CONVEX CATERPILLARS ARE SCHUR-POSITIVE

YUVAL H. KHACHATRYAN-RAZIEL

ABSTRACT. A remarkable result of Stanley shows that the set of maximal chains in the non-crossing partition lattice of type A is Schur-positive, where descents are defined by a distinguished edge labeling. A bijection between these chains and labeled trees was presented by Goulden and Yong. Using Adin-Roichman's variant of Björner's EL -labeling, we show that the subset of maximal chains in the non-crossing partition lattice of type A, whose underlying tree is a convex caterpillar, is Schur-positive.

CONTENTS

1. INTRODUCTION

A symmetric function is called Schur-positive if all the coefficients in its expansion in the basis of Schur functions are nonnegative. Determining whether a given symmetric function is Schur-positive is a major problem in contemporary algebraic combinatorics [\[19\]](#page-16-1).

With a set A of combinatorial objects, equipped with a *descent map* Des : $A \to 2^{[n-1]}$, one associates the quasi-symmetric function

$$
\mathcal{Q}(A) := \sum_{\pi \in A} \mathcal{F}_{n,\text{Des}(\pi)}
$$

Supported by Israel Science Foundation grant no. 1970/18.

where $\mathcal{F}_{n,D}$ (for $D \subseteq [n-1]$) are Gessel's fundamental quasi-symmetric functions; see Subsection [2.2](#page-4-0) for more details. The following problem is long-standing.

Problem 1.1. Given a set A, equipped with a descent map, is $Q(A)$ symmetric? In case of an affirmative answer, is it Schur-positive?

Of special interest are Schur-positive sets of maximal chains. Maximal chains in a labeled poset P are equipped with a natural descent map. A well-known conjecture of Stanley [\[16,](#page-16-2) III, Ch. 21] implies that all examples of Schur-positive labeled posets in this sense correspond to intervals in the Young lattice.

Another way to equip the set of maximal chains with a descent map is using a labeling of the edges in the Hasse diagram. A classical example of a Schur-positive set of this type, the set of all maximal chains in the non-crossing partition lattice of type A , was given by Stanley [\[17\]](#page-16-3). An EL edge-labeling of this poset was presented in an earlier work of Björner [\[3\]](#page-16-4); see also $[4, 12, 1]$ $[4, 12, 1]$ $[4, 12, 1]$.

The goal of this paper is to present an interesting set of maximal chains in the non-crossing partition lattice NC_n (equivalently: a set of edge-labeled trees) which is Schur-positive. We will use a variant of Björner's EL -labeling, presented in [\[1\]](#page-16-7).

It is well known that maximal chains in the non-crossing partition lattice may be interpreted as factorizations of the *n*-cycle $(1, 2, \ldots, n)$ into a product of $n-1$ transpositions.

Definition 1.2. A factorization $t_1 \cdots t_{n-1}$ of the n-cycle $(1, 2, \ldots, n)$ as a product of transpositions is called linearly ordered if, for every $1 \leq i \leq n-2$, t_i and t_{i+1} have a common letter.

This definition is motivated by Theorem [4.1](#page-8-0) below. Denote the set of linearly ordered factorizations of $(1, 2, \ldots, n)$ by U_n .

Proposition 1.3. For every $n \geq 1$, the number of linearly ordered factorizations of the n-cycle $(1, 2, \ldots, n)$ is

$$
|U_n|=n2^{n-3}.
$$

Our main result is

Theorem 1.4. The set of linearly ordered factorizations of the n-cycle $(1, 2, \ldots, n)$ satisfies

$$
\mathcal{Q}(U_n) = \sum_{k=0}^{n-1} (k+1) s_{(n-k,1^k)},
$$

where the descent set of any $u \in U_n$ is defined by the edge labeling of [\[1\]](#page-16-7). In particular, U_n is Schur-Positive.

It should be noted that Theorem [1.4](#page-1-0) does not follow from Stanley's proof of the Schur-positivity of the set of all maximal chains in NC_n . In fact, Stanley's action on maximal chains does not preserve linearly ordered chains.

We prove Theorem [1.4,](#page-1-0) by translating it into the language of geometric trees called convex caterpillars.

Definition 1.5. A tree is called a *caterpillar* if the subgraph obtained by removing all its leaves is a path. This path is called the spine of the caterpillar.

Definition 1.6. A *convex caterpillar* of order n is a caterpillar drawn in the plane such that

- (a) the vertices are in convex position (say, the vertices of a regular polygon) and labeled $1, \ldots, n$ clockwise;
- (b) the edges are drawn as non-crossing straight line segments; and
- (c) the spine forms a cyclic interval $(a, a+1), (a+1, a+2), \ldots, (b-1)$ $1, b)$ in $[n]$.

Denote by C_t the set of convex caterpillars of order n.

Example [1](#page-2-0).7. Figure 1 shows a convex caterpillar $c \in C_t$, with spine consisting of the edges $(8, 1)$ and $(1, 2)$, forming a cyclic interval.

FIGURE 1. A convex caterpillar and its spine

Goulden and Yong [\[7\]](#page-16-8) introduced a mapping from factorizations of $(1, 2, \ldots, n)$ to non-crossing geometric trees. This mapping is not injective: in order to recover the factorization from the tree, one has to choose a linear extension of a certain partial order on the edges, which we call the Goulden-Yong partial order; see Definition [3.2](#page-7-2) below.

In a previous work [\[9\]](#page-16-9) we proved that the Goulden-Yong order is linear if and only if the geometric tree is a convex caterpillar; see Theorem [4.1](#page-8-0) below. It follows that the Goulden-Yong map, restricted to the set U_n of linearly ordered factorizations, is a bijection onto the set C_t of convex caterpillars of order n.

Definition 1.8. The *descent set* of a linearly ordered factorization $u = (t_1, \ldots, t_{n-1}) \in U_n$ is

$$
Des(u) := \{ i \in [n-2] : t_i = (b, c) \text{ and } t_{i+1} = (b, a) \text{ with } c > a \}.
$$

Example 1.9. The convex caterpillar $c \in C t_8$, drawn in Figure [1,](#page-2-0) corresponds to the linearly ordered word

 $u = ((7, 8), (6, 8), (5, 8), (1, 8), (1, 2), (2, 4), (2, 3)) \in U_8,$

for which $Des(u) = \{1, 2, 3, 4, 6\}.$

In [\[1\]](#page-16-7), the authors define a map ϕ from the set denoted here U_n to the symmetric group \mathfrak{S}_{n-1} ; for a detailed description see Subsection [4.2](#page-11-0) below. The map ϕ is an EL-labeling of the non-crossing partition lattice. This property, relations to Björner's EL-labeling and other positivity phenomena will be discussed in another paper.

It turns out that our Definition [1.8](#page-3-2) above fits nicely with this map.

Lemma 1.10. For any $u \in U_n$,

$$
Des(\phi(u)) = Des(u).
$$

See Proposition [4.13](#page-12-1) below. We further show that the number of caterpillars with a given descent set depends only on the cardinality of the descent set.

Lemma 1.11. For every subset $J \subseteq [n-2]$,

$$
|\{c \in Ct_n : Des(c) = J\}| = |J| + 1.
$$

These two key lemmas are used to prove Theorem [1.4.](#page-1-0)

2. Background

In this section we provide the necessary definitions and historical background to explain the main results. More information can be found in the references.

2.1. Compositions, partitions and tableaux.

Definition 2.1. A weak composition of n is a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers such that $\sum_{k=1}^{\infty} \alpha_k = n$.

Definition 2.2. A *partition* of n is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\sum_{k=1}^{\infty} \lambda_k = n$. We denote $\lambda \vdash n$.

Definition 2.3. The length of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is the number of non-zero parts λ_i .

For a skew shape λ/μ , let $SYT(\lambda/\mu)$ be the set of standard Young tableaux of shape λ/μ . We use the English convention, according to which row indices increase from top to bottom (see, e.g., $[14, Ch. 2.5]$).

The *height* of a standard Young tableau T is the number of rows in T . The descent set of T is

 $Des(T) := \{i : i + 1$ appears in a lower row of T than i.

2.2. Symmetric and quasi-symmetric functions. Let $x := (x_1, x_2, \ldots)$ be an infinite sequence of commuting indeterminates. Symmetric and quasi-symmetric functions in x can be defined over various (commutative) rings of coefficients, including the ring of integers; for simplicity we define it over the field Q of rational numbers.

Definition 2.4. A symmetric function in the variables x_1, x_2, \ldots is a formal power series $f(\mathbf{x}) \in \mathbb{Q}[[\mathbf{x}]]$, of bounded degree, such that for any three sequences (of the same length k) of positive integers, $(a_1, \ldots, a_k), (i_1, \ldots, i_k)$ and (j_1, \ldots, j_k) , the coefficients of $x_{i_1}^{a_1}$ $\overline{\begin{smallmatrix} a_1 & \ldots & a_k \ i_1 & \cdots & x_{i_k} \end{smallmatrix}}$ i_k and of $x_{i_1}^{a_1}$ $\frac{a_1}{j_1} \cdots x_{j_k}^{a_k}$ $_{j_k}^{a_k}$ in f are the same:

$$
[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f.
$$

Schur functions, indexed by partitions of n , form a distinguished basis for Λ^n , the vector space of symmetric functions which are homogeneous of degree n ; see, e.g., [\[18,](#page-16-11) Corollary 7.10.6]. A symmetric function in Λ^n is *Schur-positive* if all the coefficients in its expansion in the basis $\{s_\lambda : \lambda \vdash n\}$ of Schur functions are non-negative.

The following definition of a quasi-symmetric function can be found in [\[18,](#page-16-11) 7.19].

Definition 2.5. A *quasi-symmetric function* in the variables x_1, x_2, \ldots is a formal power series $f(\mathbf{x}) \in \mathbb{Q}[[\mathbf{x}]]$, of bounded degree, such that for any three sequences (of the same length k) of positive integers, $(a_1, \ldots, a_k), (i_1, \ldots, i_k)$ and (j_1, \ldots, j_k) , where the last two are *increasing*, the coefficients of $x_{i_1}^{a_1}$ $\frac{a_1}{i_1} \cdots x_{i_k}^{a_k}$ a_k and of $x_{j_1}^{a_1}$ $\frac{a_1}{j_1} \cdots x_{j_k}^{a_k}$ $_{j_k}^{a_k}$ in f are the same:

$$
[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}]f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}]f
$$

whenever $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$.

Clearly, every symmetric function is quasi-symmetric, but not conversely: $\sum_{i < j} x_i^2 x_j$, for example, is quasi-symmetric but not symmetric.

For each subset $D \subseteq [n-1]$ define the fundamental quasi-symmetric function

$$
\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.
$$

Let β be a set of combinatorial objects, equipped with a *descent* $map \, \text{Des} : \mathcal{B} \to 2^{[n-1]}$ which associates to each element $b \in \mathcal{B}$ a subset $Des(b) \subseteq [n-1]$. Define the quasi-symmetric function

$$
\mathcal{Q}(\mathcal{B}):=\sum_{b\in\mathcal{B}}\mathcal{F}_{n,\textup{Des}(b)}.
$$

With some abuse of terminology, we say that β is Schur-positive when $\mathcal{Q}(\mathcal{B})$ is.

The following key theorem is due to Gessel.

Theorem 2.6. [\[18,](#page-16-11) Theorem 7.19.7] For every shape $\lambda \vdash n$,

$$
\mathcal{Q}(\text{SYT}(\lambda)) = s_{\lambda}.
$$

Corollary 2.7. A set B, equipped with a descent map Des : $\beta \rightarrow$ $2^{[n-1]}$, is Schur-positive if and only if there exist nonnegative integers $(m_{\lambda,\mathcal{B}})_{\lambda\vdash n}$ such that

(2.1)
$$
\sum_{b \in \mathcal{B}} \mathbf{x}^{\text{Des}(b)} = \sum_{\lambda \vdash n} m_{\lambda, \mathcal{B}} \sum_{T \in \text{SYT}(\lambda)} \mathbf{x}^{\text{Des}(T)}.
$$

There is a dictionary relating symmetric functions to characters of the symmetric group \mathfrak{S}_n . The irreducible characters of \mathfrak{S}_n are indexed by partitions $\lambda \vdash n$ and denoted χ^{λ} . The Frobenius characteristic map ch from class functions on \mathfrak{S}_n to symmetric functions is defined by $ch(\chi^{\lambda}) = s_{\lambda}$, and extended by linearity. Theorem [2.6](#page-5-1) may then be restated as follows:

$$
ch(\chi^{\lambda}) = \sum_{T \in SYT(\lambda)} \mathcal{F}_{n, Des(T)}.
$$

2.3. Maximal chains in the non-crossing partition lattice. The systematic study of noncrossing partitions began with Kreweras [\[10\]](#page-16-12) and Poupard [\[13\]](#page-16-13). Surveys of results and connections with various areas of mathematics can be found in [\[15\]](#page-16-14) and [\[2\]](#page-16-15).

A noncrossing partition of the set [n] is a partition π of [n] into nonempty blocks with the following property: for every $a < b < c < d$ in [n], if some block B of π contains a and c and some block B' of π contains b and d, then $B = B'$. Let NC_n be the set of all noncrossing partitions of [n]. Define a partial order on NC_n , by refinement: $\pi \leq \sigma$ if every block of π is contained in a block of σ . This turns NC_n into a graded lattice.

An *edge labeling* of a poset P is function from the edges of the Hasse diagram of P to the set of integers. Several different edge labelings of NC_n were defined and studied by Björner [\[3\]](#page-16-4), Stanley [\[17\]](#page-16-3), and Adin and Roichman [\[1\]](#page-16-7). Let Λ be an edge labeling of NC_{n+1} , and let F_{n+1} be the set of maximal chains in NC_{n+1} . For each maximal chain $\mathfrak{m} : \pi_0 < \pi_1 < \cdots < \pi_n$ define

$$
\Lambda^*(\mathfrak{m}) := (\Lambda(\pi_0, \pi_1), \dots, \Lambda(\pi_{n-1}, \pi_n)) \in \mathbb{N}^n,
$$

with a corresponding *descent set*

$$
Des(\mathfrak{m}) := \{ i \in [n-1] : \Lambda(\pi_{i-1}, \pi_i) > \Lambda(\pi_i, \pi_{i+1}) \}.
$$

The noncrossing partition lattice is is intimately related to cycle factorizations. The *n*-cycle $(1, 2, \ldots, n)$ can be written as a product

of $n-1$ transpositions. There is a well known bijection between such factorizations and the maximal chains in NC_{n+1} ; see, for example, [\[11,](#page-16-16) Lemma 4.3]. A classical result of Hurwitz states that the number of such factorizations is n^{n-2} [\[8,](#page-16-17) [20\]](#page-16-18), thus equal to the number of labeled trees of order n. In the next section we will describe a connection between maximal chains and geometric trees.

3. The Goulden-Yong partial order

With each sequence of $n-1$ different transpositions $w = (t_1, \ldots, t_{n-1}),$ associate a geometric graph $G(w)$ as follows. The vertex set is the set of vertices of a regular *n*-gon, labeled clockwise $1, 2, \ldots, n$. The edges correspond to the given transpositions t_1, \ldots, t_{n-1} , where the edge corresponding to a transposition $t_k = (i, j)$ is the line segment connecting vertices i and j. See Figure [2](#page-6-1) for the geometric graph $G(w)$ corresponding to $w = ((1, 4), (4, 6), (4, 5), (1, 2), (2, 3)).$

FIGURE 2. $G(w)$ for $w = ((1, 4), (4, 6), (4, 5), (1, 2), (2, 3))$

Let F_n be the set of all factorizations of the *n*-cycle $(1, 2, \ldots, n)$ into a product of $n-1$ transpositions. Write each element of F_n as a sequence (t_1, \ldots, t_{n-1}) , where $t_1 \cdots t_{n-1} = (1, 2, \ldots, n)$. The following theorem of Goulden and Yong gives necessary and sufficient conditions for a sequence of $n-1$ transpositions to belongs to F_n .

Theorem 3.1. [\[7,](#page-16-8) Theorem 2.2] A sequence of transpositions $w =$ (t_1, \ldots, t_{n-1}) belongs to F_n if and only if the following three conditions hold:

- (1) $G(w)$ is a tree.
- (2) $G(w)$ is non-crossing, namely: two edges may intersect only in common vertex.
- (3) Cyclically decreasing neighbors: For every $1 \leq i < j \leq n-1$, if $t_i = (a, c)$ and $t_j = (a, b)$ then $c >_a b$. Here \lt_a is the linear order $a <_a a + 1 <_a \cdots <_a a - 1$.

For example, the graph in Figure [2](#page-6-1) corresponds to a sequence $w \in F_6$, and indeed satisfies the conditions of Theorem [3.1.](#page-6-2)

Note that a sequence $w = (t_1, \ldots, t_{n-1}) \in F_n$ carries more information than its Goulden-Yong tree $G(w)$: it actually defines a *linear order* on the edges, with the edge corresponding to t_i preceding the edge corresponding to t_j whenever $i < j$. How much of that information can be retrieved from the tree?

Definition 3.2. Let T be a non-crossing geometric tree (namely, satisfying conditions 1 and 2 of Theorem [3.1\)](#page-6-2) on the set of vertices of a regular n-gon, labeled clockwise $1, 2, \ldots, n$. Define a relation \leq_T on the set of edges of T as follows: $(a, b) \leq_T (c, d)$ if there exists a sequence of edges $(a, b) = t_0, \ldots, t_k = (c, d)$ $(k \ge 0)$ such that for every $0 \leq i \leq k-1$, $t_i = (x, z)$ and $t_{i+1} = (x, y)$ have a common vertex x and $z >_x y$ as in condition 3 of Theorem [3.1.](#page-6-2)

Lemma 3.3. \leq_T is a partial order on the set of edges of T.

We use the following well-known fact to prove the statement.

Fact 3.4. Let R be an anti-symmetric relation on a set S such that for every $x, y \in S$ there is at most one finite sequence $x = a_0, \ldots, a_n = y$ such that $a_{i-1}Ra_i$ for every $1 \leq i \leq n$. Then the transitive closure \tilde{R} of R is anti-symmetric.

Proof of Lemma [3.3.](#page-7-3) Every finite sequence of edges in T , with the property that every two consecutive edges e and f we have $e \prec_T f$, must form a path. Now, between every two edges there is exactly one path, hence at most one sequence as above. Hence by Lemma [3.4](#page-7-4) \leq_T is anti-symmetric. It is clearly anti-reflexive, hence a strong order on the edges of T .

We call \leq_T the *Goulden-Yong partial order* corresponding to T.

Observation 3.5. For every factorization $w = (t_1, \ldots, t_n) \in F_n$, the order $t_1 < t_2 < \ldots < t_n$ is a linear extension of the Goulden-Yong order $\lt_{G(w)}$.

Example 3.6. In Figure [2,](#page-6-1) the tree $T = G(w)$ yields the partial order satisfying $(1, 4) <_T (4, 6) <_T (4, 5)$ and $(1, 4) <_T (1, 2) <_T (2, 3)$. It is not a linear order. The order $(1, 4) < (4, 6) < (4, 5) < (1, 2), (2, 3)$ is a linear extension of it.

4. Convex caterpillars

In this section we prove Theorem [1.4](#page-1-0) using the properties of convex caterpillars.

4.1. Basic properties of convex caterpillars. Let us use the following conventions. All arithmetical operations on the elements of $[n]$ will be done modulo n. [a, b] will denote the cyclic interval $\{a, a+1, \ldots, b\}$. Using this notation, for edges (a, b) , (a, c) of a geometric non-crossing tree T, we have $(a\ c) <_T (a\ b)$ if and only if $b \in [a, c]$.

The following result was proved in [\[9\]](#page-16-9). We provide a somewhat different proof, the details of which will be used later.

Theorem 4.1. [\[9,](#page-16-9) Theorem 3.2] The Goulden-Yong order on the edge set of a non-crossing geometric tree T is linear (total) if and only if T is a convex caterpillar.

The following observation follows from the fact that a linear extension of a Goulden-Yong order \leq_T on the edges of geometric non-crossing tree T corresponds to a factorization of the cycle $(1 \dots n)$ into $n-1$ transpositions.

Observation 4.2. If T is a geometric non-crossing tree and \leq_T is linear, then every two consecutive edges, viewed as transpositions in \mathfrak{S}_n , do not commute and therefore have a common vertex.

The following lemma gives sufficient conditions for \leq_T not to be linear.

Lemma 4.3. Let T be a non-crossing geometric tree. In each of the following cases, the order \leq_T is not linear.

- (1) There are edges $(a\;b),(c\;d),(e\;f)\;of\;T$ such that $(a\;b)\leq_T$ (c d), (e f) and $c, d \in [a, b-1]$ and $e, f \in [b, a-1]$.
- (2) T has edges $(a\ b), (c\ d), (e\ f)$ such that $(c\ d), (e\ f) < (a\ b)$ and $c, d \in [b+1, a]$ and $e, f \in [a+1, b]$.

Proof. We prove the first case, second one being similar by reversing directions. Suppose that \leq_T is linear and the first case holds. Note that for every $v \in [a+1, b-1]$ the edge (v, b) is smaller than (a, b) in \leq_T because of counterclockwise relation of the edges, and the same is true for any edge $(a \, v)$ with $v \in [b+1 \, a-1]$. Combining with non-crossing property of T we find that any edge that is larger than (a, b) has either end-points in $[a, b-1]$ or $[b, a-1]$. Since it has both, there must be adjacent edges with endpoints in $[a, b-1]$ and $[b, a-1]$. However they are disjoint, hence commute, contradicting the fact that \leq_T is linear \Box order. \Box

We are ready to prove Theorem [4.1.](#page-8-0)

Proof of Theorem [4.1.](#page-8-0) If T is a convex caterpillar. If the spine of T is empty, then T is a star, hence every two edges are comparable because they have a common vertex. Otherwise, let $(a a+1)$, $(a+1 a+2)$..., $(b 2 (b-1)$, $(b-1 b)$ be the spine of T. For every two edges $(k l)$ and $(k m)$ that share a common vertex k, $(k \, l) \leq_T (k \, m)$ if $m \in [k, l]$ where $[k \, l]$ denotes the cyclic interval $\{k, k+1, \ldots, l\}$ where $l-k$ and $m-k$ have values between 1 and $n-1$. Note that it is simply restatement of the fact that neighbors of k are ordered counterclockwise. Hence, every two edges in the spine are comparable with $(a\ a+1) <_T (a+1\ a+2) \cdots <_T T$ $(b-1\;b)$. It also implies that if $(k\;k+1)$ is an edge in the spine, then

for every edge (k l) that has k as an end point, (k l) $\leq_T (k \; k+1)$ and for every edge $(k + 1, m)$ that has $k + 1$ as an endpoint we have $(k k + 1) <_{T} (k + 1 m)$. It follows that if k and m are endpoints of edges in the spine with $(k k+1)$, $(k+1 k+2)$..., $(m-1 m)$, then for every edge $(k \, j)$ connected to k and every edge $(m \, l)$ connected to m we have $(k \, j) \leq_T (k \, k + 1) \leq_T \ldots (m - 1 \, m) \cdots \leq_T (m \, l)$, hence every two edges that do not have common vertex are also comparable.

To prove the converse statement, assume that \leq_T has unique linear extension. Then \leq_T is linear and we can sort the edges $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1}),$ and since every linear extension of \leq_T corresponds to decomposition of the cycle $(1 \ldots n)$ into transpositions, we can view each edge as transposition. Next, note that since \leq_T is linear, every two adjacent edges can not commute as transpositions, hence share a common vertex. Now, note that the first edge must be of form $(i i + 1)$ for some i. Assume that it $t_1 = (i \, j)$ where the length cyclic interval $[ij]$ is larger than 1 and smaller than $n-1$. Since T is a tree, there must exist a vertex k in the cyclic interval $[i+1j-1]$ and a vertex m in the cyclic interval $[j + 1i - 1]$ connected to either i or j. Note that since every two consequent edges in \leq_T must have a common vertex, t_2 must be connected to either i or j . Assume without loss of generality that $t_2 = (jk)$ for some $k \in [j+1, i-1]$. But every two consecutive edges in \leq_T must have a common vertex, and every vertex adjacent to t_2 must have vertices in the interval $[j \ i-1]$ because of the non-crossing property of \leq_T . However, this implies that the first edge in in the interval $[i \, j-1]$ has no common vertex with the edge preceding it, which means that they commute as transpositions which contradicts the fact that $\langle \nabla \rangle$ is linear.

Now i must be a leaf. For if we have an edge $(j i)$, $(i j) τ $(i i+1)$,$ contradicting the fact that $\langle \tau \rangle$ is the first edge in $\langle \tau \rangle$. m edges in $\langle \tau \rangle$ the following hold:

- (1) The end points of the first m edges in \leq_T form a cyclic interval $[i \; k]$.
- (2) The vertices $j, j+1, \ldots, i-1, i$ are leaves in T.
- (3) The edges are $(i i + 1), (i + 1 i + 2), \ldots (k 1 k)$ are edges in T and occur among the first m edges.
- (4) Every edge that has $j, j+1, \ldots, k-1$ as endpoint occurs among the first m edges.
- (5) For the m-th edge in \lt_T , $t_m = (k-1)k$ or $t_m = (k)j$.

Let t_l denote the *l*-th edge in \lt_T . The statement clearly holds for $m =$ 1. Assume that the statement holds for m . By induction hypothesis the m-th edge of \leq_T is either $(k \, j)$ or $(k-1 \, k)$ and linearity of \leq_T and the induction hypothesis t_{m+1} must have k as an endpoint, because j and k can only be endpoints of the first m edges by the hypothesis. Next we show that t_{m+1} is either $(k k + 1)$ or $(k j - 1)$. Assume $t_m = (k l)$ for $l \neq j, k + 1$. Then we have must have edges $(k \, l) \leq_T (s \, t), (u \, v)$

with $u, v \in [k \; l-1]$ and $s, t \in [l, k-1]$ contradicting Lemma [4.3.](#page-8-1) Now if $t_{m+1} = (k \; k+1)$, we are done, since the statements 1 and 2 hold by induction for m, 3 and 5 hold for $m+1$ and 4 holds because $(k k+1)$ is the maximal edge in \leq_T that has k as an endpoint. If $t_{m+1} = (k \, j-1),$ then for every $v \in [k+1, j-2]$ we have $(j-1, v) <_T (j-1, k)$ which is impossible, since $v \notin [j, k]$ contradicting the assumption. On the other hand, for every $v \in [j \; k-1]$, $(j-1 \; v)$ can not be a an edge, since by the assumption, since edges with endpoints $j, \ldots k-1$ occur among the first m edges. Hence, $j - 1$ must be a leaf. Again, it is easy to check that assumptions 1, 2, 3, 4, 5 still hold for $m + 1$. Now if we substitute m with $n-1$, we see that T must be a geometric caterpillar, because by the construction, vertices that are not leaves are $i + 1, i + 2, \ldots, k$ for some k, with edges $(i+1, i+2), \ldots, (k-1, k)$ connecting them. \Box

For example, the tree in Figure [2](#page-6-1) is a caterpillar, but not a convex one. The corresponding Goulden-Yong order is not linear.

Corollary 4.4. A non-crossing geometric tree T on n vertices is a convex caterpillar if and only if there is a unique $w \in F_n$ such that $G(w) = T$.

We shall henceforth identify a convex caterpillar $c \in C_t$ with the corresponding sequence of transpositions $(t_1, \ldots, t_{n-1}) \in F_n$.

Proposition 4.5. In a convex caterpillar $c = (t_1, \ldots, t_{n-1})$:

- (1) Any two consecutive edges t_i and t_{i+1} share a common vertex.
- (2) The first edge t_1 is of the form $(a, a + 1)$ for some a. The same holds for the last edge t_{n-1} .

Proof. The first part of the proposition follows from the proof of [4.1.](#page-8-0) The second part is simply restatement of [4.2.](#page-8-2) \Box

Definition 4.6. Let e be an edge of caterpillar c .

- (1) We say that e is a *branch* if (at least) one of its endpoints is a leaf.
- (2) We say that e is a *link* if its endpoints have cyclically consecutive labels.

By cautiously reading the proof of theorem [4.1,](#page-8-0) we get the following observation.

Observation 4.7. An edge of a convex caterpillar a c is both a link and a branch if and only if it is either the first or the last edge of c.

Lemma 4.8. Let $c = (t_1, \ldots, t_{n-1}) \in Ct_n$. The following statements hold.

- (1) The endpoints of the first k edges form a cyclic interval in $[n]$, for every $1 \leq k \leq n-1$.
- (2) If the first edge is $(i, i + 1)$ then the endpoints of the first k branches that are leaves are $i, i-1, \ldots, i-k+1$, in that order.
- (3) If the first edge is $(i, i+1)$ then the first k links are $(i, i+1), (i+1)$ $1, i + 2), \ldots, (i + k - 1, i + k).$
- (4) The product of the first k edges, viewed as transpositions, is equal to the cycle $(\ell, \ell + 1, \ldots, m)$ where ℓ is the leaf endpoint of the last branch among the first k edges and $(m-1, m)$ is the last link among the first k edges.

Proof. Parts 1, 2 and 3 follow from the proof of theorem [4.1.](#page-8-0) Part 4 follows by induction and using the fact that that if the product of first k edges (viewed as transpositions) is the cycle $(l l + 1 dots l + k)$ and where the cyclic interval is formed by the endpoints of first k edges, then the k+1-th edge is either $(l+k l+k+1)$ or $(l+k l-1)$. Multiplying these we get the desired result.

Corollary 4.9. Every $c = (t_1, \ldots, t_{n-1}) \in Ct_n$ is completely determined by its first edge t_1 and the set of indices i for which t_i is a branch.

4.2. A labeling of maximal chains. The following labeling of maximal chains in the non-crossing partition lattice was introduced by Adin and Roichmain in [\[1\]](#page-16-7) and is closely related to the the EL-labeling introduced by Björner in $[3]$. In this section we describe this labeling, denoted by ϕ . Its connection to the EL-Labeling of Björner will be discussed elsewhere.

Recall, from Definition [1.8,](#page-3-2) the notion of descent set of a convex caterpillar.

Next, we show the connection to the descents defined by the map ϕ in [\[1\]](#page-16-7). First, let us describe ϕ . For $w = (t_1, \ldots, t_{n-1}) \in F_n$ define the partial products $\sigma_j = t_j \dots t_{n-1}$ with $\sigma_n = id$. By definition $\sigma_j =$ $t_i \sigma_{i+1}$. For $1 \leq j \leq n-1$ define

$$
A_j = \{1 \le i \ n-1 : \sigma_j(i) > \sigma_{j+1}(i)\}.
$$

By the discussion preceding Definition 3.2 in [\[1\]](#page-16-7), we get the following statement.

Proposition 4.10. The following hold.

- (1) For each $1 \leq j \leq n-1, |A_j| = 1$.
- (2) The map π_w defined by

$$
\pi_w(j) = i \text{ if } A_j = \{i\}
$$

is a permutation in \mathfrak{S}_{n-1} .

Definition 4.11. [\[1,](#page-16-7) Definition 3.2] Define $\phi : F_n \to \mathfrak{S}_{n-1}$ by

$$
\phi(w)=\pi_w.
$$

Define for each $w \in F_n$:

$$
Des(w) = Des(\phi(w)).
$$

4.3. Descents of convex caterpillars. We proceed to calculate the restriction of ϕ to C_t .

Proposition 4.12. Let $c \in Ct_n$ and let $\sigma_{i+1} = (k+1...m)$. Then

$$
\phi(c)(j) = \begin{cases} l & \text{if } \sigma_{j+1}(n) \neq n \\ l & \text{if } \sigma_{j+1}(n) = n \text{ and } l < m \\ m & \text{if } \sigma_{j+1}(n) = n \text{ and } m < l \end{cases}
$$

Proof. By Lemma [4.8](#page-10-0) the product of the first $n - 1 - i$ transpositions is a cycle of form $(l \; l+1 \; \ldots m)$. However, this implies that σ_j is the cycle $(m m + 1 ... l - 1)$. Also, note that $(m - 1, m)$ is the last link among the first $n - j$ edges of c. Hence t_{i-1} equals either $(m - 1 m)$ or $(m l)$.

If $t_{j-1} = (m-1 \; m)$ and $m-1 < m$, then

$$
\sigma_{j-1}(m-1) = m > m - 1 = \sigma_j(m-1)
$$

which implies that $\phi(c)(j-1) = m-1$. $m-1 > m$ then $m-1 = n$ and

$$
\sigma_{j-1}(l-1) = n > m = \sigma_j(l-1),
$$

hence, $\phi(c)(j-1) = l-1$. If $t_{j-1} = (m \; l)$ then if $l < m$, we have $\sigma_{i-1}(l) = m > l > \sigma_i(l)$

and $\phi(c)(j-1) = l$ and if $m < l$ then

$$
\sigma_{j-1}(l-1) = l > m = \sigma_j(l-1)
$$

and $\phi(c)(j-1) = l-1$. Note, that in all four cases, we get following combinatorial description of ϕ restricted to C_t .

Proposition 4.13. The descent set of a convex caterpillar, defined as in Definition [1.8,](#page-3-2) coincides with the descent set defined via the map ϕ .

Proof. First, show that if $t_i = (a \; b)$ and $t_{i+1} = (b \; c)$ then $\phi(c)(i) >$ $\phi(c)(i + 1)$. Let $\sigma_{j+2} = (k \; k+1, \ldots, m)$ such that $(k \; k+1)$ is the first link in σ_{j+2} and $t_j = (k-1 \; k)$ is the last link among the first $j-1$ edges. There are two possibilities. We have either $b = k$ or $c = k$ Suppose that $b = k$ holds then we have $t_i = (a \ k), t_{i+1} = (c \ k)$ with $a > k$. Then by interval property of σ_j of a caterpillar we have $b = m + 1$ and $c = m + 2$ with $m + 2 > m + 1$. By proposition [4.12,](#page-12-2) $\phi(c)(j) = m + 1$, $phi(c)(j + 1) = m$ if $n \notin \{k, ..., m\}$ and $\phi(c)(j) =$ $m+2, \ \phi(c)(j+1) = m+1.$ In both cases we have $\phi(c)(j) > \phi(c)(j+1),$ hence j is a descent of $\phi(c)$. Second possibility is that $b \neq k$. In that case we have $(bc) = (k-1k)$ and $(ab) = (m+1 k-1)$ with $m+1 > k-1$. This implies that n is not contained in the interval $(k - 1 ... m + 1)$ which means that $\phi(c)(j) = m$ and $\phi(c)(j + 1) = k - 1$ and again j is a descent of $\phi(c)$.

Now assume that j is descent of ϕ . Let $\sigma_{i+2} = (k \dots m)$. Note that that there are four possibilities for t_j, t_{j+1} .

- (1) $t_j = (k-2 \; k-1), t_{j+1} = (k+1 \; k)$. In this case we have $\sigma_{j+1} = (k-1 \; k \; \dots \; m), \, \sigma_j = (k-2 \; k-1 \; \dots \; k).$ By proposition [4.12,](#page-12-2) either $\phi(c)(j+1) = k-1$ or $\phi(c)(j+1) = m$ if $k-1 = n$. We $\phi(j)(c) = m$ and or $k - 2 = m$ if $k - 2 = n$. Since we have $\phi(c)(j) > \phi(c)(j + 1)$ we can not have $\phi(j) = k - 2 > k - 1 =$ $\phi(j+1)$ because it would imply that $\phi(k-2) = n$ and this is not possible because $\phi(c)$ is permutation on $n-1$. Hence the possibilities that remain are either $\phi(j) = m > k - 1 = \phi(j + 1)$ or $\phi(j) = k - 2 > \phi(j + 1) = m$. If $\phi(j) = m > k - 1 = \phi(j + 1)$ then we have $\sigma_i = (n \; k-1), \sigma_{i+1} = (k-1 \; k)$ which means that j is a descent of c. If $\phi(j) = k - 2 > \phi(j + 1) = m$, then $\sigma(j+1) = (n\ 1)$ and $\sigma(j) = (n-1\ n)$ which again implies that j is a descent of ϕ .
- (2) $t_j = (k-1 \; k), t_{j+1} = (k \; m+1)$. Again by proposition [4.5](#page-10-1) we have either $\phi(j) = k - 1$ or $\phi(j) = m + 1$ if $\sigma(j) = (n \ 1)$ and $\phi(j + 1) = m$ or $\phi(j + 1) = m + 1$ if $\sigma_{j+2}(n) \neq n$. We must have either $\phi(c)(j) = m + 1 > m = \phi(c)(j + 1)$. In this case we have $t_i = (n\ 1)$ and $t_{i+1} - (1\ m)$. Otherwise we have $\phi(j) = k - 1 > m = \phi(j+1)$ or $\phi(j) = k - 1 > m + 1 = \phi(j+1)$. Both cases imply that $k - 1 > m$ and thus j is again descent of c.
- (3) $t_j = (k-1 \; m+1), t_{j+1} = (k-1 \; k)$. By proposition [4.5](#page-10-1) we have either $\phi(c)(j) = m$ if $k - 1 < m$ and $\phi(c)(j) = m + 1$ if $k-1 > m, m+1$. We also have $\phi(c)(j+1) = k-1$ if $k-1 < k$ and $\phi(c)(j + 1) = m$ if $t_{j+1} = (n 1) = (k - 1 k)$. Clearly, the option $\phi(c)(j) = m + 1 > k - 1 > \phi(c)(j + 1)$ is not possible because it implies $k + 1 > m + 1$ and j is a descent of ϕ . Hence we have $\phi(c)(j) = m > k - 1 = \phi(c)(j + 1)$ which implies that $m+1 > k-1$. Hence we have $t_j = (k-1 \, m+1), t_{j+1} = (k-1 \, k)$ with $m + 1 > k - 1$ which implies that j is a descent of c.
- (4) $t_j = (k \, m + 2), t_{j+1} = (k \, m + 1).$ If we have $m + 1 > m + 2$ then we have $m + 1 = n$, hence $\phi(c)(j + 1) = m = n$ and $\phi(c)(j) = 1$ which is not possible, since j is a descent. Otherwise we have either $\phi(c)(j) = m + 1, \phi(c)(j) = m$ or $\phi(c)(j) =$ $m+2, \phi(c)(j+1) = m+1$ by proposition [4.12.](#page-12-2) It is easy to check that in both cases j is also a descent of c .

 \Box

4.4. Schur-positivity of convex caterpillars.

Definition 4.14. Let $c = (t_1, \ldots, t_{n-1})$ be a convex caterpillar and let i be the index of the first edge that has 1 as its endpoint. The edge t_i is called the main edge of c and the index i is called the main index of c, denoted $I(c)$.

For example, for $c = ((4, 5), (5, 6), (3, 6), (1, 6), (1, 2))$ we have $I(c)$ 4.

Using Lemma [4.8](#page-10-0) we prove the following explicit description of the descents of a convex caterpillar c, based on $I(c)$ and on the geometry of c.

Lemma 4.15. Let $c \in Ct_n$ and $i \in [n-2]$. Then:

- (1) For $1 \leq i < I(c) 1$, i is a descent of c if and only if t_{i+1} is a branch of c.
- (2) For $i = I(c) 1$, i is always a descent of c.
- (3) For $i = I(c)$, i is a descent of c if and only if 1 is not a leaf of c.
- (4) For $I(c) < i \leq n-2$, i is a descent of c if and only if t_i is a branch of c.

Proof. We prove each case separately

- (1) First suppose that $i = I(c)$. Then $t_i = (a \; b)$ and $t_{i+1} = (b \; 1)$ for some $2 \leq a, b \leq n$. Obviously, $a > 1$ and therefore t_i is a descent.
- (2) If t_{i+1} is a branch, then $t_i = (a \; b)$, $t_{i+1} = (a \; c)$ for some $a, b, c > 1$. If (a, b) . By lemma [4.8](#page-10-0) $b = a - 1$ if (a, b) is a link or $b = c+1$ if (a, b) is a branch, the endpoints of the first $i+1$ edges form the cyclic interval $[c, a]$. Since $i < i(c)$ we $1 < c < b < a$, therefore i is a descent. On the other hand if t_{i+1} is a link then $t_i = (a, b), t_{i+1} = (b, b+1)$ and a is between $b+1$ and b in \lt_b . Because $1 < a$, we have $a < b$, thus i is not a descent.
- (3) Now if $i = I(c)$ and 1 is a leaf. Then we have $t_i = (a 1), t_{i+1} =$ $(a\;b)$. Obviously $1 < b$ and t_i is not a descent. In contrast, if t_i is a link, then $t_i = (1 \t a)$, $t_{i+1} = (1 \t b)$ and since a and b are sorted counterclockwise and are both greater than 1 in the cyclic order \leq_1 we have $b < a$, thus i is a descent.
- (4) Now suppose that $i > I(c)$. Then if t_i is a branch we have $t_i =$ $(a b) t_{i+1} = (a c)$ where b and c are ordered counterclockwise and $a < b, c < n$ which implies that $c < b$ and that t_i is a descent. On the other hand, if t_i is a link we have $t_i = (a \ a + 1), t_{i+1} =$ $(a + 1 k)$ where $k > a$, and i is not a descent.

 \Box

Combining Lemmas [4.8](#page-10-0) and [4.15](#page-14-0) and Corollary [4.9,](#page-11-1) we deduce the following key proposition.

Proposition 4.16. A convex caterpillar c is determined uniquely by $I(c)$ and $Des(c)$.

Proof. By Lemma [4.9,](#page-11-1) it suffices to show that the pair $(I(c), Des(c))$ determines the first edge and branches. Note that by observation [4.7,](#page-10-2) first and last edges are always branches.

Denote $k := I(c)$. For $i \leq k$, combine Lemmas [4.15](#page-14-0) and [4.8](#page-10-0) to determine whether the i-th edge is a branch or link. By Part 2 of Lemma [4.15](#page-14-0) we know whether 1 is a leaf or not and whether e_k is a branch or not. In both cases, applying Parts 2 and 3 of Lemma [4.8,](#page-10-0) we determine the first edge. The branches with indices larger than k are determined by Part 3 of Lemma [4.15.](#page-14-0) Hence the first edge and the branches are completely determined by the descent set and the $k = I(c)$ as desired.

The next lemma describes the possible values of $I(c)$, given the descent set of c.

Lemma 4.17. Let $c \in Ct_n$. Then either $I(c) = 1$ or $I(c) - 1 \in Des(c)$.

Proof. Let $X \subseteq [n-2]$ and suppose that $Des(c) = X$. We show that $I(c) = 1$ or $I(c) \in X + 1$. It is clear that if $I(c) \neq 1$ then there exists a $i \in \text{des}_C(1) + 1$ such that $I(c) = i$ by Part 1 of Lemma [4.15.](#page-14-0) \Box

Lemma 4.18. For every subset $J \subseteq [n-2]$ and every $i \in (1+J) \cup \{1\}$, there exists a unique $c \in Ct_n$ such that $Des(c) = J$ and $I(c) = i$.

Proof. Recall that every caterpillar is determined by its first edge and the true branches, where every $J \subseteq \{2, \ldots, n-2\}$ can appear as the set of the true branches of a caterpillar. Placing $I(c)$ after $x \in J$ results in proper set of true branches, which in turns defines a caterpillar. Now, suppose that $i \notin J$. Then $(1, 2)$ can be first edge of the leaf, since 1 is a leaf, hence 1 is not a descent, and branches correspond to the members of X. If $1 \in X$, then $(n 1)$ can be first edge, with the rest of branches defined by the descents. \Box

Corollary 4.19. For every subset $J \subseteq [n-2]$, the number of convex caterpillars with descent set J is equal to $|J|+1$.

The following observation is well known.

Observation 4.20. For every $0 \leq k \leq n-1$

 $\{\text{Des}(T) : T \in \text{SYT}(n-k, 1^k)\} = \{J \subseteq [n-1] : |J| = k\},\$

each set being obtained exactly once.

Proof of Theorem [1.4.](#page-1-0) Combine Corollary [4.19](#page-15-0) with Observation [4.20](#page-15-1) and Theorem [2.6](#page-5-1) to deduce

$$
\mathcal{Q}(Ct_n) = \sum_{k=0}^{n+1} (k+1) \sum_{\substack{J \subseteq [n-1] \\ |J|=k}} \mathcal{F}_{n,J} = \sum_{k=0}^{n+1} (k+1) s_{n-k,1^k}.
$$

Acnowledgements. This work forms part of a PhD research conducted under the supervision of Professors Ron Adin and Yuval Roichman.

REFERENCES

- [1] R. M. Adin and Y. Roichman, On maximal chains in the non-crossing partition lattice, J. Combin. Theory Ser. A 125 (2014), 18–46
- [2] D. Armstrong, Generalized noncrossing partitions and combinatorics of Coxeter groups. Mem. Amer. Math. Soc. 202 (2009).
- [3] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, Trans. Amer. Math. Soc. 260 (1980), 159–183.
- [4] A. Björner and M. Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983), 323-–341.
- [5] I. M. Gessel, Multipartite P-partitions and inner products of Schur functions, Contemp. Math. 34 (1984), 289–302.
- [6] I. M. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Combin. Theory Ser. A 64 (1993), 189–215.
- [7] I. Goulden and A. Yong, Tree-like Properties of Cycle Factorizations, J. Combin. Theory Ser. A 98 (2002), 106–117.
- [8] A. Hurwitz, Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten, Math. Ann. 39 (1891), 1–61.
- [9] Y. H. Khachtryan-Raziel, On the center of the Hurwitz graph, [arXiv:1508.02620](http://arxiv.org/abs/1508.02620).
- $[10]$ G. Kreweras, Sur les partitions non-croisées d'un cycle, Discrete Math. 1 (1972), 333–350.
- [11] J. McCammond, Noncrossing partitions in surprising locations, Amer. Math. Monthly 113 (2006), 598–610.
- [12] P. McNamara, EL-labelings, supersolvability and 0-Hecke algebra actions on posets, J. Combin. Theory Ser. A 101 (2003), 69-–89.
- [13] Y. Poupard, Etude et dénombrement paralleles des partitions non croisees d'un cycle et des coupage d'un polygone convexe, Discrete Math. 2 (1972), 279–288.
- [14] B. E. Sagan, The symmetric group. Representations, combinatorial algorithms, and symmetric functions, 2nd edition , Grad. Text in Mathematics 203. Springer-Verlag, 2001
- [15] R. Simion, Noncrossing partitions, Discrete Math. 217 (2000), 367–409.
- [16] R. P. Stanley, Ordered structures and partitions (revision of 1971 Harvard University thesis). Memoirs Amer. Math. Soc. 119 (1972).
- [17] R. P. Stanley, Parking functions and noncrossing partitions, Electronic J. Combin. 4 (1997), R20.
- [18] R. P. Stanley, Enumerative combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge Univ. Press, Cambridge, 1999.
- [19] R. P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in: Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), Amer. Math. Soc., Providence, RI, 2000, pp. 295–319.
- [20] V. Strehl, Minimal transitive products of transpositions — The reconstruction of a proof by A. Hurwitz, Sém. Lothar. Combin. 37 (1996).