BIREGULAR AND BIRATIONAL GEOMETRY OF QUARTIC DOUBLE SOLIDS WITH 15 NODES

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ABSTRACT. Three-dimensional del Pezzo varieties of degree 2 are double covers of projective space \mathbb{P}^3 branced in a quadric. In this paper we prove that if a del Pezzo variety of degree 2 has exactly 15 nodes then the corresponding quadric is a hyperplane section of the Igusa quartic or, equivalently, all such del Pezzo varieties are members of one particular linear system on the Coble fourfold. Their automorphism groups are induced from the automorphism group of Coble fourfold. Also we classify all *G*-birationally rigid varieties of such type.

Bibliography: 11 titles.

1. INTRODUCTION

In this article we work over the field of complex numbers.

Classification of rational G-Fano varieties is an important problem for classification of finite subgroups of Cremona groups (cf. [7]). Three-dimensinal G-del Pezzo varieties were partially classified by Yu. Prokhorov in [10]. In the end of the article he posed the question: which G-del Pezzo threefolds are birationally (super)rigid? Cases of rational G-del Pezzo threefolds of degree 3 and 4 were considered by the author in papers [1] and [2]. In the case of degree 2 the situation is much more complicated — it can be seen already from the classification of rational del Pezzo threefolds of degree 2 (see [4]). Such varieties have at most 16 singularities (see, for example, [10]). In the present paper we consider the case of del Pezzo threefolds of degree 2 with 15 ordinary double points. The main results of this article are the following theorems:

Theorem 1.1. Every quartic surface with 15 nodes is a hyperplane section of the Igusa quartic. Let X be a del Pezzo threefold of degree 2 with precisely 15 ordinary double points. Then the variety X is a member of a linear system $|\mathcal{L}|$ on the Coble fourfold (i.e. the cover of \mathbb{P}^4 branched in the Igusa quartic), where \mathcal{L} is a restriction of the line bundle $\mathcal{O}(1)$ on the weighted projective space $\mathbb{P}(2, 1, 1, 1, 1, 1)$, and the automorphism group of X coincides with the stabilizer of X in the automorphism group of the Coble fourfold.

Theorem 1.2. Let X be a del Pezzo threefold of degree 2 with 15 ordinary double points. Let $G \subset \operatorname{Aut}(X)$ be a subgroup such that the variety X is $G\mathbb{Q}$ -factorial and $\operatorname{rk}\operatorname{Pic}(X)^G = 1$. Then the variety X is G-birationally rigid only in the following situation: it can be given by

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the equation

$$y^{2} - 4\sum_{i=1}^{5} x_{i}^{4} + (\sum_{i=1}^{5} x_{i}^{2})^{2} = \sum_{i=1}^{5} x_{i} = 0$$

in $\mathbb{P}(2,1,1,1,1,1)$ and G is isomorphic to $S_5 \times C_2$, $A_5 \times C_2$ or S_5 (non-standard subgroup). Moreover, in this case X is G-birationally superrigid.

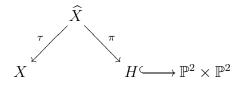
In this article we use the following notation for groups: by C_n we denote the cyclic group of order n; by D_{2n} we denote the dihedral group of order 2n; by S_n we denote the symmetric group of rank n; by A_n we denote the alternating group of rank n.

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2. Biregular geometry of del Pezzo threefolds of degree 2 with 15 nodes

Let X be a double cover of \mathbb{P}^3 branched in a quartic which has 15 ordinary double points and no other singularities or, equivalently, a del Pezzo threefold of degree 2 with 15 ordinary double points. In the sequel we will call them quartic double solids with 15 nodes. There is the following statement:

Proposition 2.1 ([10, Theorem 8.1]). The variety X can be obtained by the following diagram:



where H is a smooth divisor of bi-degree (1,1) in $\mathbb{P}^2 \times \mathbb{P}^2$, the morphism π is a blow up of four points $P_i \in H$ in general position and τ is a contraction of fifteen rational curves which has zero intersection number with the canonical class $K_{\widehat{X}}$. More presidely, the proper transforms of the following curves are contracted:

- curves of bi-degree (1,0) and (0,1) passing through P_i ;
- curves of bi-degree (1,1) passing through a pair of points P_i ;
- curve of bi-degree (2,2) passing through all the points P_i .

Definition 2.2. The Igusa quartic I is the three-dimensional quartic which can be explicitly given by the following system of equations in \mathbb{P}^5 :

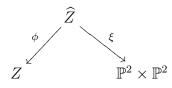
$$s_1 = 4s_4 - s_2^2 = 0,$$

where $s_j = \sum_{i=1}^{6} x_i^j$ (for our purposes such numeration of coordinates in \mathbb{P}^5 is more convenient). The Coble fourfold Z is a double cover of \mathbb{P}^4 ramified in the Igusa quartic I. In other words, Z can be explicitly given by the following system of equations in $\mathbb{P}(2, 1, 1, 1, 1, 1, 1, 1)$ with weighted homogeneous coordinates z, x_1, \ldots, x_6 :

$$s_1 = 0, \ z^2 = 4s_4 - s_2^2$$

The following structure theorem was proved recently by I. Cheltsov, A. Kuznetsov and C. Shramov:

Theorem 2.3 ([3, Theorem 1.9, Proposition 2.21]). The Coble fourfold Z can be obtained by the following diagram:



where ξ is a blow up of four points $P_i = (p_{1,i}, p_{2,i})$ in general position (without loss of generality we can assume that such points have coordinates

(1:0:0,1:0:0), (0:1:0,0:1:0), (0:0:1,0:0:1), (1:1:1,1:1:1)

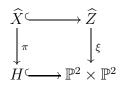
respectively). The map ϕ is a small morphism which contracts the proper transform of the following fifteen surfaces:

- eight planes of the form $p_{1,i} \times \mathbb{P}^2$ or $\mathbb{P}^2 \times p_{2,i}$;
- six quadrics of the form $l_{i,j} \times l'_{i,j}$ where $l_{i,j}$ (resp. $l'_{i,j}$) is a line in \mathbb{P}^2 passing through the points $p_{1,i}$ and $p_{1,j}$ (resp. $p_{2,i}$ and $p_{2,j}$);
- diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$.

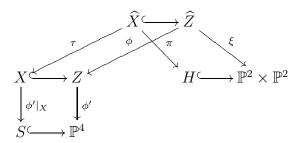
From the previous theorem and Proposition 2.1 we deduce the following proposition.

Proposition 2.4. Let X be a quartic double solid with 15 ordinary double points. Then X is isomorphic to a double cover of \mathbb{P}^3 branched in a hyperplane section of the Igusa quartic.

Proof. We will use the notation which was introduced is Proposition 2.1. We have the following commutative diagram:



where H is a smooth divisor of bi-degree (1, 1) and the morphisms ξ and π are blow ups of four points on H in general position. It is easy to see that intersections of 15 surfaces listed in Theorem 2.3 with the variety \hat{X} are precisely curves contracted by the morphism τ . Hence we have the following commutative diagram:



Note that the divisor \widehat{X} is equivalent to $-\frac{1}{3}K_{\widehat{Z}}$ and the morphism $\phi' \circ \phi$ is given by the linear system $|-\frac{1}{3}K_{\widehat{Z}}|$ (see [3, Section 2.1]), thus S is a hyperplane section of the Igusa quartic I and X is a subvariety of the Coble fourfould given by a linear equation in coordinates x_1, \ldots, x_6 . \Box

Corollary 2.5. Every quartic surface with 15 nodes is a hyperplane section of the Igusa quartic.

Remark 2.6. In the paper [11] the authors proved that a general quartic threefold with 15 ordinary double points is a hyperplane section of the Igusa quartic. From the previous corollary we see that in fact every quartic with 15 nodes has the same property.

Remark 2.7. In the sequel we will say that the variety X is a hyperplane section of the Coble fourfold, although it is not completely correct, since the corresponding linear system is not very ample.

For classification of automorphism groups of del Pezzo threefolds of degree 2 with 15 nodes we need the following well-known properties of the Igusa quartic.

Proposition 2.8. (i) The automorphism group of the Igusa quartic is isomorphic to S_6 and acts by permutations of coordinates.

(ii) The singular set of the Igusa quartic I consists of 15 lines which can be described explicitly by the following equations:

$$x_{\sigma(1)} = x_{\sigma(2)}, x_{\sigma(3)} = x_{\sigma(4)}, x_{\sigma(5)} = x_{\sigma(6)}$$

where σ is an element of the group S_6 . Such lines we denote by l_{α} where α is a partition of the set $\{1, 2, 3, 4, 5, 6\}$ into three pairs of elements.

(iii) There are exactly 10 hyperplanes whose intersection with the Igusa quartic is a quartic surface with multiplicity 2. They can be explicitly given by the following equations:

$$x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} = x_{\sigma(4)} + x_{\sigma(5)} + x_{\sigma(6)} = 0$$

where σ is an element of the group S_6 . We denote such hypersurfaces by H_β where β is a partition of the set $\{1, 2, 3, 4, 5, 6\}$ into two triples of elements.

(iv) Every hyperplane H_{β} contains exactly six lines l_{α} while every line l_{α} lies exactly on four hyperplanes H_{β} . In other words, they form a $(15_4, 10_6)$ -configuration in notation of [9]. Also, every pair of hyperplanes H_{β} contain exactly two common lines l_{α} . **Remark 2.9.** These properties can be easily deduced from the fact that the Igusa quartic is the dual variety of the Segre cubic, while singular points of the Segre cubic and planes on it form a $(15_4, 10_6)$ -configuration (see [8, §9.4.4]).

Definition 2.10. The automorphism of the $(15_4, 10_6)$ -configuration is a permutation of sets H_β and l_α which preserves the relation "a line lies on a plane".

Lemma 2.11. The automorphism group of the $(15_4, 10_6)$ -configuration is isomorphic to S_6 .

Proof. Let G be the automorphism group of the $(15_4, 10_6)$ -configuration. Obviously, the group $\operatorname{Aut}(I) \simeq S_6$ acts faithfully on the configuration of singular lines and hyperplanes, so $S_6 \subset G$. Hence it is enough to prove that $|G| \leq 720$.

Since G acts transitively on the set of lines l_{α} , we see that

$$|G| = 15 |\operatorname{Stab}(l_{((1,2)(3,4)(5,6))})|$$

The group $\operatorname{Stab}(l_{((1,2)(3,4)(5,6))})$ preserves the set of four hyperplanes H_{β} containing the line $l_{((1,2)(3,4)(5,6))}$, so

 $|G| \leqslant 15 \cdot 24 \cdot |\operatorname{Stab}(H_{((1,3,5)(2,4,6))}, H_{((1,3,6)(2,4,5))}, H_{((1,4,5)(2,3,6))}, H_{((1,4,6)(2,3,5))})|.$

The last group (let us denote it by G') fixes also six lines which lie in pairwise intersections of four fixed hyperplanes, in particular it fixes the line $l_{((1,4)(2,3)(5,6))}$. Two remaining hyperplanes passing through the line $l_{((1,4)(2,3)(5,6))}$ form a G'-invariant set. One can easily check that only the trivial element of G' fixes them. Hence we deduce that

$$G|\leqslant 15\cdot 24\cdot 2=720.$$

Proposition 2.12. Let X be a quartic double solid with 15 nodes. Let $X = Z \cap \overline{H}$ be a representation of X as a hyperplane section of the Coble fourfold. Then the automorphism group of X coincides with the stabilizer of \overline{H} in the automorphism group of the Coble fourfold Z.

Proof. Due to [3, Corollary 3.5] the automorphism group of the variety Z is isomorphic to $S_6 \times C_2$ where the group S_6 acts by permutations of coordinates x_i and C_2 acts by the change of sign of the coordinate y. The variety X is a double cover of \mathbb{P}^3 branched in a hyperplane section of the Igusa quartic. Let us denote such a hyperplane by H. Since the double cover $X \to H$ is given by the linear system $|-\frac{1}{2}K_X|$, we have a homomorphism of groups $\operatorname{Aut}(X) \to \operatorname{Aut}_{lin}(I \cap H)$ where $\operatorname{Aut}_{lin}(I \cap H)$ is the group of linear transformations of H which preserve the quartic $I \cap H$. Obviously this homomorphism is surjective and its kernel is generated by the Galois involution. Thus we need to prove that $\operatorname{Aut}_{lin}(I \cap H)$ coincides with the stabilizer of the hyperplane H in the group $\operatorname{Aut}(I) \simeq S_6$.

Intersections of lines l_{α} and hyperplanes H_{β} with the subspace H form a $(15_4, 10_6)$ -configuration of singular points of $I \cap H$ and planes which intersect $I \cap H$ in a double conic. Since the singular points are in a general enough position, the natural map from $\operatorname{Aut}_{lin}(I \cap H)$ to an automorphism group of $(15_4, 10_6)$ -configuration is an embedding. Since we have a natural isomorphism between the automorphism group of the $(15_4, 10_6)$ -configuration and the group $\operatorname{Aut}(I)$, we obtain a natural embedding $\operatorname{Aut}_{lin}(I \cap H) \hookrightarrow \operatorname{Aut}(I)$. Suppose that the image of some element of the group $\operatorname{Aut}_{lin}(I \cap H)$ in $\operatorname{Aut}(I)$ does not preserve the hyperplane H. Then this automorphism induces a linear map from H to another hyperplane H' such that the point $l_{\alpha} \cap H$ maps to $l_{\alpha} \cap H'$ for every α . Since hyperplanes Hand H' don't coincide, one has $H \cap H_{\beta} \neq H' \cap H_{\beta}$ for some index β . Note that $H_{\beta} \cap I$ is a smooth quadric surface. One can easily check that among six lines l_{α} lying on the quadric $H_{\beta} \cap I$ three lines lie in one family while another three lines lie in another family. One can easily see that every isomorphism between two conics with six marked points which are hyperplane sections of a quadric can be uniquely extended to an automorphism of the quardic which preserves families of lines. We apply this statement to the case of the map $H \cap H_{\beta} \cap I \to H' \cap H_{\beta} \cap I$. But if an automorphism of $H_{\beta} \cap I$ preserves all lines $l_{\alpha} \subset H_{\beta}$ then such an automorphism is trivial, so $H \cap H_{\beta} \cap I = H' \cap H_{\beta} \cap I$, which contradicts our assumptions.

This contradiction shows that the image of the embedding $\operatorname{Aut}_{lin}(I \cap H) \hookrightarrow \operatorname{Aut}(I)$ is contained in the stabilizer of the corresponding hyperplane $\operatorname{Stab}(H) \subset \operatorname{Aut}(I)$. Obviously, inverse statement also holds, so $\operatorname{Aut}_{lin}(I \cap H) = \operatorname{Stab}(H)$ and $\operatorname{Aut}(X) = \operatorname{Stab}(\overline{H})$. \Box

As a consequence of Proposition 2.4 and Proposition 2.12 and Corollary 2.5 we get Theorem 1.1.

3. Equivariant birational rigidity of quartic double solids with 15 nodes

Definition 3.1. Let X and Y be a varieties with an action of a finite group G. We call a rational map $f : X \dashrightarrow Y$ a G-equivariant map if there exist an automorphism τ of the group G such that the following diagram commutes for every $g \in G$:

$$\begin{array}{c} X & - \stackrel{f}{\longrightarrow} Y \\ \downarrow g & \qquad \downarrow \tau(g) \\ X & - \stackrel{f}{\longrightarrow} Y \end{array}$$

We denote the group of G-equivariant automorphisms of a G-variety X by $\operatorname{Aut}^{G}(X)$ and the group of G-equivariant birational selfmaps of a G-variety X by $\operatorname{Bir}^{G}(X)$.

Definition 3.2. A *G*-Fano variety *X* is called *G*-birationally rigid if there is no *G*-Mori fibration $X' \to Y$ such that varieties *X* and *X'* are *G*-birationally equivalent but not isomorphic. If one also has $\operatorname{Bir}^{G}(X) = \operatorname{Aut}^{G}(X)$ then *X* is called *G*-birationally superrigid.

As an application of Theorem 1.1 we classify all G-birationally rigid del Pezzo threefolds of degree 2 with 15 nodes.

Lemma 3.3. Let $X = Z \cap H$ be a quartic double solid with 15 nodes where $Z \subset \mathbb{P}(2, 1, 1, 1, 1, 1, 1)$ is a Coble fourfold. Assume that X is $\operatorname{Aut}(X)$ -birationally rigid. Then the equation of H is of the form $x_i + ax_j = 0$ where $a \neq -1$. The group $\operatorname{Aut}(X)$ in this case is isomorphic to

- $S_5 \times C_2$ if a = 0;
- $S_4 \times C_2 \times C_2$ if a = 1;
- $S_4 \times C_2$ in other cases.

Proof. Since Aut(X) always contains the Galois involution of the double cover, the variety X is always $\operatorname{Aut}(X)\mathbb{Q}$ -factorial (i.e. every $\operatorname{Aut}(X)$ -invariant divisor is a \mathbb{Q} -Cartier divisor) and Aut(X)-minimal (i.e. the rank of the invariant Picard group equals to 1). We know that $\operatorname{Aut}(X) \simeq G \times C_2$, where G is a subgroup of $\operatorname{Aut}(I) \simeq S_6$. We consider the projective space $\mathbb{P}^4 \supset I$ as a projectivization of the simplicial representation W of the group S_6 . Consequently, H is a projectivization of a four-dimensional subrepresentation $V \subset W$ of the group G. If V contains a two-dimensional subrepresentation of G then its projectivization is a G-invariant line on $\mathbb{P}(V)$. Projection from this line gives us a structure of Aut(X)-equivariant fibration by rational surfaces. We can apply an Aut(X)-equivariant resolution of singularities and Aut(X)equivariant relative minimal model program and obtain a G-Mori fiber space with the base of positive dimension which is birational to our quartic double solid, a contradiction. Hence the representation W of the group G is a direct sum either of a one-dimensional and a fourdimensional irreducible representations or of two one-dimensional and a three-dimensional irreducible representations. Also we know that there are no Aut(X)-invariant singular points on X (otherwise projection from such a point gives us a structure of an Aut(X)-equivariant conic bundle).

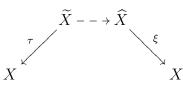
We have only the following non-abelian subgroups of S_6 which are stabilizers of hyperplanes in $\mathbb{P}(W)$: the group S_5 (for the hyperplane $x_1 = 0$), the group $S_4 \times C_2$ (for the hyperplane $x_1 \pm x_2 = 0$), the group S_4 (for the hyperplane $x_1 + ax_2 = 0$), the group $S_3 \times S_3$ (for the hyperplane $x_1 + x_2 + x_3 = 0$), the group $S_3 \times C_3$ (for the hyperplane $x_1 + \xi x_2 + \xi^2 x_3 = 0$, where $\xi^3 = 1$), the group $S_3 \times C_2$ (for the hyperplane $x_1 + x_2 + ax_3 = 0$), and the group S_3 (for the hyperplane $x_1 + ax_2 + bx_3 = 0$). For every subgroup one can calculate the character of the representation and decompose the representation W in irreducible summands. It turns out that only stabilizers of hyperplanes $\{x_i + ax_j = 0\}$ satisfy the properties mentioned above.

If a = -1 then some lines l_{α} lie on X, which is impossible since X has isolated singularities.

Proposition 3.4. In the notation of the previous proposition, assume that $a \neq 0$. Then X is not Aut(X)-birationally rigid.

Proof. Denote Aut(X) by G for simplicity. Without loss of generality we may assume that X is given by the equation $x_0 + ax_1 = 0$ in the Coble fourfold Z. If $a \neq 0$ then we have a G-invariant set which consists of three singular points

 $p_1 = (0:0:0:1:1:-1:-1), p_2 = (0:0:0:1:-1:1:-1), p_3 = (0:0:0:1:-1:-1:1)$ in coordinates $(y:x_1:x_2:x_3:x_4:x_5:x_6)$. Denote by $\pi: X \to \mathbb{P}^3$ the morphism given by the linear system $|-\frac{1}{2}K_X|$. Let l_{ij} be a line passing through $\pi(p_i)$ and $\pi(p_j)$. The preimage of l_{ij} under π consists of two curves which we denote by l'_{ij} and l''_{ij} . Let \widetilde{X} be a blow up of three singular points p_i . One can easily check that the divisor $-K_{\widetilde{X}}$ is nef and the only curves with zero intersection with it are six proper transforms of curves l'_{ij} and l''_{ij} . Let \widehat{X} be a variety which we obtain after making flops in such curves. The *G*-invariant Mori cone of the variety \widehat{X} generated by two rays and one of them is $K_{\widehat{X}}$ -negative. We need to prove that its contraction is not a divisorial contraction to X. Suppose that we have the following commutative *G*-equivariant diagram:



where τ is a blow up of points p_1, p_2 and p_3 and ξ is a contraction of a negative extremal ray. Let H be an ample generator of the group $\operatorname{Pic}(X)^G$, let $\widetilde{H} = \tau^* H$ and let \widetilde{E} be an exceptional divisor of τ . Analogously we can define $\widehat{H} = \xi^* H$ and let \widehat{E} be an exceptional divisor of ξ . Let \widehat{H}' and \widehat{E}' be the proper transforms of \widehat{H} and \widehat{E} on \widetilde{X} respectively. The group $\operatorname{Pic}(\widetilde{X})^G$ is generated by \widetilde{H} and \widetilde{E} and we have following equalities:

$$\widetilde{H}^3 = 2, \ \widetilde{H}^2 \cdot \widetilde{E} = \widetilde{H} \cdot \widetilde{E}^2 = 0, \ \widetilde{E}^3 = 6.$$

Let $\widehat{H}' = a\widetilde{H} + b\widetilde{E}$ and $\widehat{E}' = c\widetilde{H} + d\widetilde{E}$. From equalities
 $2\widetilde{H} - \widetilde{E} \sim -K_{\widetilde{X}} \sim 2\widehat{H}' - \widehat{E}'$

we deduce that c = 2a - 2 and d = 1 + 2b. Since the classes \widehat{H}' and \widehat{E}' generate the group $\operatorname{Pic}(\widetilde{X})^G$ too, the determinant of the corresponding matrix is equal to ± 1 , so we have an equality $a + 2b = \pm 1$. Also one can easily see that

$$2 = \widehat{H}^3 = \frac{1}{2}\widehat{H}^2 \cdot (-K_{\widehat{X}}) = \frac{1}{2}(\widehat{H}')^2 \cdot (-K_{\widetilde{X}}) = 2a^2 - 3b^2,$$

where the third equality follows from the fact that the anticanonical class $-K_{\tilde{X}}$ is base point free, so we can assume that it does not intersect with flopped curves. The only solution of the system of equations $2 = 2a^2 - 3b^2$ and $a + 2b = \pm 1$ in integers is a = 1, b = 0, but this contradicts our assumptions. Hence, the Sarkisov link which starts with the blow up of X in three points p_i gives us a birational transform of X to another G-Mori fiber space, thus the variety X is not G-birationally rigid. \Box

Definition 3.5. In the notation of Lemma 3.3 let a = 0. A subgroup $S_5 \subset \operatorname{Aut}(X)$ is a *twisted* subgroup, if every permutation $\sigma \in S_5$ acts as

$$(y:x_1:x_2:x_3:x_4:x_5:x_6)\mapsto (\operatorname{sign}(\sigma)y:x_{\sigma(1)}:x_{\sigma(2)}:x_{\sigma(3)}:x_{\sigma(4)}:x_{\sigma(5)}:x_6).$$

Proposition 3.6. In the notation of Lemma 3.3 let a = 0. Let G be a subgroup of

$$\operatorname{Aut}(X) \simeq S_5 \times C_2$$

such that X is a GQ-factorial and G-minimal variety. Then the variety X is G-birationally superrigid if and only if G coincides with one of the following groups: Aut(X), twisted subgroup S_5 or $A_5 \times C_2$. Moreover, in this case the variety X is G-birationally superrigid.

Proof. If the group G is contained in $S_4 \times C_2$ then the variety X is not G-birationally rigid since we have the same link as in Proposition 3.4 and the proof works in our situations without changes. If the group G is contained in $(C_5 \rtimes C_4) \times C_2$ then we have a G-invariant set of

singular points of X. One can easily check that their images on \mathbb{P}^3 are in general position and twisted cubics passing through them give us a structure of G-conic bundle on X. Indeed, for a general point of \mathbb{P}^3 we have exactly one twisted cubic as above passing through this point and general twisted cubic intersects the variety X in 5 double points and two additional points, so its preimage on X is an irreducible curve of genus 0. Also we know, that the group G cannot be a subgroup of $D_{12} \times C_2$, because such groups have only one- and two-dimensional irreducible representations while the representation W of the group G is a direct sum either of a onedimensional and a four-dimensional irreducible representations or of two one-dimensional and a three-dimensional irreducible representations (see the proof of Lemma 3.3). Hence G must be one of the following groups: A_5 , $A_5 \times C_2$, S_5 (two non-conjugate subgroups) or $S_5 \times C_2$.

Due to [10, Corollary 8.2.2] there is a natural embedding of the group $\operatorname{Aut}(X)$ into the automorphism group of the root system D_5 and only *G*-invariant vector in the corresponding lattice is the null vector (otherwise variety X is not $G\mathbb{Q}$ -factorial or *G*-minimal). The automorphism group of the root system D_5 is isomorphic to $C_2^5 \rtimes S_5$ and acts on the corresponding lattice \mathbb{Z}^5 by changes of signs and permutations of coordinates. The group A_5 and standard subgroup $S_5 \subset \operatorname{Aut}(X)$ have non-trivial invariant vector in the lattice, so *G* can not coincide with them. One can easily see, that all other subgroups satisfy the required property, and equivariant birational rigidity with respect to them were proven in [5, Theorem 4.2]. Note, that there is a mistake in this paper, more precisely, the group A_5 is not minimal, as we saw before. The pair $(X, \frac{1}{\mu}\mathcal{M})$ is canonical for every μ and every movable *G*-invariant linear subsystem $\mathcal{M} \subset |-\mu K_X|$ (see the proof of the [5, Theorem 4.2]). So the variety X is *G*-birationally superrigid by the Noether–Fano inequalities (see, for example, [6, Theorem 3.2.6]).

The proof of Theorem 1.2 is a direct consequence of Lemma 3.3, Proposition 3.4 and Proposition 3.6.

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