ORDER AND HYPER-ORDER OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We have discussed the problem of finding the condition on coefficients of f'' + A(z)f' + B(z)f = 0, $B(z)(\not\equiv 0)$ so that all non-trivial solutions are of infinite order. The hyper-order of these non-trivial solutions of infinite order is also found when $\lambda(A) < \rho(A)$ and B(z) is a transcendental entire function satisfying some conditions.

Keywords: Entire function, meromorphic function, order of growth, infinite order, complex differential equation.

2010 Mathematics Subject Classification: 34M10, 30D35.

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1. Introduction

The study of growth of solutions of complex differential equation starts with Wittich's work in Wittich [26]. For the fundamental results of complex differential equations we have consulted Hille [14] and Laine [19]. The Nevanlinna's value distribution theory plays a crucial role in investigation of complex differential equations. For the notion of value distribution theory we have consulted the standard reference Yang [31].

For an entire function f(z) the order of growth is defined as:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^{+} \log^{+} M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^{+} T(r, f)}{\log r}$$

where $M(r, f) = \max\{ |f(z)| : |z| = r \}$ is the maximum modulus of the function f(z) over the circle |z| = r and T(r, f) is the Nevanlinna characteristic function of the function f(z).

In this paper we investigate the growth of solutions $f(\not\equiv 0)$ of the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0 (1)$$

where the coefficients A(z) and $B(z) (\not\equiv 0)$ are entire functions. It is known that all solutions of the equation (1) are entire functions Laine [19]. The necessary and sufficient condition that all solutions

The research work of the author is supported by research fellowship from University Grants Commission (UGC), New Delhi.

of the equation (1) are of finite order is that the coefficients A(z) and B(z) are polynomials Laine [19]. It is easy to conclude that if any of the coefficients is a transcendental entire function then almost all solutions are of infinite order. However, there is a necessary condition for equation (1) to have a solution of finite order:

Theorem 1. [11] Suppose that f(z) be a finite order solution of the equation (1) then $T(r, B) \leq T(r, A) + O(1)$.

This implies that if equation (1) possess a solution of finite order then $\rho(B) \leq \rho(A)$. Therefore, if $\rho(A) < \rho(B)$ then all non-trivial solutions f(z) of the equation (1) are of infinite order. It is well known that above condition is not sufficient, for example: $f'' + e^{-z}f' - n^2f = 0$ has all non-trivial solutions of infinite order.

Therefore, it is interesting to find conditions on A(z) and B(z) so that all solutions $f(\not\equiv 0)$ are of infinite order. Many results have been given in this context. Gundersen [11] and Hellerstein et. al. [13] proved

Theorem 2. Let $f(\not\equiv 0)$ be a solution of the equation (1) with the coefficients satisfying

- (1) $\rho(B) < \rho(A) \le \frac{1}{2}$ or (2) A(z) is a transcendental entire functions with $\rho(A) = 0$ and B(z) is a polynomial.

then $\rho(f) = \infty$.

Further, Frei [7], Ozawa [24], Amemiya and Ozawa [1], Gundersen [9] and Langley [20] proved that all non-trivial solutions are of infinite order for the differential equation

$$f'' + Ce^{-z}f' + B(z)f = 0$$

for any nonzero constant C and for any nonconstant polynomial B(z). J. Heittokangas, J. R. Long, L. Shi, X. Wu, P. C. Wu, X. B. Wu, and Zhang gave conditions on the coefficients A(z) and B(z) so that all solutions $f(\not\equiv 0)$ are of infinite order. Their results can be found in [[22], [23], [28], [29]]. In Kumar and Saini [16] we gave conditions on coefficients and proved the following theorem:

Theorem 3. Suppose A(z) be an entire function with $\lambda(A) < \rho(A)$ and

- (1) B(z) be a transcendental entire function with $\rho(B) \neq \rho(A)$ or
- (2) B(z) be an entire function having Fabry gap

then all non-trivail solutions of the equation (1) are of infinite order.

Definition 1. An entire function $f(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$ has Fabry gap if the sequence (λ_n) satisfies

$$\frac{\lambda_n}{n} \to \infty$$

as $n \to \infty$. An entire function f(z) with Fabry gap satisfies $\rho(f) > 0$ Hayman and Rossi [12].

The concept of hyper-order were used to further investigate the growth of infinite order solutions of complex differential equations. In this context, K. H. Kwon [18] proved that:

Theorem 4. Suppose $P(z) = a_n z^n + \dots a_0$ and $Q(z) = b_n z^n + \dots b_0$ be non-constant polynomials of degree n such that either $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ (0 < c < 1), $h_1(z)$ and $h_0(z)$ be entire functions satisfying $\rho(h_i) < n$, i = 0, 1. Then every non-trivial solutions f(z) of

$$f'' + h_1 e^{P(z)} f' + h_0 e^{Q(z)} f = 0, \quad Q(z) \neq 0$$
 (2)

are of infinite order with $\rho_2(f) \geq n$.

For an entire function f(z) the hyper-order is defined in the following manner:

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ \log^+ M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}$$

C. Zongxuan [34], investigated the differential equation (2) for some special cases and proved the following theorem:

Theorem 5. Let $b \neq -1$ be any complex constant, h(z) be a non-zero polynomial. Then every solution $f(\not\equiv 0)$ of the equation

$$f'' + e^{-z}f' + h(z)e^{bz}f = 0 (3)$$

has infinite order and $\rho_2(f) = 1$.

K. H. Kwon [17] found the lower bound for the hyper-order of all solutions $f(\not\equiv 0)$ in the following theorem:

Theorem 6. [17] Suppose that A(z) and B(z) be entire functions such that (i) $\rho(A) < \rho(B)$ or (ii) $\rho(B) < \rho(A) < \frac{1}{2}$ then

$$\rho_2(f) \ge \max\{ \rho(A), \rho(B) \}$$

for all solutions $f(\not\equiv 0)$ of the equation (1).

Since the growth of an entire function with infinite order can be measured by its hyper-order. Therefore motivated from above theorems we have calculated the hyper-order of the non-trivial solutions of the equation (1) in Theorem [3] in the following theorem:

Theorem 7. Let A(z) and B(z) be entire functions of finite order satisfying the hypothesis of the Theorem [3] then all non-trivial solutions f(z) of the equation (1) have $\rho_2(f) = \max\{ \rho(A), \rho(B) \}$.

In present work, our aim is to give conditions on B(z) so that when $\rho(A) = \rho(B)$ then also the conclusion of Theorem [3] and Theorem [7] holds true. In this regard, we have proved few results.

Theorem 8. Suppose that A(z) and B(z) be transcendental entire functions satisfying $\lambda(A) < \rho(A)$ and $\mu(B) \neq \rho(A)$ then all non-trivial solutions f(z) of the equations satisfies $\rho(f) = \infty$.

Corollary 1. The conclusion of the above theorem holds true if $\mu(A) \neq \mu(B)$.

For an entire function f(z) the lower order of growth is defined as follows:

$$\mu(f) = \liminf_{r \to \infty} \frac{\log^+ \log^+ M(r, f)}{\log r} = \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}$$

In Theorem [8] and Corollary [1], the order of the coefficients A(z) and B(z) may be equal. The lower order of an entire function may be quite different from its order, for example there exists entire function f with $\mu(f) = 0$ and $\rho(f) > 0$ or $\rho(f) = \infty$ for example see Goldberg and Ostroviskii [8] (page no. 238).

The theorem below presents the hyper-order of solutions of the differential equation satisfying the conditions of the Theorem [8].

Theorem 9. Suppose that A(z) be an entire function with finite order and B(z) be a transcendental entire function with finite lower order satisfying the hypothesis of Theorem [8] then

$$\rho_2(f) = \max\{ \rho(A), \mu(B) \}$$

for all non-constant solutions f(z) of the equation (1).

Corollary 2. Suppose that A(z) and B(z) be entire functions of finite lower order such that $\lambda(A) < \rho(A)$ and $\mu(A) \neq \mu(B)$ then

$$\rho_2(f) = \max\{ \mu(A), \mu(B) \}$$
.

for all non-constant solutions f(z) of the equation (1).

Theorem 10. Suppose that A(z) be an entire function with $\lambda(A) < \rho(A)$ and B(z) be an entire function extremal to Yang's inequality such that no Borel direction of B(z) coincides with any of the critical rays of A(z). Then all non-trivial solutions f(z) of the equation (1) satisfies $\rho(f) = \infty$.

Here is an illustrative example for above theorem:

Example 1. The differential equation

$$f'' + e^{\iota z}f' + e^zf = 0$$

has all non-trivial solutions of infinite order by Theorem [10].

Theorem 11. Suppose that A(z) be an entire function with finite order and B(z) be an entire function extremal to Yang's inequality such that hypothesis of the Theorem [10] satisfied then

$$\rho_2(f) = \max\{ \rho(A), \rho(B) \}$$

for all non-trivial solutions f(z) of the equation (1).

The following theorem is motivated from Theorem [1.6] of Wu et.al. [28] where coefficients of equation (1) satisfies $\rho(A) \neq \rho(B)$.

Theorem 12. Suppose that A(z) be an entire function with $\lambda(A) < \rho(A)$ and B(z) be an entire function extremal to Denjoy's conjecture then all non-trivial solutions f of the equation (1) satisfies

$$\rho(f) = \infty$$
.

Theorem 13. Suppose that A(z) and B(z) be entire function of finite order satisfying the hypothesis of the above theorem then all non-trivial solutions f of the equation (1) satisfies

$$\rho_2(f) = \max\{ \rho(A), \rho(B) \}.$$

Definitions of Borel directions, Denjoy's conjecture, functions extremal to Yang's inequality and extremal to Denjoy's conjecture are given in the next section.

2. Preliminary Results

To make this paper self contained we mention all results which we are going to use and some additional results we have proved.

For a set $F \subset [0, \infty)$, the Lebesgue linear measure of F is defined as $m(F) = \int_F dt$ and for a set $G \subset [1, \infty)$, the logarithmic measure of G is defined as $m_1(G) = \int_G \frac{1}{t} dt$. For set $G \subset [0, \infty)$, the upper and lower logarithmic densities are defined, respectively, as follows:

$$\overline{\log dens}(G) = \limsup_{r \to \infty} \frac{m_1(G \cap [1, r])}{\log r}$$

$$\underline{\log dens}(G) = \liminf_{r \to \infty} \frac{m_1(G \cap [1, r])}{\log r}.$$

Next lemma is due to Gundersen [10] which provide the estimates for transcendental meromorphic function.

Lemma 1. Let f(z) be a transcendental meromorphic function and let $\Gamma = \{ (k_1, j_1), (k_2, j_2), \dots, (k_m, j_m) \}$ denote finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, 2, \dots, m$. Let $\alpha > 1$ and $\epsilon > 0$ be given real constants. Then the following three statements holds:

(i) there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero and there exists a constant c > 0 that depends only on α and Γ such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R(\psi_0) > 0$ so that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le c \left(\frac{T(\alpha r, f)}{r} \log^{\alpha} r \log T(\alpha r, f) \right)^{(k-j)} \tag{4}$$

If f(z) is of finite order then f(z) satisfies:

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\rho(f)-1+\epsilon)} \tag{5}$$

for all z satisfying $\arg z = \psi_0 \notin E_1$ and $|z| \geq R_0$ and for all $(k,j) \in \Gamma$

- (ii) there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure and there exists a constant c > 0 that depends only on α and Γ such that for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, inequality (4) holds.
 - If f(z) is of finite order then f(z) satisfies inequality (5), for all z satisfying $|z| \notin E_2 \cup [0,1]$ and for all $(k,j) \in \Gamma$.
- (iii) there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure and there exists a constant c > 0 that depends only on α and Γ such that for all z satisfying $|z| = r \notin E_3$ and for all $(k, j) \in \Gamma$ we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le c \left(T(\alpha r, f) r^{\epsilon} \log T(\alpha r, f) \right)^{(k-j)} \tag{6}$$

If f(z) is of finite order then

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\rho(f)+\epsilon)} \tag{7}$$

for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in \Gamma$.

Wang [25] has proved the following result using Phragmén-Lindelöf theorem.

Lemma 2. Let A(z) be an entire funtion such that $\rho(A) \in (0, \infty)$ then there exists sector $\Omega(\alpha, \beta)$ where $\alpha < \beta$ and $\beta - \alpha \ge \frac{\pi}{\rho(A)}$ such that

$$\limsup_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A)$$

for all $\theta \in (\alpha, \beta)$.

For the statement of our next lemma we need to introduce the notion of $critical\ rays$:

Definition 2. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, $a_n \neq 0$ and $\delta(P,\theta) = \Re(a_n e^{in\theta})$. A ray $\gamma = r e^{i\theta}$ is called critical ray of $e^{P(z)}$ if $\delta(P,\theta) = 0$.

The rays $\arg z = \theta$ such that $\delta(P, \theta) = 0$ divides the complex plane into 2n sectors of equal length $\frac{\pi}{n}$. Also $\delta(P, \theta) > 0$ and $\delta(P, \theta) < 0$ in the alternative sectors. Suppose that $0 \le \phi_1 < \theta_1 < \phi_2 < \theta_2 < \ldots < \phi_n < \theta_n < \phi_{n+1} = \phi_1 + 2\pi$ be 2n critical rays of $e^{P(z)}$ satisfying

 $\delta(P,\theta) > 0$ for $\phi_i < \theta < \theta_i$ and $\delta(P,\theta) < 0$ for $\theta_i < \theta < \phi_{i+1}$ where i = 1, 2, 3, ..., n. Now we fix some notations:

$$E^{+} = \{ \theta \in [0, 2\pi] : \delta(P, \theta) \ge 0 \}$$

$$E^{-} = \{ \theta \in [0, 2\pi] : \delta(P, \theta) \le 0 \}.$$

Let α , β and $r_1 > 0$, $r_2 > 0$ be such that $\alpha < \beta$ and $r_1 < r_2$ then

$$\Omega(\alpha, \beta) = \{ z \in \mathbb{C} : \alpha < \arg z < \beta \}$$

$$\Omega(\alpha, \beta; r_1, r_2) = \{ z \in \mathbb{C} : \alpha < \arg z < \beta, r_1 < |z| < r_2 \}.$$

We state following lemma which is due to Bank et.al. [2] and is useful for estimating an entire function A(z) satisfying $\lambda(A) < \rho(A)$.

Lemma 3. Let $A(z) = v(z)e^{P(z)}$ be an entire function with $\lambda(A) < \rho(A)$, where P(z) is a non-constant polynomial of degree n and v(z) is an entire function. Then for every $\epsilon > 0$ there exists $E \subset [0, 2\pi)$ of linear measure zero such that

(i) for $\theta \in E^+ \setminus E$ there exists R > 1 such that

$$|A(re^{i\theta})| \ge \exp\left((1-\epsilon)\delta(P,\theta)r^n\right)$$
 (8)

for r > R.

(ii) for $\theta \in E^- \setminus E$ there exists R > 1 such that

$$|A(re^{i\theta})| \le \exp\left((1 - \epsilon)\delta(P, \theta)r^n\right) \tag{9}$$

for r > R.

Lemma 4. [17] Let f(z) be a non-constant entire function. Then there exists a real number R > 0 such that for all r > R we have

$$\left| \frac{f(z)}{f'(z)} \right| \le r \tag{10}$$

where |z|=r.

The lemma below give property of an entire function with Fabry gap and can be found in Long [21] and Wu and Zheng [29].

Lemma 5. Let $g(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$ be an entire function of finite order with Fabry gap, and h(z) be an entire function with $\rho(h) = \sigma \in (0,\infty)$. Then for any given $\epsilon \in (0,\sigma)$, there exists a set $H \subset (1,+\infty)$ satisfying $\log \operatorname{dens}(H) \geq \xi$, where $\xi \in (0,1)$ is a constant such that for all $|z| = r \in H$, one has

$$\log M(r,h) > r^{\sigma - \epsilon}, \quad \log m(r,g) > (1 - \xi) \log M(r,g),$$

where
$$M(r,h) = \max\{\ |h(z)|: |z| = r\}$$
 , $m(r,g) = \min\{\ |g(z)|: |z| = r\}$ and $M(r,g) = \max\{\ |g(z)|: |z| = r\}$.

The following remark follows from the above lemma immediately.

Remark 1. Suppose that $g(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$ be an entire function of order $\sigma \in (0, \infty)$ with Fabry gap then for any given $\epsilon > 0$, $(0 < 2\epsilon < \sigma)$, there exists a set $H \subset (1, +\infty)$ satisfying $\log \operatorname{dens}(H) \geq \xi$, where $\xi \in (0, 1)$ is a constant such that for all $|z| = r \in H$, one has

$$|g(z)| > M(r,g)^{(1-\xi)} > \exp\left((1-\xi)r^{\sigma-\epsilon}\right) > \exp\left(r^{\sigma-2\epsilon}\right).$$

Lemma 6. [5] Let f(z) be an entire function of infinite order then

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log v(r, f)}{\log r}$$

where v(r, f) is the central index of the function f(z).

C. Zongxuan [34] provides the upper bound for the hyper-order of solutions f(z) of the equation (1).

Theorem 14. Suppose that A(z) and B(z) are entire functions of finite order. Then

$$\rho_2(f) \le \max\{ \rho(A), \rho(B) \}$$

for all solutions f(z) of the equation (1).

The following result is from Wiman-Valiron theory and we use this result to prove our next lemma which is motivated from Theorem [14].

Theorem 15. [19] Let g be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z be such that |z| = r and

$$|g(z)| > M(r,g)v(r,g)^{-\frac{1}{4}+\delta}$$

holds. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure such that

$$g^{(m)}(z) = \left(\frac{v(r,g)}{z}\right)^m (1 + o(1))g(z)$$

holds for all $m \ge 0$ and for all $r \notin F$, where v(r, g) is the central index of the function g(z).

Lemma 7. Let us suppose that A(z) and B(z) be entire functions such that $\mu(A)$ and $\mu(B)$ are finite then

$$\rho_2(f) \le \max\{ \mu(A), \mu(B) \}$$

for all solutions f of the equation (1).

Proof. Suppose max $\{ \mu(A), \mu(B) \} = \rho$. Thus for $\epsilon > 0$ we have

$$|A(re^{i\theta})| \le \exp r^{\rho + \epsilon} \tag{11}$$

and

$$|B(re^{i\theta})| < \exp r^{\rho + \epsilon} \tag{12}$$

for sufficiently large r. From Theorem [15], we choose z satisfying |z|=r and |f(z)|=M(r,f) then there exists a set $F\subset\mathbb{R}_+$ having finite logarithmic measure such that

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{v(r,f)}{z}\right)^m (1 + o(1)) \tag{13}$$

for m = 1, 2 and for all $|z| = r \notin F$, where v(r, f) is the central index of the function f(z). Thus using equation (1), (11), (12) and (13) we get

$$\left(\frac{v(r,f)}{z}\right)^{2} \left| (1+o(1)) \right| \le \exp\left(r^{\rho+\epsilon}\right) \left(\frac{v(r,f)}{z}\right) \left| (1+o(1)) \right| + \exp\left(r^{\rho+\epsilon}\right)$$
(14)

for all $|z| = r \notin F$, from here we get

$$\limsup_{r \to \infty} \frac{\log \log v(r, f)}{\log r} \le \rho + \epsilon. \tag{15}$$

Since $\epsilon > 0$ chosen is arbitrary we get $\rho_2(f) \leq \rho$.

Theorem [6] motivated us to prove following result:

Lemma 8. Suppose that A(z) and B(z) be entire function such that $\mu(A) < \mu(B)$ then

$$\rho_2(f) \ge \mu(B)$$

for all non-trivial solutions f of the equation (1).

Proof. Let $\mu < \alpha < \beta < \mu(B)$ where α and β are two real numbers.Let f be a non-trivial solutions of the equation (1). For given $\epsilon > 0$, from part (ii) of Lemma [1], there exists $E_2 \subset (1, \infty)$ with finite logarithmic measure and a constant c > 0 such that

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le c(T(2r, f))^{2k} \quad k = 1, 2$$
 (16)

for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$. From equations (1) and (16) we get

$$\exp r^{\beta} \le |B(re^{i\theta})| \le \left| \frac{f''(re^{i\theta})}{f(i\theta)} \right| + |A(re^{i\theta})| \left| \frac{f'(re^{i\theta})}{f(i\theta)} \right|$$
$$\le cT(2r, f)^4 (1 + \exp r^{\alpha})$$

for all $r \notin E_2 \cup [0, 1]$. Since $\alpha < \beta$ this implies

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge \beta$$

as $\beta \leq \mu(B)$ is arbitrary this implies

$$\rho_2(f) \ge \mu(B).$$

Lemma 9. [30] Suppose B(z) be an entire function with $\mu(B) \in [0, 1)$. Then for every $\alpha \in (\mu(B), 1)$, there exists a set $E_4 \subset [0, \infty)$ such that

$$\overline{\log dens}(E_4) \ge 1 - \frac{\mu(B)}{\alpha} \text{ and } m(r) > M(r) \cos \pi \alpha$$

for all $r \in E_4$, where $m(r) = \inf_{|z|=r} \log |B(z)|$ and $M(r) = \sup_{|z|=r} \log |B(z)|$.

The above lemma is also true for an entire function B(z) with $\rho(B) < \frac{1}{2}$. We can get next lemma easily using Lemma [9].

Lemma 10. [30] If B(z) be an entire function with $\mu(B) \in (0, \frac{1}{2})$. Then for any $\epsilon > 0$ there exists $(r_n) \to \infty$ such that

$$|B(r_n e^{i\theta})| > \exp r_n^{\mu(B) - \epsilon}$$

for all $\theta \in [0, 2\pi)$.

Lemma 11. [28] Let B(z) be an entire function with $\mu(B) \in [\frac{1}{2}, \infty)$. Then there exists a sector $\Omega(\alpha, \beta)$, $\beta - \alpha \geq \frac{\pi}{\mu(B)}$, such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{\iota\theta}|)}{\log r} \ge \mu(B)$$

for all $\theta \in \Omega(\alpha, \beta)$, where $0 \le \alpha < \beta \le 2\pi$.

Next we give definition of Borel direction and illustrate it with an example:

Definition 3. [31] For a meromorphic function f(z) of order $\rho(f) \in (0, \infty)$ in the finite plane, the ray $\arg z = \theta_0$ is called Borel direction of f of order $\rho(f)$ if for any $\epsilon > 0$, the equality

$$\limsup_{r \to \infty} \frac{\log n(\Omega(\theta_0 - \epsilon, \theta_0 + \epsilon, r), f = a)}{\log r} = \rho(f)$$

holds for every complex number a, with atmost two possible exceptions, where $n(\Omega(\theta_0 - \epsilon, \theta_0 + \epsilon, r), f = a)$ denotes the number of zeros, counting with the multiplicities, of the function f(z) - a in the region $\Omega(\theta_0 - \epsilon, \theta_0 + \epsilon, r)$

Example 2. The entire function $f(z) = e^z$ has two Borel directions namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Next result gives relation between the number of the deficient values and number of Borel directions.

Theorem 16. Let f(z) be an entire function of order $\rho(f) \in (0, \infty)$. If p is the number of its finite deficient values and q is the number of its Borel directions, then $p \leq \frac{q}{2}$.

If equality $p = \frac{q}{2}$ holds in Theorem [16] then function f(z) is called extremal to Yang's inequality. For example:

Example 3. Consider the entire function $f(z) = \int_0^z e^{-t^n} dt$ of order n has n number of finite deficient values equal to

$$a_k = e^{\frac{i2\pi k}{n}} \int_0^\infty e^{-t^n} dt, k = 0, 1, 2, \dots, n - 1$$

and 2n Borel directions equal to

$$\Phi_i = \frac{(2i-1)\pi}{2n}, i = 0, 1, 2, \dots, 2n-1.$$

Since $p = \frac{q}{2}$ therefore this function is extremal to Yang's inequality.

Next suppose that B(z) be an entire function extremal to Yang's inequality and let $\arg z = \Phi_i, \ i = 1, 2, 3, \dots, q$ denote the Borel directions of the function B(z) such that $0 \le \Phi_1 < \Phi_2 < \dots < \Phi_q < \Phi_{q+1} = \Phi_1 + 2\pi$. The following lemma is due to [27]:

Lemma 12. Suppose that B(z) be an entire function extremal to Yang's inequality and b_i , $i = 1, 2, 3, ..., \frac{q}{2}$ be the deficient values of B(z). Then for each b_i , $i = 1, 2, 3, ..., \frac{q}{2}$ there exists a corresponding sector $\Omega(\Phi_i, \Phi_{i+1})$ such that for every $\epsilon > 0$

$$\log \frac{1}{|B(z) - b_i|} > C(\Phi_i, \Phi_{i+1}, \epsilon, \delta(b_i, B))T(|z|, B)$$
(17)

holds for all $z \in \Omega(\Phi_i + \epsilon, \Phi_{i+1} - \epsilon, r, \infty)$, $C(\Phi_i, \Phi_{i+1}, \epsilon, \delta(b_i, B))$ is a positive constant depending on $\Phi_i, \Phi_{i+1}, \epsilon$ and $\delta(b_i, B)$.

Lemma 13. [23] Suppose that B(z) be an entire function extremal to Yang's inequality and there exists $\arg z = \theta$ with $\Phi_i < \theta < \Phi_{i+1}$, $1 \le j \le q$ such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} = \rho(B). \tag{18}$$

Then $\Phi_{i+1} - \Phi_i = \frac{\pi}{\rho(B)}$.

Lemma 14. [32] Suppose that f(z) be an entire function with $0 < \rho(f) < \infty$ and $\Omega(\psi_1, \psi_2)$ be a sector with $\psi_2 - \psi_1 < \frac{\pi}{\rho(f)}$. If there exists a Borel direction $\arg z = \Phi$ in $\Omega(\psi_1, \psi_2)$ then there exists at least one of the rays $\arg z = \psi_i$, i = 1 or 2 such that

$$\limsup_{r \to \infty} \frac{\log \log |f(re^{\iota\psi_i})|}{\log r} = \rho(f)$$
 (19)

Here we give a conjecture due to Denjoy [6] which gives a relation between the order of an entire function and its finte asymptotic values:

Denjoy's Conjecture: Suppose g(z) be an entire function of finite order and g has p distinct finite asymptotic values then $p \leq 2\rho(g)$.

An entire function is said to be extremal to Denjoy's conjecture if equality holds in above inequality. For example:

Example 4 ([33], page no. 210). *Let*

$$g(z) = \int_0^z \frac{\sin t^p}{t^p} dt$$

where $p \in \mathbb{N}$. Then $\rho(g) = p$ and g(z) has 2p distinct finite asymptotic values, namely

$$a_j = e^{\frac{j\pi\iota}{p}} \int_0^\infty \frac{sinr^p}{r^p} dr$$

for $j = 1, 2, \dots 2p$.

Following lemma gives the property of an entire function extremal to Denjoy's conjecture.

Lemma 15. [33] Let g(z) be an entire function extremal to Denjoy's conjecture then for any $\theta \in (0, 2\pi)$ either $\arg z = \theta$ is a Borel direction of g(z) or there exists a constant $\sigma \in (0, \frac{\pi}{4})$ such that

$$\lim_{|z| \to \infty_{z \in (\Omega(\theta - \sigma, \theta + \sigma) \setminus E_5)}} \frac{\log \log |g(z)|}{\log |z|} = \rho(g)$$
 (20)

where $E_5 \subset \Omega(\theta - \sigma, \theta + \sigma)$ such that

$$\lim_{r \to \infty} m(\Omega(\theta - \sigma, \theta + \sigma; r, \infty) \cap E_5) = 0$$

3. Proof of Theorem [7]

Proof. (1) We know that all solutions $f(\not\equiv 0)$ of the equation (1) are of infinite order, when $\rho(B) \neq \rho(A)$ by Theorem [3]. Then from part (iii) of Lemma [1] for $\epsilon > 0$, there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure such that for all z satisfying $|z| = r \notin E_3$ we have

$$\left| \frac{f''(z)}{f'(z)} \right| \le cr \left[T(2r, f) \right]^2 \tag{21}$$

where c > 0 is a constant.

If $\rho(A) < \rho(B)$ then from Theorem [6] and Theorem [14] we get that $\rho_2(f) = \max\{ \rho(A), \rho(B) \}$.

If $\rho(B) < \rho(A) = n, n \in \mathbb{N}$ then we can choose β such that $\rho(B) < \beta < \rho(A)$. Now choose $\theta \in E^+ \setminus E$ and $(r_m) \not\subset E_3$ such that equations (8), (10) and (21) are satisfied for $z_m = r_m e^{(\iota \theta)}$. Using equation (1), (8), (10) and (21) for $z_m = r_m e^{(\iota \theta)}$ we have

$$\exp\left\{ (1 - \epsilon)\delta(P, \theta)r_m^n \right\} \leq |A(r_m e^{\iota \theta})|$$

$$\leq \left| \frac{f''(r_m e^{\iota \theta})}{f'(r_m e^{\iota \theta})} \right| + |B(r_m e^{\iota \theta})| \left| \frac{f(r_m e^{\iota \theta})}{f'(r_m e^{\iota \theta})} \right|$$

$$\leq cr_m \left[T(2r_m, f) \right]^2 + \exp\left(r_m^{\beta}\right) r_m$$

since $\beta < n$ this implies that

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge \rho(A) \tag{22}$$

then from Theorem [14] and equation (22) we have

$$\rho_2(f) = \max\{ \rho(A), \rho(B) \}$$

(2) It has been proved that all non-trivial solutions f(z) of the equation (1), with A(z) and B(z) satisfying the hypothesis of the theorem, are of infinite order. Also if $\rho(A) \neq \rho(B)$ then from above part [1],

$$\rho_2(f) = \{ \rho(A), \rho(B) \}$$

Now let $\rho(A) = \rho(B) = n, n \in \mathbb{N}$. Using Lemma [1], for $\epsilon > 0$, there exists $E_3 \subset [0, \infty)$ with finite linear measure such that for all z satisfying $|z| = r \notin E_3$ we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le crT(2r, f)^{2k} \tag{23}$$

where c > 0 is a constant and $k \in \mathbb{N}$. Also from Lemma [5], for $\epsilon > 0$, there exist $H \subset (1, \infty)$ satisfying $\overline{\log dens}(H) \geq 0$ such that for all $|z| = r \in H$ we have

$$|B(z)| > \exp\left(r^{n-\epsilon}\right) \tag{24}$$

Next choose $\theta \in E^- \setminus E$, $\delta(P, \theta) < 0$ and $r_m \in H \setminus E_3$, from equations (1), (9), (23) and (24) we have

$$\exp\left(r_m^{n-\epsilon}\right) < |B(r_m e^{\iota \theta})| \le \left| \frac{f''(r_m e^{\iota \theta})}{f(r_m e^{\iota \theta})} \right| + |A(r_m e^{\iota \theta})| \left| \frac{f'(r_m e^{\iota \theta})}{f(r_m e^{\iota \theta})} \right|$$

$$\le cr_m T(2r_m, f)^4$$

$$+ \exp\left\{ (1 - \epsilon)\delta(P, \theta)r^n \right\} cr_m T(2r_m, f)^2$$

$$\le cr_m T(2r_m, f)^4 (1 + o(1)).$$

Thus we conclude that

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge n. \tag{25}$$

Using Theorem [14] and equation (25) we get

$$\rho_2(f) = \{ \rho(A), \rho(B) \}$$
.

4. Proof of Theorem [8]

Proof. If $\rho(A) = \infty$ then result follows from equation (1). Assume that $\rho(A) < \infty$.

If $\rho(A) < \mu(B)$ then result follows from Theorem [1]. Let us suppose that $\mu(B) < \rho(A)$ and f(z) be a non-trivial solution of the equation (1) with finite order. Then using part (i) of Lemma [1], for each $\epsilon > 0$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ and

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le |z|^{2\rho(f)}, \quad k = 1, 2$$
 (26)

for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$. Since $\lambda(A) < \rho(A)$ therefore $A(z) = v(z)e^{P(z)}$, where P(z) is a non-constant polynomial of degree n and v(z) is an entire function such that $\rho(v) = \lambda(A) < \rho(A)$. Then using Lemma [3], there exists $E \subset [0, 2\pi)$ with linear measure zero such that for $\theta \in E^- \setminus (E \cup E_1)$ there exists $R_1 > 1$ such that

$$|A(re^{i\theta})| \le \exp\left((1-\epsilon)\delta(P,\theta)r^n\right)$$
 (27)

for $r > R_1$.

We have following three cases on lower order of B(z):

Case 1. when $0 < \mu(B) < \frac{1}{2}$ then from Lemma [10], there exists $(r_n) \to \infty$ such that

$$|B(re^{i\theta})| > \exp\left(r^{\mu(B)-\epsilon}\right)$$
 (28)

for all $\theta \in [0, 2\pi)$ and $r > R_3$, $r \in (r_n)$.

Using equation (1), (26), (27) and (28) we have

$$\exp\left(r^{\mu(B)-\epsilon}\right) < |B(z)| \le \frac{|f''(z)|}{|f(z)|} + |A(z)| \frac{|f'(z)|}{|f(z)|}$$
$$\le r^{2\rho(f)} \{ 1 + \exp\left((1 - \epsilon)\delta(P, \theta)r^n\right) \}$$
$$= r^{2\rho(f)} \{ 1 + o(1) \}$$

for all $\theta \in E^- \setminus (E \cup E_1)$ and $r > R, r \in (r_n)$. This will conduct a contradication for sufficiently large r.

Thus all non-trivial solutions are of infinite order in this case.

Case 2. Now if $\mu(B) \geq \frac{1}{2}$ then by Lemma [11] we have that there exists a sector $\Omega(\alpha, \beta)$, $0 \leq \alpha < \beta \leq 2\pi$, $\beta - \alpha \geq \frac{\pi}{\mu(B)}$ such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} \ge \mu(B)$$
 (29)

for all $\theta \in \Omega(\alpha, \beta)$.

Since $\mu(B) < \rho(A)$ therefore there exists $\Omega(\alpha', \beta') \subset \Omega(\alpha, \beta)$ such that

for all $\phi \in \Omega(\alpha', \beta')$ we have

$$|A(re^{\iota\phi}| \le \exp\left((1-\epsilon)\delta(P,\theta)r^n\right) \tag{30}$$

for all r > R. From equation (29) we get

$$\exp\left(r^{\mu(B)-\epsilon}\right) \le |B(re^{\iota\phi})|\tag{31}$$

for $\phi \in \Omega(\alpha', \beta')$ and r > R. As done in above case, using equation (1), (26), (30) and (31) we get contradiction for sufficiently large r.

Case 3. If $\mu(B) = 0$ then from Lemma [9] for $\alpha \in (0,1)$, there exists a set $E_4 \subset [0,\infty)$ with $\overline{\log dens}(E_4) = 1$ such that

$$m(r) > M(r) \cos \pi \alpha$$

where $m(r) = \inf_{|z|=r} \log |B(z)|$ and $M(r) = \sup_{|z|=r} \log |B(z)|$. Then

$$\log|B(re^{i\theta})| > \log M(r,B) \frac{1}{\sqrt{2}} \tag{32}$$

for all $\theta \in [0, 2\pi)$ and $r \in E_4$. Now using equation (1), (26), (30) and (32) we get

$$M(r,B)^{\frac{1}{\sqrt{2}}} < |B(re^{i\theta})| \le r^{2\rho(f)} \{ 1 + \exp(1-\epsilon)\delta(P,\theta)r^n \}$$

for $\theta \notin E \cup E_1$, $\delta(P, \theta) < 0$ and r > R, $r \in E_4$. This implies that

$$\liminf_{r \to \infty} \frac{\log M(r, B)}{\log r} < \infty$$

which is not so as B(z) is an transcendental entire function. Thus non-trivial solution f with finite order of the equation (1) can not exist in this case also. Therefore all non-trivial solutions of the equation (1) are of infinite order.

5. Proof of Theorem [9]

Proof. We know that under the hypothesis of the theorem, all non-trivial solutions f(z) of the equation (1) are of infinite order. It follows from Lemma [1] that for $\epsilon > 0$, there exists a set $E_3 \subset [0, \infty)$ with finte linear measure such that

$$\left| \frac{f''(z)}{f'(z)} \right| \le cr[T(2r, f)]^2 \tag{33}$$

for all z satisfying $|z| = r \notin E_3$ when c > 0 is a constant.

If $\rho(A) < \mu(B)$ then from Theorem [6] and Lemma [7] we get that $\rho_2(f) = \max\{ \rho(A), \mu(B) \}$, for all non-trivial solutions f(z) of the equation (1).

If $\mu(B) < \rho(A)$. It is easy to choose η such that $\mu(B) < \eta < \rho(A)$. From Lemma [3], we have

$$\exp\left\{ (1 - \epsilon)\delta(P, \theta)r^n \right\} \le |A(re^{i\theta})| \tag{34}$$

for all $\theta \notin E$, $\delta(P, \theta) > 0$ and for sufficiently large r. Also

$$|B(re^{i\theta})| \le \exp r^{\eta} \tag{35}$$

for sufficiently large r and for all $\theta \in [0, 2\pi)$.

Thus from equations (1), (10), (33), (34) and (35) we have

$$\exp \left\{ (1 - \epsilon)\delta(P, \theta)r^n \right\} \leq |A(re^{i\theta})| \\
\leq \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| + |B(re^{i\theta})| \left| \frac{f(re^{i\theta})}{f'(re^{i\theta})} \right| \\
\leq cr[T(2r, f)]^2 + \exp(r^{\eta})r \\
\leq dr \exp(r^{\eta})[T(2r, f)]^2$$

for all $\theta \notin E$, $\delta(P, \theta) > 0$ and for sufficiently large r. Since $\eta < \rho(A) = n$ we have

$$\exp\{ (1 - \epsilon)\delta(P, \theta) \} \exp\{ (1 - o(1))r^n \} \le dr[T(2r, f)]^2$$
 (36)

for sufficiently large r. Thus

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge n.$$

Now using $\rho_2(f) \ge \max\{ \rho(A), \mu(B) \}$ and Lemma [7] we get the desired result.

6. Proof of Theorem [10]

Proof. If we consider the coefficients A(z) and B(z) such that $\rho(A) \neq \rho(B)$ then result follows from Theorem [3]. Therefore, it is sufficient to consider $\rho(A) = \rho(B) = n$ for some $n \in \mathbb{N}$. Suppose there exists a non-trivial solution f(z) of the equation (1) of finite order. Then using part (ii) of Lemma [1], for each $\epsilon > 0$, there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, such that

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \le |z|^{2\rho(f)}, \quad k = 1, 2$$
 (37)

for all z satisfying $|z| \notin E_2 \cup [0, 1]$.

Since B(z) is extremal to Yang's inequality therefore there exists sectors $\Omega_i(\Phi_i, \Phi_{i+1})$, $i = 1, 2, 3, \ldots, q$ such that in alternative sectors either equation (17) or equation (18) holds for the function B(z). Let $\Omega_1(\Phi_1, \Phi_2)$, $\Omega_3(\Phi_3, \Phi_4)$, ..., $\Omega_{2q-1}(\Phi_{2q-1}, \Phi_{2q})$ being the sectors such that

$$\log \frac{1}{|B(z) - b_i|} > CT(|z|, B) \tag{38}$$

holds for all $z \in \Omega_i(\Phi_i + \epsilon, \Phi_{i+1} - \epsilon, r, \infty)$, $C = C(\Phi_i, \Phi_{i+1}, \epsilon, \delta(b_i, B))$, where $\delta(b_i, B)$ is used for deficiency function of B(z), is a positive constant depending on $\Phi_i, \Phi_{i+1}, \epsilon$ and $\delta(b_i, B)$, where $i = 1, 3, \ldots, 2q - 1$.

Also, let $\Omega_2(\Phi_2, \Phi_3)$, $\Omega_4(\Phi_4, \Phi_6)$, ..., $\Omega_{2q}(\Phi_{2q}, \Phi_{2q+1})$ are the sectors for which there exists $re^{i\theta_{2i}} \in \Omega_{2i}(\Phi_{2i}, \Phi_{2i+1})$ such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta_{2i}})|}{\log r} = n \tag{39}$$

holds and $\Phi_{2i+1} - \Phi_{2i} = \frac{\pi}{n}$ where $i = 1, 2, \ldots, q$.

Now we have the following cases to be discussed:

Case 1. let us suppose that there is a Borel direction Φ of B(z) such that $\theta_i < \Phi < \phi_{i+1}$ for any $i=1,2,3,\ldots,n$ then we can easily choose ψ_1 and ψ_2 such that $\theta_i < \psi_1 < \Phi < \psi_2 < \phi_{i+1}$. It is evident from Lemma [19] that without loss of generality we can choose ψ_2 and we get

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\psi_2})|}{\log r} = n. \tag{40}$$

Thus, for $r \notin E_2 \cup [0,1]$ from equation (1), (9), (37) and (40) we have

$$\exp\left\{ r^{n-\epsilon} \right\} \leq |B(re^{\iota\psi_2})| \leq \left| \frac{f''(re^{\iota\psi_2})}{f(re^{\iota\psi_2})} \right| + |A(re^{\iota\psi_2})| \left| \frac{f'(re^{\iota\psi_2})}{f(re^{\iota\psi_2})} \right| \\
\leq r^{2\rho(f)} \left\{ 1 + \exp\left\{ 1 - \epsilon \right\} \delta(P, \psi_2) r^n \right\} \right\}$$

which is a contradiction for sufficiently large r.

Case 2. Now suppose that there is no Borel direction of B(z) contained in (θ_i, ϕ_{i+1}) for any i = 1, 2, ..., n. In this case (θ_i, ϕ_{i+1}) will be contained inside $\Omega_{2j-1}(\Phi_{2j-1}, \Phi_{2j})$, for any j = 1, 2, 3, ..., q.

Therefore for $r \notin E_2 \cup [0,1]$ and $\theta \in E^+ \cap \Omega_{2j-1}(\Phi_{2j-1}, \Phi_{2j})$, from equations (1), (8), (10), (37) and (38) we get

$$\exp \{ (1 - \epsilon)\delta(P, \theta)r^{n} \} \leq |A(re^{\iota\theta})|
\leq \left| \frac{f''(re^{\iota\theta})}{f'(re^{\iota\theta})} \right| + |B(re^{\iota\theta})| \left| \frac{f(re^{\iota\theta})}{f'(re^{\iota\theta})} \right|
\leq r^{2\rho(f)} + r \{ \exp \{ -CT(r, B) \} + |b_{2j-1}| \}
\leq r^{2\rho(f)} (1 + |b_{2j-1}| + o(1))$$

which provides a contradition for sufficiently large r.

Thus all non-trivial solutions f of the equation (1) are of infinite order.

7. Proof of the Theorem [11]

Proof. Since all non-trivial solution of the equation (1) under hypothesis are of infinite order. Therefore it follows from part (iii) of Lemma

[1] that for $\epsilon > 0$, there exists a set $E_3 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_3$ we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le cr \left[T(2r, f) \right]^{2(k-j)} \tag{41}$$

where c > 0 is a constant and $k \in \mathbb{N}$.

If $\rho(A) \neq \rho(B)$ then from part (??) of Theorem [7] we have

$$\rho_2(f) = \max\{ \rho(A), \rho(B) \}.$$

We consider $\rho(A) = \rho(B) = n$ where $n \in \mathbb{N}$. Now we have two cases to deal with

Case 1. let us suppose that there is a Borel direction Φ of B(z) such that $\theta_i < \Phi < \phi_{i+1}$ for any i = 1, 2, 3, ..., n. Thus, for $r \notin E_3$ from equation (1), (9), (40) and (41)we have

$$\exp \left\{ r^{n-\epsilon} \right\} \leq |B(re^{\iota\psi_2})| \leq \left| \frac{f''(re^{\iota\psi_2})}{f(re^{\iota\psi_2})} \right| + |A(re^{\iota\psi_2})| \left| \frac{f'(re^{\iota\psi_2})}{f(re^{\iota\psi_2})} \right| \\
\leq cr \left[T(2r,f) \right]^4 + \exp \left\{ (1-\epsilon)\delta(P,\psi_2)r^n \right\} cr \left[T(2r,f) \right]^2 \\
\leq cr \left[T(2r,f) \right]^4 (1+o(1))$$

for sufficiently large r. Thus

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge n \tag{42}$$

Then it can be seen easily from equation (42) and Theorem [14] that

$$\rho_2(f) = n$$

for all non-trivial solutions f of the equation (1).

Case 2. Now suppose that there is no Borel direction of B(z) contained in (θ_i, ϕ_{i+1}) for any i = 1, 2, ..., n. Therefore for $r \notin E_3$ and $\theta \in E^+ \cap \Omega_{2j-1}(\Phi_{2j-1}, \Phi_{2j})$, from equations (1), (8), (10), (38) and (41) we get

$$\exp \left\{ (1 - \epsilon) \delta(P, \theta) r^{n} \right\} \leq |A(re^{i\theta})|
\leq \left| \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right| + |B(re^{i\theta})| \left| \frac{f(re^{i\theta})}{f'(re^{i\theta})} \right|
\leq cr \left[T(2r, f) \right]^{2} + r \left\{ \exp \left\{ -CT(r, B) \right\}
+ |b_{2j-1}| \right\}
\leq dr \left[T(2r, f) \right]^{2} (1 + |b_{2j-1}| + o(1))$$

for sufficiently large r and for d > 0 is a constant. Thus

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge n \tag{43}$$

It follows from equation (43) and Theorem [14] that

$$\rho_2(f) = n$$

for all non-trivial solution f of the equation (1).

8. Proof of the Theorem [12]

Proof. When $\rho(A) \neq \rho(B)$ then the result holds true from Theorem [prethm]. Assume that $\rho(A) = \rho(B) = n, n \in \mathbb{N}$ and there exists a non-trivial solution f of the equation (1) of finite order. Then by part (i) of Lemma [1], for given $\epsilon > 0$ equation (26)holds true for z satisfying |z| > R and $\arg z \in E_1$. We will discuss following cases:

Case 1. suppose that the ray $\arg z = \Phi$ is a Borel direction of B(z) where $\theta_i < \Phi < \phi_{i+1}$ for some i = 1, 2, ..., n then the conclusion holds in similar manner as in Case 1. of Theorem [10].

Case 2. Suppose that $\arg z = \theta$ is not a Borel direction of B(z) for any $\theta \in (\theta_i, \phi_{i+1})$ for all i = 1, 2, ..., n then choose $\arg z = \theta \in (\theta_i, \phi_{i+1})$ for some i = 1, 2, ..., n. Then by Lemma [15] there exists $\sigma \in (0, \frac{\pi}{4})$ such that

$$\lim_{|z| \to \infty, z \in (\Omega(\theta - \sigma, \theta + \sigma) \setminus E_5)} \frac{\log \log |B(z)|}{\log |z|} = \rho(B)$$

Thus

$$\exp\left\{ r^{\rho(B)-\epsilon} \right\} \le |B(z)| \tag{44}$$

for all z satisfying $|z| = r \to \infty$ and $z \in (\Omega(\theta - \sigma, \theta + \sigma) \setminus E_5) \cap (\theta_i, \phi_{i+1}) \setminus S$, where $S = \{ z \in \mathbb{C} : \arg(z) \in E_1 \}$. Now from equations (1), (9), (26) and (44) we get a contradiction for sufficiently large r.

Thus all non-trivial solutions of the equation (1) are of infinite order.

9. Proof of the Theorem [13]

Proof. We need to consider that $\rho(A) = \rho(B) = n, n \in \mathbb{N}$. We again discuss two cases:

Case 1. suppose that the ray $\arg z = \Phi$ is a Borel direction of B(z) where $\theta_i < \Phi < \phi_{i+1}$ for some i = 1, 2, ..., n then the conclusion holds in similar manner as in Case 1. of Theorem [10].

Case 2. Suppose that $\arg z = \theta$ is not a Borel direction of B(z) for any $\theta \in (\theta_i, \phi_{i+1})$ for all i = 1, 2, ..., n then choose $\arg z = \theta \in (\theta_i, \phi_{i+1})$ for some i = 1, 2, ..., n. Then from equations (1), (4), (9) and (44) we have

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge n \tag{45}$$

From Theorem [wuthm] and equation (45) we get the desired result. \Box

10. Extention to Higher Order

This section involves linear differential equation of the form:

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + A_{k-2}f^{(k-2)} + \dots + A_0f = 0$$
 (46)

where $A_{k-1}, A_{k-2}, \ldots, A_0$ are entire functions, therefore all solutions of the equation (46) are entire function [19]. Also, all solutions of the equation (46) are of finite order if and only if all the coefficients $A_{k-1}, A_{k-2}, \ldots, A_0$ are polynomials. Thus if any of the coefficients $A_{k-1}, A_{k-2}, \ldots, A_0$ is a transcendental entire functions then there will exists a non-trivial solution of infinite order. In this section we will discuss the conditions on coefficients $A_{k-1}, A_{k-2}, \ldots, A_0$ so that all non-trivial solutions of the equation (46) are of infinite order.

For this purpose we will extend our previous results in the following manner:

Theorem 17. Suppose that there exists $A_i(z)$ such that $\lambda(A_i) < \rho(A_i)$ and $A_0(z)$ be a transcendental entire function satisfying $\mu(A_0) \neq \rho(A_i)$ and $\rho(A_j) < \mu(A_0)$ for all j = 1, 2, ..., k-1 and $j \neq i$. Then all nontrivial solutions of the equation (46) are of infinite order. Moreover, for these solutions f we have

$$\rho_2(f) \ge \mu(A_0).$$

Corollary 3. The conclusion of the above theorem also holds if $\mu(A_0) \neq \mu(A_i)$ and $\rho(A_i) < \mu(A_0)$ for all j = 1, 2, ..., k-1 and $j \neq i$.

Corollary 4. The conclusion of the theorem also holds if $\mu(A_0) \neq \mu(A_i)$ and $\mu(A_j) < \mu(A_0)$ for all j = 1, 2, ..., k-1 and $j \neq i$.

Theorem 18. Suppose that $A_1(z)$ be an entire function with $\lambda(A_1) < \rho(A_1)$ and $A_0(z)$ be an entire function extremal to Yangs inequality such that no Borel direction of $A_0(z)$ coincides with any of the critical rays of $A_1(z)$ and $\rho(A_j) < \rho(A_0)$, where j = 2, ..., k-1. Then all non-trivial solutions of the equation (46) satisfies

$$\rho(f) = \infty \quad and \quad \rho_2(f) \ge \rho(A_0).$$

Corollary 5. The conclusion of the above theorem also holds true if $\mu(A_j) < \rho(A_0)$.

Theorem 19. Suppose there exist $A_i(z)$ such that $\lambda(A_i) < \rho(A_i)$ and $A_0(z)$ be an entire function extremal to Denjoy's conjecture and $\rho(A_j) < \rho(A_0), j = 1, 2, ..., k-1, j \neq i$. Then all non-trivial solutions f of the equation (46) satisfies

$$\rho(f) = \infty \quad and \quad \rho_2(f) \ge \rho(A_0).$$

Corollary 6. The conclusion of the above theorem holds true also if $\mu(A_j) < \rho(A_0), j = 1, 2, ..., k - 1, j \neq i$.

11. Proof of the Theorem [17]

Proof. Assume that there exists a non-trivial solution f of the equation (46) with finite order then by part (iii) of Lemma [1], for given $\epsilon > 0$ there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure such that

$$\left| \frac{f^{(m)}(z)}{f(z)} \right| \le |z|^{k(\rho(f)+\epsilon)}, \quad m = 1, 2, 3, \dots, k$$
 (47)

for all z satisfying $|z| \notin E_3$.

We suppose the case when $\rho(A_i) < \mu(A_0)$ then from equations (46) and (47) we get

$$|A_0(z)| \le \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{k-1}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \dots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right|$$

$$\le |z|^{k(\rho(f)+\epsilon)} \left[1 + |A_{k-1}(z)| + \dots + |A_1(z)| \right]$$

for all z satisfying $|z| \notin E_3$. Which implies that

$$T(r, A_0) \le k(\rho(f) + \epsilon) \log r + (k-1)T(r, A_m) + O(1)$$

where $T(r, A_m) = \max\{ T(r, A_p) : p = 1, 2, ..., k-1 \}$ and $|z| = r \notin E_3$. This gives us that $\mu(A_0) \leq \mu(A_m), m = 1, 2, ..., k-1$ which is a contradiction. Thus all non-trivial solutions of the equation (46) are of infinite order.

Suppose that f be a nontrial solution of the equation (46) then by part (ii) of Lemma [1], for $\epsilon > 0$ there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure and there exists a constant c > 0 such that for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$ we have

$$\left| \frac{f^{(m)}(z)}{f(z)} \right| \le c \left(T(2r, f) \right)^{2k} \quad m = 1, 2, \dots, k$$
 (48)

Choose max{ $\rho(A_p): p=1,2,\ldots k-1$ } $<\eta<\mu(A_0)$ then from equations (46) and (48) we get

$$\exp(\mu(A_0 - \epsilon)) \le |A_0(z)| \le \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{(k-1)}| \left| \frac{f^{(k-1)}}{f(z)} \right| + \dots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \\ \le cT(2r, f)^{2k} [1 + (k-1)\exp(r^{\eta})]$$

for all z satisfying $|z| = r \notin E_2$. This will implies that

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \ge \mu(A_0).$$

Now we consider the case when $\mu(A_0) < \rho(A_i) = n$, where $n \in \mathbb{N}$ and there is a non-trivial solution f of the equation (46) of finite order then by part (i) of Lemma [1], for given $\epsilon > 0$ there exists a set $E_1 \subset [0, 2\pi)$

that has linear measure zero such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R(\psi_0) > 0$ so that for all z satisfying arg $z = \psi_0$ and $|z| \geq R_0$ we have

$$\left| \frac{f^{(m)}(z)}{f(z)} \right| \le |z|^{k\rho(f)} m = 1, 2, \dots, k$$
 (49)

Also $\rho(A_j) < \mu(A_0)$ for $j = 1, 2, ..., k - 1, j \neq i$ then we can choose $\eta > 0$ such that

$$\max\{ \rho(A_j), j = 1, 2, \dots, k - 1, j \neq i \} < \eta < \mu(A_0).$$

From above we have that

$$|A_j(z)| \le \exp r^{\eta}, \quad j = 1, 2, \dots, k - 1, j \ne i$$
 (50)

We have following cases to discuss:

Case 1. when $0 < \mu(A_0) < \frac{1}{2}$ then Lemma [10], equations (9), (46), (49) and (50) gives

$$\exp r^{\mu(A_0)-\epsilon} \le |z|^{k\rho(f)} \left[1 + \exp\left((1-\epsilon)\delta(P,\theta)r^n\right) + (k-2)\exp r^{\eta}\right]$$

for all z satisfying |z| = r > R and $\arg z \in E^- \setminus (E_1 \cup E)$. This gives a contradiction for sufficiently large r.

Case 2. Assume that $\mu(A_0) \geq \frac{1}{2}$ then by Lemma [11], equations (9), (46), (49) and (50) we get a contradiction.

Case 3. Suppose that $\mu(A_0) = 0$ then using Lemma [9], equations (9), (46), (49) and (50) we again get a contradiction.

Therefore all non-trivial solutions of the equation (46) are of infintie order.

If $\mu(A_0) = 0$ then $\rho_2(f) \geq 0$ for all non-trivial solutions f of the equation (46). Therefore we suppose that $\mu(A_0) > 0$ then using Lemma [10] (or Lemma [11]), equations (9), (46), (48) and (50) we get

$$\rho_2(f) \ge \mu(A_0)$$

for all non-trivial solutions f of the equation (46).

12. Proof of the Theorem [18]

Proof. When $\rho(A_0) \neq \rho(A_1)$ then the result follows from [16]. Thus we need to consider $\rho(A_0) = \rho(A_1) = n$ where $n \in \mathbb{N}$. Let us suppose that there exists a non-trivial solution f of the equation (46) of finite order. Then from part (ii) of Lemma [1], for $\epsilon > 0$ there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure such that

$$\left| \frac{f^{(m)}(z)}{f^{(p)}(z)} \right| \le |z|^{(m-p)\rho(f)}, \quad m, p = 0, 1, 2, \dots, k, p < m$$
 (51)

for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$.

Since $\rho(A_j) < \rho(A_0)$ for j = 2, 3, ..., k-1 then we choose $\eta > 0$ such that

$$\max\{ \rho(A_j) : j = 2, 3, \dots, k - 1 \} < \eta < \rho(A_0)$$

so that

$$|A_j(z)| \le \exp r^{\eta} \tag{52}$$

for j = 2, 3, ..., k-1. As done earlier in Theorem [10], we have following two cases to discuss:

Case 1. if there exists a Borel direction Φ of $A_0(z)$ such that $\theta_i < \Phi < \phi_{i+1}$ for i = 1, 2, ..., n then from equations (9), (40), (46), (51) and (52) we have

$$\exp^{(n-\epsilon)} \le |A_0(z)| \le \left| \frac{f^{(k)}(z)}{f(z)} \right| + |A_{(k-1)}(z)| \left| \frac{f^{(k-1)}(z)}{f(z)} \right| + \dots + |A_1(z)| \left| \frac{f'(z)}{f(z)} \right| \\ \le |z|^{k\rho(f)} [1 + (k-2) \exp(r^{\eta}) + \exp(1 - \epsilon)\delta(P, \psi_2)]$$

for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$ and $\arg z = \psi_2$. This we lead us to a contradiction for large values of r. Thus all non-trivial solutions of the equation (46) are of infinite order. From equations (9), (40), (46), (48) and (52) we have $\rho_2(f) \ge \rho(A_0)$ for all non-trivial solutions f of the equation (46).

Case 2. If there does not exists any Borel direction of $A_0(z)$ contained in (θ_i, ϕ_{i+1}) for i = 1, 2, ..., n then from equations (8), (10), (38), (46), (51) and (52) we have

$$\exp\left((1-\epsilon)\delta(P,\theta)r^{n}\right) \leq |A_{1}(z)| \leq \left|\frac{f^{(k)}(z)}{f'(z)}\right| + |A_{1}(z)| \left|\frac{f^{(k-1)}(z)}{f'(z)}\right| + \dots + |A_{0}(z)| \left|\frac{f(z)}{f'(z)}\right|$$

$$\leq |z|^{k\rho(f)} [1 + (k-2)\exp(r^{\eta}) + r(\exp(-CT(r,A_{0})) + |a_{2j-1}|)]$$

for all $|z| = r \notin E_2 \cup [0,1]$ and $\arg z = \theta \in E^+ \cap \Omega_{2j-1}(\Phi_{2j-1})$, where $a_i, i = 1, 2, \dots, \frac{q}{2}$ are deficient values of $A_0(z)$. Which will provide a contradiction for sufficiently large r.

Thus all non-trivial solutions of the equation (46) are of infinite order. From equations (8), (10), (38), (46), (48) and (52) we have $\rho_2(f) \ge \rho(A_0)$ for all non-trivial solutions f of the equation (46).

13. Proof of the Theorem [19]

Proof. If $\rho(A_i) \neq \rho(A_0)$ then result is true from [16]. Assume that $\rho(A_i) = \rho(A_0) = n, n \in \mathbb{N}$ and there exists a non-trivial solution f of the equation (46) of finite order. Then we have following two cases to

discuss:

Case 1. when the ray $\arg z = \Phi$ is a Borel direction of $A_0(z)$ where $\Phi \in (\theta_i, \phi_{i+1})$ for some i = 1, 2, ..., n. Choose $\psi_1 < \psi_2$ such that $\theta_i < \psi_1 < \phi < \psi_2 < \phi_{i+1}$ and $\psi_2 - \psi_1 < \frac{\pi}{\rho(A_i)} = \frac{\pi}{\rho(A_0)}$. Then by Lemma [19] we can have

$$\limsup_{r \to \infty} \frac{\log \log |A_0(re^{i\psi_2})|}{\log r} = \rho(A_0)$$
 (53)

Thus from equations (9), (46), (51), (52) and (53) we get contradiction for sufficiently large r.

As done in Case 1 of Theorem [18] we get $\rho_2(f) \ge \rho(A_0)$ for all non-trival solutions f of the equation (46).

Case 2. Suppose that $\arg z = \theta$ is not a Borel direction of $A_0(z)$ for any $\theta \in (\theta_i, \phi_{i+1})$ for all i = 1, 2, ..., n then choose $\arg z = \theta \in (\theta_i, \phi_{i+1})$ for some i = 1, 2, ..., n. Then Lemma [15], equations (5), (9), (46), and (50) leads us to a contradiction for sufficiently large r.

Thus all non-trivial solutions of the equation (46) are of infinite order. Also, Lemma [15], equations (4), (9), (46), and (50) gives that $\rho_2(f) \ge \rho(A_0)$ for all non-trivial solutions f of the equation (46).

Acknowledgement: I am thankful to my thesis advisor for his valuable comments and suggestions. I am also thankful to the Department of Mathematics, Deen Dayal Upadhyaya College (University of Delhi), for providing the proper research facilities.

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