

# AUTOMORPHISMS OF RELATIVE QUOT SCHEMES

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ABSTRACT. Let  $k$  be an algebraically closed field of characteristic zero. Let  $S$  be a smooth projective variety over  $k$  and let  $p_S : X \rightarrow S$  be a family of smooth projective curves over  $S$ . Let  $E$  be a vector bundle over  $X$ . For  $s \in S$  let  $X_s$  be the fibre of  $p_S$  over  $s$  and let  $E_s$  be the restriction of  $E$  to  $X_s$ . Fix  $d \geq 1$ . Let  $\mathcal{Q}(E, d) \rightarrow S$  be the relative Quot scheme parameterizing torsion quotients of  $E_s$  over  $X_s$  of degree  $d$  for all  $s \in S$ . In this article we compute the identity component of relative automorphism group scheme which parameterizes automorphisms of  $\mathcal{Q}(E, d)$  over  $S$ .

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field of characteristic zero. Let  $Y \rightarrow S$  be a smooth morphism between two projective varieties over  $k$ . Associated to this morphism we have the automorphism group scheme  $\text{Aut}(Y/S)$  which parameterizes automorphisms of  $Y$  over  $S$ . Let us denote identity component of  $\text{Aut}(Y/S)$  by  $\text{Aut}^o(Y/S)$ . It is known that  $\text{Aut}^o(Y/S)$  is an algebraic group and if  $\mathcal{T}_{Y/S}$  is the relative tangent bundle, then  $\text{Lie}(\text{Aut}^o(Y/S)) = H^0(Y, \mathcal{T}_{Y/S})$  [MO67, Theorem 3.7], [Bri18, Theorem 2.3]. We refer to [Bri14], [Bri18] for other properties of this group scheme.

We refer to [HL10, Section 2] for definitions and properties of Quot Schemes in general. The Quot Scheme which we will study in this article can be defined in the following manner. Let  $p_S : X \rightarrow S$  be a family of smooth projective curves over an algebraically closed field  $k$  of characteristic zero. Assume  $X$  and  $S$  are smooth projective varieties. Let  $E$  be a vector bundle over  $X$  of rank  $r$ . For a closed point  $s \in S$  let  $X_s$  be the fibre of  $p_S$  over  $s$  and let  $E_s$  be the restriction of  $E$  to  $X_s$ . Fix  $d \geq 1$ . Then associated to the morphism  $p_S$  and the vector bundle  $E$  we have the relative Quot scheme  $\pi_S : \mathcal{Q}(E, d) \rightarrow S$  whose closed points correspond to quotients  $E_s \rightarrow B_d$ ,  $\forall s \in S$  where  $B_d$  is a torsion sheaf of degree  $d$  over the smooth projective curve  $X_s$  [HL10, Theorem 2.2.4]. It is known that  $\mathcal{Q}(E, d)$  is a smooth projective variety [HL10, Proposition 2.2.8]. These schemes have been studied extensively. We refer the reader to [BGL94], [BDW96], [BDH15] for other properties of this scheme. In this article we compute the group scheme  $\text{Aut}^o(\mathcal{Q}(E, d)/S)$ . We recall that in the case when  $S$  is a point and  $E$  the

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2010 *Mathematics Subject Classification.* 14C05, 14J10, 14J50, 14J60, 14L15.

*Key words and phrases.* Automorphism group scheme, Quot Scheme, semistable bundle.

trivial bundle of rank  $r$  this group scheme was computed in [BDH15]. In [BM16] this group scheme was computed in another special case. We refer to Corollary 3.2 and Corollary 3.5 where these results are stated explicitly.

Over  $X$  we fix a certain ample line bundle  $\mathcal{M}$  (this line bundle is defined just before Lemma 2.12). Then the main theorem of this article is

**Theorem** (Theorem 2.15). *Suppose either  $r := \text{rank } E \geq 3$  or  $r = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $X_s$  is  $\geq 2$  for  $s \in S$ . In both of these cases we have isomorphisms*

- (1)  $\text{Aut}^o(\mathcal{Q}(E, d)) \cong \text{Aut}^o(\mathbb{P}(E)/S)$ .
- (2)  $H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \cong H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E, d)/S})$ .

As consequences of Theorem 2.15 we deduce the results of [BDH15] and [BM16] as Corollary 3.2 and Corollary 3.5. We also compute the identity component of the automorphism group scheme of the flag scheme parameterizing chains of torsion quotients of trivial bundle over a smooth projective curve (Corollary 3.4). We refer to Section 3 for more details.

## 2. MAIN THEOREM

Let us denote the projection  $X \times_S \mathcal{Q}(E, d) \rightarrow X$  by  $\pi_X$  and the projection  $X \times_S \mathcal{Q}(E, d) \rightarrow \mathcal{Q}(E, d)$  by  $p_{\mathcal{Q}}$  i.e. we have the following diagram:

$$\begin{array}{ccc} X \times_S \mathcal{Q}(E, d) & \xrightarrow{\pi_X} & X \\ \downarrow p_{\mathcal{Q}} & & \downarrow p_S \\ \mathcal{Q}(E, d) & \xrightarrow{\pi_S} & S \end{array}$$

We denote the universal quotient on  $X \times_S \mathcal{Q}(E, d)$  by

$$\pi_{\mathcal{Q}}^* E \rightarrow \mathcal{B} \rightarrow 0.$$

**Lemma 2.1.** *We have a closed immersion of algebraic groups*

$$\text{Aut}^o(\mathbb{P}(E)/S) \hookrightarrow \text{Aut}^o(\mathcal{Q}(E, d)/S).$$

*Proof.* By [MO67, Corollary 2.2] any automorphism  $g \in \text{Aut}^o(\mathbb{P}(E)/S)$  descends to an automorphism  $h \in \text{Aut}^o(X/S)$ . Therefore we have the following diagram:

$$\begin{array}{ccc} \mathbb{P}(E) \cong \mathbb{P}((g_1)^* E) & \xrightarrow{g} & \mathbb{P}(E) \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{h} & X \end{array}$$

Then  $E \cong h^* E \otimes p^* L$  for some line bundle  $L$  on  $X$ . Let us denote this isomorphism of bundles by  $\Psi_g$ . On  $X \times_S \mathcal{Q}(E, d)$  consider the quotient

$$\pi_{\mathcal{Q}}^* E \cong \pi_{\mathcal{Q}}^* h^* E \otimes \pi_{\mathcal{Q}}^* L \rightarrow (h \times id)^* \mathcal{B} \otimes \pi_{\mathcal{Q}}^* L$$

where the first isomorphism is induced from  $\Psi_g$  and the second morphism is the pullback of the universal quotient  $\pi_{\mathcal{Q}}^* E \rightarrow \mathcal{B} \rightarrow 0$  under the map  $(h \times id)$  tensored with  $\pi_{\mathcal{Q}}^* L$ . This gives a quotient of  $\pi_{\mathcal{Q}}^* E$  over  $X \times_S \mathcal{Q}(E, d)$  and

by the universal property of Quot schemes this induces an automorphism of  $\mathcal{Q}(E, d)$ . Hence, we have a homomorphism

$$\mathrm{Aut}^o(\mathbb{P}(E)/S) \rightarrow \mathrm{Aut}^o(\mathcal{Q}(E, d)/S).$$

Next we show that this homomorphism is injective. At the level of closed points the above automorphism of  $\mathcal{Q}(E, d)$  induced by  $g \in \mathrm{Aut}^o(\mathbb{P}(E)/S)$  is given by

$$[E_s \rightarrow B_d \rightarrow 0] \rightarrow [E_s \xrightarrow{\Psi_{g,s}} h^*(E_s \rightarrow B_d \rightarrow 0) \otimes L_s]$$

Suppose  $g \in \mathrm{Aut}^o(\mathbb{P}(E)/S)$  induces the identity automorphism on  $\mathcal{Q}(E, d)$ . We will show that  $g = \mathrm{id}$ . First we show that  $h = \mathrm{id}$ . Fix  $x \in X$  and let  $p_S(x) = s \in S$ . Consider any quotient

$$E_s \rightarrow \mathcal{O}_{X_s, x} / \mathfrak{m}_{X_s, x}^d$$

where  $\mathcal{O}_{X_s, x}$  is the local ring of  $X_s$  at  $x$  and  $\mathfrak{m}_{X_s, x}$  is its maximal ideal. Then under the automorphism induced by  $g$  the image of this quotient is of the form

$$E_s \rightarrow \mathcal{O}_{X_s, h(x)} / \mathfrak{m}_{X_s, h(x)}^d$$

Hence if  $g$  induces the identity automorphism of  $\mathcal{Q}(E, d)$  then  $h = \mathrm{id}$ . Next we show that  $g = \mathrm{id}$ . Let  $v \in \mathbb{P}(E)$ , let  $p(v) = x$  and  $p_S(x) = s$ . Then  $v$  corresponds to a quotient of vector spaces  $E|_x \xrightarrow{v} k$ . Since,  $h = \mathrm{id}$ ,  $p(g(v)) = x$ . is a quotient of the form  $E|_x \rightarrow k$ . Let us fix  $d-1$  degree 1 quotients  $E \xrightarrow{v_i} k_{x_i}$  for  $1 \leq i \leq d-1$  such that all  $x_i, x$  are distinct and  $p_S(x_i) = p_S(x) = s$ . Then the summation of these quotients gives us a point in  $\mathcal{Q}(E, d)$

$$E_s \rightarrow E|_x \oplus \bigoplus_{i=1}^{d-1} E|_{x_i} \rightarrow k_x \oplus \bigoplus_{i=1}^{d-1} k_{x_i}.$$

Note that each of the quotients  $E|_x \xrightarrow{h} k$  and  $E|_{x_i} \rightarrow k$  can be recovered from the above degree  $d$  quotient simply by restricting this quotient to the points  $x$  and  $x_i$  respectively. By assumption the automorphism induced by  $g$  is identity. Therefore applying the automorphism induced by  $g$  and restricting it to  $x$ , we get that  $g(v) = v$ . This completes the proof of injectivity.  $\square$

**Corollary 2.2.** *We have an inclusion of lie algebras*

$$H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \hookrightarrow H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E, d)/S}).$$

*Proof.* This follows from Lemma 2.1 and [MO67, Theorem 3.7].  $\square$

Let  $\mathcal{Z}$  be the fibered product of  $d$  copies of  $\mathbb{P}(E)$  over  $S$ . We will construct a rational map  $\Phi : \mathcal{Z} \dashrightarrow \mathcal{Q}(E, d)$ . Note that this map was already constructed in [Gan18, Section 2] in the special case when  $S = \{\mathrm{pt}\}$  and  $E = \mathcal{O}_X^r$ . First we set some notations.

**Notation 2.3.**

- (1) Let  $p : \mathbb{P}(E) \rightarrow X$  be the projection.
- (2) Let  $p_i : \mathcal{Z} \rightarrow \mathbb{P}(E)$  be the  $i$ -th projection.

- (3) For,  $i \neq j$ , let  $\Delta_{i,j} \hookrightarrow \mathcal{Z}$  be the closed subscheme given by the equation  $p_i = p_j$ .
- (4) For  $i, j$  distinct let  $\Delta_{i,j,X} \hookrightarrow \mathcal{Z}$  be the closed subscheme given by the equation  $p \circ p_i = p \circ p_j$ .
- (5) For  $i, j, k$  all distinct, let  $\Delta_{i,j,k,X} \hookrightarrow \mathcal{Z}$  be the closed subscheme given by the equation  $p \circ p_i = p \circ p_j = p \circ p_k$ .
- (6) Let  $\pi_1 : X \times_S \mathcal{Z} \rightarrow X$  and  $\pi_2 : X \times_S \mathcal{Z} \rightarrow \mathcal{Z}$  be the first and second projections respectively.
- (7) Let  $p_i \circ \pi_2 : X \times_S \mathcal{Z} \rightarrow \mathbb{P}(E)$  be denoted by  $\pi_{2,i}$ .
- (8) Let  $\Delta_i \hookrightarrow X \times_S \mathcal{Z}$  be the closed subscheme given by the equation  $\pi_1 = p \circ \pi_{2,i}$ .

We define an open set

$$\mathcal{U} := \mathcal{Z} \setminus \left( \bigcup_{i,j} \Delta_{i,j} \bigcup \bigcup_{i,j,k} \Delta_{i,j,k,X} \right).$$

Consider the following composition of morphisms over  $X \times_S \mathcal{Z}$

$$q : \pi_1^* E \rightarrow \bigoplus_{i=1}^d \pi_1^* E|_{\Delta_i} \cong \bigoplus_{i=1}^d \pi_{2,i}^* E|_{\Delta_i} \rightarrow \bigoplus_{i=1}^d \pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}.$$

Let  $u \in \mathcal{U}$ . Then  $q|_{X \times_S u}$  is the morphism

$$E \rightarrow \bigoplus_{i=1}^d E|_{p \circ p_i(u)} \rightarrow \bigoplus_{i=1}^d k_{p \circ p_i(u)}$$

where the map  $E|_{p \circ p_i(u)} \rightarrow k_{p \circ p_i(u)}$  is given by  $p_i(u) \in \mathbb{P}(E)$ . Since  $u \in \mathcal{U}$  for any  $1 \leq i \leq d$ , there can exist atmost one pair  $j \neq i$  such that  $p \circ p_i(u) = p \circ p_j(u)$ , and for such a pair  $(i, j)$ ,  $p_i(u) \neq p_j(u)$ . Hence  $q|_{X \times_S u}$  is a surjection. Therefore  $q|_{X \times_S \mathcal{U}}$  is a surjection. By universal property of  $\mathcal{Q}(E, d)$  the surjection  $q|_{X \times_S \mathcal{U}}$  induces a map

$$\Phi : \mathcal{U} \rightarrow \mathcal{Q}(E, d).$$

Then we prove the following proposition

**Proposition** (Propositon 2.14). *Suppose either  $r \geq 3$  or  $r = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $C$  is  $\geq 2$ . In both of these cases we have an isomorphism*

$$H^0(\mathcal{U}, \Phi^* \mathcal{T}_{\mathcal{Q}(E,d)/S}) = \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

To prove Proposition 2.14 we need a few lemmas. We define  $\mathcal{F}(E, d) := \ker q$ .

**Lemma 2.4.** *We have an isomorphism of vector bundles*

$$\Phi^* \mathcal{T}_{\mathcal{Q}(E,d)/S} \cong \bigoplus_{i=1}^d (\pi_2)_* \mathcal{H}om(\mathcal{F}(E, d), \pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i})|_{\mathcal{U}}.$$

*Proof.* Over  $X \times_S \mathcal{Q}(E, d)$ , we have the universal exact sequence:

$$0 \rightarrow \mathcal{A}(E, d) \rightarrow \pi_X^* E \rightarrow \mathcal{B}(E, d) \rightarrow 0.$$

Then it is known that  $\mathcal{A}(E, d)$  is a vector bundle of rank  $r$  [Gan18, Lemma 2.2] and by [HL10, Proposition 2.2.7] we have

$$\mathcal{T}_{\mathcal{Q}(E, d)/S} = (p_{\mathcal{Q}})_* \mathcal{H}om(\mathcal{A}(E, d), \mathcal{B}(E, d)).$$

Consider the following diagram:

$$\begin{array}{ccc} X \times_S \mathcal{U} & \xrightarrow{id_X \times \Phi} & X \times_S \mathcal{Q}(E, d) \\ \downarrow \pi_1 & & \downarrow p_{\mathcal{Q}} \\ \mathcal{U} & \xrightarrow{\Phi} & \mathcal{Q}(E, d) \end{array}$$

By Grauert's theorem [Har77, Corollary 12.9], We get that

$$\begin{aligned} \Phi^* \mathcal{T}_{\mathcal{Q}(E, d)/S} &= (\Phi)^* (p_{\mathcal{Q}})_* \mathcal{H}om(\mathcal{A}(E, d), \mathcal{B}(E, d)) \\ &\cong (\pi_1)_* (id_X \times \Phi)^* \mathcal{H}om(\mathcal{A}(E, d), \mathcal{B}(E, d)). \end{aligned}$$

Since  $\mathcal{A}(E, d)$  is a vector bundle, we have

$$\begin{aligned} (id_X \times_S \Phi)^* \mathcal{H}om(\mathcal{A}(E, d), \mathcal{B}(E, d)) \\ = \mathcal{H}om((id_X \times_S \Phi)^* \mathcal{A}(E, d), (id_X \times_S \Phi)^* \mathcal{B}(E, d)). \end{aligned}$$

By the definition of the map  $\Phi$  we have

$$(id_X \times_S \Phi)^* \mathcal{B}(E, d) \cong \left( \bigoplus_{i=1}^d \pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i} \right)|_{X \times_S \mathcal{U}}.$$

Also

$$\Phi^* \mathcal{A}(E, d) \cong \mathcal{F}(E, d)|_{X \times_S \mathcal{U}}.$$

since by [Gan18, Lemma 2.2]  $\mathcal{F}(E, d)|_{X \times_S \mathcal{U}}$  is again a vector bundle of rank  $r$  and there exists a surjection  $\Phi^* \mathcal{A}(E, d) \twoheadrightarrow \mathcal{F}(E, d)|_{X \times_S \mathcal{U}}$ . This completes the proof of the lemma.  $\square$

**Lemma 2.5.** *For  $1 \leq i, j \leq d$  and  $i \neq j$  we have*

$$\mathcal{H}om(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j}) = 0.$$

*Proof.* By adjunction, we have

$$\mathcal{H}om(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j}) = \mathcal{H}om(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i \cap \Delta_j}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})$$

Since  $\Delta_j$  is an integral scheme and  $\Delta_i \cap \Delta_j$  is a proper subset of  $\Delta_j$ , the later term in the above expression is zero.  $\square$

**Lemma 2.6.** *For any  $1 \leq j \leq d$  we have*

$$H^0(X \times_S \mathcal{U}, \mathcal{H}om((\pi_1 \times \pi_{2,j})^* \mathcal{F}(E, 1), \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})) = H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

*Proof.* The projection  $\pi_2$  induces isomorphism  $\Delta_j \xrightarrow{\sim} \mathcal{Z}$ . Identifying  $\Delta_j$  with  $\mathcal{Z}$  we have

$$\begin{aligned} & H^0(X \times_S \mathcal{U}, \mathcal{H}om((\pi_1 \times \pi_{2,j})^* \mathcal{F}(E, 1), \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})) \\ &= H^0(\mathcal{U}, \mathcal{H}om(p_j^* \mathcal{F}(E, 1)|_{\Delta_1}, p_j^* \mathcal{O}(1))) \\ &= H^0(\mathcal{U}, p_j^*(\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))). \end{aligned}$$

Since  $\mathcal{F}(E, 1)$  is vector bundle over  $\mathcal{Z}$  and codimension of  $\mathcal{Z} \setminus \mathcal{U} \geq 2$  we have

$$H^0(\mathcal{U}, p_j^*(\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))) = H^0(\mathcal{Z}, p_j^*(\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))).$$

Using projection formula for the morphism  $p_j$  we get that

$$H^0(\mathcal{U}, p_j^*(\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))) = H^0(\mathbb{P}(E), (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))).$$

Now over  $\mathbb{P}(E)$  we have

$$\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1) \cong \mathcal{T}_{\mathbb{P}(E)/S}.$$

This completes the proof of the lemma.  $\square$

**Lemma 2.7.** *For  $1 \leq i, j \leq d$ ,  $i \neq j$ , we have an isomorphism of sheaves:*

$$\mathcal{E}xt^1(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j}) \cong \pi_{2,i}^* \mathcal{O}(-1) \otimes \pi_{2,j}^* \mathcal{O}(1) \otimes \pi_1^* \mathcal{T}_{X/S}|_{\Delta_i \cap \Delta_j}.$$

*Proof.* Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}(-\Delta_i) \longrightarrow \mathcal{O}_{X \times_S \mathcal{Z}} \longrightarrow \mathcal{O}_{\Delta_i} \longrightarrow 0$$

Applying  $\mathcal{H}om(\cdot, \mathcal{O}_{\Delta_j})$  to the above exact sequence, we get:

$$0 \longrightarrow \mathcal{O}_{\Delta_j} \longrightarrow \mathcal{O}(\Delta_i)|_{\Delta_j} \longrightarrow \mathcal{E}xt^1(\mathcal{O}_{\Delta_i}, \mathcal{O}_{\Delta_j}) \longrightarrow 0$$

Therefore

$$\mathcal{E}xt^1(\mathcal{O}_{\Delta_i}, \mathcal{O}_{\Delta_j}) \cong \pi_1^* \mathcal{T}_{X/S}|_{\Delta_i \cap \Delta_j}.$$

and the statement follows immediately from this.  $\square$

The following corollary follows immediately from Lemma 2.7.

**Corollary 2.8.** *We have an isomorphism of sheaves on  $\mathcal{Z}$*

$$(\pi_2)_* \mathcal{E}xt^1(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j}) \cong p_i^* \mathcal{O}(-1) \otimes p_j^* \mathcal{O}(1) \otimes (p \circ p_i)^* \mathcal{T}_{X/S}|_{\Delta_{i,j,X}}.$$

**Lemma 2.9.** *Fix  $1 \leq i, j \leq d$  with  $i \neq j$ . Then for  $1 \leq k, l, m \leq d$  with  $k, l, m$  distinct we have*

- (1)  $\text{codim}(\Delta_{k,l} \cap \Delta_{i,j,X}, \Delta_{i,j,X}) \geq 2$  if  $\{k, l\} \neq \{i, j\}$ .
- (2)  $\text{codim}(\Delta_{i,j} \cap \Delta_{i,j,X}, \Delta_{i,j,X}) = r$ .
- (3) If  $\{i, j\} \not\subseteq \{k, l, m\}$  then  $\text{codim}(\Delta_{k,l,m,X} \cap \Delta_{i,j,X}, \Delta_{i,j,X}) \geq 2$ .
- (4)  $\text{codim}(\Delta_{i,j,k,X}, \Delta_{i,j,X}) = 1$ .

*Proof.* Without loss of generality we can assume  $(i, j) = (1, 2)$ . Then

$$\Delta_{1,2,X} \cong \mathbb{P}(E)_X^2 \times_S \mathbb{P}(E)_S^{d-2}.$$

- (1) Let  $\{k, l\} \cap \{1, 2\} = \emptyset$ . Without loss of generality we can assume  $(k, l) = (3, 4)$ . Then

$$\Delta_{k,l} \cap \Delta_{1,2,X} \cong \mathbb{P}(E)_X^2 \times_S \mathbb{P}(E) \times_S \mathbb{P}(E)_S^{d-4}.$$

Let  $\{k, l\} \cap \{1, 2\} = \{2\}$ . Without loss of generality we can assume  $(k, l) = (2, 3)$ . Then

$$\Delta_{k,l} \cap \Delta_{1,2,X} \cong \mathbb{P}(E)_X^2 \times_S \mathbb{P}(E)_S^{d-3}.$$

Therefore in both these cases it has codimension  $\geq 2$  in  $\Delta_{1,2,X}$ .

- (2) If  $\{k, l\} = \{1, 2\}$  then  $\Delta_{1,2} \cong \mathbb{P}(E) \times_S \mathbb{P}(E)_S^{d-2}$ . Hence it has codimension  $r$  in  $\Delta_{1,2,X}$ .
- (3) Let  $\{1, 2\} \cap \{k, l, m\} = \emptyset$ . Without loss of generality we can assume  $(k, l, m) = (3, 4, 5)$ . Then

$$\Delta_{k,l,m,X} \cap \Delta_{1,2,X} \cong \mathbb{P}(E)_X^2 \times_S \mathbb{P}(E)_X^3 \times_S \mathbb{P}(E)_S^{d-5}.$$

Let  $\{i, j\} \cap \{k, l, m\} = \{i\}$ . Without loss of generality we can assume  $k = i = 1$  and  $(l, m) = (3, 4)$ . Then

$$\Delta_{k,l,m,X} \cap \Delta_{1,2,X} \cong \mathbb{P}(E)_X^4 \times_S \mathbb{P}(E)_S^{d-4}.$$

Hence in both of these two cases it has codimension  $\geq 2$  in  $\Delta_{1,2,X}$ .

- (4)  $\Delta_{1,2,k,X} \cong \mathbb{P}(E) \times_X \mathbb{P}(E) \times_X \mathbb{P}(E) \times_S \mathbb{P}(E)_S^{d-3}$ . Hence it has codimension 1 in  $Y$ .

□

On  $\Delta_{i,j,X}$  we define the line bundle

$$\mathcal{L} := p_i^* \mathcal{O}(-1) \otimes p_j^* \mathcal{O}(1) \otimes (p \circ p_i)^* \mathcal{T}_{X/S}|_{\Delta_{i,j,X}}.$$

By Lemma 2.9 we have that  $\Delta_{i,j,k,X} \subset \Delta_{i,j,X}$  is a divisor on  $\Delta_{i,j,X}$ .

**Lemma 2.10.** *Fix  $1 \leq i, j \leq d$  with  $i \neq j$ . For any  $n \geq 0$  we have*

$$H^0(\Delta_{i,j,X}, \mathcal{L} \otimes \mathcal{O}(\sum_{k \neq i,j} n \cdot \Delta_{i,j,k,X})) = 0.$$

*Proof.* Let  $f : \Delta_{i,j,X} \rightarrow \mathbb{P}(E)_S^{d-1}$  be the product of all the projections except the  $i$ -th projection. Then by projection formula

$$f_*(\mathcal{L} \otimes \mathcal{O}(\sum_{k \neq i,j} n \cdot \Delta_{i,j,k,X})) = (f_* p_i^* \mathcal{O}(-1)) \otimes \mathcal{L}'$$

for some line bundle  $\mathcal{L}'$  on  $\mathbb{P}(E)_S^{d-1}$ . Consider the following fibered diagram:

$$\begin{array}{ccc} \Delta_{i,j,X} & \xrightarrow{p_i} & \mathbb{P}(E) \\ \downarrow f & & \downarrow p \\ \mathbb{P}(E)_S^{d-1} & \xrightarrow{g_i} & X \end{array}$$

Here  $g_i$  is the composition of  $i$ -th projection from  $\mathbb{P}(E)_S^{d-1}$  and the morphism  $p : \mathbb{P}(E) \rightarrow X$ . Since  $g_i$  is flat we have

$$f_* p_i^* \mathcal{O}(-1) = g_i^* p_* \mathcal{O}(-1).$$

Since  $p_* \mathcal{O}(-1) = 0$  we have that  $f_* p_i^* \mathcal{O}(-1) = 0$ . This completes the proof of the lemma.  $\square$

**Proposition 2.11.** *Let  $r = \text{rank } E \geq 3$ . For  $1 \leq i, j \leq d$ ,  $i \neq j$  we have*

$$H^0(X \times_S \mathcal{U}, \mathcal{E}xt^1(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})) = 0.$$

*Proof.* By Corollary 2.7 it is enough to show

$$H^0(\mathcal{U} \cap \Delta_{i,j,X}, \mathcal{L}) = 0.$$

Since  $r \geq 3$  by Lemma 2.9 we have

$$H^0(\mathcal{U} \cap \Delta_{i,j,X}, \mathcal{L}) = H^0(\Delta_{i,j,X} \setminus (\bigcup_{k \neq i,j} \Delta_{i,j,k,X}), \mathcal{L}).$$

Let  $s \in H^0(V, \mathcal{L}|_V)$ . Then for some  $n$  large enough, there exists a section  $0 \neq t \in H^0(\Delta_{i,j,X}, \mathcal{O}(\sum_{k \neq i,j} n \cdot \Delta_{1,2,k,X}))$  such that the section  $st^n$  extends to a global section of  $\mathcal{L} \otimes \mathcal{O}(\sum_{k \neq i,j} n \cdot \Delta_{1,2,k,X})$ . However by Lemma 2.10 there are no global sections of this line bundle and this completes the proof of the proposition.  $\square$

Since  $p_S$  is a projective morphism, we have a  $p_S$ -ample line bundle  $\mathcal{O}_X(1)$ . Let  $\mathcal{O}_S(1)$  be an ample line bundle on  $S$ . Then for  $a \gg 0$  the line bundle  $\mathcal{O}_X(1) \otimes \mathcal{O}_S(a)$  is an ample line bundle on  $X$ . We fix such an ample line bundle  $\mathcal{M}$  on  $X$ .

**Lemma 2.12.** *Let  $E$  be semistable with respect to  $\mathcal{M}$ ,  $\text{rank } E = 2$  and genus of  $X_s \geq 2$  for any  $s \in S$ . Fix  $1 \leq i \leq d$ . Then for any  $n \geq 0$  we have*

$$H^0(\mathcal{Z}, p_i^* \mathcal{T}_{\mathbb{P}(E)/X}^n \otimes (p \circ p_i)^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k \neq i} n \cdot \Delta_{i,k,X})) = 0.$$

*Proof.* Without loss of generality we can assume  $i = 1$ . Let us denote the  $j$ -th projection from  $(X)_S^d$  to  $X$  by  $p_{j,X}$ . We define  $X_j \subseteq (X)_S^d$  to be the closed set defined by the equation  $p_{1,X} = p_{j,X}$ . By projection formula we have

$$\begin{aligned} & \left( \prod_j (p \circ p_j)_* (p_i^* \mathcal{T}_{\mathbb{P}(E)/X}^n \otimes (p \circ p_i)^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k \neq i} n \cdot \Delta_{i,k,X})) \right) \\ &= p_{1,X}^* (S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k=2}^d n \cdot X_k). \end{aligned}$$

Now we have the following exact sequence

$$0 \longrightarrow \mathcal{O}(nX_k) \longrightarrow \mathcal{O}((n+1)X_k) \longrightarrow \mathcal{O}((n+1)X_k)|_{X_k} \longrightarrow 0$$



Note that  $\mathcal{O}((n+1)X_k)|_{X_k} = p_{1,X}^* \mathcal{T}_{X/S}^{n+1}$ . Tensoring the above exact sequence with  $p_{1,X}^*(S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* \mathcal{T}_{X/S}$  and applying  $H^0$  we get that it is enough to show

$$H^0(\mathcal{Z}, p_{1,X}^*(S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^m \mathcal{T}_{X/S}) = 0 \quad \forall n \geq 0, m \geq 1.$$

Applying projection formula for the morphism  $p_{1,X}$  we get

$$\begin{aligned} (p_{1,X})_*(p_{1,X}^*(S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^m \mathcal{T}_{X/S}) \\ = S^{2n}(E) \otimes (\det(E^\vee))^n \otimes \mathcal{T}_{X/S}^m. \end{aligned}$$

Hence it is enough to show that

$$H^0(X, S^{2n}(E) \otimes (\det(E^\vee))^n \otimes \mathcal{T}_{X/S}^m) = 0 \quad \forall n \geq 0, m \geq 1.$$

Now

$$\deg S^{2n}(E) = \binom{2+2n-1}{2} \deg E = n(2n+1) \deg E$$

and  $\text{rank } S^{2n}(E) = 2n+1$ . Therefore

$$\deg S^{2n}(E) \otimes (\det(E^\vee))^n \otimes \mathcal{T}_{X/S}^m = m(2n+1)(\deg \mathcal{T}_{X/S}).$$

Since genus of each fibre of  $X \rightarrow S$  is  $\geq 2$ ,  $\deg \mathcal{T}_{X/S} < 0$ . Hence

$$\deg S^{2n}(E) \otimes (\det(E^\vee))^n \otimes \mathcal{T}_{X/S}^m < 0.$$

Since  $E$  is semistable we have that the bundle  $S^{2n}(E) \otimes (\det(E^\vee))^n \otimes \mathcal{T}_{X/S}^m$  is also semistable with negative degree. Therefore it does not have any global section.  $\square$

**Proposition 2.13.** *Let  $r = \text{rank } E = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $C$  is  $\geq 2$ . For  $1 \leq i, j \leq d$ ,  $i \neq j$  we have*

$$H^0(X \times_S \mathcal{U}, \mathcal{E}xt^1(\pi_{2,i}^* \mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})) = 0.$$

*Proof.* By Corollary 2.7 it is enough to show

$$H^0(\mathcal{U} \cap \Delta_{i,j,X}, \mathcal{L}) = 0.$$

Define the open set

$$V := \Delta_{i,j,X} \setminus \left( \Delta_{i,j} \bigcup_{k \neq i,j}^d \Delta_{i,j,k,X} \right) \subset \Delta_{i,j,X}.$$

By Lemma 2.9 we have

$$H^0(\Delta_{i,j,X} \cap \mathcal{U}, \mathcal{L}) = H^0(V, \mathcal{L}).$$

Therefore, to show that this space vanishes, it is enough to show that

$$H^0(Y, \mathcal{L}(n(\Delta_{i,j} + \sum_{k \neq i,j} \Delta_{i,j,k,X}))) = 0 \quad \forall n \geq 0.$$

Now consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(n \cdot \Delta_{i,j}) \longrightarrow \mathcal{O}((n+1) \cdot \Delta_{i,j}) \longrightarrow \mathcal{O}(n \cdot \Delta_{i,j})|_{\Delta_{i,j}} \longrightarrow 0$$

Tensoring the above exact sequence by  $\mathcal{L}(n \cdot \sum_{k \neq i, j} \Delta_{i, j, k, X})$  and applying  $H^0$  we see that it is enough to show that

$$H^0(\Delta_{i, j}, \mathcal{L}(n(\Delta_{i, j} + \sum_{i=1}^d \Delta_{i, j, k, X})))|_{\Delta_{i, j}} = 0.$$

Note that  $\mathcal{O}(\Delta_{i, j})|_{\Delta_{i, j}} = p_i^* \mathcal{T}_{\mathbb{P}(E)/S}|_{\Delta_{i, j}}$ . Then

$$\begin{aligned} & \mathcal{L}(n(\Delta_{i, j} + \sum_{k \neq i, j} \Delta_{i, j, k, X}))|_{\Delta_{i, j}} \\ &= p_i^* \mathcal{O}(-1) \otimes p_j^* \mathcal{O}(1) \otimes (p \circ p_i)^* \mathcal{T}_{X/S} \otimes p_i^* \mathcal{T}_{\mathbb{P}(E)/X}^n \otimes \mathcal{O}(\sum_{k \neq i, j} n \cdot \Delta_{i, j, k, X})|_{\Delta_{i, j}} \\ &= p_i^* \mathcal{T}_{\mathbb{P}(E)/X}^n \otimes (p \circ p_i)^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k \neq i, j} n \cdot \Delta_{i, j, k, X})|_{\Delta_{i, j}}. \end{aligned}$$

Identifying  $\Delta_{i, j}$  with  $\mathbb{P}(E)_S^{d-1}$  the statement follows from Lemma 2.12.  $\square$

**Proposition 2.14.** *Suppose either  $r \geq 3$  or  $r = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $C$  is  $\geq 2$ . In both of these cases we have an isomorphism*

$$H^0(\mathcal{U}, \Phi^* \mathcal{T}_{\mathbb{Q}(E, d)/S}) = \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

*Proof.* By Lemma 2.4 we have

$$\Phi^* \mathcal{T}_{\mathbb{Q}(E, d)/S} \cong \bigoplus_{i=1}^d (\pi_2)_* \mathcal{H}om(\mathcal{F}(E, d), \pi_{2, i}^* \mathcal{O}(1)|_{\Delta_i})|_{\mathcal{U}}.$$

Hence for a fixed  $1 \leq j \leq d$  it is enough to show

$$H^0(X \times_S \mathcal{U}, \mathcal{H}om(\mathcal{F}(E, d), \pi_{2, j}^* \mathcal{O}(1)|_{\Delta_j})) = H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

Over  $X \times_S \mathcal{U}$  we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(E, d) & \longrightarrow & \pi_1^* E & \longrightarrow & \bigoplus_{i=1}^d \pi_{2, i}^* \mathcal{O}(1)|_{\Delta_i} \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & (\pi_1 \times \pi_{2, j})^* \mathcal{F}(E, 1) & \longrightarrow & \pi_1^* E & \longrightarrow & \pi_{2, j}^* \mathcal{O}(1)|_{\Delta_j} \longrightarrow 0 \end{array}$$

Using snake lemma for the above diagram we get the following exact sequence over  $X \times_S \mathcal{U}$

$$0 \rightarrow \mathcal{F}(E, d) \rightarrow (\pi_1 \times \pi_{2, j})^* \mathcal{F}(E, 1) \rightarrow \bigoplus_{i=1, i \neq j}^d \pi_{2, i}^* \mathcal{O}(1)|_{\Delta_i} \rightarrow 0$$

We apply  $\mathcal{H}om(\cdot, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})$  and then the the functor  $H^0$ . Now the result follows from Lemma 2.5, Lemma 2.6 and Proposition 2.13.  $\square$

**Theorem 2.15.** *Suppose either  $r := \text{rank } E \geq 3$  or  $r = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $X_s$  is  $\geq 2$  for  $s \in S$ . In both of these cases we have isomorphisms*

- (1)  $\text{Aut}^o(\mathcal{Q}(E, d)) \cong \text{Aut}^o(\mathbb{P}(E)/S)$ .
- (2)  $H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \cong H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S})$ .

*Proof.* By Corollary 2.2 we have an inclusion of lie algebras:

$$(2.16) \quad H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \hookrightarrow H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S}).$$

By Lemma 2.4 and Proposition 2.14 we have an inclusion

$$H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S}) \hookrightarrow \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

Since  $\mathcal{Z} \rightarrow \mathcal{Q}(E, d)$  is invariant under the action of the symmetric group  $S_d$  we get that this inclusion factors through

$$H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S}) \hookrightarrow \left( \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \right)^{S_d} = H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}).$$

Comparing the dimensions we get that the (2.16) is an isomorphism. Hence the inclusion in Lemma 2.1 is an isomorphism.  $\square$

### 3. APPLICATIONS

**Corollary 3.1.** *Suppose either  $r \geq 3$  or  $r = 2$ ,  $E$  is semistable with respect to  $\mathcal{M}$  and genus of  $C$  is  $\geq 2$ . Then we have the following left exact sequence of algebraic groups*

$$0 \rightarrow \text{GL}(E)/k^* \rightarrow \text{Aut}^o(\mathcal{Q}(E, d)/S) \rightarrow \text{Aut}^o(X/S)$$

*The corresponding sequence of lie algebras is given by*

$$0 \rightarrow H^0(X, \text{ad } E) \rightarrow H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S}) \rightarrow H^0(X, \mathcal{T}_{X/S})$$

*Proof.* The left exactness of the above sequences follow from Theorem 2.15 and from the fact that  $\text{Aut}^o(\mathbb{P}(E)/S)$  and its lie algebra fits into the above exact sequences.  $\square$

**Corollary 3.2.** *Let the genus of the fibres of  $X \rightarrow S$  is  $\geq 2$ . Suppose either  $r \geq 3$  or  $r = 2$  and  $E$  is semistable with respect to  $\mathcal{M}$ . Then*

- (1)  $\text{Aut}^o(\mathcal{Q}(E, d)/S) = \text{GL}(E)/k^*$ .
- (2)  $H^0(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E,d)/S}) = H^0(X, \text{ad } E)$ .

*Proof.* If genus of each fibre is  $\geq 2$  then  $(p_S)_* \mathcal{T}_{X/S} = 0$ . In particular  $H^0(X, \mathcal{T}_{X/S}) = 0$ . Hence  $\text{Aut}^o(X/S) = 0$ . Now the corollary follows from Corollary 3.1.  $\square$

Taking  $S$  to be a point and  $E = \mathcal{O}_C^r$  in Corollary 3.2 we get [BDH15, Theorem 3.1] and [BDH15, Corollary 3.2].

**Corollary 3.3.** *Let  $C$  be a smooth projective curve of genus  $\geq 2$  over an algebraically closed field  $k$  of characteristic zero. Then*

- (1)  $\text{Aut}^o(\mathcal{Q}(\mathcal{O}_C^r, d)/S) = \text{PGL}(r)$ .
- (2)  $H^0(\mathcal{Q}(\mathcal{O}_C^r, d), \mathcal{T}_{\mathcal{Q}(\mathcal{O}_C^r, d)}) = \mathfrak{sl}(r)$ .

Let  $C$  be a smooth projective curve of genus  $\geq 2$  over an algebraically closed field  $k$  of characteristic zero. Fix  $\underline{d} = (d_1, d_2, \dots, d_k) \in \mathbb{N}^k$  with  $d_1 > d_2 > \dots > d_k$  and  $r \geq 1$ . Let  $\mathcal{D}(r, \underline{d})$  be the flag scheme parametrizing chain of quotients of  $\mathcal{O}_C^r \rightarrow B_1 \rightarrow B_2 \rightarrow \dots \rightarrow B_d$  where  $B_i$  is a torsion quotient of degree  $d_i$  [HL10, 2.A.1]. It is known that  $\mathcal{D}(r, \underline{d})$  is a smooth projective variety.

**Corollary 3.4.** *We have the following isomorphisms of algebraic groups and lie algebras*

- (1)  $\text{Aut}^o(\mathcal{D}(r, \underline{d})) \cong \text{PGL}(r)$ .
- (2)  $H^0(\mathcal{D}(r, \underline{d}), \mathcal{T}_{\mathcal{D}(r, \underline{d})}) = \mathfrak{sl}(r)$ .

*Proof.* Let  $\underline{d}' := (d_2, d_3, \dots, d_k)$ . Over  $C \times \mathcal{D}(r, \underline{d}')$  we have the universal chain of filtrations:

$$\mathcal{A}(r, d_2) \subset \mathcal{A}(r, d_3) \subset \dots \subset \mathcal{A}(r, d_k) \subset \mathcal{O}_{C \times \mathcal{D}(r, \underline{d}')}^r.$$

Then  $\mathcal{D}(r, \underline{d})$  is the relative quot scheme of torsion quotients of degree  $d_1 - d_2$  of the vector bundle  $\mathcal{A}(r, d_2)$  for the map

$$C \times \mathcal{D}(r, \underline{d}') \rightarrow \mathcal{D}(r, \underline{d}).$$

By Corollary 3.2 we get that

$$H^0(\mathcal{D}(r, \underline{d}), \mathcal{T}_{\mathcal{D}(r, \underline{d})/\mathcal{D}(r, \underline{d}')}} = H^0(C \times \mathcal{D}(r, \underline{d}'), \text{ad } \mathcal{A}(r, d_2)).$$

By [Gan18, Theorem 3.2.4, Theorem 5.1] the bundle  $\mathcal{A}(r, d_2)$  is stable with respect to certain polarisations on  $C \times \mathcal{D}(r, \underline{d}')$ . Hence by Corollary 3.2 we have

$$H^0(C \times \mathcal{D}(r, \underline{d}'), \text{ad } \mathcal{A}(r, d_2)) = 0.$$

By induction on  $k$  we get that

$$H^0(\mathcal{D}(r, \underline{d}), \mathcal{T}_{\mathcal{D}(r, \underline{d})}) = H^0(C, \text{ad } \mathcal{O}_C^r) = \mathfrak{sl}(r).$$

This completes the proof of the corollary.  $\square$

Let  $C$  be a smooth projective curve over an algebraically closed field of characteristic zero. In [BM16] the authors computed the identity component of automorphism group scheme of a certain generalized quot scheme  $\mathcal{Q}_C(r, d_p, d_z)$ . We recall the definition of this scheme: Fix  $r \geq 2, d_p, d_z \geq 1$ . Consider the quot scheme  $\mathcal{Q}(\mathcal{O}_C^r, d_p)$  and the universal kernel bundle  $\mathcal{A}(r, d_p)$  over  $C \times \mathcal{Q}(\mathcal{O}_C^r, d_p)$ . Then  $\mathcal{Q}(r, d_p, d_z)$  is defined as the relative Quot scheme associated to the projection  $C \times \mathcal{Q}(\mathcal{O}_C^r, d_p) \rightarrow \mathcal{Q}(\mathcal{O}_C^r, d_p)$  and the bundle

$\mathcal{A}(r, d_p)^\vee$ . By [Gan18, Theorem 3.2.4]  $\mathcal{A}(r, d_p)$  is stable with respect to certain polarisations. Hence  $H^0(C \times \mathcal{Q}(r, d_p), \text{ad } \mathcal{A}(r, d_p)^\vee) = 0$ . Now from Theorem 2.15 we get the result proved in [BM16]:

**Corollary 3.5.** [BM16, Theorem 2.1] *Let  $C$  be a smooth projective curve of genus  $\geq 2$  over an algebraically closed field  $k$  of characteristic zero. We have the following isomorphisms of algebraic groups and lie algebras*

- (1)  $\text{Aut}^o(\mathcal{Q}(r, d_p, d_z)) \cong \text{PGL}(r)$ .
- (2)  $H^0(\mathcal{Q}(r, d_p, d_z), \mathcal{T}_{\mathcal{Q}(r, d_p, d_z)}) = \mathfrak{sl}(r)$ .

**Corollary 3.6.** *Let  $C$  be a smooth projective curve over an algebraically closed field  $k$ . Let  $E$  be a vector bundle of rank  $\geq 3$  over  $C$ . Fix  $d \geq 1$ . Let  $\mathcal{Q}(E, d)$  be the quot scheme of torsion quotients of  $E$  of degree  $d$ . Then we have*

- (1) *If genus of  $C = 0$ , i.e.  $C \cong \mathbb{P}^1$ , then*

$$\text{Aut}^o(\mathcal{Q}(E, d)) = \text{PGL}(2, k) \times \text{GL}(E)/k^* .$$

- (2) *If genus of  $C = 1$  and if  $E$  is semistable then we have the following sequence of algebraic groups*

$$0 \rightarrow \text{GL}(E)/k^* \rightarrow \text{Aut}^o(\mathcal{Q}(E, d)) \rightarrow \text{Aut}^o(C) \rightarrow 0 .$$

- (3) *If  $E$  is not semistable, then  $\text{Aut}^o(\mathcal{Q}(E, d)) = \text{GL}(E)/k^*$ .*

*Proof.* If  $C \cong \mathbb{P}^1$  then any vector bundle  $E$  admits a  $\text{GL}(2)$  linearisation, in particular we have a homomorphism  $\text{GL}(2) \rightarrow \text{Aut}^o(\mathcal{Q}(E, d))$ . This homomorphism factors through  $\text{PGL}(2, k)$  and gives a section to the map  $\text{Aut}^o(\mathcal{Q}(E, d)) \rightarrow \text{PGL}(2, k)$ . Therefore the left exact sequence in Corollary 3.1 is exact in this case and it splits.

From now on we assume that genus of  $C$  is 1 i.e.  $C$  is an elliptic curve. Recall that a bundle  $E$  is called semi-homogeneous if  $\text{Aut}^o(\mathbb{P}(E)) \rightarrow \text{Aut}^o(C) = C$  is surjective ([Muk78, Definition 5.2]). By [Muk78, Proposition 6.13] every semi-homogenous bundle is semistable. Hence (3) follows from Corollary 3.1. Let us assume  $E$  is semistable. Then  $E \cong \bigoplus E_i$ , where  $E_i$  are indecomposable of slope  $\mu(E_i) = \mu(E)$ . By [Ati57, Theorem 10] any indecomposable bundle over  $C$  is semi-homogenous and therefore by [Muk78, Proposition 6.9] we have that  $E$  is semi-homogenous. Now (2) follows from Corollary 3.1.  $\square$

## REFERENCES

- [Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.* (3), 7:414–452, 1957. doi:10.1112/plms/s3-7.1.414.
- [BDH15] Indranil Biswas, Ajneet Dhillon, and Jacques Hurtubise. Automorphisms of the quot schemes associated to compact Riemann surfaces. *Int. Math. Res. Not. IMRN*, (6):1445–1460, 2015. doi:10.1093/imrn/rnt259.
- [BDW96] Aaron Bertram, Georgios Daskalopoulos, and Richard Wentworth. Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians. *J. Amer. Math. Soc.*, 9(2):529–571, 1996. doi:10.1090/S0894-0347-96-00190-7.

- [BGL94] Emili Bifet, Franco Ghione, and Maurizio Letizia. On the Abel-Jacobi map for divisors of higher rank on a curve. *Math. Ann.*, 299(4):641–672, 1994. doi:10.1007/BF01459804.
- [BM16] Indranil Biswas and Sukhendu Mehrotra. Automorphisms of the generalized quot schemes. *Adv. Theor. Math. Phys.*, 20(6):1473–1484, 2016. doi:10.4310/ATMP.2016.v20.n6.a6.
- [Bri14] Michel Brion. On automorphisms and endomorphisms of projective varieties. In *Automorphisms in Birational and Affine Geometry*, volume 79 of *Springer Proceedings in Mathematics and Statistics*, pages 59–81. Springer, Cham, 2014. doi:10.1007/978-3-319-05681-4\_4.
- [Bri18] Michel Brion. Notes on automorphism groups of projective varieties, 2018. URL [https://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos\\_final.pdf](https://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf).
- [Gan18] Chandranandan Gangopadhyay. Stability of sheaves over Quot schemes. *Bull. Sci. Math.*, 149:66–85, 2018. doi:10.1016/j.bulsci.2018.08.001.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. doi:10.1017/CBO9780511711985.
- [MO67] Hideyuki Matsumura and Frans Oort. Representability of group functors, and automorphisms of algebraic schemes. *Invent. Math.*, 4:1–25, 1967. doi:10.1007/BF01404578.
- [Muk78] Shigeru Mukai. Semi-homogeneous vector bundles on an Abelian variety. *J. Math. Kyoto Univ.*, 18(2):239–272, 1978. doi:10.1215/kjm/1250522574.

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