Adelic geometry on arithmetic surfaces I: idelic and adelic interpretation of the Deligne pairing

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Abstract

For an arithmetic surface $X \to B = \operatorname{Spec} O_K$ the Deligne pairing $\langle\ ,\ \rangle :\ \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \operatorname{Pic}(B)$ gives the "schematic contribution" to the Arakelov intersection number. We present an idelic and adelic interpretation of the Deligne pairing; this is the first crucial step for a full idelic and adelic interpretation of the Arakelov intersection number.

For the idelic approach we show that the Deligne pairing can be lifted to a pairing $\langle\ ,\ \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \operatorname{Pic}(B)$, where $\ker(d^1_\times)$ is an important subspace of the two dimensional idelic group $\mathbf{A}_X^\times$. On the other hand, the argument for the adelic interpretation is entirely cohomological.

0 Introduction

0.1 Background

Problems of topological nature on $(\mathbb{Q}, |\cdot|)$ (where $|\cdot|$ is the usual euclidean absolute value), are commonly solved after an embedding of $\mathbb{Q}$ in its completion $\mathbb{R}$ with respect to $|\cdot|$. In this way we can take advantage of the completeness properties of $\mathbb{R}$ and the density of $\mathbb{Q}$ inside $\mathbb{R}$. Any other absolute value which doesn't give the standard euclidean topology on $\mathbb{Q}$ is equivalent (in the sense of absolute values, see footnote) to the p-adic absolute value $|\cdot|_p$ for any prime p, so it makes sense to embed $\mathbb{Q}$ densely in its completion $\mathbb{Q}_p$ with respect to $|\cdot|_p$. The above way of reasoning can be easily generalized for any number field K, and adeles were introduced in 1930s by Chevalley in order to consider simultaneously all the completions of K with respect all possible places[1]. It is not very useful to study simply the product $\prod_{\mathfrak{p}} K_{\mathfrak{p}}$ of all completions because the resulting space is "too big" and fails to be a locally compact additive group. For any non-archimedean place $\mathfrak{p}$ let $\mathcal{O}_{\mathfrak{p}}$ be the closed unit ball in $K_{\mathfrak{p}}$ under the standard representative of the place $\mathfrak{p}$, then the ring of adeles is defined as a subset of $\prod_{\mathfrak{p}} K_{\mathfrak{p}}$, namely:

$$\mathbf{A}_K := \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}}$$

[1] A place $\mathfrak{p}$ is an equivalence class of absolute values on K where two absolute values are declared equivalent if they generate the same topology.

where $\prod'$ is the restricted products with respect to the additive subgroups $\{\mathcal{O}_\mathfrak{p}\colon \mathfrak{p} \text{ is non-archimedean}\}$. Note that for archimedean places, the unit ball is not an additive subgroup. The most important features of $\mathbf{A}_K$ were well described in [24] and consist mainly in the fact that $\mathbf{A}_K$ is a locally compact additive group (so it admits a Haar measure), K is discrete in $\mathbf{A}_K$, and the quotient $\mathbf{A}_K/K$ is compact. Moreover the Pontryagin dual $\widehat{\mathbf{A}_K}$ has a very simple description and $\mathbf{A}_K \cong \widehat{\mathbf{A}_K}$.

The multiplicative version of the adelic theory is the idelic theory, and the group of ideles attached to K is defined as:

$$\mathbf{A}_K^\times = \prod_{\mathfrak{p}}{}' K_\mathfrak{p}^\times$$

where the restricted product is taken with respect to the subgroups $\mathcal{O}_\mathfrak{p}^\times := \{x \in \mathcal{O}_\mathfrak{p}\colon |x|_\mathfrak{p} = 1\}$. In this case $\mathcal{O}_\mathfrak{p}^\times$ has a group structure also in the archimedean case. But a number field K can be seen as the function field of the nonsingular arithmetic curve $B = \operatorname{Spec} O_K$, where O_K is the ring of integers of K, and we know that there is a bijection between points of the completed curve $\widehat{B}$ in the sense of Arakelov geometry and places of K. Therefore the ring of adeles attached to K can be described in a more geometric way related to $\widehat{B}$:

$$\mathbf{A}_{\widehat{B}} := \prod_{b\in\widehat{B}}{}' K_b = \mathbf{A}_K$$

where K_b is still the local field attached to the "point" b. So, classical adelic theory can be deduced from 1-dimensional arithmetic geometry. We can adopt a similar approach but starting from 1-dimensional algebraic geometry: fix a nonsingular algebraic projective curve X over a perfect field k with function field denoted by $k(X)$; then to each closed point $x \in X$ we can associate a non-archimedean local field K_x with its valuation ring denoted by $\mathcal{O}_x$. The ring of adeles associated to X is then:

$$\mathbf{A}_X := \prod_{x\in X}{}' K_x\,;$$

In this case $\mathbf{A}_X$ is not a locally compact additive group unless k is a finite field. Each K_x is endowed with a structure of locally linearly compact k-vector space (in the sense of [14]), therefore $\mathbf{A}_X$ is again locally linearly compact and one can show similarly to the arithmetic case that: $k(X)$ is discrete in $\mathbf{A}_X$, the quotient $\mathbf{A}_X/k(X)$ is a linearly compact k-vector space and $\mathbf{A}_X$ is a self dual k-vector space. In other words, from a topological point of view, the passage from arithmetic theory to algebraic theory implies that we substitute the theory of compactness of groups with the theory of linear compactness of vector spaces. In both arithmetic and geometric 1-dimensional case, adelic and idelic theory give a generalization of the intersection theory (i.e. the theory of degree of divisors):

- Ideles can be easily seen as a generalization of line bundles (resp. Arakelov line bundles), so it is natural to give an extension of the theory of divisors (resp. Arakelov divisors) from an idelic point of view.

- For an algebraic curve X and any divisor $D \in \operatorname{Div}(X)$ we can define a subspace $\mathbf{A}_X(D) \subset \mathbf{A}_X$ and a complex $\mathcal{A}_X(D)$. The cohomology of $\mathcal{A}_X(D)$ is equal to the usual Zariski cohomology $H^i(X, \mathscr{O}_X(D))$, therefore we can give an interpretation of $\deg(D)$ in terms of the characteristic of $\mathcal{A}_X(D)$ which will be called the adelic characteristic. For a completed arithmetic curve $\widehat{B}$ we cannot define a complex $\mathcal{A}_{\widehat{B}}(\widehat{D})$ associated to an Arakelov divisor $\widehat{D}$, since for archimedean points closed unit balls are not additive groups. However, one can recover the Arakelov degree of $\widehat{D}$ as the product of volumes (with respect to an opportune choice of Haar measures) of certain closed balls in K_b.

The above theory remains valid also in the case of singular curves, because by normalization we can always reduce to the nonsingular case.

The first attempt to construct a 2-dimensional adelic/idelic theory from 2-dimensional geometry was partially made by Parshin in [20], but he treated only rational adeles (i.e. a subset of the actual ring of adeles) for algebraic surfaces. A more structured approach to the adelic theory for algebraic surfaces was given in [21], still with a few mistakes[2], and it can be summarized in the following way: let's fix a nonsingular, projective surface $(X, \mathscr{O}_X)$ over a perfect field k, then to each "flag" $x \in y$ made of a closed point x inside an integral curve $y \subset X$ we can associate the ring $K_{x,y}$ which will be a 2-dimensional

[2]for example in [21] the definition of the object K_x, and consequently of the subspace A_{02}, is wrong (compare with section 1.2 for more details).

local field if y is nonsingular at x, or a finite product of 2-dimensional local fields if we have a singularity. Roughly speaking a 2-dimensional local field is a local field whose residue field is again a local field (see subsection 1.1), and in our case $K_{x,y}$ carries two distinct levels of discrete valuations: there is the discrete valuation associated to the containment $x \in y$ and the discrete valuation associated to $y \subset X$. Formally, $K_{x,y}$ is obtained through a process of successive localisations and completions starting from $\mathscr{O}_{X,x}$. With the symbol $\mathcal{O}_{x,y}$ we denote the product of valuation rings inside $K_{x,y}$. Similarly to the 1-dimensional theory, we perform a "double restricted product", first over all points ranging on a fixed curve, and then over all curves in X, in order to obtain the ring of adeles for surfaces:

$$\mathbf{A}_X := \prod_{\substack{x\in y\\ y\subset X}}'' K_{x,y} \subset \prod_{\substack{x\in y\\ y\subset X}} K_{x,y}\,.$$

The topology on $K_{x,y}$ can be defined thanks to the construction by completions and localisations, and by starting with the standard $\mathfrak{m}_x$-adic topology on $\mathscr{O}_{X,x}$, then the topology on $\mathbf{A}_X$ can be defined canonically. The idelic group attached to X is $\mathbf{A}_X^\times$.

For 2-dimensional local fields like $K_{x,y}$ there is a well known theory of differential forms and residues (see for example [25]); one can globalise the constructions in order to obtain a k-character $\xi^\omega : \mathbf{A}_X \to k$ associated to a rational differential form $\omega \in \Omega^2_{k(X)|k}$ and the differential pairing:

$$\begin{aligned} d_\omega : \mathbf{A}_X \times \mathbf{A}_X &\to k \\ (\alpha,\beta) &\mapsto \xi^\omega(\alpha\beta)\,. \end{aligned}$$

In [9] it is shown that: ξ^ω induces the self duality in the category of k-vector spaces, of $\mathbf{A}_X$, the subspace $\mathbf{A}_X/k(X)^\perp$ is linearly compact (orthogonal spaces are calculated with respect to d_ω) and $k(X)$ is discrete in $\mathbf{A}_X$.

Both $\mathbf{A}_X$ and $\mathbf{A}_X^\times$ carry some important subspaces which in turn lead to the construction of certain complexes $\mathcal{A}_X(D)$, for a divisor D, and $\mathcal{A}_X^\times$ (respectively the "adelic complex" and the "idelic complex"). The cohomology of such complexes can be calculated by geometric methods thanks to the following important results:

$$H^i(\mathcal{A}_X^\times) \cong H^i(X, \mathscr{O}_X^\times)\,, \tag{0.1}$$

$$H^i(\mathcal{A}_X(D)) \cong H^i(X, \mathscr{O}_X(D))\,. \tag{0.2}$$

For a proof of (0.1) and (0.2) see respectively [6] and [9]. Again, idelic and adelic theory give an extension of the intersection theory on X:

- In [21] it is shown that the group $\mathrm{Div}(X)$ can be lifted to a subspace if $\mathbf{A}_X^\times$ and the intersection pairing on $\mathrm{Div}(X)$ can be extended at the level of ideles.
- In [9] it is shown that the characteristic of the complex $\mathcal{A}_X(D)$ can be used to redefine the intersection pairing between two divisors in terms of adeles, even without using isomorphism (0.2). Such a theory gives an alternative approach to the Riemann-Roch theorem for algebraic surfaces.

Most of the 2-dimensional constructions outlined above are true for any 2-dimensional, Noetherian, regular scheme, so in particular for arithmetic surfaces. However, the adelic theory associated to an arithmetic surface $X \to B = \operatorname{Spec} O_K$ (K is a number field) is more complicated and less developed. Locally, the rings $K_{x,y}$ have a completely different structure between each other as 2-dimensional local fields, depending whether y is horizontal or vertical. Moreover, there was the global issue of interpreting the archimedean data of the completed surface $\widehat{X}$, in the sense of Arakelov geometry, in an adelic and idelic way. A first definition of the "full" (or completed) ring of adeles $\mathbf{A}_{\widehat{X}}$ has been given only recently in [8].

The purpose of this series of papers is to study the adelic and idelic theory for a completed arithmetic surface and eventually obtain a 2-dimensional generalisation of the Tate thesis.

Remark 0.1. In [2] Beilinson shortly described how to attach a n-dimensional adelic theory to any n-dimensional Noetherian scheme in a very abstract functorial way. Reworks and clarifications of this approach are [11] and [19, 8]. In particular in [19, 8.4 and 8.5] it is proved that for dimensions 1 and 2 our explicit theory agrees with Beilinson theory of adeles.

0.2 Results in this paper

In this first paper we explain the "schematic part" of the idelic and adelic lift of the Arakelov intersection number (i.e. ignoring the fibres at infinity), whereas a full account of the theory will be published subsequently together with other results.

Given two Arakelov divisors $\widehat{D} = D + \sum_\sigma \alpha_\sigma X_\sigma$ and $\widehat{E} = E + \sum_\sigma \beta_\sigma X_\sigma$, where $D, E \in \mathrm{Div}(X)$, one piece of the Arakelov intersection number $\widehat{D}.\widehat{E}$ is obtained thanks to the Deligne pairing $\langle \mathscr{O}_X(D), \mathscr{O}_X(E)\rangle \in \mathrm{Pic}(B)$. We define $d^1_\times$ to be an arrow of the idelic complex $\mathcal{A}^\times_X$ associated to X, then we construct an idelic Deligne pairing:

$$\langle\ ,\ \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{Pic}(B)$$

which descends to the Deligne pairing through the composition:

$$\ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathrm{Pic}(X) \times \mathrm{Pic}(X)\,.$$

This will be the arithmetic version of Parshin idelic lift given in [21]. Our approach will be from local to global: the main idea consists in globalising Kato's local symbol for 2-dimensional local fields containing a local field (see [12] or [15]), which is the generalisation of the usual tame symbol for valuation fields (see appendix A). In the arithmetic framework given by the arithmetic surface $\varphi : X \to B$, we have the following constructions: for any point x sitting on a curve $y \subset X$ we define a ring, which is a finite sum of 2-dimensional local fields, denoted by $K_{x,y}$; moreover K_b is the local field associated to the point $b \in B$ such that $\varphi(x) = b$. Then Kato's symbol translates into a skew symmetric, bilinear map:

$$(\ ,\ )_{x,y} : K^\times_{x,y} \times K^\times_{x,y} \to K^\times_b\,.$$

Roughly speaking, by composing it with the valuation v_b on K_b and by summing over all flags $x \in y$ such that $\varphi(x) = b$, we show that we obtain a well defined integer n_b. By repeating the argument for each $b \in B$ we obtain a divisor $\sum_{b\in B} n_b[b]$. At this point we prove that such a pairing descends to the Deligne pairing.

The adelic theory is very similar to the geometric case and the crucial point consists in considering the arithmetic analogue of the Euler-Poincare characteristic of coherent sheaves, i.e. the determinant of cohomology. We use the cohomological properties of the adelic complex of the base scheme B in order to give the definition of the adelic determinant of cohomology. Then it is enough to use the fact that the Deligne pairing can be expressed in terms of the (adelic) determinant of cohomology.

Overview of the contents. Section 1 is a quick review of adelic geometry for arithmetic surfaces, where just by simplicity, we ignore the contribution of fibres at infinity. A more comprehensive introduction to adelic geometry can be found in [6]. In section 2 and 3 we construct respectively the idelic and adelic Deligne pairing. Finally, appendix A is just a collection of the basic notions of algebraic K-theory needed in this paper and appendix B is a review of the main features of the determinant of cohomology.

Basic notations. All rings are considered commutative and unitary. When we pick a point x in a scheme X we generally mean a *closed point* if not otherwise specified, also all sums $\sum_{x\in X}$ are meant to be "over all closed points of X". The cardinality of a set T is denoted as $\#(T)$. If F is a field, then $\overline{F}$ doesn't denote the algebraic closure. For a morphisms of schemes $f : X \to S$, the schematic preimage of $s \in S$ is X_s. Sheaves are denoted with the "mathscr" lateχ font; particular the structure sheaf of a scheme X is $\mathscr{O}_X$ (note the difference with the font $\mathcal{O}$). For any $\mathscr{O}_X$-module $\mathscr{F}$ and any $D \in \mathrm{Div}(X)$ we put $\mathscr{F}(D) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)$. The notation $\det(\cdot)$ is used for the notion of "determinant" in the category of free modules over a ring and in the category of free $\mathscr{O}_X$-modules; the exact meaning will be clear from the context. If K is a number field and $X \to \operatorname{Spec} O_K$ is an integral scheme over the ring of integers O_K, then the function field of X is denoted by $K(X)$. Finally it is important to point out that the letter K will denote different mathematical objects in this paper (and in different contexts), so the reader should check at the beginning of each section its specific meaning from time to time.

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1 Review of 2-dimensional adelic geometry

1.1 Abstract local theory

Let's recall the definition of local field:

Definition 1.1. A *local field* (or a *1-dimensional local field*) F, is one of the fields listed below:

(1) $F = \mathbb{R}$ endowed with the usual real absolute value$|\cdot|$.

(2) $F = \mathbb{C}$ endowed with the usual complex absolute value $||\cdot||$.

(3) F is a complete discrete valuation field (the valuation is surjective) such that the residue field $\overline{F}$ is a perfect field. The valuation ring of F is denoted as $\mathcal{O}_F$ and its maximal ideal is $\mathfrak{p}_F$. Moreover if v is the valuation on F, then the absolute value is given by $|x|_v := q^{-v(x)}$, where $q = \#(\overline{F})$ if $\overline{F}$ is a finite field, and $q = e := \exp(1)$ otherwise.

If F is of type (1) or (2), it is an *archimedean local field* otherwise it is a *non-archimedean local field.* A local field is topologized with the topology induced by the absolute value. A morphism between local fields is a continuous field homomorphism.

Remark 1.2. According to our definition, a non-archimedean local field endowed with its natural topology is in general not locally compact.

Remember that if F is a non-archimedean local field there exists only one surjective complete valuation on it (see [19, Theorem 1.4]). A higher local field is a simple generalization of definition 1.1: given a complete discrete valuation field F, it might happen that the residue field $F^{(1)} := \overline{F}$ is again a complete discrete valuation field; by taking one more time the residue field we have the field $F^{(2)}$. In other words, a complete discrete valuation field might originate a potentially infinite sequence of fields $\{F^{(i)}\}_{i\geq 0}$ such that $F^{(0)} = F$ and $F^{(i+1)} = \overline{F^{(i)}}$. Each $F^{(i)}$ is called the *i-th residue field.*

Definition 1.3. A *n-dimensional local field*, for $n \geq 2$, is a complete discrete valuation field F admitting sequence of residue fields $\{F^{(i)}\}_{i>0}$ such that $F^{(n-1)}$ is a local field. If $F^{(n-1)}$ is an archimedean local field, then F is called *archimedean*, otherwise we say that F is *non-archimedean.* F has *mixed characteristic* if $\mathrm{char}(F) \neq \mathrm{char}(\overline{F})$.

Example 1.4. The simplest n-dimensional local field is the field of iterated Laurent series over a perfect field K:

$$F = K((t_1))\dots((t_n))\,.$$

If $f = \sum a_j t_n^j \in F$, with $a_i \in K((t_1))\dots((t_{n-1}))$, we have the complete discrete valuation defined by $v(f) = \min\{j\colon a_j \neq 0\}$. The valuation ring is $\mathcal{O}_F = K((t_1))\dots((t_{n-1}))[[t_n]]$ and the residue field is $F^{(1)} = K((t_1))\dots((t_{n-1}))$. Clearly $F^{(n)} = K$. When $n = 2$, the elements of $K((t_1))((t_2))$ are the formal power series $\sum_{i,j} a_{i,j} t_1^i t_2^j$ such that $a_{i,j} = 0$ when the indexes i and j are chosen in the following way: let's plot the couples (j,i) as a lattice on the plane, then we select a semiplane like the one which is not coloured in figure 1. The coordinate j is bounded from right, whereas the coordinate i is bounded from above by a descending staircase line.

Remark 1.5. The above definition of dimension for a n-dimensional local field might seem quite counter-intuitive, indeed a n-dimensional local field can also be a m-dimensional local field for $m \neq n$. For instance $F = K((t_1))\dots((t_n))$ is m-dimensional for any $m = 1,\dots,n$. For our purposes it will be clear from the context which dimension we want to take in account. Often it is convenient to consider the maximum amongst all possible dimensions (when it exists).

Remark 1.6. Note in the case of archimedean n-dimensional local fields the n-th residue field doesn't exist.

Let's give a less trivial example of higher local field:

Example 1.7. Let (K, v_K) be a non-archimedean local field and consider the following set of (double) formal series:

$$K\{\{t\}\} := \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \colon a_j \in K,\ \inf_j v_K(a_j) > -\infty,\ \lim_{j\to-\infty} a_j = 0 \right\}$$

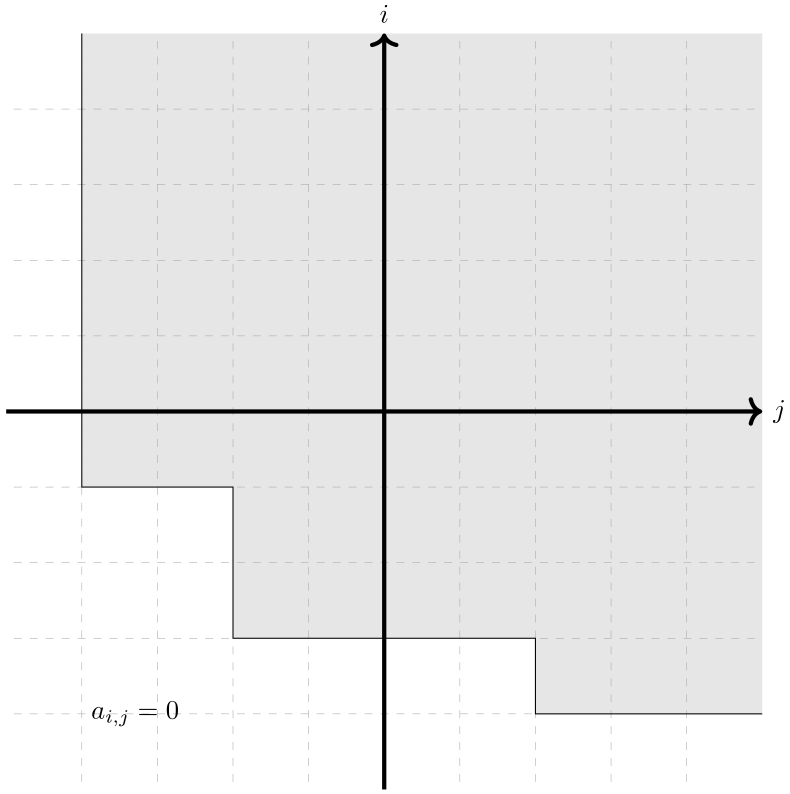


Figure 1: A cartesian diagram showing the lattice of couples (j, i) corresponding to the coefficients $a_{i,j}$ of a power series in $\sum_{i,j} a_{i,j} t_1^i t_2^j \in K((t_1))((t_2))$.

Addition and multiplication in $K\{\{t\}\}$ are defined in the following way:

$$\sum_{j=-\infty}^{\infty} a_j t^j + \sum_{j=-\infty}^{\infty} b_j t^j = \sum_{j=-\infty}^{\infty} (a_j + b_j) t^j \tag{1.1}$$

$$\sum_{j=-\infty}^{\infty} a_j t^j \cdot \sum_{j=-\infty}^{\infty} b_j t^j = \sum_{j=-\infty}^{\infty} \left(\sum_{r=-\infty}^{\infty} a_r b_{j-r} \right) t^j \tag{1.2}$$

and $K\{\{t\}\}$ becomes a field. Note that the series with index r in equation (1.2) is actually a convergent series in K. We can also define the following discrete valuation v on $K\{\{t\}\}$:

$$v\left(\sum_{j=-\infty}^{\infty} a_j t^j \right) := \inf_j v_K(a_j)\,. \tag{1.3}$$

It is not difficult to verify that v is a well defined valuation and $K\{\{t\}\}$ is complete with respect to v. Let's now analyse the structure of $F = K\{\{t\}\}$ as valuation field:

$$\mathcal{O}_F = \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \in K\{\{t\}\} \colon a_j \in \mathcal{O}_K \right\}$$

$$\mathfrak{p}_F = \left\{ \sum_{j=-\infty}^{\infty} a_j t^j \in K\{\{t\}\} \colon a_j \in \mathfrak{p}_K \right\}$$

Consider the surjective homomorphism:

$$\begin{array}{rccc} \pi : & \mathcal{O}_F & \to & \overline{K}((t)) \\ & \sum a_j t^j & \mapsto & \sum \overline{a_j} t_j \end{array}$$

where clearly $\overline{a_j}$ is the natural image of a_j in $\overline{K}$. Now it is evident that π induces an isomorphism $\overline{F} \cong \overline{K}((t))$. In other words F has a structure of 2-dimensional local field such that $F^{(1)} = \overline{K}((t))$ and $F^{(2)} = \overline{K}$. Clearly such a construction can be iterated several times to get the field:

$$K\{\{t_1\}\}\ldots\{\{t_n\}\}\,.$$

For example if $K = \mathbb{Q}_p$, then $K\{\{t\}\}$ is a 2-dimensional local field of mixed characteristic.

Remember that we have the following classical classification theorem for local fields:

Theorem 1.8 (Classification theorem for local fields)**.** *Let F be a local field:*

(1) *When F is archimedean, then $F = \mathbb{R}$ or $F = \mathbb{C}$.*

(2) *When F is not archimedean there are two cases:*

  ($2a$) *If $\operatorname{char} F = \operatorname{char} \overline{F}$, then $F \cong \overline{F}((t))$.*

  ($2b$) *If $\operatorname{char} F \neq \operatorname{char} \overline{F} = p$, then F is isomorphic to K_p which denotes a finite extension of $\mathbb{Q}_p$.*

Proof. (1) is true just by definition. For (2) see for example [10, II.5]. □

Such a classification can be extended for higher local fields, in particular any n-dimensional local field can be obtained by "combining" the higher local fields presented in examples 1.4 and 1.7:

Theorem 1.9 (Classification theorem for n-dimensional local fields)**.** *Let F be a n-dimensional local field with $n \geq 2$.*

(1) *If $\operatorname{char} F = \operatorname{char} F^{(1)} = \ldots = \operatorname{char} F^{(n-1)}$, then*

$$F \cong F^{(n-1)}((t_1))\ldots((t_{n-1}))$$

and $F^{(n-1)}$ is isomorphic to one of the four fields listed in theorem 1.8.

(2) *If $r \in \{2, 3, \ldots n\}$ is the unique number such that $\operatorname{char} F^{(n-r)} \neq \operatorname{char} F^{(n-r+1)} = p$, then:*

  ($2a$) *When $r \neq n$, F is isomorphic to a finite extension of:*

$$K_p\{\{t_1\}\}\ldots\{\{t_{r-1}\}\}((t_r))\ldots((t_{n-1}))\,.$$

  ($2b$) *When $r = n$ (i.e. in the mixed characteristic case), F is isomorphic to a finite extension of:*

$$K_p\{\{t_1\}\}\ldots\{\{t_{n-1}\}\}\,.$$

Proof. See [19, Theorem 2.18]. □

In this paper we will focus mainly on 2-dimensional local fields, so let's give a table with all possible 2-dimensional local fields by using the classification theorem:

2-dimensional local fields			
Geometric	Arithmetic		Archimedean
$(0,0,0)$, (p,p,p)	$(0,p,p)$	$(0,0,p)$	
$K((t_1))((t_2))$ with K perfect	finite extension of $K_p\{\{t\}\}$	$K_p((t))$	$\mathbb{C}((t))$ or $\mathbb{R}((t))$

(1.4)

For a non-archimedean local field (F, v), we have the notion of *local parameter* ϖ which is any generator of the maximal ideal $\mathfrak{p}_F$, or equivalently any element such that $v(\varpi) = 1$. Clearly we have the (recursive) generalization for n-dimensional local fields.

Definition 1.10. Let F be a non-archimedean n-dimensional local field, then a *sequence of local parameters for F* is a n-tuple $(\varpi_1, \ldots, \varpi_n) \in F \times \ldots \times F$ satisfying the following properties:

- ϖ_n is a local parameter for F.

- $(\varpi_1, \ldots, \varpi_{n-1}) \in \mathcal{O}_F \times \ldots \times \mathcal{O}_F$ and the sequence of natural projections $(\overline{\varpi}_1, \ldots, \overline{\varpi}_{n-1})$ is a sequence of local parameters for the residue field $\overline{F}$.

One can obtain a sequence of local parameters, by applying the following algorithm: choose any local parameter for $F^{(n-1)}$, then pick any of its liftings in F, this will be ϖ_1. Choose choose any local parameter for $F^{(n-2)}$, then pick any of its liftings in F, this will be ϖ_2, etc. Let's give another basic definition:

Definition 1.11. Let F be a n-dimensional local field and put $\mathcal{O}_F^{(0)} := F$, then we define recursively the *j-th valuation ring* (for $j \geq 1$):

$$\mathcal{O}_F^{(j)} := \left\{ x \in \mathcal{O}_F : \overline{x} \in \mathcal{O}_{\overline{F}}^{(j-1)} \right\}$$

It is clear that $\mathcal{O}_F^{(1)} = \mathcal{O}_F$. For the algebraic properties of $\mathcal{O}_F^{(j)}$ the reader can check [19, 3].

From now on in this section we assume that F is a 2-dimensional local field such that:

- $\operatorname{char} F = 0$ and $\operatorname{char} F^{(2)} = p$.
- F is endowed with a ST-ring[3] topology and there exists a mixed characteristic local field K with a fixed embedding $K \hookrightarrow F$ of ST-rings (K has the discrete valuation topology).

In this case we say that F is an *arithmetic 2-dimensional local field over K*. The presence of the local field K inside F comes from the theory of arithmetic surfaces and it will be explained in section 2.

Equal characteristic. Let F be an arithmetic 2-dimensional local field such that $\operatorname{char} \overline{F} = 0$.

Definition 1.12. The *coefficient field of F (with respect to K)* is the algebraic closure of K inside F and it is denoted as k_F.

The coefficient field k_F is a finite extension of K and moreover $\overline{F} = k_F$. In particular $F \cong k_F((t))$. The valuation field F is naturally endowed with the usual tame symbol $(\ ,\ )_F : F^\times \times F^\times \to k_F^\times$, so we can obtain the Kato symbol (or two dimensional tame symbol) by simply composing it with the field norm map:

Definition 1.13. The *Kato symbol* for F (with respect to K) is given by:

$$(\ ,\ )_{F|K} : N_{k_F|K} \circ (\ ,\ )_F : F^\times \times F^\times \to K^\times .$$

Mixed characteristic. Now we assume that F is an arithmetic 2-dimensional local field of mixed characteristic. By the classification theorem $\mathbb{Q}_p$ is contained in F and we have the notion of constant field of F which replaces the one of coefficient field:

Definition 1.14. The *constant field of F* is the algebraic closure of $\mathbb{Q}_p$ in F, and it is denoted by k_F.

Remark 1.15. Note that the definition of the constant field doesn't depend on K so it makes sense for any 2-dimensional local field of mixed characteristic. Of course it might happen that $K = \mathbb{Q}_p$.

Since K is a finite extension of $\mathbb{Q}_p$ (by the 1-dimensional classification theorem), we know that k_F is an intermediate field between K and F. The constant field k_F is a finite extension of $\mathbb{Q}_p$ (so also a finite extension of K).

Definition 1.16. We say that an arithmetic 2-dimensional local field of mixed characteristic F is *standard* if there is a k_F isomorpshism $F \cong k_F\{\{t\}\}$. When an isomorphism is given, we say that we have fixed a *parametrization* of F.

We will study standard fields first and extend any result for a generic F thanks to the following result:

Proposition 1.17. *There exists a standard field L contained in F such that: $[F : L] < \infty$, $k_F = k_L$ and $\overline{F} = \overline{L}$.*

Proof. See [18, Lemma 2.14]. □

[3] A ST-ring (semi-topological ring) is just a ring endowed with a linear topology as additive group such that the multiplication for any fixed element is continuous. Note that a ST-ring is not necessarily a topological ring. See [25] for more details.

So, from now on in this subsection we fix L to be a standard field contained in F with the properties described in proposition 1.17. Clearly we have the following field extensions that need to be kept always in mind (we mark the finite extensions with the superscript $\mathfrak{f}$):

$$\mathbb{Q}_p \subseteq^{\mathfrak{f}} K \subseteq^{\mathfrak{f}} k_L = k_F \subseteq L \cong k_L\{\{t\}\} \subseteq^{\mathfrak{f}} F\,. \tag{1.5}$$

Finally, we want to define the Kato symbol for F and the strategy is the usual one: we start from $k_L\{\{t\}\}$ and we extend our arguments to F by checking that everything is independent from parametrizations and from the choice of the standard fields. We will heavily use some K-theoretic notions developed in appendix A.

Fix just for the moment $L = k_L\{\{t\}\}$, then we define:

$$(\ ,\ )_{L|K} : L^\times \times L^\times \xrightarrow{\{\,,\,\}} K_2(L) \longrightarrow \widehat{K}_2(L) \xrightarrow{-\operatorname{res}_L^{(2)}} \widehat{K}_1(k_L) = k_L^\times \xrightarrow{N_{k_L|K}} K^\times \tag{1.6}$$

where:

- $\{\,,\,\}$ is the natural projection arising from the definition of $K_2(L)$ (see proposition A.2).
- The morphism $K_2(L) \to \widehat{K}_2(L)$ is the map given by the construction of $\widehat{K}_2(L)$ as projective limit (see equation (A.2)).
- $\operatorname{res}_L^{(2)}$ is the higher Kato residue map constructed in theorem A.8. Note that $\widehat{K}_1(k_L) = k_L^\times$ because k_L is already complete.

Moreover by simplicity we use the following notation:

$$\partial_L : K_2(L) \longrightarrow \widehat{K}_2(L) \xrightarrow{-\operatorname{res}_L^{(2)}} k_L^\times\,. \tag{1.7}$$

Remark 1.18. [15] gives an explicit description of $\operatorname{res}_L^{(2)}$ which involves winding numbers.

Definition 1.19. Let L be a generic standard field and fix a parametrization: $p : L \to k_L\{\{t\}\}$ then we define:

$$(\ ,\ )_{L|K} : L^\times \times L^\times \xrightarrow{\{\,,\,\}} K_2(L) \xrightarrow{K_2(p)} K_2(k_L\{\{t\}\}) \xrightarrow{\partial_{k_L\{\{t\}\}}} k_L^\times \xrightarrow{N_{k_L|K}} K^\times$$

and we put $\partial_L := \partial_{k_L\{\{t\}\}} \circ K_2(p)$.

Proposition 1.20. *Let L be a standard field, then the definition of $(\ ,\ )_{L|K}$ doesn't depend on the parametrization of L.*

Proof. See [15, Corollary 3.7]. □

At this point we are ready to give the general definition of the Kato symbol:

Definition 1.21. Let F an arithmetic 2-dimensional local field and let L be a standard field contained in F, then the *Kato symbol for F* (with respect to K) is given by:

$$(\ ,\ )_{F|K} : F^\times \times F^\times \xrightarrow{\{\,,\,\}} K_2(F) \xrightarrow{K_2(N_{F|L})} K_2(L) \xrightarrow{\partial_L} k_L^\times \xrightarrow{N_{k_L|K}} K^\times \tag{1.8}$$

Proposition 1.22. *The definition of $(\ ,\ )_{F|K}$ doesn't depend on the choice of L inside F.*

Proof. See [12, Proposition 3]. □

1.2 Adelic geometry

Let's fix $B = \operatorname{Spec} O_K$ for a number field K; $\varphi : X \to B$ is a B-scheme satisfying the following properties:

- X is two dimensional, integral, and regular. The generic point of X is η and the function field of X is denoted by $K(X)$.
- φ is proper and flat.
- The generic fibre, denoted by X_K, is a geometrically integral, smooth, projective curve over K. The generic point of B is denoted by ξ.

We say that X is an *arithmetic surface over* B. We consider the set of all possible *flags* $x \in Y \subset X$ where x is a closed point of X contained in an integral curve Y.

From now on a curve Y on X will always be an integral curve and its unique generic point will be denoted with the letter y. By simplicity we will often identify Y with its generic point y, which means that by an abuse of language and notation we will use sentences like "let $y \subset X$ be a curve on X..." or "let $x \in y \subset X$ be a flag on X...". In other words y is considered as a scheme or as a point depending on the context.

Definition 1.23. Fix a closed point $x \in X$, then:

- $\mathcal{O}_x := \widehat{\mathscr{O}_{X,x}}$. It is a Noetherian, complete, regular, local, domain of dimension 2 with maximal ideal $\widehat{\mathfrak{m}_x}$.
- $K'_x := \operatorname{Frac} \mathcal{O}_x$.
- $K_x := K(X)\mathcal{O}_x \subseteq K'_x$. Notice that this is not a field.

For a curve $y \subset X$ we put:

- $\mathcal{O}_y := \widehat{\mathscr{O}_{X,y}}$. It is a complete DVR with maximal ideal $\widehat{\mathfrak{m}_y}$.
- $K_y := \operatorname{Frac} \mathcal{O}_y$. It is a complete discrete valuation field with valuation ring $\mathcal{O}_y$. The valuation is denoted by v_y.

For any point $b \in B$ we put:

- $\mathcal{O}_b := \widehat{\mathscr{O}_{B,b}}$. It is a complete DVR.
- $K_b := \operatorname{Frac} \mathcal{O}_b$. It is a local field with finite residue field. The valuation is denoted by v_b.

Fix a flag $x \in y \subset X$, then we have a surjective local homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x}$ with kernel $\mathfrak{p}_{y,x}$ induced by the closed embedding $y \subset X$ (note that $\mathfrak{p}_{y,x}$ is a prime ideal of height 1).

The inclusion $\mathscr{O}_{X,x} \subset \mathcal{O}_x$ induces a morphism of schemes $\phi : \operatorname{Spec} \mathcal{O}_x \to \operatorname{Spec} \mathscr{O}_{X,x}$ and we define the *local branches of* y *at* x as the elements of the set

$$y(x) := \phi^{-1}(\mathfrak{p}_{y,x}) = \{ \mathfrak{z} \in \operatorname{Spec} \mathcal{O}_x : \mathfrak{z} \cap \mathscr{O}_{X,x} = \mathfrak{p}_{y,x} \} .$$

If $y(x)$ contains only an element, we say that y is unbranched at x.

Remark 1.24. If x is a cusp point on y, one can show that y unbranched at x.

Definition 1.25. Let $\mathfrak{z} \in y(x)$ be a local branch of a curve y at point x, then let's define the field

$$K_{x,\mathfrak{z}} := \operatorname{Frac} \left(\widehat{(\mathcal{O}_x)_{\mathfrak{z}}} \right) .$$

in other words: we localise $\mathcal{O}_x$ at the prime ideal $\mathfrak{z}$, then we complete it at its maximal ideal and finally we take the fraction field. By convenience we put $\mathcal{O}_{x,\mathfrak{z}} := \widehat{(\mathcal{O}_x)_{\mathfrak{z}}}$.

The proof of the following proposition relies on some basic commutative algebra results:

Proposition 1.26. *Let $x \in y \subset X$ be a flag and let $\mathfrak{z} \in y(x)$. Then $K_{x,\mathfrak{z}}$ is a 2-dimensional valuation field such that $\mathcal{O}_{K_{x,\mathfrak{z}}} = \mathcal{O}_{x,\mathfrak{z}}$ and $K^{(2)}_{x,\mathfrak{z}}$ is a finite extension of $k(x)$.*

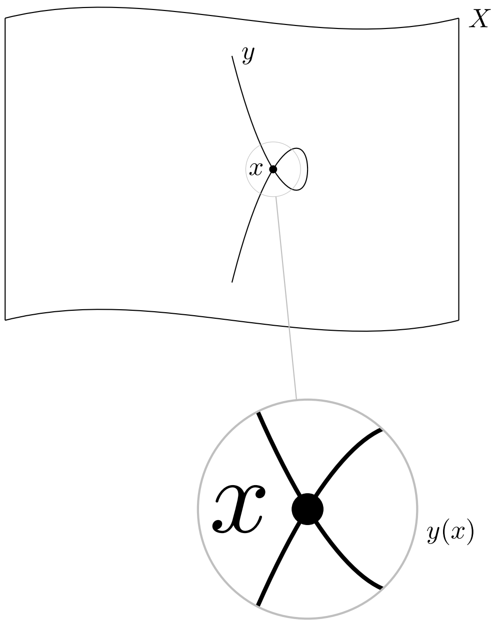


Figure 2: Informally the local branches of y at x can be depicted in the following way: consider a small neighbourhood of x, then each distinct "piece of y" that we see passing through x corresponds to a local branch $\mathfrak{z}$. In this particular case y has a simple node at x, so 2 local branches at x.

Proof. First of all $\operatorname{ht}\mathfrak{z} \geq \operatorname{ht}\mathfrak{p}_{y,x} = 1$, but if $\operatorname{ht}\mathfrak{z} = 2$ then $\mathfrak{z}$ is the maximal ideal of $\mathcal{O}_x$ and we have that $\mathfrak{z} \cap \mathscr{O}_{X,x} = \mathfrak{m}_x$, a contradiction. Therefore $\operatorname{ht}\mathfrak{z} = 1$ and $\dim\left(\mathcal{O}_x\right)_{\mathfrak{z}} = 1$. It follows that $\widehat{\left(\mathcal{O}_x\right)_{\mathfrak{z}}}$ is a Noetherian, complete, local, domain of dimension 1, i.e. a complete DVR which is the valuation ring of the complete discrete valuation field $K_{x,\mathfrak{z}}$. The residue field of $K_{x,\mathfrak{z}}$ is by definition:

$$K_{x,\mathfrak{z}}^{(1)} := \left(\mathcal{O}_x\right)_{\mathfrak{z}} / \mathfrak{z}\left(\mathcal{O}_x\right)_{\mathfrak{z}} = \operatorname{Frac}\left(\mathcal{O}_x/\mathfrak{z}\right) .$$

Note that $\mathcal{O}_x/\mathfrak{z}$ is a Noetherian, complete, local domain of dimension 1 (in general we may lose the regularity by passing to the quotient). Consider the normalisation $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ of $\mathcal{O}_x/\mathfrak{z}$; the domain $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is obviously normal and again Noetherian and complete. Moreover by Nagata theorem (see [3, Ch. IX, 4, no 2, Theorem 2]) $\mathcal{O}_x/\mathfrak{z}$ is a Japanese ring, therefore in particular $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a finite $\mathcal{O}_x/\mathfrak{z}$-module. Now [7, Corollary 7.6] implies that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is also local, and by summing up all the listed property we can conclude that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a complete DVR with fraction field $\operatorname{Frac}\left(\mathcal{O}_x/\mathfrak{z}\right)$. This proves that $K_{x,\mathfrak{z}}^{(1)}$ is a complete valuation field.
It remains to show only that the second residue field $K_{x,\mathfrak{z}}^{(2)}$ is a finite extension of $k(x)$. By definition $K_{x,\mathfrak{z}}^{(2)}$ is the residue field of the local ring $\widetilde{\mathcal{O}_x/\mathfrak{z}}$, but we already know that $\widetilde{\mathcal{O}_x/\mathfrak{z}}$ is a finite $\mathcal{O}_x/\mathfrak{z}$-module, so $K_{x,\mathfrak{z}}^{(2)}$ is a finite extension of:

$$\left(\mathcal{O}_x/\mathfrak{z}\right) / \left(\widehat{\mathfrak{m}_x}/\mathfrak{z}\right) \cong \mathcal{O}_x/\widehat{\mathfrak{m}_x} \cong \mathscr{O}_{X,x}/\mathfrak{m}_x = k(x) .$$

□

It is not amongst the purposes of this paper to treat the topology of $K_{x,\mathfrak{z}}$, but it is enough to know that there are several ways to topologise $K_{x,\mathfrak{z}}$, some of them are equivalent, and we end up with a structure of ST-ring on $K_{x,\mathfrak{z}}$. See for example [4, 1.] for a survey about topologies on $K_{x,\mathfrak{z}}$.

Definition 1.27. Let $x \in y \subset X$ be a flag and let $\mathfrak{z} \in y(x)$, then we put $E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}}$ and $k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}$.

$$\begin{array}{ccccc}
K_{x,\mathfrak{z}} & \supset & \mathcal{O}_{x,\mathfrak{z}} := \mathcal{O}_{K_{x,\mathfrak{z}}} & \supset & \mathcal{O}^{(2)}_{x,\mathfrak{z}} := \mathcal{O}^{(2)}_{K_{x,\mathfrak{z}}} \\
 & & \downarrow & & \downarrow \\
 & & E_{x,\mathfrak{z}} := K^{(1)}_{x,\mathfrak{z}} & \supset & \mathcal{O}_{E_{x,\mathfrak{z}}} \\
 & & & & \downarrow \\
 & & & & k_{\mathfrak{z}}(x) := K^{(2)}_{x,\mathfrak{z}}
\end{array}$$

The valuation on $K_{x,\mathfrak{z}}$ is $v_{x,\mathfrak{z}}$ and the valuation on $E_{x,\mathfrak{z}}$ is $v^{(1)}_{x,\mathfrak{z}}$. Moreover:

$$K_{x,y} := \prod_{\mathfrak{z} \in y(x)} K_{x,\mathfrak{z}}\,, \quad \mathcal{O}_{x,y} := \prod_{\mathfrak{z} \in y(x)} \mathcal{O}_{x,\mathfrak{z}}\,, \quad \mathcal{O}^{(2)}_{x,y} := \prod_{\mathfrak{z} \in y(x)} \mathcal{O}^{(2)}_{x,\mathfrak{z}}\,,$$

$$E_{x,y} := \prod_{\mathfrak{z} \in y(x)} E_{x,\mathfrak{z}}\,, \quad k_y(x) := \prod_{\mathfrak{z} \in y(x)} k_{\mathfrak{z}}(x)\,.$$

Remark 1.28. Remember from commutative algebra the following chain of implications:

$$(A \text{ regular local}) \Rightarrow (A \text{ a UFD}) \Rightarrow (\text{Any prime } \mathfrak{p} \text{ s.t. } \mathrm{ht}(\mathfrak{p}) = 1 \text{ is principal})\,.$$

So, $\mathscr{O}_{X,x}$ is a UFD and $\mathfrak{p}_{y,x}$ is principal, but also $\mathcal{O}_x$ is a UFD and $\mathfrak{z}$ is principal.

Proposition 1.29. *Let $\mathfrak{p}_{y,x} = (\varpi_y)$ for $\varpi_y \in \mathscr{O}_{X,x}$, then we can choose the uniformizing parameter for $K_{x,\mathfrak{z}}$ to be ϖ_y.*

Proof. We show that ϖ_y generates the maximal ideal of $\mathcal{O}_{x,\mathfrak{z}}$. First of all we notice that the ring $\mathscr{O}_{X,x}/\varpi_y\mathscr{O}_{X,x} \cong \mathscr{O}_{y,x}$ is reduced, and this implies that $\widehat{\mathscr{O}_{y,x}} = \mathcal{O}_x/\varpi_y\mathcal{O}_x$ is reduced too. By remark 1.28 ϖ_y has a unique factorization $\varpi_y = p_1 \dots p_m$ in $\mathcal{O}_x$, and all the $p_i's$ are distinct prime elements thanks to the fact that $\mathcal{O}_x/\varpi_y\mathcal{O}_x$ is reduced. Again remark 1.28 implies that $\mathfrak{z} = (p_j)$ for some index j. Any element of $\mathfrak{z}(\mathcal{O}_x)_{\mathfrak{z}}$ can be written as $\frac{p_j a}{b}$ with $b \notin \mathfrak{z}$ but:

$$\frac{p_j a}{b} = \frac{p_1 \dots p_m a}{p_1 \dots p_{j-1} p_{j+1} \dots p_m b} = \frac{\varpi_y a}{p_1 \dots p_{j-1} p_{j+1} \dots p_m b}$$

Since $p_1 \dots p_{j-1} p_{j+1} \dots p_m b \notin \mathfrak{z}$, we can conclude that ϖ_y generates the prime ideal $\mathfrak{z}(\mathcal{O}_x)_{\mathfrak{z}}$ of $(\mathcal{O}_x)_{\mathfrak{z}}$. □

Corollary 1.30. *If ϖ_y is a uniformizing parameter for the complete valuation field K_y, then it is a uniformizing parameter for $K_{x,\mathfrak{z}}$.*

Proof. It follows from proposition 1.29 and the fact that $(\mathscr{O}_{X,x})_{\mathfrak{p}_{y,x}} \cong \mathscr{O}_{X,y}$. □

Remark 1.31. Fix a flag $x \in y \subset X$. The local homomorphism $\mathscr{O}_{X,x} \to \mathscr{O}_{y,x}$ induces a local homomorphism $\mathcal{O}_x \to \widehat{\mathscr{O}_{y,x}}$ which gives a bijective correspondence between the ideals in $y(x)$ and the minimal prime ideals of $\widehat{\mathscr{O}_{y,x}}$. Hence the ring of adeles of the curve y is recovered in the following way:

$$\mathbf{A}_y = \prod_{x \in y}{}' E_{x,y}\,.$$

Proposition 1.32. *Let's denote with $v_{x,\mathfrak{z}}$ the valuation of $K_{x,\mathfrak{z}}$ and with v_y the valuation of K_y. Then the restriction of $v_{x,\mathfrak{z}}$ to K_y is equal to v_y.*

Proof. By remark 1.31 we deduce that $E_{x,\mathfrak{z}}$ contains $k(y)$, which is in turns the residue field of K_y, so the claims follows directly from corollary 1.30. □

The structure of $K_{x,\mathfrak{z}}$ depends on the nature of the curve y and we can distinguish two cases:

y is a vertical curve. If $\varphi(y) = b \in B$, then y is a projective curve over the finite field $k(b)$; we assume that $k(b)$ has characteristic p. $K_{x,\mathfrak{z}}$ has characteristic 0 since we have the embeddings $\mathbb{Q} \subset K \subset K(X) \subset K_{x,\mathfrak{z}}$ and the residue field $E_{x,\mathfrak{z}}$ has characteristic p since $k(b) \subset k(y) \subset E_{x,\mathfrak{z}}$. We conclude that $K_{x,\mathfrak{z}}$ is a two dimensional local field of type $(0, p, p)$ and by the classification theorem we have that $K_{x,\mathfrak{z}}$ is a finite extension of $K_p\{\{t\}\}$ where K_p is a finite extension of $\mathbb{Q}_p$.

y **is a horizontal curve.** In this case $K_{x,\mathfrak{z}}$ has still characteristic 0, but we have the embedding $K \subseteq k(y)$ given by the surjective map $y \to B$; therefore $E_{x,\mathfrak{z}}$ has characteristic 0. Moreover, if $\varphi(x) = b$, the local homomorphism $\varphi_x^{\#} : \mathscr{O}_{B,b} \to \mathscr{O}_{X,x}$ induces a field embedding $k(b) \subseteq k(x)$ and this implies that $k_{\mathfrak{z}}(x)$ has characteristic p. We conclude that $K_{x,\mathfrak{z}}$ is a two dimensional local field of type $(0, 0, p)$ and by the classification theorem we have that $K_{x,\mathfrak{z}} \cong K_p((t))$.

If $\varphi(x) = b$ we have an induced embedding $K_b \hookrightarrow K_{x,\mathfrak{z}}$, so we can conclude that $K_{x,\mathfrak{z}}$ is an arithmetic 2-dimensional local field over K_b and we can apply the local theory developed in subsection 1.1.

The ring of adeles $\mathbf{A}_X$ will be the result of a "glueing" of the local data $\{K_{x,y}\}_{x\in y\subset X}$ where the couple (x, y) runs amongst all flags in X. The glueing procedure will be described precisely, but roughly speaking we will define $\mathbf{A}_X$ inside the big product of rings

$$\mathbf{A}_X \subset \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y}$$

as a sort of "double restricted product".

First "restricted product": the rings $\mathbb{A}_y^{(r)}$ and $\mathbb{A}_y$. In this paragraph we fix a curve $y \subset X$, and denote with $\mathfrak{J}_{x,y}$ the Jacobson radical of $\mathcal{O}_{x,y}$.

Definition 1.33. Let's put:

$$\mathbb{A}_y^{(0)} = \left\{ \begin{array}{l} (\alpha_{x,y})_{x\in y} \in \prod_{x\in y} \mathcal{O}_{x,y} \colon \forall s > 0,\ \alpha_{x,y} \in \mathcal{O}_x + \mathfrak{J}_{x,y}^s \\ \text{for all but finitely many } x \in y. \end{array} \right\} \subset \prod_{x\in y} \mathcal{O}_{x,y}$$

then for any $r \in \mathbb{Z}$

$$\mathbb{A}_y^{(r)} := \widehat{\mathfrak{m}}_y^r \mathbb{A}_y^{(0)} \subset \prod_{x\in y} K_{x,y}\,.$$

Clearly $\mathbb{A}_y^{(r)} \supseteq \mathbb{A}_y^{(r+1)}$ and $\bigcap_{r\in\mathbb{Z}} \mathbb{A}_y^{(r)} = 0$. Moreover we define

$$\mathbb{A}_y := \bigcup_{r\in\mathbb{Z}} \mathbb{A}_y^{(r)}\,.$$

Remark 1.34. We have the inclusion $\mathbb{A}_y \subset \prod_{x\in y} K_{x,y}$, therefore we can interpret $\mathbb{A}_y$ as a "restricted product" of the rings $K_{x,y}$ for y fixed and $x \in y$. Thus we can write:

$$\mathbb{A}_y = \prod_{x\in y}{}' K_{x,y}$$

where $\prod'$ here is just a piece of notation without any formal meaning.

Each $\mathbb{A}_y^{(r)}$ can be endowed with a ind/pro linear topology, and $\mathbb{A}_y$ can be seen as linear direct limit of the topological groups $\mathbb{A}_y^{(r)}$. More details about such topologies will be given in the second paper of the series.

Second "restricted product": the ring $\mathbf{A}_X$. The construction of $\mathbb{A}_y$ can be seen as a way to take the restricted product of $\prod_{x\in y} K_{x,y}$. The final step in order to construct the ring of adeles $\mathbf{A}_X$ is to take the restricted product of the groups $\mathbb{A}_y$ over all the curves in X with respect to the subgroups $\mathbb{A}_y^{(0)}$.

Definition 1.35.

$$\mathbf{A}_X := \left\{ (\beta_y)_{y\subset X} \in \prod_{y\subset X} \mathbb{A}_y \colon \beta_y \in \mathbb{A}_y^{(0)} \text{ for all but finitely many } y \right\} \subset \prod_{\substack{x\in y,\\ y\subset X}} K_{x,y}\,.$$

In a more suggestive way, we write by commodity

$$\mathbf{A}_X = \prod_{\substack{x \in y \\ y \subset X}}'' K_{x,y}$$

where the symbol "$\prod''$" is just a piece of notation which remembers that we are taking a "double restricted product".

Remark 1.36. It is fundamental to recall that $\mathbf{A}_X$ is *not* the full ring of adeles associated to the completed surface $\widehat{X}$, bcause we didn't take in account the fibres at infinity.

In order to topologise $\mathbf{A}_X$ we need to recall the description of the restricted product, by means of categorical limits, for linearly topologised groups. Let $\{G_i\}_{i\in I}$ a set of linearly topologised groups and for any i let $H_i \subset G_i$ be a subgroup endowed with the subspace topology. We denote the family of finite subsets of I as $\mathcal{P}_f(I)$; it forms a directed set with the relation $J \subseteq J'$. For any $J \in \mathcal{P}_f(I)$ define

$$G_J := \prod_{i \in J} G_i \times \prod_{i \notin J} H_i \,,$$

if $J \subseteq J'$ the identity in each factor induces an embedding $G_J \hookrightarrow G_{J'}$, thus we have a direct system $\{G_J\}_J$ and it is easy to see that

$$\prod_i{}' G_i = \varinjlim_J G_J \,,$$

where $\prod_i' G_i$ is the usual restricted product of the G_i with respect to the subgroups H_i. At this point, on each G_J we put the product topology and $\prod_i' G_i$ is endowed with the linear direct limit topology. By definition $\mathbf{A}_X$ is the restricted product of the groups $\mathbb{A}_y$ with respect to the subgroups $\mathbb{A}_y^{(0)}$ for any $y \subset X$. Therefore we endow $\mathbf{A}_X$ with the topology described above.

We now introduce some important subspaces in order to construct the adelic complexes associated to the surface X. Here the definitions are made "by hands", but such subspaces can be recovered as a particular case of the general theory of Beilinson adeles (see [19, 8]). First of all let's consider the following diagonal embeddings:

$$K_x \subset \prod_{y \ni x} K_{x,y}, \quad K_y \subset \prod_{x \in y} K_{x,y} \,,$$

so we can consider:

$$\prod_{x \in X} K_x \subset \prod_{\substack{x \in y \\ y \subset X}} K_{x,y}, \quad \prod_{y \subset X} K_y \subset \prod_{\substack{x \in y \\ y \subset X}} K_{x,y} \,.$$

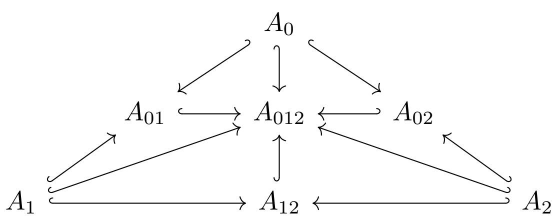
Let's define:

$$A_{012} := \mathbf{A}_X \,; \quad A_{12} := \mathbf{A}_X \cap \prod_{\substack{x \in y \\ y \subset X}} \mathcal{O}_{x,y} = \prod_{y \subset X} \mathbb{A}_y^{(0)};$$

$$A_{02} := \mathbf{A}_X \cap \prod_{x \in X} K_x \,; \quad A_2 := \mathbf{A}_X \cap \prod_{x \in X} \mathcal{O}_x \,; \quad A_{01} := \mathbf{A}_X \cap \prod_{y \subset X} K_y \,;$$

$$A_1 := \mathbf{A}_X \cap \prod_{y \subset X} \mathcal{O}_y \,; \quad A_0 := K(X)$$

The containment relations are depicted in the following diagram:



and we have the adelic complex:

$$\begin{array}{llll} \mathcal{A}_X : & A_0\oplus A_1\oplus A_2 \xrightarrow{d^0} A_{01}\oplus A_{02}\oplus A_{12} \xrightarrow{\quad d^1 \quad} A_{012} \\ & (a_0,a_1,a_2) \longmapsto (a_0-a_1, a_2-a_0, a_1-a_2) \\ & \qquad\qquad\qquad (a_{01},a_{02},a_{12}) \longmapsto a_{01}+a_{02}+a_{12}\,. \end{array} \tag{1.9}$$

If $D=\sum_{y\subset X} n_y[y]$ is a divisor of X we can define the subgroups

$$\mathbf{A}_X(D) := \prod_{y\subset X} \mathbb{A}_y^{(-n_y)}\,.$$

Note that $\mathbf{A}_X(D)$ is a well defined subgroup of $\mathbf{A}_X$ because $n_y=0$ for all but finitely many y. Let's define the subspaces

$$A_{12}(D) := A_{012}\cap \mathbf{A}_X(D) = \mathbf{A}_X(D)\,.$$

$$A_1(D) := A_{01}\cap \mathbf{A}_X(D);\quad A_2(D) := A_{02}\cap \mathbf{A}_X(D);$$

in order to get the complex

$$\mathcal{A}_X(D):\quad A_0\oplus A_1(D)\oplus A_2(D) \xrightarrow{d^0_D} A_{01}\oplus A_{02}\oplus A_{12}(D) \xrightarrow{d^1_D} A_{012} \tag{1.10}$$

such that the maps are the same of those in equation (1.9). Furthermore note that $\mathcal{A}_X=\mathcal{A}_X(0)$. There is also the idelic version of complex (1.9):

$$\begin{array}{llll} \mathcal{A}_X^\times : & A_0^\times\oplus A_1^\times\oplus A_2^\times \xrightarrow{d^0_\times} A_{01}^\times\oplus A_{02}^\times\oplus A_{12}^\times \xrightarrow{d^1_\times} A_{012}^\times=\mathbf{A}_X^\times \\ & (a_0,a_1,a_2) \longmapsto (a_0a_1^{-1}, a_2a_0^{-1}, a_1a_2^{-1}) \\ & \qquad\qquad\qquad (a_{01},a_{02},a_{12}) \longmapsto a_{01}a_{02}a_{12} \end{array} \tag{1.11}$$

and we have a well defined surjective map:

$$\begin{array}{rcl} p:\ \ker(d^1_\times) & \to & \mathrm{Div}(X) \\ (\alpha,\beta,\alpha^{-1}\beta^{-1}) & \mapsto & \displaystyle\sum_{y\subset X} v_y(\alpha_{x,y})[y]\,. \end{array}$$

2 Idelic Deligne pairing

Let's still consider the arithmetic surface $\varphi: X\to B$, and let's denote with $\mathbf{A}_B$ and $\mathbf{A}_B^\times$ respectively the nonarchimedean parts of the one dimensional adeles and ideles associated to the base B (in other words we are not considering the archimedean places of K). Remember that for any two divisors D,E on X with no common components we have:

$$i_x(D,E) := \mathrm{length}_{\mathcal{O}_{X,x}}\, \mathcal{O}_{X,x}/\left(\mathcal{O}_X(-D)_x+\mathcal{O}_X(-E)_x\right)\,.$$

The Deligne pairing is a bilinear and symmetric map:

$$\langle\,,\,\rangle:\ \mathrm{Pic}(X)\times\mathrm{Pic}(X)\to\mathrm{Pic}(B)$$

which was introduced for the first time in [5]. More details about the construction of $\langle\,,\,\rangle$ can be found in [6, D.2.3]. First of all let's see how the Deligne pairing can be lifted to a pairing at the level of divisors on X and with target in $\mathrm{Pic}(B)$:

Proposition 2.1. *There exists a unique pairing*

$$[[\,,\,]]\colon \operatorname{Div}(X)\times\operatorname{Div}(X)\to\operatorname{Pic}(B)$$

satisfying the following properties:

(1) *It is bilinear and symmetric.*

(2) *It descends to the Deligne pairing*

$$\langle\,,\,\rangle : \operatorname{Pic}(X)\times\operatorname{Pic}(X)\to\operatorname{Pic}(B)\,.$$

(3) *If $D,E\in\operatorname{Div}(X)$ are two divisors with no common components then $[[D,E]]$ is equal to the class of the divisor $\langle D,E\rangle$ in $\operatorname{Pic}(B)$ (here the bracket $\langle D,E\rangle$ denotes the pushforward through φ of the 0-cycle collecting all the local intersection numbers between D and E). In other words:*

$$[[D,E]]=\sum_{x\in D\cap E}[k(x):k(\varphi(x))]i_x(D,E)\,[\varphi(x)]\in\operatorname{Pic}(B)\,.$$

Proof. For any $D,E\in\operatorname{Div}(X)$ it is enough to put:

$$[[D,E]]:=\langle\mathscr{O}_X(D),\mathscr{O}_X(E)\rangle$$

where on the right hand side we have the Deligne pairing between invertible sheaves. Uniqueness follows from properties (1)-(3) and what is commonly called "the moving lemma" ([16, Proposition 9.1.11]). ☐

At this point we will try to work in complete analogy to the geometric case and we will use the Kato symbol defined in section 1.1 to obtain the map, denoted below with a question mark, which makes the following diagram commutative:

$$\begin{array}{ccc}
\ker(d^1_\times)\times\ker(d^1_\times) & & \mathbf{A}_B^\times \\
\downarrow{\scriptstyle p\times p} & \searrow{\scriptstyle ?} & \downarrow \\
\operatorname{Div}(X)\times\operatorname{Div}(X) & \xrightarrow{[[\,,\,]]} & \\
\downarrow & & \downarrow \\
\operatorname{Pic}(X)\times\operatorname{Pic}(X) & \xrightarrow{\langle\,,\,\rangle} & \operatorname{Pic}(B)\cong\operatorname{CH}^1(B)
\end{array}\tag{2.1}$$

As usual, fix a flag $x\in y$ with $\mathfrak{z}\in y(x)$ and assume that $\varphi(x)=b$, then we define

$$(\,,\,)_{x,\mathfrak{z}}:=(\,,\,)_{K_{x,\mathfrak{z}}|K_b}:K_{x,\mathfrak{z}}^\times\times K_{x,\mathfrak{z}}^\times\to K_b^\times$$

where $(\,,\,)_{K_{x,\mathfrak{z}}|K_b}$ is the Kato symbol defined in section 1.1. Remember that depending on whether y is horizontal or vertical, we have a different expression for $(\,,\,)_{x,\mathfrak{z}}$. Then we put:

$$(\,,\,)_{x,y}:=\prod_{\mathfrak{z}\in y(x)}(\,,\,)_{x,\mathfrak{z}}:K_{x,y}^\times\times K_{x,y}^\times\to K_b^\times\,.$$

It is important to point out that $(\,,\,)_{x,y}$ maps $\mathcal{O}_{x,y}^\times\times\mathcal{O}_{x,y}^\times$ to $\mathcal{O}_b^\times$.

Proposition 2.2. *The pairing $(\,,\,)_{x,y}$ is a skew-symmetric bilinear form on $K_{x,y}^\times$ satisfying the following properties:*

(1) *Let $r,s\in K_x^\times$, then for all but finitely many curves y containing x we have that $(r,s)_{x,y}=1$ and moreover $\prod_{y\ni x}(r,s)_{x,y}=1$.*

(2) *Let y be a vertical curve and let $r,s\in K_y^\times$, then $\prod_{x\in y}(r,s)_{x,y}=1$. In particular $(r,s)_{x,y}\in\mathcal{O}_b^\times$ for all but finitely many $x\in y$.*

Proof. Skew symmetry and bilinearity are clear. See [15, Theorem 4.3] for (1); note that in [15] the proof is made for $r,s\in K(X)^\times$, but it is easy to see that it actually works also for $r,s\in K_x^\times$. See [15, Theorem 5.1] for (2). ☐

Definition 2.3. The *idelic Deligne pairing*

$$\langle\, ,\, \rangle_i : \ker(d^1_\times) \times \ker(d^1_\times) \to \mathrm{CH}^1(B)$$

is given by:

$$(r,s) \mapsto \langle r,s\rangle_i := \sum_{b\in B} n_b(r,s)[b] \in \mathrm{CH}^1(B) \tag{2.2}$$

such that:

$$n_b(r,s) := \sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\gamma_{x,y},\beta_{x,y})_{x,y}\right) \tag{2.3}$$

for $r = (\alpha,\beta,\alpha^{-1}\beta^{-1})$, $s = (\gamma,\delta,\gamma^{-1}\delta^{-1}) \in \ker(d_1^\times)$ and where v_b is the complete discrete valuation on K_b. It is crucial to emphasize the fact the we consider $\sum_{b\in B} n_b(r,s)[b]$ in its linear equivalence class in $\mathrm{CH}^1(B)$ and not just as a divisor. By simplicity of notation we avoid to mention the canonical map $\mathrm{Div}(B) \to \mathrm{CH}^1(B)$.

One can verify that definition 2.3 makes sense:

Proposition 2.4. *The summations (2.2) and (2.3) are finite.*

Proof. Thanks to the second adelic restricted product over curves it is enough to check it only for a fixed nonsingular vertical curve $y \subset X_b$. Let's write $\beta_{x,y} = f s_{x,y} \in K(X)^\times \mathcal{O}_x^\times$, then

$$(\gamma_{x,y},\beta_{x,y})_{x,y} = (\gamma_{x,y}, f)_{x,y}(\gamma_{x,y}, s_{x,y})_{x,y}\,. \tag{2.4}$$

[15, Lemma 5.2] shows that $\prod_{x\in y}(\gamma_{x,y}, f)_{x,y}$ is convergent, and this means that $(\gamma_{x,y}, f)_{x,y} \in \mathcal{O}_b^\times$ for all but finitely many $x \in y$. Moreover if $p = \mathrm{char}\, k(b)$, then p is a uniformizing parameter for $K_{x,y}$, so $\gamma_{x,y} = p^r c_{x,y}$ with $c_{x,y} \in \mathcal{O}_{x,y}^\times$. Thus:

$$(\gamma_{x,y}, s_{x,y})_{x,y} = (p^r, s_{x,y})_{x,y}(c_{x,y}, s_{x,y})_{x,y}$$

Obviously $(c_{x,y}, s_{x,y})_{x,y} \in \mathcal{O}_b^\times$, so in order to finish the proof it remains to show that $(p^r, s_{x,y})_{x,y}$ lies in $\mathcal{O}_b^\times$ too. Just for simplicity of calculations let's assume that $K_{x,y}$ is a standard field and $K_{x,y} = K_p\{\{t\}\}$ (the argument works easily also for non standard fields). By the explicit expression of the Kato's residue homomorphism (cf. [15, equation (8)]) we can calculate that:

$$(p^r, s_{x,y})_{x,y} = N_{K_p|K_b}\left(p^{rw}\right)$$

where $w \in \mathbb{Z}$ is the winding number associated to $s_{x,y}$ (see [15, equation (7)] for details). But we know that $s_{x,y} \in \mathcal{O}_x^\times = \mathcal{O}_{K_p}^\times + t\mathcal{O}_{K_p}[[t]]$, thus:

$$s_{x,y} = a + t\sum_{i\geq 0} a_i t^i = a\left(1 + t\sum_{i\geq 0}\frac{a_i}{a}t^i\right)\,.$$

It follows that $w = 0$ and the proof is complete. □

Remark 2.5. For any $b \in B$ we have the following decomposition for the big product (2.3):

$$\sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\, ,\,)_{x,y}\right) = \sum_{\substack{y\subset X_b,\\ x\in y}} v_b\left((\, ,\,)_{x,y}\right) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b\left((\, ,\,)_{x,y}\right)\,.$$

We put $\langle\, ,\, \rangle_i$ as the undetermined function in diagram (2.1) and we have the following fundamental result:

Theorem 2.6. *Consider the notation of diagram (2.1). The pairing $\langle\, ,\, \rangle_i$ satisfies the following properties:*

1. *It is bilinear and symmetric.*
2. *Let $r, s, r', s' \in \ker(d^1_\times)$ such that $p(r) = p(r')$ and $p(s) = p(s')$, then $\langle r,s\rangle_i = \langle r',s'\rangle_i$.*

(3) *It descends naturally to a pairing* $H^1(\mathcal{A}_X^\times) \times H^1(\mathcal{A}_X^\times) \to \mathrm{Pic}(B)$.

Proof. Let's fix $r = (\alpha, \beta, \alpha^{-1}\beta^{-1})$, $s = (\gamma, \delta, \gamma^{-1}\delta^{-1}) \in \ker(d^1_\times)$; moreover we can fix $b \in B$ and work componentwise.
(1) Bilinearity is clear. We will show that as elements of $\mathrm{Div}(B)$ we have $\langle r, s\rangle_i = \langle s, r\rangle_i + (f)$ with $f \in K^\times$. For any flag $x \in y$: $\alpha^{-1}_{x,y}\beta^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$ and $\gamma^{-1}_{x,y}\delta^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$ so we have that:

$$\begin{aligned} 0 = &\sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\alpha^{-1}_{x,y}\beta^{-1}_{x,y}, \gamma^{-1}_{x,y}\delta^{-1}_{x,y})_{x,y}\right) = \\ = &\underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b\left((\alpha_{x,y}, \gamma_{x,y})_{x,y}\right)}_{(i)} + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y})}_{(ii)} + \\ &+ \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\beta_{x,y}, \gamma_{x,y})_{x,y})}_{(iii)} + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\beta_{x,y}, \delta_{x,y})_{x,y})}_{(iv)} . \end{aligned} \tag{2.5}$$

Now we analyze in detail the underbraced terms in equation (2.5): for (i) we have the following decomposition thanks to remark 2.5:

$$\begin{aligned} (i) = &\sum_{\substack{y\subset X_b,\\ x\in y}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) = \\ &\overset{(\text{prop.}2.2(2))}{=} 0 + \sum_{x\in X_b} \sum_{\substack{y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) . \end{aligned}$$

By definition we have that $(ii) = n_b(s, r)$ and $(iii) = -n_b(r, s)$. Finally:

$$(iv) = \sum_{x\in X_b} \sum_{y\ni x} (\beta_{x,y}, \delta_{x,y})_{x,y} =^{(\text{prop.}2.2(1))} 0$$

By substituting in equation (2.5) we conclude that

$$n_b(r, s) = n_b(s, r) + \sum_{\substack{x\in X_b,\\ y\ni x,\\ y \text{ horiz.}}} v_b((\alpha_{x,y}, \gamma_{x,y})_{x,y}) . \tag{2.6}$$

Let y be an horizontal curve and let $x \in y$ such that $\varphi(x) = b$, then the coefficient field of $K_{x,\mathfrak{z}}$ is $k(y)_x$. The two dimensional valuation $v_{x,\mathfrak{z}}$ extends the valuation v_y on $k(y)$ and moreover that the norm $N_{k(y)_x|K_b}$ extends $N_{k(y)|K}$. It follows that $(\ ,\ )_{x,\mathfrak{z}}$ extends the one dimensional tame symbol

$$(\ ,\ )_y := N_{k(y)|K} \circ (\ ,\ )_{k(y)} : K_y^\times \times K_y^\times \to k(y)^\times \to K^\times .$$

This means that for any two elements $u, v \in K_y$, where y is horizontal, we have that:

$$(u, v)_{x,y} = (u, v)_y \in K$$

for any $x \in y$. Therefore we can rewrite equation (2.6):

$$n_b(r, s) = n_b(s, r) + \sum_{y \text{ horiz.}} v_b((\alpha_{x,y}, \gamma_{x,y})_y) . \tag{2.7}$$

Let's put $f = \prod_{y \text{ horiz.}} (\alpha_{x,y}, \gamma_{x,y})_y \in K^\times$, then equation (2.7) implies the following equality:

$$\langle r, s\rangle_i = \langle s, r\rangle_i + \sum_{b\in B} v_b(f)[b] = \langle s, r\rangle_i + (f) .$$

(2) Let $r' = (\alpha', \beta', (\alpha')^{-1}(\beta')^{-1})$ and $s' = (\gamma', \delta', (\gamma')^{-1}(\delta')^{-1})$. Since $p(r) = p(r')$ and $p(s) = p(s')$, then $v_y(\alpha_{x,y}) = v_y(\alpha'_{x,y})$ and $v_y(\gamma_{x,y}) = v_y(\gamma'_{x,y})$. This means that $\gamma'_{x,y} = f_{x,y}\gamma_{x,y}$ and $\alpha'_{x,y} = g_{x,y}\alpha_{x,y}$

for $f_{x,y}, g_{x,y} \in \mathcal{O}_y^\times$ (for any $x \in y$). Then we have the following chain of equalities depending on what we showed in claim (1):

$$n_b(r', s') = \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}\gamma_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} ((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\gamma_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + n_b(r', s) = \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + n_b(s, r') + v_b(f) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((g_{x,y}\alpha_{x,y}, \delta_{x,y})_{x,y}) + v_b(f) = (*)$$

Where $f \in K^\times$.

$$(*) = \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) + \sum_{\substack{y\subset X_b,\\ x\in y}} v_b((g_{x,y}, \delta_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y}, \delta_{x,y})_{x,y}) + v_b(f) =$$

$$= \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y})}_{(i)} + \underbrace{\sum_{\substack{x\in X_b,\\ y\ni x}} v_b((g_{x,y}, \delta_{x,y})_{x,y})}_{(ii)} + n_b(r, s) + v_b(fg)\,.$$

Note that in the last line we used the fact that $n_b(s, r) = n_b(r, s) + v_b(g)$ for $g \in K^\times$. We have to show that the terms (i) and (ii) are valuations at b of elements of $K^\times$. Since $(\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y} \in \mathcal{O}^\times_{x,y}$, we have:

$$0 = \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, (\alpha')^{-1}_{x,y}(\beta')^{-1}_{x,y})_{x,y}) =$$

$$= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \alpha'_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((f_{x,y}, \beta'_{x,y})_{x,y}) =$$

$$= v_b(h) + \sum_{\substack{y\subset X_b,\\ x\in y}} v_b((f_{x,y}, \beta'_{x,y})_{x,y})\,.$$

with $h = \prod_{y \text{ horiz.}} (f_{x,y}, \alpha'_{x,y})_y \in K^\times$. For (ii) the argument is similar.

(3) Let $r, s \in \mathrm{im}(d^0_\times)$. It means that $\alpha = lm^{-1}$, $\beta = tl^{-1}$, $\gamma = uv^{-1}$, $\delta = zu^{-1}$ for $l, u \in A_0^\times = K(X)^\times$, $m, v \in A_1^\times$ and $t, z \in A_2^\times$. So:

$$\begin{aligned} n_b(r, s) &= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((uv^{-1}_{x,y}, t_{x,y}l^{-1})_{x,y}) = \\ &= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((u, t_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((u, l^{-1})_{x,y}) + \\ &+ \sum_{\substack{x\in X_b,\\ y\ni x}} v_b(v^{-1}_{x,y}, t_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((v^{-1}_{x,y}, l^{-1})_{x,y})\,. \end{aligned} \tag{2.8}$$

Now it is enough to appeal to one of the arguments previously used to conclude that each summand of equation (2.8) is either 0 or of the form $v_b(f)$ for $f \in K^\times$. It means that $\langle r, s\rangle_i = 0$ in $\mathrm{CH}^1(B)$. ☐

We want to give an alternative formula for the coefficient $n_b(r,s)$. Notice that:

$$
\begin{aligned}
&- \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y},\gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y}) = \\
&= \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y},\gamma_{x,y})_{x,y}) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y},\delta_{x,y})_{x,y}) = \\
&= v_b(f) + \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y},\delta_{x,y})_{x,y}) = v_b(f) + n_b(s,r) = v_b(fg) + n_b(r,s)
\end{aligned}
\tag{2.9}
$$

for $f,g\in K^\times$. Therefore, we can also express:

$$
n_b(r,s) = - \sum_{\substack{x\in X_b,\\ y\ni x}} v_b((\alpha_{x,y},\gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y})\,. \tag{2.10}
$$

In particular if y is a horizontal curve:

$$
- v_b((\alpha_{x,y},\gamma_{x,y}^{-1}\delta_{x,y}^{-1})_{x,y}) = v_y(\alpha_{x,y}) v_b\left(N_{k(y)_x|K_b}\left(\overline{\gamma_{x,y}^{-1}\delta_{x,y}^{-1}}\right)\right)\,. \tag{2.11}
$$

The following lemmas are fundamental in order to understand the relationship between $\langle r,s\rangle_i$ and Deligne pairing.

Lemma 2.7. *Let $X_b\subset X$ the fiber over $b\in B$ and assume that X_b has at least two irreducible components. If $D\subset X_b$ is an integral curve, then there exists a divisor $D'\sim D$ such that D' doesn't have components contained in X_b.*

Proof. Consider Γ running amongst all irreducible components of X_b, then put

$$
S := \bigcup_{\substack{\Gamma\subset X_b\\ \Gamma\neq D}} (\Gamma\cap D)\,.
$$

By the moving lemma we can find $D'\sim D$ not passing by S. It is clear by the definition of S that D' cannot have vertical components contained in X_b. □

Lemma 2.8. *Let D,E two prime divisors on X and let $x\in D\cap E$ a nonsingular point for both D and E. Moreover let $d_x,e_x\in \mathscr{O}_{X,x}$ be the local equations at x of D and E respectively. Then we have the equality:*

$$
v_{x,D}^{(1)}(\overline{e_x}) = i_x(D,E)
$$

where $v_{x,D}^{(1)}: E_{x,D}^\times\to\mathbb{Z}$ is the one dimensional valuation and $\overline{e_x}\in E_{x,D}^\times$ is the natural projection through the map $\mathcal{O}_{x,D}\to E_{x,D}^\times$.

Proof. Put $y=D$ and $v=v_{x,D}^{(1)}$. First of all notice that $\mathscr{O}_{X,x}\subseteq\mathscr{O}_{X,y}$, therefore $\overline{e_x}\in\mathscr{O}_{y,x}$ and it is the image on the natural map $\mathscr{O}_{X,x}\to\mathscr{O}_{y,x}\subset k(y)$. We have to show that $v(\overline{e_x}) = \mathrm{length}_{\mathscr{O}_{X,x}}\frac{\mathscr{O}_{X,x}}{(d_x,e_x)}$, but we know that $\mathscr{O}_{y,x}=\frac{\mathscr{O}_{X,x}}{d_x\mathscr{O}_{X,x}}$, thus

$$
\mathrm{length}_{\mathscr{O}_{X,x}}\frac{\mathscr{O}_{X,x}}{(d_x,e_x)} = \mathrm{length}_{\mathscr{O}_{y,x}}\frac{\mathscr{O}_{y,x}}{\overline{e_x}\mathscr{O}_{y,x}} = v(\overline{e_x})\,.
$$

□

Theorem 2.9. *If $r,s\in\ker(d^1_\times)$ such that $D=p(r)$ and $E=p(s)$ are two nonsingular prime divisors on X with no common components, then $\langle r,s\rangle_i = [[D,E]]$.*

Proof. Fix $r=(\alpha,\beta,\alpha^{-1}\beta^{-1})$, $s=(\gamma,\delta,\gamma^{-1}\delta^{-1})$. We want to show that it is enough to restrict to the case when either D or E is horizontal. In any case, by theorem 2.6(2) we always choose $\delta_{x,y}$ in the following way: $\delta_{x,y} = 1$ if $x \notin D \cap E$ and $\delta_{x,y} = t_x^{-1}$, where $t_x \in \mathscr{O}_{X,x}$ is the local equation of E at x, if $x \in D \cap E$. For any $y \neq D$, $\alpha_{x,y} \in \mathcal{O}_y^\times$, since $p(a) = D$, therefore:

$$n_b(r,s) = -\sum_{x \in D \cap X_b} v_b((\alpha_{x,D}, \gamma_{x,D}^{-1}\delta_{x,D}^{-1})_{x,D})\,. \tag{2.12}$$

If $D \subseteq X_b$ and $E \subseteq X_{b'}$ with $b \neq b'$, then by proposition 2.2(2) and the choice of $\delta_{x,y}$ we have:

$$n_b(r,s) = -\sum_{x \in D \cap X_b} v_b((\alpha_{x,D}, \gamma_{x,D}^{-1})_{x,D}) = 0\,. \tag{2.13}$$

So in such a particular case $\langle r,s\rangle_i = [[D,E]] = 0$.
If $D, E \in X_b$ we can apply lemma 2.7 and find a divisor $D' = \sum_j n_j\Gamma_j \sim D$ such that $\Gamma_j \not\subset X_b$. Clearly

$$[[\Gamma_j, E]] = \sum_j n_j[[\Gamma_j, E]]$$

therefore from now on we can restrict our calculation to the case where either D or E is horizontal. By symmetry we can fix D to be horizontal and we denote with $K(D)$ its function field. In this case we have an explicit expression given by equation (2.11):

$$n_b(r,s) = \sum_{x \in D \cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{\gamma_{x,D}^{-1}t_x}\right)\right) = \sum_{x \in D \cap X_b} v_b\left(N_{K(D)|K}\left(\overline{\gamma_{x,D}^{-1}t_x}\right)\right) =$$
$$= \sum_{x \in D \cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{\gamma_{x,D}^{-1}}\right)\right) + \sum_{x \in D \cap E \cap X_b} v_b\left(N_{K(D)_x|K_b}\left(\overline{t_x}\right)\right)\,.$$

Now by the theory of extensions of valuation fields (see [10, II(2.5)]), we know that if $v_x := v_{x,D}^{(1)}$ is the valuation on $K(D)_x$, then:

$$v_x = \frac{1}{[k(x):k(b)]} v_b \circ N_{K(D)_x|K_b}\,.$$

Therefore we obtain:

$$n_b(r,s) = \sum_{x \in D \cap X_b} [k(x):k(b)] v_x\left(\overline{\gamma_{x,D}^{-1}}\right) + \sum_{x \in D \cap E \cap X_b} [k(x):k(b)] v_x\left(\overline{t_x}\right)\,. \tag{2.14}$$

Put by simplicity $f = \overline{\gamma_{x,D}^{-1}} \in K(D)^\times$, consider the restricted morphism of arithmetic curves $\varphi : D \to B$ and the principal divisor $(f) \in \mathrm{Princ}(D)$, then:

$$\varphi_*((f)) = \sum_{b \in B}\left(\sum_{x \in D \cap X_b}[k(x):k(b)]v_x(f)\right)[b]\,.$$

Moreover $v_x\left(\overline{t_x}\right) = i_x(D,E)$ by lemma 2.8. Equation 2.14 implies that in $\mathrm{Div}(B)$ we have the following equality:

$$\langle r,s\rangle_i = \varphi_*((f)) + [[D,E]]\,.$$

But by [16, 7 Remark 2.19] we know that $\varphi_*((f)) = \left(N_{K(D)|K}(f)\right) \in \mathrm{Princ}(B)$, so the proof is complete. □

We obtained the idelic representation of Deligne pairing:

Corollary 2.10. *Diagram (2.1) is commutative.*

Proof. For any two divisors $D, E \in \mathrm{Div}(X)$ define the pairing:

$$\Theta(D,E) := \langle r', s'\rangle_i$$

for a choice of $r', s' \in \ker(d_\times^1)$ such that $p(r') = D$ and $p(s') = E$. By theorem 2.6(2) Θ is well defined and moreover by 2.6(1), 2.6(3) and 2.9 we can conclude that $\Theta(D,E) = D.E$. Thus for any $a, b \in \ker(d_\times^1)$ we have that:

$$\langle r,s\rangle_i = \Theta(p(r), p(s)) = p(r).p(s)\,.$$

□

3 Adelic Deligne pairing

A clever and quick adelic interpretation of intersection theory on algebraic surfaces is given in [9] and the strategy is very simple: first of all one defines the adelic Euler-Poincare characteristic $\chi_a(\cdot)$ which associates an integer to any divisor on the surface (or more in general to any invertible sheaf) by using just data coming from the adelic complex. Then the adelic intersection pairing is defined accordingly to equation (B.1) by using $\chi_a(\cdot)$ instead of the usual Euler-Poincare characteristic. Here we try to follow the same approach; so, it is evident that we have to define the adelic determinant of the cohomology

$$\det_a R\varphi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(B)$$

which should be a function involving only adelic data, and then the adelic Deligne pairing according to theorem B.3.

For any coherent sheaf $\mathscr{F}$ on B and any closed point $b \in B$, we define the following objects:

$$K_b(\mathscr{F}) := \mathscr{F}_\xi \otimes_K K_b = (\mathscr{F}_b \otimes_{\mathscr{O}_{B,b}} K) \otimes_K K_b = \mathscr{F}_b \otimes_{\mathscr{O}_{B,b}} K_b\,,$$

$$\mathcal{O}_b(\mathscr{F}) := \mathscr{F}_b \otimes_{\mathscr{O}_{B,b}} \mathcal{O}_b\,.$$

$$\mathbf{A}_B(\mathscr{F}) := \prod_{b\in B}{}' K_b(\mathscr{F})$$

where the restricted product is taken with respect to the rings $\mathcal{O}_b(\mathscr{F})$, and

$$\mathbf{A}_B(\mathscr{F})(0) := \prod_{b\in B} \mathcal{O}_b(\mathscr{F})\,.$$

Moreover recall that we have the following one dimensional adelic complex given by:

$$\begin{array}{rrcl} \mathcal{A}_B(\mathscr{F}): \quad 0 \to \mathscr{F}_\xi \oplus \mathbf{A}_B(\mathscr{F})(0) & \to & \mathbf{A}_B(\mathscr{F}) \to 0 \\ (f,(\alpha_b)_b) & \mapsto & (f-\alpha_b)_b \end{array}$$

It is important to point out that we want to consider $\mathcal{A}_B(\mathscr{F})$ as a complex of O_K-modules in the natural way.

Definition 3.1. Let D be a divisor on X satisfying proposition B.1. For any invertible sheaf $\mathscr{L}$ on B we put by simplicity $\mathscr{G} := \varphi_*\mathscr{L}(D)$ and $\mathscr{H} := \varphi_*(\mathscr{L}(D)/\mathscr{L})$. Then the *adelic determinant of cohomology* is given by:

$$\det_a R\varphi_*(\mathscr{L}) := \det\left(H^0(\mathcal{A}_B(\mathscr{G}))\right) \otimes \left(\det\left(H^0(\mathcal{A}_B(\mathscr{H}))\right)\right)^*$$

Where with $*$ we denote the algebraic dual. Note that $\det_a R\varphi_*(\mathscr{L})$ is a O_K-module, but by abuse of notation we can consider it as an element in $\mathrm{Pic}(B)$ after taking the associated sheaf $(\det_a R\varphi_*(\mathscr{L}))^\sim$ (see [16, 5.1.2]). In other words we omit the operator $\sim$ by simplicity of notations.

Definition 3.2. The *adelic Deligne pairing* between two invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ on B is defined as:

$$\langle \mathscr{L}, \mathscr{M}\rangle_a :=$$
$$\det_a R\varphi_*(\mathscr{O}_X) \otimes (\det_a R\varphi_*(\mathscr{L}))^{-1} \otimes (\det_a R\varphi_*(\mathscr{M}))^{-1} \otimes \det_a R\varphi_*(\mathscr{L}\otimes\mathscr{M})$$

It is immediate to verify that the definition of the adelic Deligne pairing coincides with the usual Deligne pairing by using equation (B.3). Indeed thanks to [11] $H^0(\mathcal{A}_B(\mathscr{G})) \cong H^0(B,\mathscr{G})$ and $H^0(B,\mathscr{G})^\sim \cong \mathscr{G}$ by affine Serre's theorem (see [22, II,4]). Obviously the same holds for $\mathscr{F}$.

Appendices

A Topics in K-theory

Algebraic K-theory is a very wide subject with a long history. It can be approached in many different ways and several links can be build between all approaches (see for example [23]). This appendix is not a short introduction to algebraic K-theory, but just a mere collection of definition and notations needed in this text.

Definition A.1. Let G be an abelian group, and fix an integer ≥ 1. A *r-Steinberg map* is an homomorphism of $\mathbb{Z}$-modules $f : (F^\times)^{\oplus_r} \to G$ such that $f(a_1, \ldots, a_r) = 0$ whenever there exist two indexes i, j such that $i \neq j$ and $a_i + a_j = 1$.

Let's denote with $\mathbf{St}(r)$ the category whose objects are the r-Steinberg maps $f : (F^\times)^{\oplus_r} \to G$ and the morphisms are the commutative diagrams:

$$\begin{array}{ccc} (F^\times)^{\oplus_r} & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & \nearrow_{\phi} & \\ H & & \end{array}$$

where ϕ is a group homomorphism.

Proposition A.2. *The category* $\mathbf{St}(r)$ *has the initial object.*

Proof. We construct the initial objects by hands. Let's define

$$K_r(F) := \underbrace{F^\times \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} F^\times}_{r \text{ times}} / S$$

where S is the (multiplicative) subgroup generated by the following set:

$$\{a_1 \otimes \ldots \otimes a_r \colon a_i + a_j = 1 \text{ for some } i \neq j\}\,.$$

The natural image of a pure tensor $a_1 \otimes \ldots \otimes a_r$ in $K_r(F)$ is denoted by $\{a_1, \ldots, a_r\}$. Clearly we have an induced map:

$$\begin{array}{rccc} \{\ \}: & (F^\times)^{\oplus_r} & \to & K_r(F) \\ & (a_1, \ldots, a_r) & \mapsto & \{a_1, \ldots, a_r\} \end{array}$$

At this point it is straightforward to see that $\{\ \} : (F^\times)^{\oplus_r} \to K_r(F)$ is the initial object for $\mathbf{St}(r)$. ☐

Definition A.3. For $r = 0$ we put $K_0(F) := \mathbb{Z}$ and in general we call the group $K_r(F)$ constructed in proposition A.2 *the r-th K-group of F*. Note that $K_1(F) = F^\times$. The map $\{\ \} : (F^\times)^{\oplus_r} \to K_r(F)$ is called the r-th symbol map and in the cases $r = 0, 1$ it is just the identity.

Remark A.4. The groups introduced in definitons A.3 are usually called Milnor K-groups and the standard notation is K_r^M. However in this text we can simplify the notation.

The construction $K_r(\ )$ is functorial, in fact let $f : F^\times \to L^\times$ be a group homomorphism, then the composition:

$$(F^\times)^{\oplus r} \xrightarrow{f^{\oplus r}} (L^\times)^{\oplus r} \xrightarrow{\{\ ,\ \}} K_r(L)$$

is evidently a Steinberg map. By the universal property it induces a morphism $K_r(f) : K_r(F) \to K_r(L)$.

When F is a complete discrete valuation field there exists a nice relationship between K-groups of F and K-groups of the residue fields:

Theorem A.5. *Let F be a discrete valuation field (not necessarily complete) then there is a unique group homomorphism:*

$$\partial_r : K_r(F) \to K_{r-1}(\overline{F})$$

satisfying the following property:

$$\partial_r(\{x_1, \ldots, x_{r-1}, \varpi\}) = \{\overline{x_1}, \ldots, \overline{x_{r-1}}\}$$

for any local parameter ϖ of F and any $x_1, \ldots, x_{r-1} \in \mathcal{O}_F^\times$.

Proof. See [17]. ☐

Definition A.6. The map ∂_r described in theorem A.5 is called *the (Milnor) r-th boundary map.*

Consider the *tame symbol* for a complete discrete valuation field (F, v):

$$\begin{aligned} (\,,\,)_F : F^\times \times F^\times &\to \overline{F}^\times \\ (a,b) &\mapsto (a,b)_F = (-1)^{v(a)v(b)}\,\overline{a^{v(b)}b^{-v(a)}}\,. \end{aligned} \tag{A.1}$$

We have a nice description of the boundary map ∂_2 in relation to the tame symbol. By the universal property of $K_2(F)$, the tame symbol $(\,,\,)_F$ induces a unique map $\Psi : K_2(F) \to \overline{F}^\times = K_1(F)$ such that $\Psi(\{\,,\,\}) = (\,,\,)_F$. Let $a \in \mathcal{O}_F^\times$ and let ϖ be a local parameter for F, then $\Psi(\{a,\varpi\}) = (a,\varpi)_F = \overline{a}$; this actually means that $\partial_2 = \Psi$. In other words the 2-nd boundary map for a complete discrete valuation field is exactly the map induced naturally by the tame symbol.

For a discrete valuation field F (not necessarily complete) we have the multiplicative group $U_F^{(i)} := 1 + \mathfrak{p}_F^i$ for $i \geq 1$ and we have also the K-theoretic version of it:

$$U^i K_r(F) := \{\{a_1 \dots a_r\} \in K_r(F) \colon a_j \in U_F^{(i)}\ \forall j = 1, \dots, r\}$$

and we put:

$$\widehat{K}_r(F) := \varprojlim_i K_r(F)/U^i K_r(F)\,. \tag{A.2}$$

Clearly we have a natural homomorphism $K_r(F) \to \widehat{K}_r(F)$ and moreover if $\widehat{F}$ is the completion of F there is an isomorphism $\widehat{K}_r(F) \cong \widehat{K}_r(\widehat{F})$. Now put $L = \mathrm{Frac}\,(\mathcal{O}_F[[t]])$, for any prime ideal $\mathfrak{p}$ of height 1 in $\mathcal{O}_F[[t]]$ we have that $\mathcal{O}_F[[t]]_\mathfrak{p}$ is a discrete valuation ring and in particular $F\{\{t\}\}$ is the completion of L at $\mathfrak{p} = \mathfrak{p}_F\mathcal{O}_F[[t]]$. Consider the set:

$$\mathfrak{S} := \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F[[t]]) \colon \mathrm{ht}\,\mathfrak{p} = 1, \mathfrak{p} \neq \mathfrak{p}_F\mathcal{O}_F[[t]]\}\,,$$

and for any $\mathfrak{p} \in \mathfrak{S}$ let's denote with $\partial_r^{(\mathfrak{p})} : K_r(L) \to K_{r-1}(k(\mathfrak{p}))$ the r-th boundary map relative to the valuation defined by $\mathfrak{p}$.

Definition A.7. For $r \geq 1$, the r-th (Kato) residue map on L is given by the following composition:

$$\mathrm{res}_L^{(r)} : K_r(L) \xrightarrow{(\partial_r^{(\mathfrak{p})})_{\mathfrak{p}\in\mathfrak{S}}} \bigoplus_{\mathfrak{p}\in\mathfrak{S}} K_{r-1}(k(\mathfrak{p})) \xrightarrow{\sum_{\mathfrak{p}\in\mathfrak{S}} K_r(N_{k(\mathfrak{p})|F})} K_{r-1}(F)$$

Theorem A.8. *The r-th residue map satisfies:*

$$\mathrm{res}_L^{(r)}\left(U^{(i)}K_r(L)\right) \subseteq U^{(i)}K_{r-1}(F) \quad \forall i \geq 1$$

therefore it induces a homomorphism:

$$\mathrm{res}_{F\{\{t\}\}}^{(r)} : \widehat{K}_r(F\{\{t\}\}) \cong \widehat{K}_r(L) \to \widehat{K}_{r-1}(F)\,.$$

Proof. See [12, Theorem 1]. □

B Determinant of cohomology

For an algebraic surface Z over a field k, intersection theory can be introduced by using the Euler-Poincare characteristic $\chi_k : \mathrm{Coh}(Z) \to \mathbb{Z}$ "restricted" to $\mathrm{Pic}(Z)$. In fact, the intersection number between two invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ on Z can be calculated by the following formula:

$$\mathscr{L}.\mathscr{M} := \chi_k(\mathscr{O}_Z) - \chi_k(\mathscr{L}^{-1}) - \chi_k(\mathscr{M}^{-1}) + \chi_k(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}) \tag{B.1}$$

In the Arakelov setting given by the arithmetic surface $\varphi : X \to B$, we want to define a map $\mathrm{Coh}(X) \to \mathrm{Pic}(B)$ such that, when we take the "restriction" to $\mathrm{Pic}(X)$, we obtain a formula, similar to (B.1), relating our map to the Deligne pairing. In other words, we would like to have the arithmetic equivalent notion of the Euler-Poincare characteristic. The answer to our query will be the determinant of the cohomology, denoted by $\det R\varphi_*$, and in this section we are going to construct it step by step.

The first thing to notice is that χ_k is a cohomological object and in the case of $\varphi : X \to B$ the "relative cohomology" is captured by the higher direct image functors $R^i\varphi_*$. By keeping in mind that the output of the determinant of the cohomology should be an invertible sheaf on the base B, in analogy with the definition of χ_k, the most reasonable definition should be something like:

$$\det R\varphi_*(\mathscr{F}) := \bigotimes_{j\geq 0}(\det R^j\varphi_*\mathscr{F})^{(-1)^j} = \det \varphi_*(\mathscr{F}) \otimes (\det R^1\varphi_*(\mathscr{F}))^{-1} \tag{B.2}$$

Unfortunately equation (B.2) doesn't make any sense in general, since the higher direct images $R^j\varphi_*\mathscr{F}$ are not locally free sheaves, so we cannot take the determinant. However, we will cook up a definition of $\det R\varphi_*(\mathscr{F})$ which agrees with equation (B.2) when $R^j\varphi_*\mathscr{F}$ are locally free.

The following proposition is fundamental:

Proposition B.1. *There exists an effective divisor D on X which doesn't contain any fibre of φ such that for any coherent sheaf $\mathscr{F}$ on X we get an exact sequence:*

$$0 \to \varphi_*\mathscr{F} \to \varphi_*\mathscr{F}(D) \to \varphi_*(\mathscr{F}(D)/\mathscr{F}) \to R^1\varphi_*\mathscr{F} \to 0 \tag{B.3}$$

such that $\varphi_*\mathscr{F}(D)$ and $\varphi_*(\mathscr{F}(D)/\mathscr{F})$ are both locally free sheaves on B.

Proof. See [1, XIII section 4.] or [13, VI, Lemma 1.1]. □

Definition B.2. Let $\mathscr{F} \in \mathrm{Coh}(X)$ and let D be a divisor as in proposition B.1; *the determinant of the cohomology of* $\mathscr{F}$ is:

$$\det R\varphi_*(\mathscr{F}) := \det \varphi_*(\mathscr{F}(D)) \otimes (\det \varphi_*(\mathscr{F}(D)/\mathscr{F}))^{-1} \in \mathrm{Pic}(B)$$

Moreover $\det R\varphi_*(\mathscr{F})$ doesn't depend on the choice of D (for the proof of this statement see [1, XIII section 4.] or [13, VI]).

Now we want to show that if $R^0\varphi_*\mathscr{F} = \varphi_*\mathscr{F}$ and $R^1\varphi_*\mathscr{F}$ are both locally free, then $\det R\varphi_*(\mathscr{F})$ is given by equation (B.2). Consider the exact sequence of equation (B.3), put

$$f : \varphi_*\mathscr{F}(D) \to \varphi_*(\mathscr{F}(D)/\mathscr{F}),$$

$$g : \varphi_*(\mathscr{F}(D)/\mathscr{F}) \to R^1\varphi_*\mathscr{F},$$

and $\mathscr{G} = \mathrm{im}(f) = \ker(g)$. Then we get the following two short exact sequences of locally free sheaves:

$$0 \to \varphi_*\mathscr{F} \to \varphi_*\mathscr{F}(D) \xrightarrow{f} \mathscr{G} \to 0; \tag{B.4}$$

$$0 \to \mathscr{G} \to \varphi_*(\mathscr{F}(D)/\mathscr{F}) \xrightarrow{g} R^1\varphi_*\mathscr{F} \to 0. \tag{B.5}$$

At this point we use the properties of the determinant on short exact sequences and we obtain:

$$\det R\varphi_*(\mathscr{F}) = \det \varphi_*(\mathscr{F}(D)) \otimes (\det \varphi_*(\mathscr{F}(D)/\mathscr{F}))^{-1} \cong$$
$$\cong \det \varphi_*\mathscr{F} \otimes \det \mathscr{G} \otimes (\det \mathscr{G})^{-1} \otimes (\det R^1\varphi_*(\mathscr{F}))^{-1} \cong$$
$$\cong \det \varphi_*\mathscr{F} \otimes \det R^1\varphi_*(\mathscr{F}))^{-1}.$$

The relationship between the determinant of cohomology and Deligne pairing is given by the following theorem:

Theorem B.3. *Let $\mathscr{L}, \mathscr{M}$ be two invertible sheaves on X, then*

$$\langle \mathscr{L}, \mathscr{M} \rangle \cong \det R\varphi_*(\mathscr{O}_X) \otimes (\det R\varphi_*(\mathscr{L}))^{-1} \otimes (\det R\varphi_*(\mathscr{M}))^{-1} \otimes \det R\varphi_*(\mathscr{L} \otimes \mathscr{M}).$$

Proof. See [1, XIII, Theorem 5.8]. □

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