using a suitable coordinate transformation [10]. Besides, the discretized models of continuous-time systems may also fall into this category by appropriate selection of the sampling time. We will see this in detail for Euler discretization in the next section. The following theorem shows that under this assumption, the maximum admissible Lipschitz constant is achievable through an LMI optimization over γ_d.

Theorem 2. *Consider the system (1)-(2). The observer error dynamics (5) is (globally) asymptotically stable with maximum admissible Lipschitz constant γ_d^*, if there exist scalars $\varepsilon > 0, \xi > 1$, fixed matrix $Q > 0$ and matrices $P > 0$ and G such that the following LMI optimization problem has a solution*

$$\min(\xi)$$

s.t.

$$\begin{bmatrix} P - Q - \varepsilon I & A_d^T P - C_d^T G^T \\ PA_d - GC_d & P \end{bmatrix} > 0 \tag{32}$$

$$\begin{bmatrix} \Psi_2 I & P \\ P & \Psi_2 I \end{bmatrix} > 0 \tag{33}$$

where

$$\Psi_2 = \frac{1}{3}\left[\lambda_{\min}(Q)\,\xi - \lambda_{\max}(Q)\right]. \tag{34}$$

once the problem is solved:

$$L = P^{-1}G \tag{35}$$

$$\gamma_d^* \triangleq \max(\gamma_d) = \frac{1}{\xi} \tag{36}$$

Proof: Having the same Lyapunov function candidate it follows that, $\triangle V$ is given by (11). Knowing $\gamma_d < 1$, (18) reduces to

$$\gamma_d < \frac{\lambda_{\min}(Q)}{\left[2\bar{\sigma}(A_d - LC_d) + 1\right]\lambda_{\max}(P)} \tag{37}$$

where Q is the same as before. Based on (28), it can be written

$$[2\bar{\sigma}(A_d - LC_d) + 1]\,\lambda_{\max}(P) < 3\bar{\sigma}(P) + \bar{\sigma}(Q). \tag{38}$$

From the above, we have

$$\frac{\lambda_{\min}(Q)}{[2\bar{\sigma}(A_d - LC_d) + 1]\,\lambda_{\max}(P)} > \frac{\lambda_{\min}(Q)}{3\bar{\sigma}(P) + \lambda_{\max}(Q)}.$$

Eventually, a sufficiency condition for (37) is

$$\gamma_d < \frac{\lambda_{\min}(Q)}{3\bar{\sigma}(P) + \lambda_{\max}(Q)} \rightarrow \bar{\sigma}(P) < \frac{\lambda_{\min}(Q)}{3\gamma_d} - \frac{1}{3}\lambda_{\max}(Q) \tag{39}$$

which, by means of Schur's complement lemma is equivalent to the second LMI in Theorem 2. △

Remark 1. The purpose of Theorem 2 is two-fold. (i) to find a gain matrix "L" that stabilizes the observer error dynamics, and (ii) to maximized γ_d. Dropping the maximization of γ_d still renders a stable observer. In this case the proposed LMI optimization reduces to an LMI feasibility problem (namely; satisfying the constraints) which is easier. The only parameter to be chosen in both cases is the positive definite matrix Q.

Remark 2- Nonlinear Uncertainty. The advantage of maximization of γ_d is that if the maximum admissible Lipschitz constant achieved by Theorem 1, γ_d^*, is greater than the actual Lipschitz constant of the system, γ_d, then the proposed observer can tolerate some nonlinear uncertainty. Consider the system with nonlinear uncertainty as below:

$$F_\Delta(x,u) \triangleq F(x,u) + \Delta F(x,u) \tag{40}$$

$$x(k+1) = A_d x(k) + F_\Delta(x,u) \tag{41}$$

$$y(k) = C_d x(k). \tag{42}$$

Suppose the additive nonlinear uncertainty is Lipschitz with unknown Lipschitz constant $\Delta\gamma_d$. According to the Theorem 1, $F_\Delta(x(k), u(k))$ can be any Lipschitz nonlinear function with Lipschitz constant less than or equal to γ_d^*, i.e.:

$$\|F_\Delta(x_1,u) - F_\Delta(x_2,u)\| \leq \gamma_d^* \|x_1 - x_2\|. \tag{43}$$

On the other hand

$$\begin{aligned}
&\|F_\Delta(x_1,u) - F_\Delta(x_2,u)\| = \|F(x_1,u) + \Delta F(x_1,u) \cdots \\
&\cdots - F(x_2,u) - \Delta F(x_2,u)\| \\
&\leq \|F(x_1,u) - F(x_2,u)\| + \|\Delta F(x_1,u) - \Delta F(x_2,u)\| \\
&\leq \gamma_d \|x_1 - x_2\| + \Delta\gamma_d \|x_1 - x_2\|
\end{aligned}$$

So, there must be:

$$\gamma_d + \Delta\gamma_d \leq \gamma_d^* \rightarrow \Delta\gamma_d \leq \gamma_d^* - \gamma_d. \tag{44}$$

This means that the proposed observer is robust against any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal to $\gamma_d^* - \gamma_d$.

4 Observer Design For Nonlinear Sampled-Data Systems Via Euler Approximation

In usual, given a continuous nonlinear model, an exact discretization can not be found in closed form, thus originating the need of approximate discrete-time models. A framework for nonlinear observer design based on approximated models has been recently proposed in [5]. In this section, our focus will be on Euler approximation which is an important case because it is easy to derive and it doesn't change the structure of the original nonlinear model. Following the notation of [5], we consider the following continuous-time system

$$\begin{aligned} \dot{x} &= Ax + f(x,u) \\ y &= Cx \end{aligned} \tag{45}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. We assume that system has an equilibrium point at the origin and $f(x,u)$ is locally Lipschitz with the Lipschitz constant γ_c. The family of exact discretizations of (45) is:

$$\begin{aligned} x(k+1) &= A_d x(k) + F_T^e(x(k), u(k)) \\ y(k) &= C_d x(k) \end{aligned} \tag{46}$$

index T means the discretization is dependent to the sampling time, T. To compute (46) we need a closed-form solution of (45) over the sampling intervals *[k T,(k+1)T)*, which is hard to obtain or even impossible. However, it is realistic to assume that a family of approximate discrete-time models is available

$$\begin{aligned} x^a(k+1) &= A_d^a x(k) + F_T^a(x(k), u(k)) \\ y(k) &= C_d x^a(k). \end{aligned} \tag{47}$$

Then for the Euler approximation we have

$$A_d^a = I + AT \tag{48}$$

$$F_T^a(x^a(k), u(k)) = Tf(x^a(k), u(k)). \tag{49}$$

Similar to (4), the proposed observer is

$$\hat{x}^a_{k+1} = A^a_d \hat{x}^a_k + F^a_T(\hat{x}^a_k, u_k) + L(y_k - C_d \hat{x}^a_k). \tag{50}$$

Before expressing our result, we recall two aspects from [5], *consistency* and *semiglobal practicality*. The definitions are omitted here due to space limitations. According to the verifiable consistency conditions given in [30], if the trajectories of a continuous-time Lipschitz nonlinear system are bounded, then the Euler approximation is consistent (one-step consistent) with the exact discrete-time model.

Based on (49), the Lipschitz constant of the Euler approximation is $\gamma_d = T\gamma_c$. Again, we assume $\gamma_d < 1$. This is even less restrictive than in section 2, because here T directly multiplies γ_c and can be chosen sufficiently small. The following theorem shows that how the algorithm proposed in Theorem 2 can be used to design an observer using Euler approximate discrete-time model guaranteing observer practical convergence when applied to the (unknown) exact model, by the appropriate selection of Q.

Theorem 3. *The observer (50) designed using the Euler approximate model (49) is semiglobal practical in T with the maximum admissible Lipschitz constant γ^*_d, if the trajectories of (45) are bounded and there exist scalars $\varepsilon > 0$, $\xi > 1$, fixed matrix $Q > 0$ and matrices $P > 0$ and G such that the LMI optimization problem (32)-(33) has a solution where $\lambda_{\min}(Q) = T$.*

Proof. Consider the same Lyapunov function used in Theorems 1 and 2:

$$V_T(e(k)) = [x^a(k) - \hat{x}^a(k)]^a \, P \, [x^a(k) - \hat{x}^a(k)] \tag{51}$$

then

$$\begin{aligned} \|V_T(e_{k_1}) - V_T(e_{k_2}))\| &= \left\| e^T_{k_1} P e_{k_1} - e^T_{k_2} P e_{k_2} \right\| \\ &= \left\| [e(k_1) - e(k_2)]^T \, P \, [e(k_1) + e(k_2)] \right\| \\ &\leq \lambda_{\max}(P) \|e(k_1) + e(k_2)\| \|e(k_1) - e(k_2)\| \end{aligned} \tag{52}$$

by the definition of observer error, the observer error is finite (note that the convergence of the observer states to the states of the Euler approximate model has already been achieved by virtue of Theorem 2, so

$$\exists \, M \in (0, \infty) \, , \;\; \lambda_{\max}(P) \, \|e(k_1) + e(k_2)\| \leq M \tag{53}$$

thus, from the above:

$$\|V_T(e(k_1)) - V_T(e(k_2))\| \leq M \|e(k_1) - e(k_2)\| \tag{54}$$

Similar to what we did it section 2, we have

$$F^a_{k,T} \triangleq F^a_T(x^a(k), u(k)),\ \hat{F}^a_{k,T} \triangleq F^a_T(\hat{x}^a(k), u(k)) \tag{55}$$

$$\begin{aligned} V_T(e_{k+1}) - V_T(e_k) &= -e_k^T Q e_k + 2e_k^T (A_d - LC_d)^T \cdots \\ &\cdots P(F^a_{k,T} - \hat{F}^a_{k,T}) + (F^a_{k,T} - \hat{F}^a_{k,T})^T P(F^a_{k,T} - \hat{F}^a_{k,T}) \\ &\leq -\lambda_{\min}(Q)\|e_k\|^2 + 2\bar{\sigma}(A^a_d - LC_d)\lambda_{\max}(P)\gamma_d\|e_k\|^2 \\ &\quad + \gamma_d^2 \lambda_{\max}(P)\|e_k\|^2. \end{aligned} \tag{56}$$

Using (37) and (24), it can be written

$$\begin{aligned} &[-\lambda_{\min}(Q) + 2\bar{\sigma}(A^a_d - LC_d)\lambda_{\max}(P)\gamma_d]\|e_k\|^2 \\ &\qquad \leq -\frac{\lambda_{\min}(Q)\|e_k\|^2}{2\bar{\sigma}(A_d - LC_d) + 1} \leq -\frac{\lambda_{\min}(Q)\|e_k\|^2}{2\sqrt{1 + \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}} + 1} \end{aligned} \tag{57}$$

substituting (57) into (56) and knowing that $\lambda_{\min}(Q) = T$ and $\gamma_d = T\gamma_c$, we will have

$$\begin{aligned} \frac{V_T(e(k+1)) - V_T(e(k))}{T} &\leq \\ -\frac{\|e_k\|^2}{2\sqrt{1 + \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}} + 1} &+ T.\gamma_c^2 \lambda_{\max}(P) \|e_k\|^2 . \end{aligned} \tag{58}$$

Now, we define the following functions:

$$\alpha_1(\|e_k\|) \triangleq \lambda_{\min}(P) \|e_k\|^2 \tag{59}$$

$$\alpha_2(\|e_k\|) \triangleq \lambda_{\max}(P) \|e_k\|^2 \tag{60}$$

$$\alpha_3(\|e_k\|) \triangleq \frac{1}{2\sqrt{1 + \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}} + 1} \|e_k\|^2 \tag{61}$$

$$\rho_0(T) \triangleq T,\ \gamma_0(\|e_k\|) \triangleq \gamma_c^2 \lambda_{\max}(P) \|e_k\|^2 \tag{62}$$

$$\gamma_1(\|x_k\|) \triangleq 0,\ \gamma_2(\|u_k\|) \triangleq 0. \tag{63}$$

Then, the following can be written

$$\alpha_1(\|e_k\|) \leq V_T(e(k)) \leq \alpha_2(\|e_k\|) \tag{64}$$

$$\begin{aligned} \frac{V_T(e(k+1)) - V_T(e(k))}{T} &\leq -\alpha_3(\|e_k\|) \\ &+ \rho_0(T) [\gamma_0(\|e_k\|) + \gamma_1(\|x_k\|) + \gamma_2(\|u_k\|)] \end{aligned} \tag{65}$$

where $\alpha_1(.)$, $\alpha_2(.)$,$\alpha_3(.)$ and $\rho_0(.)$ are in class-$\mathcal{K}_\infty$ and $\gamma_0(.)$, $\gamma_1(.)$ and $\gamma_2(.)$ are nondecreasing functions. Finally, since the trajectories of (47) are bounded for the Euler approximation, the Euler approximate model is consistent with the exact model (45). It follows that all conditions of Theorem 1 in [5] are satisfied and the proposed observer is semiglobal practical in T. $\triangle$

Remark 3. Q is not necessarily equal to TI, nevertheless, it can be figured out from (58) that to have a better convergence rate, $\lambda_{\max}(Q)$ must to be as small as possible

$$\lambda_{\max}(Q) = \lambda_{\min}(Q) = T \Leftrightarrow \; Q = TI. \tag{66}$$

Remark 4: Nonlinear Uncertainty. Similar to Remark 2, in section 2, the observer is robust against any additive Lipschitz nonlinear uncertainty with Lipschitz constant less than or equal to $\gamma_c^* - \gamma_c$.

5 Nonlinear H_∞ Observer Synthesis

In this section we extend the result of the previous section by proposing a new nonlinear robust H_∞ observer design method. Consider the system

$$x(k+1) = A_d x(k) + F(x(k), u(k)) + B_d w(k) \tag{67}$$

$$y(k) = C_d x(k) \tag{68}$$

where $w(t) \in \ell_2[0, \infty)$ is an unknown exogenous disturbance. Suppose that

$$z(k) = He(k) \tag{69}$$

stands for the controlled output for error state where H is a known matrix. Our purpose is to design the observer parameter L such that the observer error dynamics is asymptotically stable and the following specified H_∞ norm upper bound is simultaneously guaranteed.

$$\|z\| \leq \mu \|w\|. \tag{70}$$

The following theorem introduces a new method for nonlinear robust H_∞ observer design.

Theorem 4. *Consider Lipschitz nonlinear system (67)-(68) with given Lipschitz constant γ_d, along with the observer (4). The observer error dynamics is (globally) asymptotically stable with minimum $\mathfrak{L}_2$ gain, μ^*, if there* *exist scalars* $\epsilon > 0$ *and* $\zeta > 0$*, fixed matrix* $Q > 0$ *and matrices* $P > 0$ *and* G *such that the following LMI optimization problem has a solution.*

$$\min(\zeta)$$

$$\begin{bmatrix} P - Q - \varepsilon I & A_d^T P - C_d^T G^T \\ PA_d - GC_d & P \end{bmatrix} > 0 \tag{71}$$

$$\begin{bmatrix} \Psi_1 I & P \\ P & \Psi_1 I \end{bmatrix} > 0 \tag{72}$$

$$\begin{bmatrix} \Lambda_2 I & \frac{1}{2}\left[2(\gamma_d + 1)\Psi_1 + \lambda_{\max}(Q)\right] I \\ \star & B_d^T P B_d - \zeta I \end{bmatrix} < 0 \tag{73}$$

where Ψ_1 is as in (8) and $\Lambda_2 = H^T H - Q + \gamma_d \left[3\Psi_1 + \lambda_{\max}(Q)\right]$.*Once the problem is solved* $L = P^{-1}G$ *and* $\mu^* \triangleq \min(\mu) = \sqrt{\zeta}$.

Proof: Consider the same Lyapunov function candidate as before, thus,

$$\begin{aligned} \Delta V = {} & e_k^T (A_d - LC_d)^T P (A_d - LC_d) e_k \\ & + 2e_k^T (A_d - LC_d)^T P (F_k - \hat{F}_k) - e_k^T P e_k \\ & + (F_k - \hat{F}_k)^T P (F_k - \hat{F}_k) + 2w_k^T B_d^T P (A_d - LC_d) e_k \\ & + 2w_k^T B_d^T P (F_k - \hat{F}_k) e_k + w_k^T B_d^T P B_d w_k \end{aligned}$$

where the first four terms are the same as those found in Theorem 1, and the next three terms are due to the disturbance w. If $w_k = 0$, ΔV is given by (11) so the LMIs (71) and (72) guarantee the asymptotic stability. If $w \neq 0$, we have that

$$\begin{aligned} w_k^T B_d^T P (A_d - LC_d) e_k &\leq w_k^T B_d^T \bar{\sigma}(P) \bar{\sigma}(A_d - LC_d) e_k \\ w_k^T B_d^T P (F_k - \hat{F}_k) &\leq w_k^T B_d^T \bar{\sigma}(P) \gamma_d e_k \end{aligned}$$

from the above and using (16), (17) and (28), we have:

$$\begin{aligned} \Delta V \leq {} & e_k^T \left[-Q + \gamma_d \left(3\lambda_{\max}(P) + \lambda_{\max}(Q)\right)\right] e_k \\ & + w_k^T B_d^T \left[2\lambda_{\max}(P)(\gamma_d + 1) + \lambda_{\max}(Q)\right] e_k \\ & + w_k^T B_d^T P B_d w_k. \end{aligned} \tag{74}$$

Now, define

$$J \triangleq \sum_{k=0}^{\infty} \left[z(k)^T z(k) - \mu^2 w(k)^T w(k)\right]. \tag{75}$$

So, $J < \sum_{k=0}^{\infty} \left[z(k)^T z(k) - \mu^2 w(k)^T w(k) + \Delta V \right]$. Thus, a sufficient condition for $J \le 0$ is that

$$\forall\, k \in [0,\infty), \qquad z^T z - \mu^2 w^T w + \Delta V \le 0. \tag{76}$$

We have

$$\begin{aligned} z(k)^T z(k) - \mu^2 w(k)^T w(k) + \Delta V \le\ & e_k^T [H^T H - Q \cdots \\ & \cdots + \gamma_d \left(3\lambda_{\max}(P) + \lambda_{\max}(Q)\right)] e_k \\ & + w_k^T B_d^T \left[2\lambda_{\max}(P)(\gamma_d + 1) + \lambda_{\max}(Q)\right] e_k \\ & + w_k^T (B_d^T P B_d - \mu^2 I) w_k. \end{aligned} \tag{77}$$

So a sufficient condition for $J \le 0$ is that the right hand side of the above inequality be negative. Then

$$z^T z - \mu^2\ w^T w \le 0 \Rightarrow \|z\| \le \mu |w\| \tag{78}$$

substituting $\lambda_{\max}(P)$ from (30) into (77) and defining $\zeta = \mu^2$, the LMI (73) is obtained. $\triangle$

Remark 5. For the Euler approximation, according to Theorem 3, if $\lambda_{\min}(Q) = T$ then the proposed H_∞ observer will be semiglobal practical in T. In this case, if we choose $Q = TI$ as suggested in Remark 3, then it is clear that LMIs (72) and (73) can be simplified. Furthermore, having $\gamma_d < 1$, we can first maximize the admissible Lipschitz constant using Theorem 2, and then minimize μ for the maximized γ_d, using Theorem 4. In this case, according to Remark 2, robustness against nonlinear uncertainty is also guaranteed.

Now we show the usefulness of this method through a design example.

Example: Consider the following continuous-time nonlinear system and its Euler approximation

$$\begin{aligned} &\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} x_1^3 \\ -6x_1^5 - 6x_1^2 x_2 - 2x_1^4 - 2x_1^3 \end{bmatrix} \\ &y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x \\ &x^a(k+1) = (I + AT)x^a(k) + Tf(x^a(k)) \\ &y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x^a(k). \end{aligned}$$

It is well-known that the polynomial type nonlinearities are locally Lipschitz. $f(x)$ is Lipschitz in the following region

$$\mathcal{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0.3 \right\}$$

with Lipschitz constant $\gamma_c = 0.6109$. We assume $T = 0.1$ sec and design observer (50). Using Theorem 2, we have

$$\gamma_c^* = 0.67.$$

Now, using Theorem 4, with $H = 0.25I,\ \gamma_d = \gamma_d^* = T\gamma_c^*,\ B = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, we get

$$\mu^* = 0.1308,\ L = \begin{bmatrix} 1.0497 & 0.3588 \end{bmatrix}^T.$$

Figure 1, shows the state trajectories for the continuous-time system along with their estimations made by an observer which uses the Euler-approximate model. Simulation is done for 10 seconds (100 samples) in the presence of a Gaussian disturbance with zero mean and standard deviation 0.01. It can be seen in the figure that after 3 seconds, the true and estimated states are almost identical.

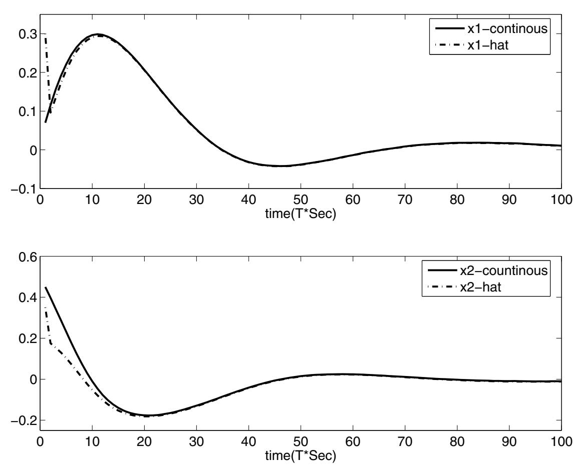

Figure 2: The true and estimated states

5.1 Higher Order Approximate Models

Under certain conditions, the Lipschitz contiguity is preserved under second order approximate discretization as studied in [31]. In most practical applications, first or second order discretization should be enough, specially since the sampling time can be selected small enough to ensure desired bounds on the approximation error. Furthermore, the expressions involving higher-order approximate models rapidly become very complicated. In particular, higher-order partial derivatives require tensor analysis of higher-orders. Under the ZOH assumption, similar to the approach given in [32], we have:

$$\begin{aligned}
x(k+1) &= x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!} \frac{d^l x}{dt^l}|_{t_k} \\
&= x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!} \frac{d^{l-1}}{dt^{l-1}}[Ax + f(x,u)]|_{t_k} \\
&= x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!}[A\frac{d^{l-1}x}{dt^{l-1}} + \frac{d^{l-1}}{dt^{l-1}} f(x,u)]|_{t_k}.
\end{aligned} \tag{79}$$

where

$$\begin{cases} \frac{d}{dt} f(x,u) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial u} \cdot \frac{du}{dt} \\ \frac{d^n}{dt^n} f(x,u) = \frac{d}{dt}[\frac{d^{n-1}}{dt^{n-1}} f(x,u)], \quad n \geq 2 \end{cases} \tag{80}$$

Under the ZOH assumption, $\frac{du}{dt} = 0$ in each sampling interval and thus:

$$\begin{aligned}
&x(k+1) = x(k) + \sum_{l=1}^{\infty} \frac{T^l}{l!}[A\frac{d^{l-1}x}{dt^{l-1}} + \frac{d^{l-1}}{dt^{l-1}} f(x,u)]|_{t_k} \\
&\frac{d}{dt} f(x,u) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt}, \qquad \frac{d^n f}{dt^n} = \frac{d}{dt}(\frac{d^{n-1} f}{dt^{n-1}}), \ n \geq 2.
\end{aligned} \tag{81}$$

The first order approximation, ($l = 1$) leads to the well-known Euler approximate model.

The robust nonlinear observer design results can be extended to higher-order approximate models as well, which is a topic for further research.

6 Conclusion

In this paper, a new algorithm for robust H_∞ nonlinear observer design for nonlinear discrete-time systems was proposed based on an LMI approach. The observer is robust in the sense that it can achieve convergence to the true state, despite nonlinear model uncertainty with guaranteed exogenous disturbance rejection ratio. In addition, when the exact discrete-time model of the system is not available, the same algorithm can still be used for the Euler approximated model. In the proposed algorithms, the admissible Lipschitz constant and the disturbance attenuation level can be maximized through LMI optimization. These features make the proposed algorithm an efficient design method.

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