Fully coupled mean-field FBSDEs with jumps and related optimal control problems

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Abstract

This paper study a type of fully coupled mean-field forward-backward stochastic differential equations with jumps under the monotonicity condition, including the existence and the uniqueness of the solution of our equation as well as the continuity property of the solutions with respect to the parameters. Then we establish the stochastic maximum principle for the corresponding optimal control problems and give the applications to mean-variance portfolio problems and linear-quadratic problems, respectively.

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1 Introduction

Forward-backward stochastic differential equations (FBSDEs, for short) have attracted significant attention because of their wide range of applications, from solving nonlinear partial differential equations (PDEs, for short), pricing American options to describing some optimization problems (refer to, [12]). Inspired by the introduction of a recursive stochastic utility function in [7], Antonelli [1] first investigated the existence and the uniqueness of the solution of FBSDEs driven by Brownian motion with requiring the small enough Lipschitz constant of the coefficients. In order to deal with fully coupled FBSDEs on an arbitrarily given time interval, Ma, Protter, Yong [10] introduced a "four-step scheme" approach which combines probability methods and PDE methods. Using this method, they obtained the existence and the uniqueness of the solution with deterministic and non-degenerate diffusion coefficients. Peng and Wu [15] used a purely probabilistic continuation method to study fully coupled FBSDEs with additional monotonicity condition on the coefficients. There are also many other methods to study the solution of FBSDEs, see Delarue [6] and Zhang [20] for numerical approaches, Ma, et al. [11] for a unified approach, etc. For more details about fully coupled FBSDEs, the readers also refer to Ma and Yong [12], or Yong [19] and the references therein.

On the other hand, mean-field limits are widely applied to many diverse areas such as statistical physics, quantum mechanics and quantum chemistry. Based on this, Buckdahn, et al. [5] obtained a new type of BSDEs, namely mean-field BSDEs. In [4], Buckdahn, Li and Peng made an in-depth study of such

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type of BSDEs and got the existence and the uniqueness of the solution of mean-field BSDEs, as well as a comparison theorem. They also established the link between the solution of this mean-field BSDEs and some nonlocal PDEs. Min, Peng, Qin [13] generalized their work to fully coupled mean-field FBSDEs cases. Barles, Buckdahn, Pardoux [3] studied a new type of BSDEs driven by Brownian motion and a Poisson random measure, namely BSDEs with jumps and showed the connection with a system of parabolic integro-PDEs. Royal [16] gave a strict comparison theorem for BSDEs with jumps and the relation to non-linear expectation. Li, Min [9] investigated a new type of mean-field BSDEs with jumps, namely mean-field BSDEs with jumps involving value function and obtained the related dynamic programming principle.

Inspired by the above works, one of our aim is to study a type of fully coupled mean-field FBSDEs with jumps. To the best of our knowledge, no corresponding works have been done until now. To be more specific, we consider the following fully coupled mean-field FBSDEs with jumps:

$$\begin{cases} dx(t) = \displaystyle\int_E E'\Big[b\big(t,\lambda(t,e),(\lambda(t,e))'\big)\Big]\lambda(de)dt + \int_E E'\Big[\sigma\big(t,\lambda(t,e),(\lambda(t,e))'\big)\Big]\lambda(de)dB_t \\ \qquad\qquad + \displaystyle\int_E E'\Big[h\big(t,\lambda(t-,e),(\lambda(t-,e))',e\big)\Big]\widetilde{\mu}(dt,de), \\ -dy(t) = \displaystyle\int_E E'\Big[f\big(t,\lambda(t,e),(\lambda(t,e))'\big)\Big]\lambda(de)dt - z(t)dB_t - \int_E k(t,e)\widetilde{\mu}(dt,de),\ t\in[0,T], \\ x(0) = a, \\ y(T) = E'[\Phi(x(T),(x(T))')], \end{cases} \tag{1.1}$$

where

$$\lambda(t,e) = \big(x(t),y(t),z(t),k(t,e)\big),\ \ \lambda(t-,e) = \big(x(t-),y(t-),z(t),k(t,e)\big),$$

b,σ,h,f,Φ are mappings with appropriate dimensions, $T\geq 0$ is an arbitrarily fixed number. Under the classical assumption (H3.1) and monotonicity assumption (H3.2), the existence and the uniqueness of the solution of our fully coupled mean-field FBSDEs with jumps (1.1) are obtained by using a purely probabilistic continuation method (See, Theorem 3.1). Furthermore, we study the continuity of the solution of equation (1.1) relying on parameters under our assumptions (See, Theorem 4.1).

Another aim of this paper is to study the related optimal control problems for the controlled fully coupled mean-field FBSDEs with jumps (1.1) in a Markovian framework. Our motivation of this part is followed from many theoretical works and a wide range of applications with respect to the stochastic maximum principle of the stochastic control problems under jump-diffusion framework. Framstad, Oksendal, Sulem [8] proved a sufficient maximum principle for the optimal control of jump diffusions and gave applications to optimization problems in a financial market. Oksendal, Sulem [14] and Shi, Wu [18] studied the maximum principle for optimal control of FBSDEs with jumps by using different approaches, respectively. Shen, Siu [17] generalized their work to mean-field cases.

Let us give the specific description of our control problem, where the dynamic has the following form

$$\begin{cases} dx^v(t) = \displaystyle\int_E E'[b\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dt + \int_E E'[\sigma\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dB_t \\ \qquad\qquad + \displaystyle\int_E E'[h\big(t,\pi^v(t-,e),v(t),e\big)]\widetilde{\mu}(dtde), \\ -dy^v(t) = \displaystyle\int_E E'[f\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dt - z^v(t)dB_t - \int_E k^v(t,e)\widetilde{\mu}(dtde), \\ x^v(0) = a,\quad y^v(T) = E'[\Phi\big(x^v(T),(x^v(T))'\big)], \end{cases} \tag{1.2}$$

where

$$\pi^v(t,e) = \big(x^v(t),y^v(t),z^v(t),k^v(t,e),(x^v(t))',(y^v(t))',(z^v(t))',(k^v(t,e))'\big),$$

$$\pi^v(t-,e) = \big(x^v(t-),y^v(t-),z^v(t),k^v(t,e),(x^v(t-))',(y^v(t-))',(z^v(t))',(k^v(t,e))'\big),$$

and the cost functional has the following form

$$J(v(\cdot)) = E\Big[\int_0^T \int_E E'\big[g\big(t, \pi^v(t,e), v(t)\big)\big]\lambda(de)dt + E'[\varphi\big(x^v(T), (x^v(T))'\big)] + \gamma\big(y^v(0)\big)\Big], \tag{1.3}$$

where all coefficients of the dynamic and the cost functional are given deterministic functions (See, Section 5 for more details). Our control domain is convex and we get the necessary and sufficient condition for the optimality of the control with the help of a convex perturbation (See, Theorem 5.1 and 5.2). Moreover, we apply these results to a mean-variance portfolio selection mixed with a mean-field recursive utility and a linear-quadratic optimal control problem, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce the framework of our study and some results on mean-field forward and backward SDEs with jumps. In Section 3, we prove the existence and the uniqueness of solution of fully coupled mean-field FBSDEs with jumps.We present the continuity of solutions of our equation with respect to the parameters in Section 4. Section 5 is devoted to discussing the necessary and sufficient condition of the optimal control problem for the related fully coupled mean-field FBSDEs with jumps. In Section 6 we give two applications to illustrate the results of Section 5. A corresponding lemma (used in the proof of Theorem 3.1) and its proof are given in Appendix.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space which is the completed product of the Wiener space $(\Omega_1, \mathcal{F}_1, P_1)$ and the Poisson space $(\Omega_2, \mathcal{F}_2, P_2)$:

• $(\Omega_1, \mathcal{F}_1, P_1)$ is a classical Wiener space, where $\Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$ is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$ with value 0 in time 0, $\mathcal{F}_1$ is the completed Borel σ-algebra over Ω_1, and P_1 is the Wiener measure such that $B_s(\omega) = \omega_s$, $s \in \mathbb{R}_+$, $\omega \in \Omega_1$, and $B_{-s}(\omega) = \omega(-s)$, $s \in \mathbb{R}_+$, $\omega \in \Omega_1$, are two independent d-dimensional Brownian motions. The natural filtration $\{\mathcal{F}_s^B, s \geq 0\}$ is generated by $\{B_s\}_{s\geq 0}$ and augmented by all P_1-null sets, i.e.,

$$\mathcal{F}_s^B = \sigma\{B_r, r \in (-\infty, s]\} \vee \mathcal{N}_{P_1},\ \ s \geq 0.$$

• $(\Omega_2, \mathcal{F}_2, P_2)$ is a Poisson space. We denote by $p:\ D_p \subset \mathbb{R} \to E$ the point functions, where D_p is a countable subset of the real line $\mathbb{R}$, $E = \mathbb{R}^l \setminus \{0\}$ is equipped with its Borel σ-field $\mathcal{B}(E)$. We introduce the counting measure $\mu(p, dtde)$ on $\mathbb{R} \times E$ as follows:

$$\mu(p, (s,t] \times \Delta) = \sharp\{r \in D_p \cap (s,t] : p(r) \in \Delta\},\ \Delta \in \mathcal{B}(E),\ s,t \in \mathbb{R},\ s < t,$$

where $\sharp$ denotes the cardinal number of the set. We identify the point function p with $\mu(p,\cdot)$. Let Ω_2 be the set of all point functions p on E, and $\mathcal{F}_2$ be the smallest σ-field on Ω_2. The coordinate mappings $p \to \mu(p, (s,t] \times \Delta)$, $s,t \in \mathbb{R}$, $s < t$, $\Delta \in \mathcal{B}(E)$, are measurable with respect to $\mathcal{F}_2$. On the measurable space $(\Omega_2, \mathcal{F}_2)$ we consider the probability measure P_2 such that the canonical coordinate measure $\mu(p, dtde)$ becomes a Poisson random measure with the compensator $\widehat{\mu}(dtde) = dt\lambda(de)$; the process $\{\widetilde{\mu}((s,t] \times A) = (\mu - \widehat{\mu})((s,t] \times A)\}_{s\leq t}$ is a martingale, for any $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. Here λ is supposed to be a σ-finite measure on $(E, \mathcal{B}(E))$ with $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$. The filtration $\{\mathcal{F}_t^\mu\}_{t\geq 0}$ generated by the coordinate measure μ is introduced by setting:

$$\dot{\mathcal{F}}_t^\mu = \sigma\{\mu((s,r] \times \Delta) : -\infty < s \leq r \leq t, \Delta \in \mathcal{B}(E)\},\ t \geq 0,$$

and taking the right-limits $\mathcal{F}_t^\mu = (\bigcap_{s>t} \dot{\mathcal{F}}_s^\mu) \vee \mathcal{N}_{P_2}$, $t \geq 0$, augmented by all the P_2-null sets. At last, we set $(\Omega, \mathcal{F}, P) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$, where $\mathcal{F}$ is completed with respect to P, and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by

$$\mathcal{F}_t := \mathcal{F}_t^B \otimes \mathcal{F}_t^\mu,\ t \geq 0, \text{ augmented by all P-null sets.}$$

For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. Fix $T > 0$, we also shall introduce the following three spaces of processes which will be used frequently in what follows:

$\mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R}) := \{(\psi_t)_{0\leq t\leq T}$ real-valued $\mathbb{F}$-adapted càdlàg process : $E[\sup_{0\leq t\leq T}|\psi_t|^2] < +\infty\}$;

$\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^n) := \{(\psi_t)_{0\leq t\leq T}$ $\mathbb{R}^n$-valued $\mathbb{F}$-progressively measurable process : $||\psi||^2 = E[\int_0^T |\psi_t|^2 dt] < +\infty\}$;

$\mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^n) := \{K : \Omega \times [0,T] \times E \to \mathbb{R}^n$ $\mathcal{P}\otimes\mathcal{B}(E)$-measurable mapping : $|K|^2_{L^2(\lambda)} = E[\int_0^T \int_E |K_t(e)|^2 \lambda(de)dt] < +\infty\}$.[1]

For the reader's convenience, let us first introduce the framework of mean-field SDEs with jumps and mean-field BSDEs with jumps which will be used in the follows. For more details we refer to [9].

Let $(\bar{\Omega},\bar{\mathcal{F}},\bar{P}) = (\Omega',\mathcal{F}',P')\otimes(\Omega,\mathcal{F},P) = (\Omega,\mathcal{F},P)\otimes(\Omega,\mathcal{F},P)$ be the (non-completed) product of $(\Omega,\mathcal{F},P)$ with itself. Let us endow the product space $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ with the filtration $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t = \mathcal{F}\otimes\mathcal{F}_t, 0 \leq t \leq T\}$.

Given a random variable ξ over $(\Omega,\mathcal{F},P)$, we denote by ξ' its (under $\bar{P}$) independent copy on $(\Omega',\mathcal{F}',P')$: $\xi'(\omega) = \xi(\omega)$, $\omega \in \Omega'(=\Omega)$. Extending ξ, ξ' canonically to $\bar{\Omega}$, $\xi(\omega',\omega) = \xi(\omega)$, $\xi'(\omega',\omega) = \xi'(\omega')$, $(\omega,\omega') \in \bar{\Omega} = \Omega'\times\Omega$, we have for all nonnegative Borel functions $f : \mathbb{R}^2 \to \mathbb{R}_+$, $E'[f(\xi',\xi)] = \int_{\Omega'} f(\xi'(\omega'),\xi)P'(d\omega') = E[f(\xi,x)]|_{x=\xi}$.

The driving coefficient of our mean-field BSDE with jumps is a mapping

$$f = f(\bar{\omega},t,y,z,k,y',z',k') : \bar{\Omega}\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m) \to \mathbb{R}^m$$

which is $\bar{\mathcal{P}}$-measurable, for each (y,z,k,y',z',k') in $\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)$. Moreover, we also make the following assumptions on f:

(i) There exists a constant $C \geq 0$ such that, $\bar{P}$-a.s., for all $t \in [0,T]$, $y_1,y_2,y_1',y_2' \in \mathbb{R}^m$, $z_1,z_2,z_1',z_2' \in \mathbb{R}^{m\times d}$, $k_1,k_2,k_1',k_2' \in L^2(E,\mathcal{B}(E),\lambda;\mathbb{R}^m)$,

$$|f(t,y_1,z_1,k_1,y_1',z_1',k_1') - f(t,y_2,z_2,k_2,y_2',z_2',k_2')| \leq C(|y_1-y_2| + |y_1'-y_2'| + |z_1-z_2| + |z_1'-z_2'| + |k_1-k_2|_{L^2(\lambda)} + |k_1'-k_2'|_{L^2(\lambda)}).$$

(ii) $|f(\cdot,0,0,0,0,0,0)| \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$. (H2.1)

Lemma 2.1. *Under the assumption* (H2.1)*, for any random variable* $\xi \in L^2(\Omega,\mathcal{F}_T,P)$*, the mean-field BSDE with jumps*

$$y(t) = \xi + \int_t^T E'[f(s,y(s),z(s),k(s),y(s)',z(s)',k(s)')]ds - \int_t^T z(s)dB_s - \int_t^T\int_E k(s,e)\tilde{\mu}(ds,de),\ 0 \leq t \leq T, \tag{2.1}$$

has a unique adapted solution

$$(y(t),z(t),k(t))_{t\in[0,T]} \in \mathcal{S}^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\times\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{m\times d})\times\mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m).$$

For the proof the readers may refer to [9].

Remark 2.1. *From above notions, the generator of above mean-field BSDE has to be understood as follows*

$$\begin{aligned}E'[f(s,y(s),z(s),k(s),y(s)',z(s)',k(s)')](\omega) &= E'[f(s,y(s,\omega),z(s,\omega),k(s,\omega),(y(s))',(z(s))')]\\ &= \int_\Omega f(\omega',\omega,s,y(s,\omega),z(s,\omega),y(s,\omega'),z(s,\omega'))P(d\omega'),\quad \omega\in\Omega.\end{aligned}$$

[1] $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-predictable sub-sets of $\Omega\times[0,T]$.

Remark 2.2. *If we assume*
(i) *For each fixed* $(x, x', e) \in \mathbb{R}^n \times \mathbb{R}^n \times E$, $b(\cdot, x, x')$, $\sigma(\cdot, x, x')$ *and* $\gamma(\cdot, x, x', e)$ *are continuous in* t*;*
(ii) *There exists a* $C > 0$ *such that, for all* $t \in [0, T]$, $x_1, x_2, x_1', x_2' \in \mathbb{R}^n$,

$$|b(t, x_1, x_1') - b(t, x_2, x_2')| + |\sigma(t, x_1, x_1') - \sigma(t, x_2, x_2')| \leq C(|x_1 - x_2| + |x_1' - x_2'|);$$

(iii) *There exists* ρ: $E \to \mathbb{R}^+$ *with* $\int_E \rho^2(e)\lambda(de) < +\infty$, *such that, for any* $t \in [0, T]$, $x_1, x_2, x_1', x_2' \in \mathbb{R}^n$ *and* $e \in E$,

$$\begin{aligned} &|\gamma(t, x_1, x_1', e) - \gamma(t, x_2, x_2', e)| \leq \rho(e)(|x_1 - x_2| + |x_1' - x_2'|), \\ &|\gamma(t, 0, 0, e)| \leq \rho(e). \end{aligned}$$

Then, for any random variable $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, *the following mean-field SDE with jumps:*

$$x(s) = \zeta + \int_t^s E'[b(r, x(r), (x(r))')]dr + \int_t^s E'[\sigma(r, x(r), (x(r))')]dB_r + \int_t^s \int_E E'[\gamma(r, x(r-), (x(r-))', e)]\widetilde{\mu}(dr, de), \tag{2.2}$$

has a unique adapted solution $x \in S^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$.

For more details, the reader is referred to, e.g., [9].

3 Mean-field FBSDE with jumps: Existence and uniqueness

We consider the following fully coupled mean-field forward-backward stochastic differential equations with jumps:

$$\begin{cases} dx(t) = \displaystyle\int_E E'[b(t, x(t), y(t), z(t), k(t, e), (x(t))', (y(t))', (z(t))', (k(t, e))')]\lambda(de)dt \\ \qquad + \displaystyle\int_E E'[\sigma(t, x(t), y(t), z(t), k(t, e), (x(t))', (y(t))', (z(t))', (k(t, e))')]\lambda(de)dB_t \\ \qquad + \displaystyle\int_E E'[h(t, x(t-), y(t-), z(t), k(t, e), (x(t-))', (y(t-))', (z(t))', (k(t, e))', e)]\widetilde{\mu}(dt, de), \\ -dy(t) = \displaystyle\int_E E'[f(t, x(t), y(t), z(t), k(t, e), (x(t))', (y(t))', (z(t))', (k(t, e))')]\lambda(de)dt \\ \qquad - z(t)dB_t - \displaystyle\int_E k(t, e)\widetilde{\mu}(dt, de), \ t \in [0, T], \\ x(0) = a, \\ y(T) = E'[\Phi(x(T), (x(T))')], \end{cases} \tag{3.1}$$

where the coefficients:

$$\begin{aligned} &b : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \to \mathbb{R}^n, \\ &\sigma : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \to \mathbb{R}^{n\times d}, \\ &h : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times E \to \mathbb{R}^n, \\ &f : \bar{\Omega} \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \to \mathbb{R}^m, \\ &\Phi : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m. \end{aligned}$$

Given an $m \times n$ full-rank matrix G. We use the following notations

$$\lambda = \begin{pmatrix} x \\ y \\ z \\ k \end{pmatrix}, \quad \widetilde{\lambda} = \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \\ \widetilde{k} \end{pmatrix}, \quad \mathbf{A}(\mathbf{t}, \lambda, \widetilde{\lambda}, \mathbf{e}) = \begin{pmatrix} -G^T f(t, \lambda, \widetilde{\lambda}) \\ Gb(t, \lambda, \widetilde{\lambda}) \\ G\sigma(t, \lambda, \widetilde{\lambda}) \\ Gh(t, \lambda, \widetilde{\lambda}, e) \end{pmatrix}$$

where $G\sigma = (G\sigma_1, \cdots, G\sigma_d)$. We use the standard inner product and Euclidean norm in $\mathbb{R}^{m\times d}$.

Definition 3.1. *A quadruple of processes* (X, Y, Z, K) *is called an adapted solution of mean-field FBSDE with jumps (3.1), if* $(X, Y, Z, K) \in \mathcal{H}^2_{\mathbb{F}}(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0, T; \mathbb{R}^m)$ *and satisfies equation (3.1).*

We assume that

(H3.1)
(i) $A(t, \lambda, \widetilde{\lambda}, e)$ is uniformly Lipschitz with respect to $\lambda, \widetilde{\lambda}$;
(ii) The coefficients (b, σ, h, f) are uniformly Lipschitz in $(x, y, z, k, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{k})$;
(iii) for each λ, $\widetilde{\lambda}$, $A(\cdot, \lambda, \widetilde{\lambda})$ is in $\mathcal{M}^2_{\mathbb{F}}(0, T)$;
(iv) $\Phi(x, \widetilde{x})$ is uniformly Lipchitz with respect to $x, \widetilde{x} \in \mathbb{R}^n$;
(v) for each $(x, y, z, k, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{k}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m$,
$\Phi(x, \widetilde{x}) \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_T, \bar{P}; \mathbb{R}^m)$;
b, σ, h, f are $\bar{\mathbb{F}}$-progressively measurable;
$h(\cdot, 0, 0, 0, 0, 0, 0, 0, 0, \cdot) \in \mathcal{K}^2_{\mathbb{F},\lambda}(0, T)$.

We also need the following monotonicity assumptions. For any $\lambda = (x, y, z, k)^T$, $\widetilde{\lambda} = (\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{k})^T$, $\bar{\lambda} = (\bar{x}, \bar{y}, \bar{z}, \bar{k})^T$, $\widehat{l} = l - \bar{l}$, where $l = x, y, z, k, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{k}$, respectively, it holds, $\bar{P}$-a.s.,

(H3.2)
(i) $\int_E \langle A(t, \lambda, \widetilde{\lambda}, e) - A(t, \bar{\lambda}, \widetilde{\lambda}, e), \lambda - \bar{\lambda}\rangle \lambda(de) \leq -\beta_1 |\widehat{x}|^2 - \beta_2(|\widehat{y}|^2 + |\widehat{z}|^2) - \beta_3 \int_E |\widehat{k}(e)|^2 \lambda(de)$,
(ii) $\langle \Phi(x, \widetilde{x}) - \Phi(\bar{x}, \widetilde{x}), G(x - \bar{x})\rangle \geq \mu_1 |\widehat{x}|^2$,

where β_1, β_2, β_3 and μ_1 are given nonnegative constants with

$$(1)\ \beta_1 - L_A C_0 > 0,\ \beta_2 - L_A C_0 \geq 0,\ \beta_3 - L_A \geq 0, \mu_1 - L_\Phi \lambda_1 > 0,$$

or

$$(2)\ \beta_1 - L_A C_0 = 0,\ \beta_2 - L_A C_0 > 0,\ \beta_3 - L_A > 0,\ \mu_1 - L_\Phi \lambda_1 > 0,$$

where L_A, L_Φ are the Lipschitz constants of A, Φ with respect to $\widetilde{\lambda}$, $\widetilde{x}$, respectively; C_0 and λ_1 satisfy $\int_E 1\lambda(de) \leq C_0$ and $|G\widehat{l}(T)| \leq \lambda_1 |\widehat{l}(T)|$, respectively.

Then we have the following main result in this section.

Theorem 3.1. *We assume (H3.2) and (H3.2) hold, then mean-field FBSDE with jumps (3.1) has a unique adapted solution* (X, Y, Z, K).

Proof. We first prove the uniqueness of the solution. Let $\lambda(t, e) = (x(t), y(t), z(t), k(t, e))$ and $\bar{\lambda}(t, e) = (\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{k}(t, e))$ be two solutions of equation (3.1). We set $\widehat{l} = l - \bar{l}$, where $l = x(t), y(t), z(t), k(t, e), \widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t), \widetilde{k}(t, e)$, respectively. Applying Itô's formula to $\langle G\widehat{x}(s), \widehat{y}(s)\rangle$, we get

$$\begin{aligned} &E\Big\langle E'[\Phi(x(T), (x(T))')] - E'[\Phi(\bar{x}(T), (\bar{x}(T))')], G(x(T) - \bar{x}(T))\Big\rangle \\ &= E\int_0^T \int_E \Big\langle E'[A(t, \lambda(t, e), (\lambda(t, e))', e)] - E'[A(t, \bar{\lambda}(t, e), (\bar{\lambda}(t, e))', e)], \lambda(t, e) - \bar{\lambda}(t, e)\Big\rangle \lambda(de)dt. \end{aligned}$$

From (H3.2) the monotonicity assumptions of Φ and A, we get

$$\begin{aligned} (\mu_1 - L_\Phi \lambda_1)E[|\widehat{x}(T)|^2] \leq\ & -E\int_0^T \Big[\beta_1 |\widehat{x}(t)|^2 + \beta_2(|\widehat{y}(t)|^2 + |\widehat{z}(t)|^2) + \beta_3 \int_E |\widehat{k}(t, e)|^2 \lambda(de)\Big]dt \\ & + L_A C_0 E\int_0^T \Big[|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2\Big]dt + L_A E\int_0^T \int_E |\widehat{k}(t, e)|^2 \lambda(de)dt. \end{aligned} \tag{3.2}$$

(1) When $\beta_1 - L_A C_0 > 0$, $\beta_2 - L_A C_0 \geq 0$, $\beta_3 - L_A \geq 0$, $\mu_1 - L_\Phi \lambda_1 > 0$, from (3.2) we can get

$$|\widehat{x}(t)|^2 = 0, dtdP\text{-a.e.},\ |\widehat{x}(T)|^2 = 0, P\text{-a.s.}$$

Thus, $\Phi(x(T), (x(T))') = \Phi(\bar{x}(T), (\bar{x}(T))')$, $\bar{P}$-a.s. Therefore, from Lemma 2.1 it follows that

$$||\widehat{y}||_{\mathcal{S}^2_{\mathbb{F}}} = 0,\ ||\widehat{z}||_{\mathcal{H}^2_{\mathbb{F}}} = 0,\ ||\widehat{k}||_{\mathcal{K}^2_{\mathbb{F},\lambda}} = 0.$$

(2) When $\beta_1 - L_A C_0 = 0,\ \beta_2 - L_A C_0 > 0,\ \beta_3 - L_A > 0,\ \mu_1 - L_\Phi \lambda_1 > 0$, from (3.2) we can get

$$||\widehat{y}||_{\mathcal{S}^2_{\mathbb{F}}} = 0,\ ||\widehat{z}||_{\mathcal{H}^2_{\mathbb{F}}} = 0,\ ||\widehat{k}||_{\mathcal{K}^2_{\mathbb{F},\lambda}} = 0,\ x(T) = \bar{x}(T),\ P\text{-a.s.}$$

From the uniqueness of solutions of mean-field SDEs with jumps (refer to [9], or Remark 2.2), we get $x(t) = \bar{x}(t)$, P-a.s., for all $t \in [0, T]$.

We now prove the existence of the solution. For this we introduce the following mean-field FBSDEs with jumps parameterized by $\alpha \in [0, 1]$:

$$\begin{cases} dx^\alpha(t) = \Big[\alpha \int_E E'[b(t, \chi^\alpha(t, e))]\lambda(de) + E'[\phi(t)]\Big]dt + \Big[\alpha \int_E E'[\sigma(t, \chi^\alpha(t, e))]\lambda(de) + E'[\psi(t)]\Big]dB_t \\ \qquad\qquad + \int_E \Big[\alpha E'[h(t, \chi^\alpha(t-, e))] + E'[\varphi(t, e)]\Big]\widetilde{\mu}(dt, de), \\ -dy^\alpha(t) = \Big[(1-\alpha)\beta_1 G x^\alpha(t) + \alpha \int_E E'[f(t, \chi^\alpha(t, e))]\lambda(de) + E'[\gamma(t)]\Big]dt - z^\alpha(t)dB_t - \int_E k^\alpha(t, e)\widetilde{\mu}(dt, de), \\ x^\alpha(0) = a, \\ y^\alpha(T) = \alpha E'[\Phi(x^\alpha(T), (x^\alpha(T))')] + (1-\alpha)G x^\alpha(T) + \xi, \end{cases} \tag{3.3}$$

where $\chi^\alpha(t, e) = (x^\alpha(t), y^\alpha(t), z^\alpha(t), k^\alpha(t, e), (x^\alpha(t))', (y^\alpha(t))', (z^\alpha(t))', (k^\alpha(t, e))')$, $\chi^\alpha(t-, e) = (x^\alpha(t-), y^\alpha(t-), z^\alpha(t), k^\alpha(t, e), (x^\alpha(t-))', (y^\alpha(t-))', (z^\alpha(t))', (k^\alpha(t, e))', e)$; $\phi,\ \psi$ and γ are given processes in $\mathcal{H}^2_{\mathbb{F}}(0, T)$ with values in $\mathbb{R}^n,\ \mathbb{R}^{n\times d}$ and $\mathbb{R}^m$, respectively; $\varphi \in \mathcal{K}^2_{\mathbb{F},\lambda}(0, T; \mathbb{R}^n)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, P)$.

When $\alpha = 0$, from the existence and the uniqueness of the solutions of Mckean-Vlasov equation with jumps and mean-field BSDE with jumps we know equation (3.3) has a unique solution. Then from Lemma 7.1 in Appendix, there exists a positive constant δ_0 depending on Lipschitz constants, $\beta_1,\ \beta_2,\ \beta_3,\ \mu_1,\ \lambda_1$ and T, such that, for every $\delta \in [0, \delta_0]$, equation (3.3) for $\alpha = \delta$ has a unique solution. We can repeat this process N times where $1 \le N\delta_0 \le 1 + \delta_0$. It means that, in particular, mean-field FBSDE (3.3) for $\alpha = 1$ has a unique solution, i.e., (3.1) has a unique solution.
The proof is complete. ∎

Remark 3.1. *We note that the existence and the uniqueness of the solution of our equation (3.1) can also be obtained if the monotonicity assumption (H3.2) in Theorem 3.1 is changed by the following form*

(H3.3) (i) $\int_E \langle A(t, \lambda, \widetilde{\lambda}, e) - A(t, \bar{\lambda}, \widetilde{\lambda}, e), \lambda - \bar{\lambda}\rangle \lambda(de) \ge \beta_1|\widehat{x}|^2 + \beta_2(|\widehat{y}|^2 + |\widehat{z}|^2) + \beta_3 \int_E |\widehat{k}(e)|^2\lambda(de)$;
(ii) $\langle \Phi(x, \widetilde{x}) - \Phi(\bar{x}, \widetilde{x}), G(x - \bar{x})\rangle \le -\mu_1|\widehat{x}|^2$;

where $\beta_1,\ \beta_2,\ \beta_3$ and μ_1 are given nonnegative constants with $\beta_1 - L_A C_0 > 0,\ \beta_2 - L_A C_0 \ge 0,\ \beta_3 - L_A \ge 0,\ \mu_1 - L_\Phi\lambda_1 > 0$, or $\beta_1 - L_A C_0 = 0,\ \beta_2 - L_A C_0 > 0,\ \beta_3 - L_A > 0,\ \mu_1 - L_\Phi\lambda_1 > 0$, where $L_A,\ L_\Phi,\ C_0$ and λ_1 are the same as those in (H3.2).

The proof of this result is similar to that of Theorem 3.1 but one need to notice that the equation (3.3) should be changed into the following form

$$\begin{cases} dx^\alpha(t) = \Big[\alpha \int_E E'[b(t, \chi^\alpha(t, e))]\lambda(de) + E'[\phi(t)]\Big]dt + \Big[\alpha \int_E E'[\sigma(t, \chi^\alpha(t, e))]\lambda(de) + E'[\psi(t)]\Big]dB_t \\ \qquad\qquad + \int_E \Big[\alpha E'[h(t, \chi^\alpha(t, e), e)] + E'[\varphi(t, e)]\Big]\widetilde{\mu}(dt, de), \\ -dy^\alpha(t) = \Big[-(1-\alpha)\beta_1 G x^\alpha(t) + \alpha \int_E E'[f(t, \chi^\alpha(t, e))]\lambda(de) + E'[\gamma(t)]\Big]dt - z^\alpha(t)dB_t - \int_E k^\alpha(t, e)\widetilde{\mu}(dt, de), \\ x^\alpha(0) = a, \\ y^\alpha(T) = \alpha E'[\Phi(x^\alpha(T), (x^\alpha(T))')] - (1-\alpha)G x^\alpha(T) + \xi. \end{cases}$$

Remark 3.2. *(i) When* Φ *does not depends on* x, $\widetilde{x}$, *i.e.,* $\Phi(x,\widetilde{x})=\xi\in L^2(\Omega,\mathcal{F}_T,P)$ *is given, for the existence and the uniqueness of the solution of mean-field FBSDE (3.1), the monotonicity assumption (H3.2) can be weakened as*

$$\int_E\langle A(t,\lambda,\widetilde{\lambda},e)-A(t,\bar{\lambda},\widetilde{\lambda},e),\lambda-\bar{\lambda}\rangle\lambda(de)\leq-\beta_1|\widehat{x}|^2-\beta_2|\widehat{y}|^2;$$

where β_1 *and* β_2 *are given nonnegative constants with* $\beta_1-C_0L_A\geq 0,\ \beta_2-C_0L_A\geq 0$ *(the equalities can not be established at the same time),* L_A *is the Lipchitz constants of* A *with respect to* $\widetilde{\lambda}$.

(ii) When σ *does not depends on* z, z', k, k', *the mean-field FBSDE (3.1) also has a unique adapted solution, but the monotonicity (H3.2) should be weakened as*

$$\begin{array}{ll}\text{(i)} & \int_E\langle A(t,\lambda,\widetilde{\lambda},e)-A(t,\bar{\lambda},\widetilde{\lambda},e),\lambda-\bar{\lambda}\rangle\lambda(de)\leq-\beta_1|\widehat{x}|^2;\\ \text{(ii)} & \langle\Phi(x,\widetilde{x})-\Phi(\bar{x},\widetilde{x}),G(x-\bar{x})\rangle\geq\mu_1|\widehat{x}|^2,\end{array}$$

where β_1 *and* μ *are given nonnegative constants with* $\beta_1>L_A+2L_AC_{L_g,T}C_0^2$, $\mu_1>L_\Phi\lambda_1+8C_{L_g,T}L_\Phi^2L_AC_0$ *(* $C_{L_g,T}:=\exp\{[C_0(4L_f+12L_f^2+8L_f^2C_0)+1]T\}$*).*

Example 3.1. *We consider*

$$\begin{cases}dx(t)=E'[-y'(t)-2y(t)]dt+E'[-z'(t)-2z(t)]dB_t+\displaystyle\int_E E'[-k'(t,e)-2k(t,e)]\widetilde{\mu}(dt,de),\ t\in[0,T],\\ -dy(t)=E'[x'(t)+2x(t)]dt-z(t)dB_t-\displaystyle\int_E k(t,e)\widetilde{\mu}(dt,de),\ t\in[0,T],\\ x(0)=1,\\ y(T)=E'[x'(T)+2x(T)].\end{cases}$$

We can take $\beta_1=\beta_2=\beta_3=2$, $\mu_1=2$, $C_0=1$, $L_A=1$, $L_\Phi=1$, *from Theorem* 3.1, *we know it has a unique solution.*

We now give an example to explain that the assumption (H3.2) is necessary for Theorem 3.1, i.e., if the coefficients of our equation do not satisfy (H3.2), the solution of equation (3.1) may not exist.

Example 3.2. *We take* $m=n=d=1$ *here. We consider*

$$\begin{cases}dx(t)=E[y(t)]dt+dB_t+\displaystyle\int_E k(t,e)\widetilde{\mu}(dt,de),\ \ t\in[0,\frac{3}{4}\pi],\\ -dy(t)=E[x(t)]dt-z(t)dB_t-\displaystyle\int_E k(t,e)\widetilde{\mu}(dt,de),\ \ t\in[0,\frac{3}{4}\pi],\\ x(0)=1,\ y(\frac{3}{4}\pi)=-E[x(\frac{3}{4}\pi)],\ \ t\in[0,\frac{3}{4}\pi].\end{cases}\tag{3.4}$$

It's easy to check this equation does not satisfy (H3.2), we point out that it also does not exist an adapted solution. In fact, if $(x,y,z,k)_{0\leq t\leq\frac{3}{4}\pi}$ *is the solution of mean-field FBSDE (3.4), then* $(E[x(t)],E[y(t)])$ *is the solution of the following ordinary differential equation (ODE, for short):*

$$\begin{cases}\dot{X}=Y,\ \dot{Y}=-X,\\ X(0)=1,\ Y(\frac{3}{4}\pi)=-X(\frac{3}{4}\pi),t\in[0,\frac{3}{4}\pi].\end{cases}\tag{3.5}$$

But we know this ODE has no solution, therefore there is no adapted solution of (3.4).

4 Continuity property on the parameters

In this section we will discuss the continuity of the solution of equation (3.1) depending on parameters. We consider the following mean-field FBSDEs with coefficients $(b_\alpha, \sigma_\alpha, h_\alpha, f_\alpha, \Phi_\alpha),\ \alpha \in \mathbb{R}$:

$$\begin{cases} dx^\alpha(t) = \displaystyle\int_E E'[b_\alpha(t,\chi^\alpha(t,e))]\lambda(de)dt + \int_E E'[\sigma_\alpha(t,\chi^\alpha(t,e))]\lambda(de)dB_t + \int_E E'[h_\alpha(t,\chi^\alpha(t-,e))]\widetilde{\mu}(dt,de), \\ -dy^\alpha(t) = \displaystyle\int_E E'[f_\alpha(t,\chi^\alpha(t,e))]\lambda(de)dt - z^\alpha(t)dB_t - \int_E k^\alpha(t,e)\widetilde{\mu}(dt,de), \\ x^\alpha(0) = a, \\ y^\alpha(T) = E'[\Phi_\alpha(x^\alpha(T),(x^\alpha(T))')], \end{cases} \tag{4.1}$$

where

$$\chi^\alpha(t,e) = (x^\alpha(t), y^\alpha(t), z^\alpha(t), k^\alpha(t,e), (x^\alpha(t))', (y^\alpha(t))', (z^\alpha(t))', (k^\alpha(t,e))'),$$
$$\chi^\alpha(t-,e) = (x^\alpha(t-), y^\alpha(t-), z^\alpha(t), k^\alpha(t,e), (x^\alpha(t-))', (y^\alpha(t-))', (z^\alpha(t))', (k^\alpha(t,e))', e),$$

and the mappings $b_\alpha, \sigma_\alpha, h_\alpha, f_\alpha, \Phi_\alpha, A_\alpha = (-G^T f_\alpha, Gb_\alpha, G\sigma_\alpha, Gh_\alpha)^T$ satisfy (H3.1) and (H3.2), for each $\alpha \in \mathbb{R}$. Then, from Theorem 3.1 we know mean-field FBSDE (4.1) has a unique solution $(x^\alpha, y^\alpha, z^\alpha, k^\alpha)$ for each $\alpha \in \mathbb{R}$.
Let us give some more assumptions.

(H4.1) (i) The coefficients $(b_\alpha, \sigma_\alpha, h_\alpha, f_\alpha, \Phi_\alpha), \alpha \in \mathbb{R}$, are uniformly Lipschitz in $(x, y, z, k, \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{k})$;
(ii) The mappings $\alpha \mapsto (b_\alpha, \sigma_\alpha, h_\alpha, f_\alpha, \Phi_\alpha),\ \alpha \in \mathbb{R}$, are continuous respectively.

Then we have the following continuity property.

Theorem 4.1. *Let the coefficients $(b_\alpha, \sigma_\alpha, h_\alpha, f_\alpha, \Phi_\alpha), \alpha \in \mathbb{R}$, satisfy (H3.1), (H3.2) and (H4.1), and the associated solution of mean-field FBSDE with jumps (4.1) is denoted by $(x^\alpha, y^\alpha, z^\alpha, k^\alpha)$. Then, the mappings*

$$\alpha \mapsto (x^\alpha, y^\alpha, z^\alpha, k^\alpha, x^\alpha(T)) : \mathbb{R} \mapsto \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m) \times L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$$

is continuous.

Proof. For simplicity of notations, we only prove the continuity of the solutions $(x^\alpha, y^\alpha, z^\alpha, k^\alpha, x^\alpha(T))$ of mean-field FBSDE (4.1) at $\alpha = 0$. We want to prove that $(x^\alpha, y^\alpha, z^\alpha, k^\alpha, x^\alpha(T))$ converges to $(x^0, y^0, z^0, k^0, x^0(T))$ in $\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m) \times L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ as α tends to 0. We set $\lambda^\alpha(t,e) = (x^\alpha(t), y^\alpha(t), z^\alpha(t), k^\alpha(t,e))$, and $\widehat{\lambda}(t,e) = \lambda^\alpha(t,e) - \lambda^0(t,e) = (\widehat{x}(t), \widehat{y}(t), \widehat{z}(t), \widehat{k}(t,e)) = (x^\alpha(t) - x^0(t), y^\alpha(t) - y^0(t), z^\alpha(t) - z^0(t), k^\alpha(t,e)) - k^0(t,e))$, then from (4.1) we know

$$\begin{cases} d\widehat{x}(t) = \displaystyle\int_E E'\Big[b_\alpha(t,\lambda^\alpha(t,e),(\lambda^\alpha(t,e))') - b_0(t,\lambda^0(t,e),(\lambda^0(t,e))')\Big]\lambda(de)dt \\ \qquad + \displaystyle\int_E E'\Big[\sigma_\alpha(t,\lambda^\alpha(t,e),(\lambda^\alpha(t,e))') - \sigma_0(t,\lambda^0(t,e),(\lambda^0(t,e))')\Big]\lambda(de)dB_t \\ \qquad + \displaystyle\int_E E'\Big[h_\alpha(t,\lambda^\alpha(t,e),(\lambda^\alpha(t,e))',e) - h_0(t,\lambda^0(t,e),(\lambda^0(t,e))',e)\Big]\widetilde{\mu}(dt,de), \\ -d\widehat{y}(t) = \displaystyle\int_E E'\Big[f_\alpha(t,\lambda^\alpha(t,e),(\lambda^\alpha(t,e))') - f_0(t,\lambda^0(t,e),(\lambda^0(t,e))')\Big]\lambda(de)dt \\ \qquad - \widehat{z}(t)dB_t - \displaystyle\int_E \widehat{k}(t,e)\widetilde{\mu}(dt,de), \\ \widehat{x}(0) = 0, \\ \widehat{y}(T) = E'\Big[\Phi_\alpha(x^\alpha(T),(x^\alpha(T))') - \Phi_0(x^0(T),(x^0(T))')\Big]. \end{cases} \tag{4.2}$$

From assumptions (H3.1), (H3.2) and (H4.1), and standard estimates of $\widehat{x}(t)$ and $(\widehat{y}(t),\widehat{z}(t),\widehat{k}(t))$, we get

$$\sup_{0\le t\le T}E|\widehat{x}(t)|^2\le C_1E\int_0^T\Big(|\widehat{y}(t)|^2+|\widehat{z}(t)|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt+C_1\bar{E}\int_0^T\int_E\Big[|\widehat{b}(t,e)|^2+|\widehat{\sigma}(t,e)|^2\Big]\lambda(de)dt; \tag{4.3}$$

$$\begin{aligned}&E\int_0^T\Big(|\widehat{y}(t)|^2+|\widehat{z}(t)|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt\\ &\le C_1\Big\{E\int_0^T|\widehat{x}(t)|^2dt+E|\widehat{x}(T)|^2+\bar{E}\int_0^T\int_E|\widehat{f}(t,e)|^2\lambda(de)dt+\bar{E}[|\widehat{\Phi}(T)|^2]\Big\},\end{aligned} \tag{4.4}$$

here C_1 depends on the Lipchitz constants of $(b_\alpha,\sigma_\alpha,h_\alpha,f_\alpha)$, constant C_0 and T, where

$$\begin{aligned}\widehat{b}(t,e)&=b_\alpha(t,\lambda^0(t,e),(\lambda^0(t,e))')-b_0(t,\lambda^0(t,e),(\lambda^0(t,e))'),\\ \widehat{\sigma}(t,e)&=\sigma_\alpha(t,\lambda^0(t,e),(\lambda^0(t,e))')-\sigma_0(t,\lambda^0(t,e),(\lambda^0(t,e))'),\\ \widehat{h}(t,e)&=h_\alpha(t,\lambda^0(t,e),(\lambda^0(t,e))',e)-h_0(t,\lambda^0(t,e),(\lambda^0(t,e))',e),\\ \widehat{f}(t,e)&=-f_\alpha(t,\lambda^0(t,e),(\lambda^0(t,e))')+f_0(t,\lambda^0(t,e),(\lambda^0(t,e))'),\\ \widehat{\Phi}(T)&=\Phi_\alpha(x^0(T),(x^0(T))')-\Phi_0(x^0(T),(x^0(T))').\end{aligned}$$

Applying Itô's formula to $\langle G\widehat{x}(t),\widehat{y}(t)\rangle$ it yields

$$\begin{aligned}&E\big\langle E'[\Phi_\alpha(x^\alpha(T),(x^\alpha(T))')-\Phi_\alpha(x^0(T),(x^0(T))')],G\widehat{x}(T)\big\rangle\\ &\quad+E\big\langle E'[\Phi_\alpha(x^0(T),(x^0(T))')-\Phi_0(x^0(T),(x^0(T))')],G\widehat{x}(T)\big\rangle\\ &=E\int_0^T\int_E E'\big\langle A_\alpha(t,\lambda^\alpha(t,e),(\lambda^\alpha(t,e))',e)-A_\alpha(t,\lambda^0(t,e),(\lambda^0(t,e))',e),\widehat{\lambda}(t,e)\big\rangle\lambda(de)dt\\ &\quad+E\int_0^T\int_E E'\Big[\langle G\widehat{x}(t),\widehat{f}(t,e)\rangle+\langle G^T\widehat{y}(t),\widehat{b}(t,e)\rangle+\langle G^T\widehat{z}(t),\widehat{\sigma}(t,e)\rangle+\langle G^T\widehat{k}(t,e),\widehat{h}(t,e)\rangle\Big]\lambda(de)dt.\end{aligned}$$

With the help of (H3.2) and the Lipschitz properties of A_α and Φ_α, we have

$$\begin{aligned}&(\mu_1-L_{\Phi_\alpha}\lambda_1)E|\widehat{x}(T)|^2+(\beta_1-C_0L_{A_\alpha})E\int_0^T|\widehat{x}(t)|^2dt+(\beta_2-C_0L_{A_\alpha})E\int_0^T(|\widehat{y}(t)|^2+|\widehat{z}(t)|^2)dt\\ &\quad+(\beta_3-L_{A_\alpha})E\int_0^T\int_E|\widehat{k}(t,e)|^2\lambda(de)dt\\ &\le C_2E\Big[E'|\widehat{\Phi}(T)|^2+\int_0^T\int_E E'\Big(|\widehat{b}(t,e)|^2+|\widehat{f}(t,e)|^2+|\widehat{\sigma}(t,e)|^2\Big)\lambda(de)dt\Big]\\ &\quad+\delta\Big[E|\widehat{x}(T)|^2+E\int_0^T\Big(|\widehat{x}(t)|^2+|\widehat{y}(t)|^2+|\widehat{z}(t)|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt\Big],\end{aligned} \tag{4.5}$$

for any $\delta>0$. Since $\beta_1-C_0L_{A_\alpha}>0$, $\beta_2-C_0L_{A_\alpha}\ge0$, $\beta_3-L_{A_\alpha}\ge0$, $\mu_1-L_{\Phi_\alpha}\lambda_1>0$ $\big($the situation of $\beta_1-C_0L_{A_\alpha}=0$, $\beta_2-C_0L_{A_\alpha}>0$, $\beta_3-L_{A_\alpha}>0$, $\mu_1-L_{\Phi_\alpha}\lambda_1>0$ can be similar discussed$\big)$, from (4.5) we have

$$\begin{aligned}&(\mu_1-L_{\Phi_\alpha}\lambda_1)E|\widehat{x}(T)|^2+(\beta_1-C_0L_{A_\alpha})E\int_0^T|\widehat{x}(t)|^2dt\\ &\le C_2E\Big[E'|\widehat{\Phi}(T)|^2+\int_0^T\int_E E'\Big(|\widehat{b}(t,e)|^2+|\widehat{\sigma}(t,e)|^2+|\widehat{h}(t,e)|^2+|\widehat{f}(t,e)|^2\Big)\lambda(de)dt\Big]\\ &\quad+\delta\Big[E|\widehat{x}(T)|^2+E\int_0^T\Big(|\widehat{x}(t)|^2+|\widehat{y}(t)|^2+|\widehat{z}(t)|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt\Big].\end{aligned} \tag{4.6}$$

Using (4.4) and (4.6) we can take sufficiently small δ such that

$$\begin{aligned} & E|\widehat{x}(T)|^2+E\int_0^T\Big(|\widehat{x}(t)|^2+|\widehat{y}(t)|^2+|\widehat{z}|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt \\ \leq\ & C\bar{E}\Big[|\widehat{\Phi}(T)|^2+\int_0^T\int_E\Big(|\widehat{b}(t,e)|^2+|\widehat{\sigma}(t,e)|^2+|\widehat{h}(t,e)|^2+|\widehat{f}(t,e)|^2\Big)dt\Big], \end{aligned} \tag{4.7}$$

here the constant C only depends on $C_1,\ C_2,\ \beta_1,\ \mu_1,\ L_{A_\alpha},\ L_{\Phi_\alpha}$.
Hence, we have that $(x^\alpha,y^\alpha,z^\alpha,k^\alpha,x^\alpha(T))$ converges to $(x^0,y^0,z^0,k^0,x^0(T))$ in $\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d})\times\mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m)\times L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n)$ as α tends to 0. ∎

5 Maximum principle for the controlled fully coupled mean-field FBSDEs with jumps

We consider the following controlled fully coupled mean-field forward-backward SDEs with jumps:

$$\begin{cases} dx^v(t)= \displaystyle\int_E E'[b\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dt+\int_E E'[\sigma\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dB_t \\ \qquad\qquad +\displaystyle\int_E E'[h\big(t,\pi^v(t-,e),v(t),e\big)]\widetilde{\mu}(dtde), \\ -dy^v(t)= \displaystyle\int_E E'[f\big(t,\pi^v(t,e),v(t)\big)]\lambda(de)dt-z^v(t)dB_t-\int_E k^v(t,e)\widetilde{\mu}(dtde), \\ x^v(0)=a,\quad y^v(T)=E'[\Phi\big(x^v(T),(x^v(T))'\big)], \end{cases} \tag{5.1}$$

where

$$\pi^v(t,e)=\big(x^v(t),y^v(t),z^v(t),k^v(t,e),(x^v(t))',(y^v(t))',(z^v(t))',(k^v(t,e))'\big),$$
$$\pi^v(t-,e)=\big(x^v(t-),y^v(t-),z^v(t),k^v(t,e),(x^v(t-))',(y^v(t-))',(z^v(t))',(k^v(t,e))'\big).$$

Let U be a nonempty convex subset of $\mathbb{R}^k$, we define the admissible control set

$$\mathcal{U}_{ad}=\{v(\cdot)\in\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^k)|v(t)\in U,\ 0\leq t\leq T,\ \bar{P}\text{-a.s.}\}.$$

We now define the following cost functional:

$$J(v(\cdot))=E\Big[\int_0^T\int_E E'\big[g\big(t,\pi^v(t,e),v(t)\big)\big]\lambda(de)dt+E'[\varphi\big(x^v(T),(x^v(T))'\big)]+\gamma\big(y^v(0)\big)\Big], \tag{5.2}$$

where

$$\begin{aligned} & b:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times U\to\mathbb{R}^n, \\ & \sigma:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times U\to\mathbb{R}^{n\times d}, \\ & h:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times U\times E\to\mathbb{R}^n, \\ & f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times U\to\mathbb{R}^m, \\ & g:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times U\to\mathbb{R}, \\ & \Phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^m\ ,\ \varphi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\ ,\ \gamma:\mathbb{R}^m\to\mathbb{R}. \end{aligned}$$

Our stochastic optimal control problem is to minimize the cost functional $J(v(\cdot))$ over all admissible controls. An admissible control $u(\cdot)$ is called an optimal control if the cost functional $J(v(\cdot))$ attains the minimum at $u(\cdot)$. Equation (5.1) is called the state equation, the solution $(x(\cdot),y(\cdot),z(\cdot),k(\cdot,\cdot))$ corresponding to $u(\cdot)$ is called the optimal trajectory.
We assume

$$\text{(H5.1)}\begin{cases} \text{(i)} & b,\ \sigma,\ h,\ f,\ g,\ \Phi,\ \varphi \text{ and } \gamma \text{ are continuously differentiable to } (x,\ y,\ z,\ k,\ \widetilde{x},\ \widetilde{y},\ \widetilde{z},\ \widetilde{k},\ v); \\ \text{(ii)} & \text{The derivatives of } b,\ \sigma,\ h,\ f,\ \Phi \text{ are bounded;} \\ \text{(iii)} & \text{The derivatives of } g \text{ are bounded by } C(1+|x|+|y|+|z|+|k|+|\widetilde{x}|+|\widetilde{y}|+|\widetilde{z}|+|\widetilde{k}|+|v|); \\ \text{(iv)} & \text{The derivatives of } \varphi \text{ and } \gamma \text{ are bounded by } C(1+|x|+|\widetilde{x}|) \text{ and } C(1+|y|),\ \text{respectively;} \\ \text{(v)} & \text{For any given admissible control } v(\cdot),\ \text{the coefficients satisfy (H3.1) and (H3.2).} \end{cases}$$

Let $u(\cdot)$ be an optimal control and $(x(\cdot), y(\cdot), z(\cdot), k(\cdot,\cdot))$ be the corresponding optimal trajectory. Let $v(\cdot)$ be such that $u(\cdot)+v(\cdot)\in\mathcal{U}_{ad}$. Since U is convex, we may choose the perturbation

$$u_\rho(\cdot)=u(\cdot)+\rho v(\cdot)\in\mathcal{U}_{ad},$$

for any $0\le\rho\le 1$.

To simplify the form of the following variational equation (5.3), variational inequality (5.7) and adjoint equation (5.8), we introduce the following notations:

$$\begin{aligned}
&\theta(t,e)=\big(t,x(t),y(t),z(t),k(t,e),(x(t))',(y(t))',(z(t))',(k(t,e))',u(t)\big),\\
&\theta(t-,e)=\big(t,x(t-),y(t-),z(t),k(t,e),(x(t-))',(y(t-))',(z(t))',(k(t,e))',u(t),e\big),\\
&\rho(t,e)=\big(t,(x(t))',(y(t))',(z(t))',(k(t,e))',x(t),y(t),z(t),k(t,e),(u(t))'\big),\\
&\rho(t-,e)=\big(t,(x(t-))',(y(t-))',(z(t))',(k(t,e))',x(t-),y(t-),z(t),k(t,e),(u(t))',e\big).
\end{aligned}$$

We denote by $(x_\rho(\cdot),\ y_\rho(\cdot),\ z_\rho(\cdot),k_\rho(\cdot,\cdot))$ the trajectory corresponding to u_ρ. Then we have the following convergence result.

Lemma 5.1. *Under the assumption (H5.1), it holds*

$$\lim_{\rho\to0}\frac{x_\rho(t)-x(t)}{\rho}=x^1(t),\quad \lim_{\rho\to0}\frac{y_\rho(t)-y(t)}{\rho}=y^1(t),\ \lim_{\rho\to0}\frac{z_\rho(t)-z(t)}{\rho}=z^1(t),\ \text{ in } \mathcal{H}^2_{\mathbb{F}}(0,T).$$

$$\lim_{\rho\to0}\frac{k_\rho(t,e)-k(t,e)}{\rho}=k^1(t,e),\ \text{ in } \mathcal{K}^2_{\mathbb{F}\lambda}(0,T),$$

where $(x^1(\cdot),y^1(\cdot),z^1(\cdot),k^1(\cdot,\cdot))$ is the unique solution of the following variational equation:

$$\left\{\begin{aligned}
dx^1(t)=&\int_E E'\big\{b_x(\theta(t,e))x^1(t)+b_y(\theta(t,e))y^1(t)+b_z(\theta(t,e))z^1(t)+b_k(\theta(t,e))k^1(t,e)+b_v(\theta(t,e))v(t)\\
&+b_{\widetilde{x}}(\theta(t,e))(x^1(t))'+b_{\widetilde{y}}(\theta(t,e))(y^1(t))'+b_{\widetilde{z}}(\theta(t,e))(z^1(t))'+b_{\widetilde{k}}(\theta(t,e))(k^1(t,e))'\big\}\lambda(de)dt\\
&+\int_E E'\big\{\sigma_x(\theta(t,e))x^1(t)+\sigma_y(\theta(t,e))y^1(t)+\sigma_z(\theta(t,e))z^1(t)+\sigma_k(\theta(t,e))k^1(t,e)+\sigma_v(\theta(t,e))v(t)\\
&+\sigma_{\widetilde{x}}(\theta(t,e))(x^1(t))'+\sigma_{\widetilde{y}}(\theta(t,e))(y^1(t))'+\sigma_{\widetilde{z}}(\theta(t,e))(z^1(t))'+\sigma_{\widetilde{k}}(\theta(t,e))(k^1(t,e))'\big\}\lambda(de)dB_t\\
&+\int_E E'\big\{h_x(\theta(t-,e))x^1(t)+h_y(\theta(t-,e))y^1(t)+h_z(\theta(t-,e))z^1(t)+h_k(\theta(t-,e))k^1(t,e)\\
&+h_v(\theta(t-,e))v(t)+h_{\widetilde{x}}(\theta(t-,e))(x^1(t))'+h_{\widetilde{y}}(\theta(t-,e))(y^1(t))'+h_{\widetilde{z}}(\theta(t-,e))(z^1(t))'\\
&+h_{\widetilde{k}}(\theta(t-,e))(k^1(t,e))'\big\}\widetilde{\mu}(dtde),\\
-dy^1(t)=&\int_E E'\big\{f_x(\theta(t,e))x^1(t)+f_y(\theta(t,e))y^1(t)+f_z(\theta(t,e))z^1(t)+f_k(\theta(t,e))k^1(t,e)+f_v(\theta(t,e))v(t)\\
&+f_{\widetilde{x}}(\theta(t,e))(x^1(t))'+f_{\widetilde{y}}(\theta(t,e))(y^1(t))'+f_{\widetilde{z}}(\theta(t,e))(z^1(t))'+f_{\widetilde{k}}(\theta(t,e))(k^1(t,e))'\big\}\lambda(de)dt\\
&-z^1(t)dB_t-\int_E k^1(t,e)\widetilde{\mu}(dtde),\\
x^1(0)=&0,\ \ y^1(T)=E'[\Phi_x\big(x(T),(x(T))'\big)x^1(T)+\Phi_{\widetilde{x}}\big(x(T),(x(T))'\big)(x^1(T))'].
\end{aligned}\right.\tag{5.3}$$

Remark 5.1. *(i)When $l=b,\sigma,h,f,\Phi$, respectively, l_x is the partial derivative of $l(t,x,y,z,\widetilde{x},\widetilde{y},\widetilde{z},k,\widetilde{k},v)$ with respect to x; $l_{\widetilde{x}}$ is the partial derivative of $l(t,x,y,z,\widetilde{x},\widetilde{y},\widetilde{z},k,\widetilde{k},v)$ with respect to $\widetilde{x}$. Similar to $l_y,l_z,l_k,l_{\widetilde{y}},l_{\widetilde{z}},l_{\widetilde{k}},l_v$.*
(ii)From (H5.1), it is easy to verify that equation (5.3) satisfies (H3.1) and (H3.2), then there exists a unique solution (x^1,y^1,z^1,k^1) of linear mean-field FBSDE (5.3).

Proof. Let $\widehat{x}(t)=x_\rho(t)-x(t)$, $\widehat{y}(t)=y_\rho(t)-y(t)$, $\widehat{z}(t)=z_\rho(t)-z(t)$, $\widehat{k}(t,e)=k_\rho(t,e)-k(t,e)$. Then

$$\begin{cases} d\widehat{x}(t)=\displaystyle\int_E E'\Big[b\big(t,\pi_\rho(t,e),u_\rho(t)\big)-b\big(t,\pi(t,e),u(t)\big)\Big]\lambda(de)dt\\ \qquad\quad+\displaystyle\int_E E'\Big[\sigma\big(t,\pi_\rho(t,e),u_\rho(t)\big)-\sigma\big(t,\pi(t,e),u(t)\big)\Big]\lambda(de)dB_t\\ \qquad\quad+\displaystyle\int_E E'\Big[h\big(t,\pi_\rho(t-,e),u_\rho(t),e\big)-h\big(t,\pi(t-,e),u(t),e\big)\Big]\widetilde{\mu}(dtde),\\ -d\widehat{y}(t)=\displaystyle\int_E E'\Big[f\big(t,\pi_\rho(t,e),u_\rho(t)\big)-f\big(t,\pi(t,e),u(t)\big)\Big]\lambda(de)dt\\ \qquad\quad-\widehat{z}(t)dB_t-\displaystyle\int_E\widehat{k}(t,e)\widetilde{\mu}(dtde),\\ \widehat{x}(0)=0,\ \widehat{y}(T)=E'\Big[\Phi\big(x_\rho(T),(x_\rho(T))'\big)-\Phi\big(x(T),(x(T))'\big)\Big], \end{cases}\tag{5.4}$$

where $\pi(t,e)=\pi^u(t,e)$, $\pi_\rho(t,e)=\pi^{u_\rho}(t,e)$, $\pi(t-,e)$ and $\pi_\rho(t-,e)$ are similarly defined. From Theorem 4.1, it is easy to know that $\big(\widehat{x}(\cdot),\widehat{y}(\cdot),\widehat{z}(\cdot),\widehat{k}(\cdot,\cdot)\big)$ converges to 0 in $\big(M_{\mathbb{F}}^2(0,T)\big)^3\times\mathcal{K}^2_{\mathbb{F}\lambda}(0,T)$ as ρ tends to 0. Now, we define $\Delta l(t)=\frac{l_\rho(t)-l(t)}{\rho}$, $l=x,y,z$, $\Delta k(t,e)=\frac{k_\rho(t,e)-k(t,e)}{\rho}$. Then, from (5.4) we have

$$\begin{cases} d\Delta x(t)=\displaystyle\int_E E'\Big[\bar{b}\big(t,\Delta\pi(t,e),v(t)\big)\Big]\lambda(de)dt+\int_E E'\Big[\bar{\sigma}\big(t,\Delta\pi(t),v(t)\big)\Big]\lambda(de)dB_t\\ \qquad\quad+\displaystyle\int_E E'\Big[\bar{h}\big(t,\Delta\pi(t-,e),v(t),e\big)\Big]\widetilde{\mu}(dtde),\\ -d\Delta y(t)=\displaystyle\int_E E'\Big[\bar{f}\big(t,\Delta\pi(t,e),v(t)\big)\Big]\lambda(de)dt-\Delta z(t)dB_t-\int_E\Delta k(t,e)\widetilde{\mu}(dtde),\\ \Delta x(0)=0,\ \Delta y(T)=E'\Big[M(T)\Delta x(T)+N(T)(\Delta x(T))'\Big], \end{cases}\tag{5.5}$$

where

$$\begin{aligned} \Delta\pi(t,e)&=(\Delta x(t),\ \Delta y(t),\ \Delta z(t),\ \Delta k(t,e),\ (\Delta x(t))',\ (\Delta y(t))',\ (\Delta z(t))',\ (\Delta k(t,e))'),\\ \Delta\pi(t-,e)&=(\Delta x(t-),\ \Delta y(t-),\ \Delta z(t),\ \Delta k(t,e),\ (\Delta x(t-))',\ (\Delta y(t-))',\ (\Delta z(t))',\ (\Delta k(t,e))'),\\ &\bar{b}(t,\ x,\ y,\ z,\ k,\ \widetilde{x},\ \widetilde{y},\ \widetilde{z},\ \widetilde{k},\ v)\\ &=A(t,e)x+B(t,e)y+C(t,e)z+D(t,e)k+E(t,e)\widetilde{x}+F(t,e)\widetilde{y}+G(t,e)\widetilde{z}+H(t,e)\widetilde{k}+I(t,e)v, \end{aligned}$$

and

$$\begin{aligned} A(t,e)\Delta x(t)&=\frac{1}{\rho}\Big[b\big(t,x_\rho(t),y_\rho(t),\cdots,u_\rho(t)\big)-b\big(t,x(t),y_\rho(t),\cdots,u_\rho(t)\big)\Big],\\ B(t,e)\Delta y(t)&=\frac{1}{\rho}\Big[b\big(t,x(t),y_\rho(t),z_\rho(t),\cdots,u_\rho(t)\big)-b\big(t,x(t),y(t),z_\rho(t),\cdots,u_\rho(t)\big)\Big],\\ C(t,e)\Delta z(t)&=\frac{1}{\rho}\Big[b\big(t,x(t),y(t),z_\rho(t),k_\rho(t,e),\cdots,u_\rho(t)\big)-b\big(t,x(t),y(t),z(t),k_\rho(t,e),\cdots,u_\rho(t)\big)\Big],\\ D(t,e)\Delta k(t,e)&=\frac{1}{\rho}\Big[b\big(t,\cdots,z(t),k_\rho(t,e),(x_\rho(t))',\cdots\big)-b\big(t,\cdots,z(t),k(t,e),(x_\rho(t))',\cdots\big)\Big],\\ E(t,e)(\Delta x(t))'&=\frac{1}{\rho}\Big[b\big(t,\cdots,k(t,e),(x_\rho(t))',(y_\rho(t))',\cdots\big)-b\big(t,\cdots,k(t,e),(x(t))',(y_\rho(t))',\cdots\big)\Big],\\ F(t,e)(\Delta y(t))'&=\frac{1}{\rho}\Big[b\big(t,\cdots,(x(t))',(y_\rho(t))',(z_\rho(t))',\cdots\big)-b\big(t,\cdots,(x(t))',(y(t))',(z_\rho(t))',\cdots\big)\Big],\\ G(t,e)(\Delta z(t))'&=\frac{1}{\rho}\Big[b\big(t,\cdots,(y(t))',(z_\rho(t))',(k_\rho(t,e))',u_\rho(t)\big)-b\big(t,\cdots,(y(t))',(z(t))',(k_\rho(t,e))',u_\rho(t)\big)\Big],\\ H(t,e)(\Delta k(t,e))'&=\frac{1}{\rho}\Big[b\big(t,\cdots,(z(t))',(k_\rho(t,e))',u_\rho(t)\big)-b\big(t,\cdots,(z(t))',(k(t,e))',u_\rho(t)\big)\Big], \end{aligned}$$

$$I(t,e)v(t) = \frac{1}{\rho}\Big[b\big(t,\cdots,(k(t,e))', u_\rho(t)\big) - b\big(t,\cdots,(k(t,e))', u(t)\big)\Big],$$
$$M(T)\Delta x(T) = \frac{1}{\rho}\Big[\Phi\big(x_\rho(T),(x_\rho(T))'\big) - \Phi\big(x(T),(x_\rho(T))'\big)\Big],$$
$$N(T)(\Delta x(T))' = \frac{1}{\rho}\Big[\Phi\big(x(T),(x_\rho(T))'\big) - \Phi\big(x(T),(x(T))'\big)\Big],$$

where $\bar\sigma$, $\bar h$, $\bar f$ are similarly defined. From (H5.1) and the fact $(\widehat{x}(\cdot),\widehat{y}(\cdot),\widehat{z}(\cdot),\widehat{k}(\cdot,\cdot))$ converges to 0 in $\big(\mathcal{H}^2_{\mathbb{F}}(0,T)\big)^3\times\mathcal{K}^2_{\mathbb{F}\lambda}(0,T)$ as ρ tends to 0, we know

$$\lim_{\rho\to 0}[A(t,e)-b_x\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[B(t,e)-b_y\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[C(t,e)-b_z\big(\theta(t,e)\big)]=0,$$
$$\lim_{\rho\to 0}[D(t,e)-b_k\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[E(t,e)-b_{\widetilde{x}}\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[F(t,e)-b_{\widetilde{y}}\big(\theta(t,e)\big)]=0,$$
$$\lim_{\rho\to 0}[G(t,e)-b_{\widetilde{z}}\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[H(t,e)-b_{\widetilde{k}}\big(\theta(t,e)\big)]=0,\quad \lim_{\rho\to 0}[I(t,e)-b_v\big(\theta(t,e)\big)]=0,$$

and

$$\lim_{\rho\to 0}\Big\{\bar b\big(t,\Delta\pi(t,e),v(t)\big) - b_x\big(\theta(t,e)\big)\Delta x(t) - b_y\big(\theta(t,e)\big)\Delta y(t) - b_z\big(\theta(t,e)\big)\Delta z(t) - b_k\big(\theta(t,e)\big)\Delta k(t,e)$$
$$- b_{\widetilde{x}}\big(\theta(t,e)\big)(\Delta x(t))' - b_{\widetilde{y}}\big(\theta(t,e)\big)(\Delta y(t))' - b_{\widetilde{z}}\big(\theta(t,e)\big)(\Delta z(t))' - b_{\widetilde{k}}\big(\theta(t,e)\big)(\Delta k(t,e))' - b_v\big(\theta(t,e)\big)v(t)\Big\} = 0,$$

$\bar\sigma$, $\bar h$, $\bar f$, $\Delta y(T)$ have similar results. From the uniqueness of the solution of equation (5.3), we know $(\Delta x(\cdot),\ \Delta y(\cdot),\ \Delta z(\cdot),\ \Delta k(\cdot,\cdot))$ converges to $(x^1(\cdot),\ y^1(\cdot),\ z^1(\cdot),\ k^1(\cdot,\cdot))$ in $\big(\mathcal{H}^2_{\mathbb{F}}(0,T)\big)^3\times\mathcal{K}^2_{\mathbb{F}\lambda}(0,T)$ as ρ tends to 0. ∎

Because $u(\cdot)$ is an optimal control, then

$$\rho^{-1}[J(u(\cdot)+\rho v(\cdot)) - J(u(\cdot))] \geq 0. \tag{5.6}$$

Using the similar approach of Lemma 5.1, from (5.6) we have the following results.

Lemma 5.2. *We suppose (H5.1) holds. Then, the following variational inequality holds:*

$$E\Big\{\int_0^T\int_E E'[g_x\big(\theta(t,e)\big)x^1(t) + g_y\big(\theta(t,e)\big)y^1(t) + g_z\big(\theta(t,e)\big)z^1(t) + g_k\big(\theta(t,e)\big)k^1(t,e) + g_{\widetilde{x}}\big(\theta(t,e)\big)(x^1(t))'$$
$$+ g_{\widetilde{y}}\big(\theta(t,e)\big)(y^1(t))' + g_{\widetilde{z}}\big(\theta(t,e)\big)(z^1(t))' + g_{\widetilde{k}}\big(\theta(t,e)\big)(k^1(t,e))' + g_v\big(\theta(t,e)\big)v(t)]\lambda(de)dt$$
$$+ E'[\varphi_x\big(x(T),(x(T))'\big)x^1(T) + \varphi_{\widetilde{x}}\big(x(T),(x(T))'\big)(x^1(T))'] + \gamma_y(y(0))y^1(0)\Big\} \geq 0. \tag{5.7}$$

Now we introduce the following adjoint mean-field FBSDE with jumps to equation (5.3):

$$
\begin{cases}
dp(t) = -\displaystyle\int_E E'\Big\{b_y^T(\theta(t,e))q(t) + \sigma_y^T(\theta(t,e))m(t) + h_y^T\big(\theta(t-,e)\big)n(t,e) - f_y^T\big(\theta(t,e)\big)p(t) \\
\qquad + g_y\big(\theta(t,e)\big) + b_{\widetilde{y}}^T(\rho(t,e))(q(t))' + \sigma_{\widetilde{y}}^T(\rho(t,e))(m(t))' + h_{\widetilde{y}}^T\big(\rho(t-,e)\big)(n(t,e))' \\
\qquad - f_{\widetilde{y}}^T\big(\rho(t,e)\big)(p(t))' + g_{\widetilde{y}}\big(\rho(t,e)\big)\Big\}\lambda(de)dt \\
\qquad - \displaystyle\int_E E'\Big\{b_z^T(\theta(t,e))q(t) + \sigma_z^T(\theta(t,e))m(t) + h_z^T\big(\theta(t-,e)\big)n(t,e) - f_z^T\big(\theta(t,e)\big)p(t) \\
\qquad + g_z\big(\theta(t,e)\big) + b_{\widetilde{z}}^T(\rho(t,e))(q(t))' + \sigma_{\widetilde{z}}^T(\rho(t,e))(m(t))' + h_{\widetilde{z}}^T\big(\rho(t-,e)\big)(n(t,e))' \\
\qquad - f_{\widetilde{z}}^T\big(\rho(t,e)\big)(p(t))' + g_{\widetilde{z}}\big(\rho(t,e)\big)\Big\}\lambda(de)dB_t \\
\qquad - \displaystyle\int_E E'\Big\{b_k^T(\theta(t,e))q(t) + \sigma_k^T(\theta(t,e))m(t) + h_k^T\big(\theta(t-,e)\big)n(t,e) - f_k^T\big(\theta(t,e)\big)p(t) \\
\qquad + g_k\big(\theta(t,e)\big) + b_{\widetilde{k}}^T(\rho(t,e))(q(t))' + \sigma_{\widetilde{k}}^T(\rho(t,e))(m(t))' + h_{\widetilde{k}}^T\big(\rho(t-,e)\big)(n(t,e))' \\
\qquad - f_{\widetilde{k}}^T\big(\rho(t,e)\big)(p(t))' + g_{\widetilde{k}}\big(\rho(t,e)\big)\Big\}\widetilde{\mu}(dtde), \\
-dq(t) = \displaystyle\int_E E'\Big\{b_x^T(\theta(t,e))q(t) + \sigma_x^T(\theta(t,e))m(t) + h_x^T\big(\theta(t-,e)\big)n(t,e) - f_x^T\big(\theta(t,e)\big)p(t) \\
\qquad + g_x\big(\theta(t,e)\big) + b_{\widetilde{x}}^T(\rho(t,e))(q(t))' + \sigma_{\widetilde{x}}^T(\rho(t,e))(m(t))' + h_{\widetilde{x}}^T\big(\rho(t-,e)\big)(n(t,e))' \\
\qquad - f_{\widetilde{x}}^T\big(\rho(t,e)\big)(p(t))' + g_{\widetilde{x}}\big(\rho(t,e)\big)\Big\}\lambda(de)dt - m(t)dB_t - \displaystyle\int_E n(t,e)\widetilde{\mu}(dtde), \\
p(0) = -\gamma_y(y(0)), \\
q(T) = E'[\varphi_x\big(x(T),(x(T))'\big) + \varphi_{\widetilde{x}}\big((x(T))',x(T)\big) \\
\qquad - \Phi_x\big(x(T),(x(T))'\big)p(T) - \Phi_{\widetilde{x}}\big((x(T))',x(T)\big)(p(T))'].
\end{cases}
\tag{5.8}
$$

From Theorem 3.1, we know there exists a unique quadruple $(p(\cdot), q(\cdot), m(\cdot), n(\cdot,\cdot))$ satisfying (5.8). We define the Hamiltonian function H as follows:

$$
\begin{aligned}
&H(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v,p,q,m,n,e) = \langle q, b(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v)\rangle + \langle m, \sigma(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v)\rangle \\
&+ \langle n, h(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v,e)\rangle - \langle p, f(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v)\rangle + g(t,x,y,z,k,\widetilde{x},\widetilde{y},\widetilde{z},\widetilde{k},v).
\end{aligned}
\tag{5.9}
$$

Then we have the following maximum principle.

Theorem 5.1. *Let $u(\cdot)$ be an optimal control and let $(x(\cdot),\ y(\cdot),\ z(\cdot),\ k(\cdot,\cdot))$ be the corresponding trajectory. Then, we have*

$$
\int_E E'\big\langle H_v\big(t,\pi(t,e),\ u(t),\ p(t),\ q(t),\ m(t),\ n(t,e),\ e\big),\ v-u(t)\big\rangle\lambda(de) \geq 0, \forall\ v \in U, dtdP\text{-a.e.}, \tag{5.10}
$$

where $\pi(t,e) = (x(t),y(t),z(t),k(t,e),(x(t))',(y(t))',(z(t))',(k(t,e))')$, $(p(\cdot),q(\cdot),m(\cdot),n(\cdot,\cdot))$ is the solution of the adjoint equation (5.8).

Proof. Applying Itô's formula to $\langle x^1(t), q(t)\rangle + \langle y^1(t), p(t)\rangle$, from equations (5.3) and (5.8), (H3.1), (H3.2) and (H5.1), with the help of (5.7) and (5.9), for $v(\cdot)$ such that $u(\cdot)+v(\cdot) \in \mathcal{U}_{ad}$, we get

$$
E\int_0^T \int_E E'\langle H_v\big(t,\pi(t,e),\ u(t),\ p(t),\ q(t),\ m(t),\ n(t,e),\ e\big), v(t)\rangle\lambda(de)dt \geq 0. \tag{5.11}
$$

Denote $H_v(t,e) = H_v\big(t,\pi(t,e),\ u(t),\ p(t),\ q(t),\ m(t),\ n(t,e),\ e\big)$. For any $\bar{v}(\cdot) \in \mathcal{U}_{ad}$, we define

$$
v(s) = \begin{cases} \bar{v}(s) - u(s), & s \in [t, t+\varepsilon], \\ 0, & \text{otherwise}. \end{cases}
$$

Then from (5.11) we get

$$\frac{1}{\varepsilon}E\int_t^{t+\varepsilon}\int_E E'\langle H_v(s,e),\bar v(s)-u(s)\rangle\lambda(de)dt\geq 0. \tag{5.12}$$

Putting $\varepsilon\to 0$, we have $E\int_E E'\langle H_v(t,e),\bar v(t)-u(t)\rangle\lambda(de)\geq 0$, *a.e.* Then, let $\bar v(t)=vI_A+u(t)I_{A^c}$, for $A\in\mathcal{F}_t$ and $v\in U$, we can get that

$$\begin{aligned}0\leq &E\int_E E'\langle H_v(t,e),\bar v(t)-u(t)\rangle\lambda(de)=E[\int_E E'\langle H_v(t,e),v-u(t)\rangle\lambda(de)\cdot I_A]\\ =&E[\int_E E'\langle H_v(t,e),v-u(t)\rangle\lambda(de)|\mathcal{F}_t]=\int_E E'\langle H_v(t,e),v-u(t)\rangle\lambda(de),\ \text{dtdP-a.e.}\end{aligned} \tag{5.13}$$

∎

We now study assumptions, under which the necessary condition (5.10) becomes a sufficient one.

Theorem 5.2. *(Sufficient conditions for the optimality of the control) Let (H5.1) hold and the control $u(\cdot)$ satisfies (5.10), where $(p(\cdot),q(\cdot),m(\cdot),n(\cdot,\cdot))$ is the solution of the adjoint equation (5.8). We further assume that the following convexity conditions:*
(1) $\Phi(x,\widetilde x)=ax+b\widetilde x,\ a,b\in\mathbb{R}^n$;
(2) φ is convex with respect to $x,\widetilde x$;
(3) γ is convex with respect to y;
(4) Hamiltonian function H is convex with respect to $(x,y,z,k,\widetilde x,\widetilde y,\widetilde z,\widetilde k,v)$.
Then u is an optimal control.

Proof. For any $v\in\mathcal{U}_{ad}$, from (5.2) we have

$$\begin{aligned}J(v(\cdot))-J(u(\cdot))=E\Big[\int_0^T\int_E E'\big[g\big(t,\pi^v(t,e),v(t)\big)-g\big(t,\pi(t,e),u(t)\big)\big]\lambda(de)dt\\ +E'[\varphi\big(x^v(T),(x^v(T))'\big)-\varphi\big(x(T),(x(T))'\big)]+\gamma\big(y^v(0)\big)-\gamma\big(y(0)\big)\Big].\end{aligned} \tag{5.14}$$

Since φ is convex in $x,\tilde x$ and γ is convex in y, we get

$$\begin{aligned}&\gamma\big(y^v(0)\big)-\gamma\big(y(0)\big)\geq\gamma_y\big(y(0)\big)\big(y^v(0)-y(0)\big),\\ &\varphi\big(x^v(T),(x^v(T))'\big)-\varphi\big(x(T),(x(T))'\big)\\ &\ \geq\varphi_x\big(x(T),(x(T))'\big)\big(x^v(T)-x(T)\big)+\varphi_{\widetilde x}\big(x(T),(x(T))'\big)\big((x^v(T))'-(x(T))'\big).\end{aligned} \tag{5.15}$$

Observe that $EE'[\varphi_{\widetilde x}\big(x(T),(x(T))'\big)\big((x^v(T))'-(x(T))'\big)]=EE'[\varphi_{\widetilde x}\big((x(T))',x(T)\big)\big(x^v(T)-x(T)\big)]$, from (5.14) and (5.15), we obtain

$$\begin{aligned}J(v(\cdot))-J(u(\cdot))\geq&E\Big[\int_0^T\int_E E'\big[g\big(t,\pi^v(t,e),v(t)\big)-g\big(t,\pi(t,e),u(t)\big)\big]\lambda(de)dt\\ &+E'[\varphi_x\big(x(T),(x(T))'\big)+\varphi_{\widetilde x}\big((x(T))',x(T)\big)]\big(x^v(T)-x(T)\big)+\gamma_y\big(y(0)\big)\big(y^v(0)-y(0)\big)\Big].\end{aligned} \tag{5.16}$$

Denote $H_x(t,e):=H_x\big(t,x(t),y(t),z(t),k(t,e),(x(t))',(y(t))',(z(t))',(k(t,e))',u(t),p(t),q(t),m(t),n(t,e),e\big)$, $H_{\widetilde x}(t,e)$, $H_y(t,e)$, $H_{\widetilde y}(t,e)$, $H_z(t,e)$, $H_{\widetilde z}(t,e)$, $H_k(t,e)$, $H_{\widetilde k}(t,e)$, $H_v(t,e)$ are similarly defined. Applying Itô's formula to $q(t)\big(x^v(t)-x(t)\big)$ and taking the expectation, we obtain

$$\begin{aligned}&EE'[\varphi_x\big(x(T),(x(T))'\big)\big(x^v(T)-x(T)\big)+\varphi_{\widetilde x}\big((x(T))',x(T)\big)\big(x^v(T)-x(T)\big)]\\ &=E\{E'[a\big(x^v(T)-x(T)\big)+b\big((x^v(T))'-(x(T))'\big)]p(T)\}\\ &-E\int_0^T\int_E E'[\big(x^v(t)-x(t)\big)H_x(t,e)+\big(x^v(t)-x(t)\big)'H_{\widetilde x}(t,e)]\lambda(de)dt\\ &+E\int_0^T\int_E\Big[q(t)E'[b(t,\pi^v(t,e),v(t))-b(t,\pi(t,e),u(t))]+m(t)E'[\sigma(t,\pi^v(t,e),v(t))\\ &-\sigma(t,\pi(t,e),u(t))]+n(t,e)E'[h\big(t,\pi^v(t-,e),v(t),e\big)-h\big(t,\pi(t-,e),u(t),e\big)]\Big]\lambda(de)dt.\end{aligned} \tag{5.17}$$

Applying Itô's formula to $p(t)\big(y^v(t)-y(t)\big)$ and taking the expectation, we obtain

$$
\begin{aligned}
&E\big\{p(T)\cdot E'[a\big(x^v(T)-x(T)\big)+b\big((x^v(T))'-(x(T))'\big)]+\gamma_y(y(0))(y^v(0)-y(0))\big\}\\
&=-E\int_0^T\int_E E'\Big[\big(y^v(t)-y(t)\big)H_y(t,e)+\big(y^v(t)-y(t)\big)'H_{\widetilde{y}}(t,e)+\big(z^v(t)-z(t)\big)H_z(t,e)\\
&+\big(z^v(t)-z(t)\big)'H_{\widetilde{z}}(t,e)+\big(k^v(t,e)-k(t,e)\big)H_k(t,e)+\big(k^v(t,e)-k(t,e)\big)'H_{\widetilde{k}}(t,e)\Big]\lambda(de)dt\\
&-E\int_0^T\int_E p(t)E'[f\big(t,\pi^v(t,e),v(t)\big)-f\big(t,\pi(t,e),u(t)\big)]\lambda(de)dt.
\end{aligned}
\tag{5.18}
$$

Then, from (5.16), (5.17) and (5.18) we have

$$
\begin{aligned}
&J(v(\cdot))-J(u(\cdot))\geq -E\int_0^T\int_E E'\Big[\big(x^v(t)-x(t)\big)H_x(t,e)+\big(x^v(t)-x(t)\big)'H_{\widetilde{x}}(t,e)+\big(y^v(t)-y(t)\big)H_y(t,e)\\
&+\big(y^v(t)-y(t)\big)'H_{\widetilde{y}}(t,e)+\big(z^v(t)-z(t)\big)H_z(t,e)+\big(z^v(t)-z(t)\big)'H_{\widetilde{z}}(t,e)+\big(k^v(t,e)-k(t,e)\big)H_k(t,e)\\
&+\big(k^v(t,e)-k(t,e)\big)'H_{\widetilde{k}}(t,e)\Big]\lambda(de)dt+E\int_0^T\int_E E'\Big[H\big(t,\pi^v(t,e),v(t),p(t),q(t),m(t),n(t,e),e\big)\\
&-H\big(t,\pi(t,e),u(t),p(t),q(t),m(t),n(t,e),e\big)\Big]\lambda(de)dt.
\end{aligned}
\tag{5.19}
$$

From the convexity of H, we know

$$
\begin{aligned}
&H\big(t,\pi^v(t,e),v(t),p(t),q(t),m(t),n(t,e),e\big)-H\big(t,\pi(t,e),u(t),p(t),q(t),m(t),n(t,e),e\big)\\
&\geq\big(x^v(t)-x(t)\big)H_x(t,e)+\big(y^v(t)-y(t)\big)H_y(t,e)+\big(z^v(t)-z(t)\big)H_z(t,e)+\big(k^v(t,e)-k(t,e)\big)H_k(t,e)\\
&+\big(x^v(t)-x(t)\big)'H_{\widetilde{x}}(t,e)+(y^v(t)-y(t))'H_{\widetilde{y}}(t,e)+(z^v(t)-z(t))'H_{\widetilde{z}}(t,e)+(k^v(t,e)-k(t,e))'H_{\widetilde{k}}(t,e)\\
&+(v(t)-u(t))H_v(t,e).
\end{aligned}
\tag{5.20}
$$

From (5.19) and (5.20), we get

$$
J(v(\cdot))-J(u(\cdot))\geq E\int_0^T\int_E E'[(v(t)-u(t))H_v(t,e)]\lambda(de)dt. \tag{5.21}
$$

Combined with the maximum condition (5.10), we obtain the desired result. ∎

6 Applications

6.1 Application to mean-variance portfolio selection mixed with a mean-field recursive utility

In this section, we study a mean-variance portfolio selection mixed with a mean-field recursive utility functional optimization problem applying the maximum principle derived in Section 5. We suppose that there is a financial market consisting of two investment possibilities:
(i) a risk-free security (e.g., a bond), where the price $S_0(t)$ at time t is given by

$$
dS_0(t)=\rho_tS_0(t)dt,\ S_0(0)\geq 0, \tag{6.1}
$$

where ρ_t is a bounded deterministic function.
(ii) a risky security (e.g., a stock), where the price $S_1(t)$ at time t is given by

$$
dS_1(t)=S_1(t-)\Big[\mu_tdt+\sigma_tdB_t+\int_E\eta(t,e)\widetilde{\mu}(dedt)\Big],\ S_1(0)>0, \tag{6.2}
$$

where $\mu_t \neq 0, \sigma_t \neq 0,\ \eta(t,e)$ are bounded deterministic functions and $\mu_t > \rho_t$. We also assume that $\eta(t,e) > -1$, for all t and $e \in E$ such that $S_1(t) > 0$.
Assume that $\theta(t) = (\theta_0(t), \theta_1(t))$ is a portfolio which represents the number of units at time t of the risk-free and the risky security. Then the corresponding wealth process $x(t)$ is given by

$$x^\theta(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t),\ t \geq 0. \tag{6.3}$$

We also assume the portfolio is self-financing, that is,

$$x^\theta(t) = x^\theta(0) + \int_0^t \theta_0(s)dS_0(s) + \int_0^t \theta_1(s-)dS_1(s),\ t \geq 0. \tag{6.4}$$

Let $v(t) = \theta_1(t)S_1(t)$ denote the amount invested in the risky security. Then from (6.3) and (6.4), we get the wealth dynamics:

$$dx^v(t) = [\rho_t x^v(t) + (\mu_t - \rho_t)v(t)]dt + \sigma_t v(t)dB_t + \int_E \eta(t,e)v(t-)\widetilde{\mu}(dtde), \tag{6.5}$$

where $x^v(0) = x_0$ is given.

We consider a investor, endowed with initial wealth $x_0 > 0$, who chooses at each time t his or her portfolio strategy $v(t)$. The investor's object is to find an admissible portfolio strategy $v(\cdot) \in \mathcal{U}_{ad}$ which maximizes the following expected utility functional:

$$J(v(\cdot)) = E[-\frac{1}{2}(x^v(T) - a)^2] + y^v(t)|_{t=0}, \tag{6.6}$$

where

$$\begin{aligned} y^v(t) = E\Big[&\gamma x^v(T) + \widetilde{\gamma}E[x^v(T)] \\ &+ \int_t^T \alpha\rho_s x^v(s) + \widetilde{\alpha}\rho_s E[x^v(s)] + (\mu_s - \rho_s)v(s) - \beta y^v(s) - \widetilde{\beta}E[y^v(s)]ds|\mathcal{F}_t\Big],\ t \in [0,T], \end{aligned} \tag{6.7}$$

with nonnegative constants $a, \gamma, \widetilde{\gamma}, \alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$. Notice that the investor's utility functional consists of two parts: One part is the terminal reward

$$E[-\frac{1}{2}(x^v(T) - a)^2];$$

The other part is a mean-field recursive utility functional with generator $f(t, x, \widetilde{x}, y, \widetilde{y}, v) = \alpha\rho_t x + \widetilde{\alpha}\rho_t\widetilde{x} + (\mu_t - \rho_t)v - \beta y - \widetilde{\beta}\widetilde{y}$. Mean-field recursive utility is an extension to mean-field (and jumps) of the classical recursive utility concept of Duffie and Epstein [7] (i.e., $\widetilde{\alpha} = \widetilde{\beta} = \widetilde{\gamma} \equiv 0$ in (6.7)), the interested reader can referred to [2] and the references therein for the concept of mean-field recursive utility.

Remark 6.1. *When only the terminal part is considered for the utility functional, Framstad, et al. [8] solved the above mean-variance portfolio selection by using the sufficient maximum principle in Example 4.1. In addition to the terminal utility functional, Shi, Wu [18] also considered a recursive utility functional for the mean-variance portfolio selection problem. Then, we generalize their recursive utility to mean-field cases in our model, that is we consider mean-variance portfolio selection mixed with a mean-field recursive utility functional.*

We now apply the result of Section 5 to solve the above optimization problem (6.5)-(6.6). In fact, in our jump-diffusion framework, the wealth process $x^v(\cdot)$ in (6.5) and mean-field recursive utility process $y^v(\cdot)$ in (6.7) can be regarded as the solution of the following mean-field FBSDEs with jumps:

$$\begin{cases} dx^v(t) =[\rho_t x^v(t) + (\mu_t - \rho_t)v(t)]dt + \sigma_t v(t)dB(t) + \int_E \eta(t,e)v(t-)\widetilde{\mu}(dtde), \\ -dy^v(t) =\Big[\alpha\rho_t x^v(t) + \widetilde{\alpha}\rho_t E[x^v(t)] + (\mu_t - \rho_t)v(t) - \beta y^v(t) - \widetilde{\beta}E[y^v(t)]\Big]dt \\ \qquad\qquad - z^v(t)dB(t) - \int_E k^v(t,e)\widetilde{\mu}(dtde), \\ x^v(0) =x_0,\ y^v(T) = \gamma x^v(T) + \widetilde{\gamma}E[x^v(T)], \end{cases} \tag{6.8}$$

and the optimization problem can be rewritten as

$$\mathcal{J}(u(\cdot)) = \inf_{v\in\mathcal{U}_{ad}}\mathcal{J}(v(\cdot)), \tag{6.9}$$

where $\mathcal{J}(v(\cdot)) = -J(v(\cdot))$.

It is easy to verify that all the assumptions in Section 5 are satisfied for this problem. The related adjoint equations (5.8) become the following form

$$\begin{cases} dp(t) = -(\beta+\widetilde{\beta})p(t)dt, \\ -dq(t) = \rho_t[q(t)-(\alpha+\widetilde{\alpha})p(t)]dt - m(t)dB(t) - \int_E n(t,e)\widetilde{\mu}(dtde), \\ p(0)=1,\ q(T) = x(T) - a - (\alpha+\widetilde{\alpha})p(T). \end{cases} \tag{6.10}$$

Obviously, $p(t) = \exp\{-(\beta+\widetilde{\beta})t\}$, $0\le t\le T$. The related Hamiltonian function has the following form

$$\begin{aligned} &H(t,x,y,\widetilde{x},\widetilde{y},v;p,q,m,n,e) \\ =&q[\rho_t x + (\mu_t-\rho_t)v] + m\sigma_t v + n\eta(t,e)v - p[\alpha\rho_t x + \widetilde{\alpha}\rho_t\widetilde{x} + (\mu_t-\rho_t)v - \beta y - \widetilde{\beta}\widetilde{y}]. \end{aligned} \tag{6.11}$$

Since this is a linear expression of v, we get from (5.10)

$$\big(q(t)-p(t)\big)(\mu_t-\rho_t) + m(t)\sigma_t + \int_E n(t,e)\eta(t,e)\lambda(de) = 0. \tag{6.12}$$

We set $q(t) = \varphi_t x(t) + \psi_t$, where φ_t, ψ_t are deterministic differential functions which will be specified below. Then, from (6.10) we get

$$-\rho_t q(t) + (\alpha+\widetilde{\alpha})\rho_t p(t) = \dot{\varphi}_t x(t) + \dot{\psi}_t + \varphi_t\rho_t x(t) + \varphi_t(\mu_t-\rho_t)u(t), \tag{6.13}$$

and

$$\begin{aligned} m(t) &= \varphi_t\sigma_t u(t), \\ n(t,e) &= \varphi_t\eta(t,e)u(t). \end{aligned} \tag{6.14}$$

Substituting (6.14) into (6.12), we have

$$u(t) = \frac{(\mu_t-\rho_t)(-\varphi_t x(t) - \psi_t + p(t))}{\varphi_t\Lambda_t}, \tag{6.15}$$

where $\Lambda_t = \sigma_t^2 + \int_E \eta^2(t,e)\lambda(de)$. On the other hand, from (6.13) we get

$$u(t) = \frac{-x(t)\dot{\varphi}_t - 2\rho_t x(t)\varphi_t - \dot{\psi}_t - \rho_t\psi_t + (\alpha+\widetilde{\alpha})p(t)\rho_t}{\varphi_t(\mu_t-\rho_t)}. \tag{6.16}$$

By comparing (6.15) and (6.16), we obtain the following ordinary differential equation

$$\begin{cases} \dot{\varphi}_t + \big(2\rho_t - \dfrac{(\mu_t-\rho_t)^2}{\Lambda_t}\big)\varphi_t = 0, \quad \varphi_T = 1, \\ \dot{\psi}_t + \big(\rho_t - \dfrac{(\mu_t-\rho_t)^2}{\Lambda_t}\big)\psi_t - (\alpha+\widetilde{\alpha})\rho_t^2 + \dfrac{(\mu_t-\rho_t)^2}{\Lambda_t}p(t) = 0, \quad \psi_T = -a-(\alpha+\widetilde{\alpha})p(T). \end{cases} \tag{6.17}$$

Then we obtain

$$\varphi_t = \exp\{\int_t^T \big(2\rho_s - \frac{(\mu_s-\rho_s)^2}{\Lambda_s}\big)ds\},\ 0\le t\le T, \tag{6.18}$$

and

$$\begin{aligned} \psi_t =& [-a-(\alpha+\widetilde{\alpha})p(T)]\exp\{\int_t^T \big(\rho_s - \frac{(\mu_s-\rho_s)^2}{\Lambda_s}\big)ds\} - \int_t^T \Big[\big(\frac{(\mu_s-\rho_s)^2}{\Lambda_s}p(s) - (\alpha+\widetilde{\alpha})\rho_s^2\big) \\ &\exp\{\int_s^T (\frac{(\mu_r-\rho_r)^2}{\Lambda_r} - \rho_r)dr\}\Big]ds\cdot\exp\{\int_t^T (\rho_s - \frac{(\mu_s-\rho_s)^2}{\Lambda_s})ds\},\ 0\le t\le T. \end{aligned} \tag{6.19}$$

Finally, by combining the above discussion and Theorem 5.2, we obtain the following theorem.

Theorem 6.1. *The optimal solution u of our mean-variance portfolio selection mixed with a mean-field recursive utility (6.5) and (6.6) is given (in feedback form) by*

$$u(t) = \frac{(\mu_t - \rho_t)(-\varphi_t x(t) - \psi_t + p(t))}{\varphi_t \Lambda_t}, \tag{6.20}$$

where $\Lambda_t = \sigma_t^2 + \int_E \eta^2(t,e)\lambda(de)$, $p(t) = \exp\{-(\beta + \widetilde{\beta})t\}$, φ_t *and* ψ_t *are given by (6.18) and (6.19), respectively.*

6.2 Application to linear-quadratic optimal control problem

Now we consider an example of linear-quadratic stochastic control problem. The dynamic of our problem is the following linear mean-field FBSDEs with jumps

$$\begin{cases} dx^v(t) = \{ax^v(t) + \widetilde{a}E[x^v(t)]\}dt + \{bx^v(t) + Bv(t)\}dB_t + \int_E \{L(e)v(t)\}\widetilde{\mu}(dtde), \\ -dy^v(t) = \{cx^v(t) + \widetilde{c}E[x^v(t)] + ly^v(t) + \widetilde{l}E[y^v(t)] + Dv(t)\}dt - z^v(t)dB_t - \int_E k^v(t,e)\widetilde{\mu}(dtde), \\ x^v(0) = a, \quad y^v(T) = x^v(T), \end{cases} \tag{6.21}$$

where $a, \widetilde{a}, b, B, c, \widetilde{c}, l, \widetilde{l}, D$ are constants, $L(e)$ is bounded deterministic function and $v \in \mathcal{U}_{ad}$.

The cost functional is a quadratic one, and it has the form

$$J(v(\cdot)) = \frac{1}{2}\int_0^T RE[x^v(t)]^2 dt + \frac{1}{2}NE[x^v(T)]^2 + \frac{1}{2}QE[y^v(0)]^2, \tag{6.22}$$

where R, N, Q are positive constants. Then the related Hamiltonian function has the following form

$$H(x,\widetilde{x},y,\widetilde{y},v,p,q,m,n,e) = q(ax + \widetilde{a}\widetilde{x}) + m(bx + Bv) + nL(e)v - p(cx + \widetilde{c}\widetilde{x} + ly + \widetilde{l}\widetilde{y} + Dv) + \frac{1}{2}Rx^2. \tag{6.23}$$

The adjoint equation can be written as

$$\begin{cases} dp(t) = (l + \widetilde{l})p(t)dt, \\ -dq(t) = \{aq(t) + bm(t) - cp(t) + Rx(t) + \widetilde{a}E[q(t)] - \widetilde{c}p(t)\}dt - m(t)dB_t - \int_E n(t,e)\widetilde{\mu}(dtde), \\ p(0) = -Qy(0), \quad q(T) = Nx(T) - p(T). \end{cases} \tag{6.24}$$

Then, $p(t) = -Qy(0)\exp{(l + \widetilde{l})t}$, $t \in [0,T]$.
From (5.10), we have

$$Bm(t) + \int_E n(t,e)L(e)\lambda(de) - p(t)D = 0. \tag{6.25}$$

We assume

$$q(t) = \phi(t)x(t) + \psi(t)E[x(t)] + \theta(t), \tag{6.26}$$

where $\phi(t)$, $\psi(t)$, $\theta(t)$ are deterministic differentiable functions. Applying Itô's formula to (6.26), from (6.21) we have

$$\begin{aligned} dq(t) = &\Big\{ \big[\dot{\phi}(t) + a\phi(t)\big]x(t) + \big[\widetilde{a}\phi(t) + \dot{\psi}(t) + (a + \widetilde{a})\psi(t)\big]E[x(t)] + \dot{\theta}(t)\Big\}dt \\ &+ \phi(t)[bx(t) + Bu(t)]dB_t + \int_E \phi(t)L(e)u(t)\widetilde{\mu}(dtde). \end{aligned} \tag{6.27}$$

Compared with (6.24), we obtain

$$m(t) = \phi(t)[bx(t) + Bu(t)], \tag{6.28}$$

$$n(t,e) = \phi(t)L(e)u(t), \tag{6.29}$$

$$-aq(t)-bm(t)+(c+\widetilde{c})p(t)-Rx(t)-\widetilde{a}E[q(t)] = \big[\dot{\phi}(t)+a\phi(t)\big]x(t)+\big[\widetilde{a}\phi(t)+\dot{\psi}(t)+(a+\widetilde{a})\psi(t)\big]E[x(t)]+\dot{\theta}(t) \tag{6.30}$$

Substituting (6.28), (6.29) into (6.25), we get

$$\phi(t)u(t) = \frac{1}{\Lambda}[p(t)D - Bb\phi(t)x(t)], \tag{6.31}$$

where $\Lambda = B^2 + \int_E L^2(e)\lambda(de)$. Then from (6.30) and (6.35), we have

$$\begin{aligned}&\Big[\dot{\phi}(t) + (2a + b^2 - \frac{B^2b^2}{\Lambda})\phi(t) + R\Big]x(t) + \Big[\dot{\psi}(t) + (2a + 2\widetilde{a})\psi(t) + 2\widetilde{a}\phi(t)\Big]E[x(t)] \\ &\qquad + \dot{\theta}(t) + (a+\widetilde{a})\theta(t) + \frac{bBD}{\Lambda}p(t) - (c+\widetilde{c})p(t) = 0.\end{aligned} \tag{6.32}$$

Noting the terminal condition in (6.24), we get

$$\begin{aligned}&\dot{\phi}(t) + (2a + b^2 - \frac{B^2b^2}{\Lambda})\phi(t) + R = 0,\ \phi(T) = N,\\ &\dot{\psi}(t) + (2a + 2\widetilde{a})\psi(t) + 2\widetilde{a}\phi(t) = 0,\ \psi(T) = 0,\\ &\dot{\theta}(t) + (a+\widetilde{a})\theta(t) + \frac{bBD}{\Lambda}p(t) - (c+\widetilde{c})p(t) = 0,\ \theta(T) = -p(T).\end{aligned} \tag{6.33}$$

The solutions of these equations are

$$\begin{aligned}&\phi(t) = [N + \frac{R}{2a^2+b^2-\frac{b^2B^2}{\Lambda}}]\exp\{(2a^2+b^2-\frac{b^2B^2}{\Lambda})(T-t)\} - \frac{R}{2a^2+b^2-\frac{b^2B^2}{\Lambda}},\\ &\psi(t) = \exp\{-2(a+\widetilde{a})t\}\int_t^T 2\widetilde{a}\phi(s)\exp\{2(a+\widetilde{a})s\}ds,\\ &\theta(t) = -p(T)exp\{(a+\widetilde{a})(T-t)\} - \int_t^T [c+\widetilde{c} - \frac{bBD}{\Lambda}]p(s)\exp\{(a+\widetilde{a})s\}ds.\end{aligned} \tag{6.34}$$

Finally, since the assumptions of Theorem 5.2 are satisfied in our case, we get the following result.

Theorem 6.2. *The optimal solution u of our linear-quadratic control problem (6.21) and (6.22) is given (in feedback form) by*

$$u(t) = \frac{1}{\Lambda\phi(t)}[p(t)D - Bb\phi(t)x(t)], \tag{6.35}$$

where $\Lambda = B^2 + \int_E L^2(e)\lambda(de)$, $p(t) = -Qy(0)\exp{(l+\widetilde{l})t}$, $\phi(t)$ is given by (6.34).

7 Appendix

Lemma 7.1. *Assume (H3.1) and (H3.2) hold. If for an $\alpha_0 \in [0,1)$ there exists a solution $(x^{\alpha_0}, y^{\alpha_0}, z^{\alpha_0}, k^{\alpha_0})$ of equation (3.3), then there exists a positive constant δ_0 such that, for each $\delta \in [0,\delta_0]$ there exists a solution $(x^{\alpha_0+\delta}, y^{\alpha_0+\delta}, z^{\alpha_0+\delta}, k^{\alpha_0+\delta})$ of mean-field FBSDE with jumps (3.3) for $\alpha = \alpha_0 + \delta$.*

Proof. Since there exists a (unique) solution of equation (3.3) for every $\phi \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $\gamma \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, $\psi \in \mathcal{H}^2_{\mathbb{F}}(0,T,\mathbb{R}^{n\times d})$, $\varphi \in \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^n)$, $\alpha_0 \in [0,1)$, then for each $x(T) \in L^2(\Omega,\mathcal{F}_T,P)$ and a quadruple $(\lambda(t,e))_{0\le t\le T} = (x(t),y(t),z(t),k(t,e))_{0\le t\le T} \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+m+m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m)$ and $\delta > 0$, the following mean-field FBSDE with jumps

$$
\begin{cases}
dX(t) = \Big[\alpha_0 \int_E E'[b(t,\Lambda(t,e),(\Lambda(t,e))')]\lambda(de) + \delta \int_E E'[b(t,\lambda(t,e),(\lambda(t,e))')]\lambda(de) + E'[\phi(t)]\Big]dt \\
\qquad + \Big[\alpha_0 \int_E E'[\sigma(t,\Lambda(t,e),(\Lambda(t,e))')]\lambda(de) + \delta \int_E E'[\sigma(t,\lambda(t,e),(\lambda(t,e))')]\lambda(de) + E'[\psi(t)]\Big]dB_t \\
\qquad + \int_E \Big[\alpha_0 E'[h(t,\Lambda(t,e),(\Lambda(t,e))',e)] + \delta E'[h(t,\lambda(t,e),(\lambda(t,e))',e)] + E'[\varphi(t,e)]\Big]\widetilde{\mu}(dt,de), \\
-dY(t) = \Big[(1-\alpha_0)\beta_1 GX(t) + \alpha_0 \int_E E'[f(t,\Lambda(t,e),(\Lambda(t,e))')]\lambda(de) + \delta\big(-\beta_1 Gx(t) \\
\qquad + \int_E E'[f(t,\lambda(t,e),(\lambda(t,e))')]\lambda(de)\big) + E'[\gamma(t)]\Big]dt - Z(t)dB_t - \int_E K(t,e)\widetilde{\mu}(dt,de), \\
X(0) = a, \\
Y(T) = \alpha_0 E'[\Phi(X(T),(X(T))')] + (1-\alpha_0)GX(T) + \delta\Big(E'[\Phi(x(T),(x(T))')] - Gx(T)\Big) + \xi,
\end{cases}
\tag{7.1}
$$

exists a unique solution

$$
(\Lambda(t,e))_{0\le t\le T} = (X(t),Y(t),Z(t),K(t,e))_{0\le t\le T} \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+m+m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m).
$$

We now prove that the mapping $I_{\alpha_0+\delta}$ defined by

$$
I_{\alpha_0+\delta}(\lambda \times x(T)) = \Lambda \times X(T) :
$$

$\mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+m+m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m) \times L^2(\Omega,\mathcal{F}_T,P) \mapsto \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+m+m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m) \times L^2(\Omega,\mathcal{F}_T,P)$

is a contraction when δ is small enough. For any $\bar\lambda = (\bar x,\bar y,\bar z,\bar k) \in \mathcal{H}^2_{\mathbb{F}}(0,T;\mathbb{R}^{n+m+m\times d}) \times \mathcal{K}^2_{\mathbb{F},\lambda}(0,T;\mathbb{R}^m)$ and $\bar x(T) \in L^2(\Omega,\mathcal{F}_T,P)$, we denote

$$
\bar\Lambda \times \bar X(T) = I_{\alpha_0+\delta}(\bar\lambda \times \bar x(T)),\ \ \widehat\lambda = (\widehat x,\widehat y,\widehat z,\widehat k) = (x-\bar x, y-\bar y, z-\bar z, k-\bar k),
$$

$$
\widehat\Lambda = (\widehat X,\widehat Y,\widehat Z,\widehat K) = (X-\bar X, Y-\bar Y, Z-\bar Z, K-\bar K).
$$

Applying Itô's formula to $\langle G\widehat X(t),\widehat Y(t)\rangle$ it yields

$$
\begin{aligned}
& E\Big\langle \alpha_0 E'[\Phi(X(T),(X(T))') - \Phi(\bar X(T),(\bar X(T))')] + (1-\alpha_0)G\widehat X(T) \\
&\quad + \delta\Big(E'[\Phi(x(T),(x(T))') - \Phi(\bar x(T),(\bar x(T))')] - G\widehat x(T)\Big), G\widehat X(T)\Big\rangle \\
=\ & E\int_0^T \int_E \langle \alpha_0 E'[A(t,\Lambda(t,e),(\Lambda(t,e))') - A(t,\bar\Lambda(t,e),(\bar\Lambda(t,e))')], \widehat\Lambda(t,e)\rangle \lambda(de)dt \\
&\quad - E\int_0^T (1-\alpha_0)\beta_1 \langle G\widehat X(t), G\widehat X(t)\rangle dt + E\int_0^T \delta\beta_1 \langle G\widehat X(t), G\widehat x(t)\rangle dt \\
&\quad + E\int_0^T \int_E \delta E'\Big[\langle G\widehat X(t),\widehat f(t,e)\rangle + \langle G^T\widehat Y(t),\widehat b(t,e)\rangle + \langle G^T\widehat Z(t),\widehat\sigma(t,e)\rangle + \langle G^T\widehat K(t,e),\widehat h(t,e)\rangle\Big]\lambda(de)dt,
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat b(t,e) &= b(t,\lambda(t,e),(\lambda(t,e))') - b(t,\bar\lambda(t,e),(\bar\lambda(t,e))'), \\
\widehat\sigma(t,e) &= \sigma(t,\lambda(t,e),(\lambda(t,e))') - \sigma(t,\bar\lambda(t,e),(\bar\lambda(t,e))'), \\
\widehat h(t,e) &= h(t,\lambda(t,e),(\lambda(t,e))',e) - h(t,\bar\lambda(t,e),(\bar\lambda(t,e))',e), \\
\widehat f(t,e) &= -f(t,\lambda(t,e),(\lambda(t,e))') + f(t,\bar\lambda(t,e),(\bar\lambda(t,e))').
\end{aligned}
$$

From the assumptions (H3.1) and (H3.2), we know
(1) if $\beta_1 - C_0 L_A = 0,\ \mu_1 - L_\Phi \lambda_1 > 0,\ \beta_2 - C_0 L_A > 0,\ \beta_3 - C_0 L_A > 0$, then we have

$$\begin{aligned}
&E\Big[\int_0^T (|\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de))dt\Big]\\
&\leq \delta C_2 E\Big\{\int_0^T \Big(|\widehat{X}(t)|^2 + |\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de)\Big)dt + |\widehat{X}(T)|^2 + |\widehat{x}(T)|^2\\
&\quad + \int_0^T \Big(|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt\Big\}.
\end{aligned} \tag{7.2}$$

On the other hand, from standard technique to the forward equation for $\widehat{X}(t) = X(t) - \bar{X}(t)$, we get

$$\begin{aligned}
\sup_{0\leq t\leq T} E[|\widehat{X}(t)|^2] \ &\leq \delta C_2 E\Big[\int_0^T \Big(|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt\Big]\\
&\quad + C_2 E\Big[\int_0^T \Big(|\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de)\Big)dt\Big].
\end{aligned} \tag{7.3}$$

From (7.2) and (7.3) we get

$$\begin{aligned}
&E\Big\{\int_0^T \Big(|\widehat{X}(t)|^2 + |\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de)\Big)dt + |\widehat{X}(T)|^2\Big\}\\
&\leq \bar{C}\delta E\Big\{\int_0^T \Big(|\widehat{x}(t)| + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt + |\widehat{x}(T)|^2\Big\}.
\end{aligned} \tag{7.4}$$

Here the constant $\bar{C}$ depends on the Lipschitz constants, $\lambda_1,\ \beta_1,\ \beta_2,\ \beta_3,\ C_0$ and T.
(2) If $\beta_1 - C_0 L_A > 0,\ \beta_2 - C_0 L_A \geq 0,\ \beta_3 - L_A \geq 0,\ \mu_1 - L_\Phi \lambda_1 > 0$, then we have

$$\begin{aligned}
&E[|\widehat{X}(T)|^2] + E[\int_0^T |\widehat{X}(t)|^2 dt]\\
&\leq \delta C_1 E\Big\{\int_0^T \Big(|\widehat{X}(t)|^2 + |\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{(K)}(t,e)|^2\lambda(de)\Big)dt + |\widehat{X}(T)|^2\\
&\quad + \int_0^T \Big(|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt + |\widehat{x}(T)|^2\Big\}.
\end{aligned} \tag{7.5}$$

From the standard estimate of the mean-field BSDE part, we get

$$\begin{aligned}
&E\Big[\int_0^T \Big(|\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de)\Big)dt\Big]\\
&\leq C_1\delta E\Big\{\int_0^T \Big(|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt + |\widehat{x}(T)|^2\Big\} + C_1\Big\{E\int_0^T |\widehat{X}(t)|^2 dt + E|\widehat{X}(T)|^2\Big\}.
\end{aligned} \tag{7.6}$$

Here the constant C_1 depends on the Lipschitz constants, $\lambda_1,\ \beta_1,\ \mu_1,\ C_0,\ \alpha_0$, and T.
From (7.5), (7.6) and the standard estimate of $\widehat{X}(t)$, it follows that, for the sufficiently small $\delta > 0$,

$$\begin{aligned}
&E\Big\{\int_0^T \Big(|\widehat{X}(t)|^2 + |\widehat{Y}(t)|^2 + |\widehat{Z}(t)|^2 + \int_E |\widehat{K}(t,e)|^2\lambda(de)\Big)dt + |\widehat{X}(T)|^2\Big\}\\
&\leq \bar{C}\delta E\Big\{\int_0^T \Big(|\widehat{x}(t)|^2 + |\widehat{y}(t)|^2 + |\widehat{z}(t)|^2 + \int_E |\widehat{k}(t,e)|^2\lambda(de)\Big)dt + |\widehat{x}(T)|^2\Big\}.
\end{aligned} \tag{7.7}$$

Here the constant $\bar{C}$ depends only on the Lipschitz constants, $\lambda_1,\ \beta_1,\ \mu_1,\ \alpha_0$ and T.
From above all, we now choose $\delta_0 = \frac{1}{2\bar{C}}$ in (7.4) and (7.7). Obviously, for every fixed $\delta \in [0, \delta_0]$, the mapping

$I_{\alpha_0+\delta}$ is a contraction in the sense that

$$E\Big\{\int_0^T\Big(|\widehat{X}(t)|^2+|\widehat{Y}(t)|^2+|\widehat{Z}(t)|^2+\int_E|\widehat{K}(t,e)|^2\lambda(de)\Big)dt+|\widehat{X}(T)|^2\Big\}$$
$$\leq\frac{1}{2}E\Big\{\int_0^T\Big(|\widehat{x}(t)|^2+|\widehat{y}(t)|^2+|\widehat{z}(t)|^2+\int_E|\widehat{k}(t,e)|^2\lambda(de)\Big)dt+|\widehat{x}(T)|^2\Big\}.$$

It means immediately that this mapping has a unique fixed point

$$\Lambda^{\alpha_0+\delta}=(X^{\alpha_0+\delta},Y^{\alpha_0+\delta},Z^{\alpha_0+\delta},K^{\alpha_0+\delta}),$$

which is the solution of equation (3.3) for $\alpha=\alpha_0+\delta$. ∎

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