

# THE CLASSIFICATION OF HYPERELLIPTIC THREEFOLDS

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ABSTRACT. We complete the classification of hyperelliptic threefolds, describing in an elementary way the hyperelliptic threefolds with group  $D_4$ . These are algebraic and form an irreducible 2-dimensional family.

## INTRODUCTION

A Generalized Hyperelliptic Manifold  $X$  is defined to be a quotient  $X = T/G$  of a complex torus  $T$  by the free action of a finite group  $G$  which contains no translations. We say that  $X$  is a Generalized Hyperelliptic Variety if moreover the torus  $T$  is projective, i.e., it is an Abelian variety  $A$ .

The main purpose of the present paper is to complete the classification of the Generalized Hyperelliptic Manifolds of complex dimension three. The cases where the group  $G$  is Abelian were classified by H. Lange in [La01], using work of Fujiki [Fu88] and the classification of the possible groups  $G$  given by Uchida and Yoshihara in [UY76]: the latter authors showed that the only possible non Abelian group is the dihedral group  $D_4$  of order 8.

This case was first excluded but it was later found that it does indeed occur (see [CD18] for an account of the story and of the role of the paper [DHS08]). Our paper is fully self-contained and show that the family described in [CD18] gives all the possible hyperelliptic threefolds with group  $D_4$ .

Our main theorem is the following

**Theorem 0.1.** *Let  $T$  be a complex torus of dimension 3 admitting a fixed point free action of the dihedral group*

$$G := D_4 := \langle r, s \mid r^4 = 1, s^2 = 1, (rs)^2 = 1 \rangle,$$

*such that  $G = D_4$  contains no translations.*

*Then  $T$  is algebraic. More precisely, there are two elliptic curves  $E, E'$  such that:*

*(I)  $T$  is a quotient  $T := T'/H$ ,  $H \cong \mathbb{Z}/2$ , where*

$$T' := E \times E \times E' =: E_1 \times E_2 \times E_3,$$

$$H := \langle \omega \rangle, \quad \omega := (h + k, h + k, 0) \in T'[2],$$

*and  $h, k$  are 2-torsion element  $h, k \in E[2]$ , such that  $h, k \neq 0, h + k \neq 0$ ;*

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(II) there is an element  $h' \in E'$  of order precisely 4, such that, for  $z = (z_1, z_2, z_3) \in T'$ :

$$\begin{aligned} r(z) &= (z_2, -z_1, z_3 + h') = R(z_1, z_2, z_3) + (0, 0, h'), \\ s(z) &= (z_1 + h, -z_2 + k, -z_3) = S(z_1, z_2, z_3) + (h, k, 0). \end{aligned}$$

Conversely, the above formulae give a fixed point free action of the dihedral group  $G = D_4$  which contains no translations.

In particular, we have the following normal form:

$$\begin{aligned} E &= \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau'), \quad \tau, \tau' \in \mathcal{H} := \{z \in \mathbb{C} | \text{Im}(z) > 0\}, \\ h &= 1/2, k = \tau/2, h' = 1/4 \\ r(z_1, z_2, z_3) &:= (z_2, -z_1, z_3 + 1/4) \\ s(z_1, z_2, z_3) &:= (z_1 + 1/2, -z_2 + \tau/2, -z_3). \end{aligned}$$

In particular, the Teichmüller space of hyperelliptic threefolds with group  $D_4$  is isomorphic to the product  $\mathcal{H}^2$  of two upper halfplanes.

### 1. PROOF OF THE MAIN THEOREM

We use the following notation:  $T = V/\Lambda$  is a complex torus of dimension 3, which admits a free action of the group

$$G = \langle r, s | r^4 = s^2 = (rs)^2 = 1 \rangle \cong D_4,$$

such that the complex representation  $\rho: G \rightarrow \text{GL}(3, \mathbb{C})$  is faithful.

A first observation is that the complex representation  $\rho$  of  $G$  must contain the 2-dimensional irreducible representation  $V_1$  of  $G$  (else,  $\rho$  would be a direct sum of 1-dimensional representations: this, by the assumption on the faithfulness of  $\rho$ , would imply that  $G$  is Abelian, a contradiction).

Hence we have a splitting

$$V = V_1 \oplus V_2,$$

where  $V_2$  is 1-dimensional, and we can choose an appropriate basis so that, setting  $R := \rho(r), S := \rho(s)$ , we are left with the two cases

$$\begin{aligned} \text{Case 1: } R &= \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \\ \text{Case 2: } R &= \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}. \end{aligned}$$

which are distinguished by the multiplicity of the eigenvalue 1 of  $S$ .

Indeed  $R$  is necessarily of the form above, since the freeness of the  $G$ -action implies that  $\rho(g)$  must have eigenvalue 1 for every  $g \in G$ .

**Lemma 1.1.** *In both Cases 1 and 2, the complex torus  $T = V/\Lambda$  is isogenous to a product of three elliptic curves,  $T \sim_{\text{isog.}} E_1 \times E_2 \times E_3$ , where  $E_i \subset T$ , for  $i = 1, 2, 3$  and  $E_1$  and  $E_2$  are isomorphic elliptic curves. In other words, writing  $E_j = W_j/\Lambda_j$ , the complex torus  $T$  is isomorphic to*

$$(E_1 \times E_1 \times E_3)/H, \quad H = \Lambda/(\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3).$$

*Proof.* Let  $I$  be the identity of  $T$ .

In Case 1, we set  $E_1 := \ker(S - I)^0 = \text{im}(S + I)$ ,  $E_3 := \ker(R - I)^0$  and  $E_2 := R(E_1)$  (here, the superscript zero denotes the connected component of the identity). Then it is clear that  $E_1 \cong E_2$ , and that  $T$  is isogenous to  $E_1 \times E_2 \times E_3$ .

In Case 2, we define similarly  $E_2 := \ker(S + I)^0 = \text{im}(S - I)$ ,  $E_3 := \ker(R - I)_0$  and  $E_1 := R(E_2)$ . We obtain again  $E_1 \cong E_2$ , and that  $T$  is isogenous to  $E_1 \times E_2 \times E_3$ . □

**Lemma 1.2.** *Writing  $E_j = W_j/\Lambda_j$ , the following statements hold.*

- (1) *In Case 1, the lattice  $\Lambda_2$  is equal to  $W_2 \cap \Lambda$ .*
- (2) *In Case 2, the lattice  $\Lambda_1$  is equal to  $W_1 \cap \Lambda$ .*

*Proof.* (1) Obviously,  $E_2 = R(E_1) = W_2/R(\Lambda_1)$ , i.e.,  $\Lambda_2 = R(\Lambda_1) \subset W_2 \cap \Lambda$ . On the other hand,  $R(W_2 \cap \Lambda) \subset W_1 \cap \Lambda = \Lambda_1$ , and applying the automorphism  $R$  of  $\Lambda$  gives  $W_2 \cap \Lambda \subset R(\Lambda_1) = \Lambda_2$ .

(2) Here,  $E_1 = R(E_2) = W_1/R(\Lambda_2)$ , i.e.,  $\Lambda_1 = R(\Lambda_2) \subset W_1 \cap \Lambda$ . For the converse inclusion, observe  $R(W_1 \cap \Lambda) \subset W_2 \cap \Lambda = \Lambda_2$ , and applying  $R$  yields again the result. □

We can now choose coordinates on  $V$  such that  $r$  is induced by a transformation of the form

$$r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3),$$

by choosing as the origin in  $V_1$  a fixed point of the restriction of  $r$  to  $V_1$ .

We can now view  $r, s$  as affine self maps of  $T$  induced by affine self maps of  $E_1 \times E_2 \times E_3$  of the form

$$r(z_1, z_2, z_3) = (z_2, -z_1, z_3 + c_3),$$

$$s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, \pm z_3 + a_3),$$

and sending the subgroup  $H$  to itself.

**Lemma 1.3.** *The freeness of the action of the powers of  $r$  is equivalent to:  $H$  contains no element with last coordinate equal to  $c_3$ , or  $2c_3$ .*

*Moreover,  $(0, 0, 4c_3) \in H$ .*

*Proof.*  $r(z) = z$  is equivalent to  $(z_1 - z_2, z_1 + z_2, -c_3) \in H$ . However, the endomorphism

$$(z_1, z_2) \mapsto (z_1 - z_2, z_1 + z_2)$$

of  $E_1 \times E_2$  is surjective, hence  $H$  cannot contain any element with last coordinate equal to  $c_3$ .

Since  $r^2(z) = (-z_1, -z_2, z_3 + 2c_3)$ ,  $r^2(z) = z$  is equivalent to  $(-2z_1, -2z_2, 2c_3) \in H$ , and we reach the similar conclusion that  $H$  cannot contain any element with last coordinate equal to  $2c_3$ .

Finally, the condition that  $r^4$  is the identity is equivalent to  $(0, 0, 4c_3) \in H$ . □

**Proposition 1.1.** *Case 2 does not occur.*

*Proof.* Since we assume that

$$s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, z_3 + a_3),$$

and that  $s^2$  is the identity, it must be

$$(2a_1, 0, 2a_3) \in H.$$

Consider now  $rs$ :

$$rs(z) = (-z_2 + a_2, -z_1 - a_1, z_3 + a_3 + c_3).$$

The condition that  $(rs)^2$  is the identity is equivalent to:

$$(a_1 + a_2, -(a_1 + a_2), 2(a_3 + c_3)) \in H.$$

This condition, plus the previous one, imply that

$$(a_2 - a_1, -(a_1 + a_2), 2c_3) \in H,$$

contradicting Lemma 1.3. □

Henceforth we shall assume that we are in Case 1, and we can choose the origin in  $E_3$  so that

$$s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, -z_3).$$

**Lemma 1.4.** *If*

$$s(z_1, z_2, z_3) := (z_1 + a_1, -z_2 + a_2, -z_3),$$

*then*

$$(2a_1, 0, 0) \in H$$

*and  $H$  contains no element of the form*

$$(a_1, w_2, w_3).$$

*Proof.* The first condition is equivalent to  $s^2$  being the identity, while the second is equivalent to the condition that  $s$  acts freely, since  $s(z) = z$  is equivalent to  $(a_1, -2z_2 + a_2, -2z_3) \in H$ . □

**Proposition 1.2.** For each  $\lambda \in \Lambda$  there exist  $\lambda' \in \Lambda$ ,  $\lambda_1 \in \Lambda_1$ ,  $\lambda_2 \in \Lambda_2$ ,  $\lambda_3 \in \Lambda_3$ , such that

$$2\lambda = \lambda_1 + \lambda', \quad 2\lambda' = \lambda_2 + \lambda_3$$

More precisely, we even have:

$$\Lambda \subset (1/2)\Lambda_1 + (1/2)\Lambda_2 + (1/4)\Lambda_3.$$

*Proof.* Let  $\lambda \in \Lambda$ : we can write

$$2\lambda = \underbrace{(I + S)\lambda}_{=:\lambda_1 \in \Lambda_1} + \underbrace{(I - S)\lambda}_{=:\lambda' \in \Lambda}.$$

Furthermore, since  $\lambda' \in \text{im}(I - S)$ , we obtain

$$2\lambda' = \underbrace{(I + R^2)\lambda'}_{=:\lambda_3 \in \Lambda_3} + \underbrace{(I - R^2)\lambda'}_{=:\lambda_2 \in \Lambda \cap W_2 = \Lambda_2}.$$

Hence,  $\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{4} + \frac{\lambda_3}{4}$  for unique  $\lambda_j \in \Lambda_j$ .

Applying the automorphism  $R$  of  $\Lambda$  and the unicity of the  $\lambda_j$  yields the result, since  $R$  exchanges  $\Lambda_1$  and  $\Lambda_2$ .

□

**Proposition 1.3.** We have

$$\Lambda \subset (1/2)\Lambda_1 + (1/2)\Lambda_2 + (1/2)\Lambda_3.$$

*Proof.* For  $\lambda \in \Lambda$  we can write  $\lambda = \frac{\lambda_1}{2} + \frac{\lambda_2}{2} + \frac{\lambda_3}{4}$  for unique  $\lambda_j \in \Lambda_j$ . We now use the property

$$E_i \hookrightarrow T \Rightarrow \forall (0, 0, d) \in H, \quad d = 0.$$

Indeed,  $2\lambda = \lambda_1 + \lambda_2 + \frac{\lambda_3}{2}$ , hence  $(0, 0, [\frac{\lambda_3}{2}]) \in H$  and  $\frac{\lambda_3}{2} = 0$  in  $E_3$ . Equivalently, there is an element  $\lambda'_3 \in \Lambda_3$  with

$$\frac{\lambda_3}{4} = \frac{\lambda'_3}{2}.$$

□

**Lemma 1.5.** Consider the transformation  $rs$ :

$$rs(z) = (-z_2 + a_2, -z_1 - a_1, -z_3 + c_3).$$

The condition that its square is the identity amounts to

$$(a_1 + a_2, -(a_1 + a_2), 0) \in H,$$

while the freeness of its action is equivalent to the fact that  $H$  contains no element of the form

$$(w_1 - a_2, w_1 + a_1, w_3) \Leftrightarrow \forall (d_1, d_2, d_3) \in H: \quad d_1 + a_2 \neq d_2 - a_1.$$

*Proof.* The first condition is straightforward, while the freeness of the action is equivalent to the non existence of  $(z_1, z_2, z_3)$  such that

$$(z_1 + z_2 - a_2, z_2 + z_1 + a_1, 2z_3 - c_3) \in H.$$

As usual, we observe that for each  $w_1, w_3$  there exist  $z_1, z_2, z_3$  with  $z_1 + z_2 = w_1, 2z_3 - c_3 = w_3$ .

□

We put together the conclusions of Lemmas 1.3, 1.4, 1.5,

- (i)  $(0, 0, 4c_3) \in H$
  - (ii)  $(2a_1, 0, 0) \in H$
  - (iii)  $(a_1 + a_2, -a_1 - a_2, 0) \in H$ , hence also  $(a_1 - a_2, a_1 + a_2, 0) \in H$ .
- (1)  $H$  contains no element of the form  $(w_1, w_2, c_3)$ ,
  - (2) nor of the form  $(w_1, w_2, 2c_3)$
  - (3) nor of the form  $(a_1, w_2, w_3)$
  - (4) nor of the form  $(w_1, w_2, w_3)$  with  $w_1 + a_2 = w_2 - a_1$ .

It follows from (iii) and (3) that  $a_2 \neq 0$ . While the condition that each element of  $H$  which has two coordinates equal to zero is indeed zero (since  $E_i$  embeds in  $T$ !) imply

$$2a_1 = 0, 4c_3 = 0.$$

By conditions (1), (2), (3) the elements  $a_1, c_3$  have respective orders exactly 2, 4. Moreover:

- (4) and (i) imply that  $a_1 + a_2 \neq 0$
- (ii), (iii) and the fact that  $H$  has exponent 2 implies  $2a_2 = 2a_1 = 0$ ,  $2a_1 + 2a_2 = 0$ . Hence  $a_1 \neq a_2$  are nontrivial 2-torsion elements.

We have thus obtained the desired elements

$$h := a_1, k := a_2, h' := c_3.$$

It suffices to show that  $H$  is generated by  $\omega := (h + k, h + k, 0) = (a_1 + a_2, a_1 + a_2, 0)$ .

Observe first that  $\omega \in H$ , by condition (iii).

Condition (4) implies that the first coordinate of an element of  $H$  must be a multiple of  $(a_1 + a_2)$ : since it cannot equal  $a_1$ , by condition (3), and if it equals  $a_2$ , we can add  $\omega$  and obtain an element of  $H$  with first coordinate  $a_1$ . Using  $R$ , we infer that both coordinates must be a multiple of  $(a_1 + a_2)$ . Possibly adding  $\omega$ , we may assume that  $w_1 = 0$ : then by (4) we conclude that also  $w_2 = 0$ . Finally, the condition that each element of  $H$  which has two coordinates equal to zero is indeed zero, show that  $H$  is then generated by  $\omega$ , as we wanted to show.

The last assertions of the main theorem follow now in a straightforward way (see [CC17] concerning general properties of Teichmüller spaces of hyperelliptic manifolds).

#### REFERENCES

- [CC17] F. CATANESE, P. CORVAJA: Teichmüller spaces of generalized hyperelliptic manifolds. Complex and symplectic geometry, 39-49, Springer INdAM Ser., 21, Springer, Cham (2017).
- [CD18] F. CATANESE, A. DEMLEITNER: Hyperelliptic Threefolds with group  $D_4$ , the Dihedral group of order 8. Preprint (2018), arXiv:1805.01835.
- [DHS08] K. DEKIMPE, M. HALENDA, A. SZCZEPAŃSKI: Kähler flat manifolds. J. Math. Soc. Japan 61 (2009), no. 2, 363-377.
- [Fu88] A. FUJIKI: Finite automorphism groups of complex tori of dimension two. Publ. Res. Inst. Math. Sci., 24 (1988), 1-97.
- [La01] H. LANGE: Hyperelliptic varieties. Tohoku Math. J. (2) 53 (2001), no. 4, 491-510.
- [UY76] K. UCHIDA, H. YOSHIHARA: Discontinuous groups of affine transformations of  $\mathbb{C}^3$ . Tohoku Math. J. (2) 28 (1976), no. 1, 89-94.

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