

REMARK ON CONTROLLABILITY TO TRAJECTORIES OF A SIMPLIFIED FLUID-STRUCTURE ITERATION MODEL.

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ABSTRACT. We prove the exact controllability result to trajectories of a simplified model of motion of a rigid body in fluid flow. Unlike a previously known results such a trajectory does not need to be a stationary solution.

1. INTRODUCTION AND MAIN RESULTS

The paper is concerned with the following controllability problem: In the bounded domain $Q = (0, T) \times \Omega$, $\Omega = [a, b]$, $-\infty < a < 0 < b < +\infty$, $x = (x_0, x_1)$ we consider the system of semilinear heat equations

$$(1.1) \quad \begin{aligned} G_1(x, w_1) &= \rho_1 \partial_{x_0} w_1 - a_1 \partial_{x_1}^2 w_1 + b_1 \partial_{x_1} w_1 + c_1 w_1 \\ &+ g_1(x, w_1, \partial_{x_1} w_1) = f_1 + \chi_\omega u \quad \text{in } Q_+ = (0, T) \times (0, b), \end{aligned}$$

$$(1.2) \quad \begin{aligned} G_2(x, w_2) &= \rho_2 \partial_{x_0} w_2 - a_2 \partial_{x_1}^2 w_2 + b_2 \partial_{x_1} w_2 + c_2 w_2 \\ &+ g_2(x, w_2, \partial_{x_1} w_2) = f_2 \quad \text{in } Q_- = (0, T) \times (a, 0). \end{aligned}$$

On the interface $[0, T] \times \{0\}$ functions w_1, w_2 are connected through the boundary conditions

$$(1.3) \quad w_1(x_0, 0) - w_2(x_0, 0) = (\partial_{x_1} w_1 - \partial_{x_1} w_2 - M \partial_{x_0} w_1)(x_0, 0) + r(x_0) \quad \text{on } [0, T],$$

where r is a given function, M is a positive constant. On the lateral boundary of cylinder Q functions w_1, w_2 satisfy the Dirichlet boundary conditions

$$(1.4) \quad w_1(x_0, b) = w_2(x_0, a) = 0 \quad \text{on } [0, T].$$

The initial condition is

$$(1.5) \quad w(0, x_1) = w_0(x_1) \quad \text{on } \Omega.$$

Here

$$w = \begin{cases} w_1 & \text{for } x \in Q_+, \\ w_2 & \text{for } x \in Q_-. \end{cases}$$

Function $u(x)$ is the control distributed over domain $Q_\omega = (0, T) \times \omega$, $\omega = (d, b)$, $d \in (0, b)$: $\text{supp } u \subset Q_\omega$. One of the physical applications of system (1.1)-(1.4) is the rigid body moving through the fluid flow, where w_j is velocity of the fluid flow, M is the mass of the rigid body, $\int_0^{x_0} w_1(\tilde{x}_0, 0) d\tilde{x}_0 + h_0$ is the position of the body (see [19] for details of the model.) We are

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looking for locally distributed control u such that at moment T for the given target function w_2 we have:

$$(1.6) \quad w(T, x_1) = w_2(x_1) \quad \text{on } \Omega.$$

We make the following standard assumptions:

$$(1.7) \quad \rho_1, a_1, b_1 \in C^1(\bar{Q}_+), \quad \rho_2, a_2, b_2 \in C^1(\bar{Q}_-), \quad c_1 \in L^\infty(0, T; L^2(0, b)), \quad c_2 \in L^\infty(0, T; L^2(a, 0)),$$

there exist a positive constant α such that

$$(1.8) \quad \rho_1(x) \geq \alpha > 0, \quad a_1(x) \geq \alpha > 0 \quad \forall x \in Q_+, \quad \rho_2(x) \geq \alpha > 0, \quad a_2(x) \geq \alpha > 0 \quad \forall x \in Q_-,$$

$$(1.9) \quad g_1 \in C^2(\bar{Q}_+ \times \mathbb{R}^1 \times \mathbb{R}^1), \quad g_2 \in C^2(\bar{Q}_- \times \mathbb{R}^1 \times \mathbb{R}^1).$$

there exist constants C_1, \dots, C_2 independent of x and ξ_i , and $p_j \geq 1, j \in \{1, 2, 3\}$ such that

$$(1.10) \quad \begin{aligned} |g_i(x, \xi_1, \xi_2)| &\leq C(1 + |\xi_1|^{p_1} + |\xi_1|^{p_1}|\xi_2|), \quad |\partial_{\xi_1} g_i(x, \xi_1, \xi_2)| \leq C(1 + |\xi_1|^{p_1-1} + |\xi_1|^{p_2-1}|\xi_2|), \\ |\partial_{\xi_2} g_i(x, \xi_1, \xi_2)| &\leq C(1 + |\xi_1|^{p_3}) \quad \forall (x, \xi_1, \xi_2) \in Q \times \mathbb{R}^2 \quad \text{and} \quad \forall i \in \{1, 2\}. \end{aligned}$$

Remark. *The nonlinear term $g_1(x, u, \partial_{x_1} u) = g_2(x, u, \partial_{x_1} u) = u \partial_{x_1} u$ satisfies (1.9) and (1.10).*

Since it is well known that the controllability problem (1.1)-(1.6) can not be solved for an arbitrary target function w_2 we introduce the additional condition:

Condition 1. *There exist a pair $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2) \in H^{1,2}(Q_+) \times H^{1,2}(Q_-)$ and control $\mathbf{u} \in L^2(Q)$, $\text{supp } \mathbf{u} \subset \bar{Q}_\omega$ such that*

$$(G_1(x, \mathbf{w}_1), G_2(x, \mathbf{w}_2)) = (f_1 + \mathbf{u}, f_2),$$

$$(\mathbf{w}_1 - \mathbf{w}_2)(x_0, 0) = (\partial_{x_1} \mathbf{w}_1 - \partial_{x_1} \mathbf{w}_2 - M \partial_{x_0} \mathbf{w}_1)(x_0, 0) - r(x_0) = 0 \quad \text{on } [0, T],$$

$$\mathbf{w}(\cdot, b) = \mathbf{w}(\cdot, a) = 0 \quad \text{on } [0, T], \quad \mathbf{w}(T, \cdot) = w_2.$$

Our main result is the following

Theorem 1.1. *Let $f_1 \in L^2(Q_+)$, $f_2 \in L^2(Q_-)$, $r \in L^2(0, T)$. Suppose that assumptions (1.6)-(1.10) and Condition 1 for functions f_1, f_2, r holds true, $w_0 \in H_0^1(\Omega)$. Then there exists a positive $\epsilon > 0$ such that if*

$$\|w_0 - \mathbf{w}(\cdot, 0)\|_{H_0^1(\Omega)} \leq \epsilon$$

the controllability problem (1.1)-(1.6) has solution $(w, u) \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-) \cap C^0(0, T; H_0^1(\Omega)) \times L^2(Q)$, $\text{supp } u \subset \bar{Q}_\omega$.

Theorem 1.1 was established for the case $\mathbf{w} \equiv 0$ with control located at both ends in [5] and at one end in [12].

Another physical application of the controllability problem (1.1)-(1.6) describes to rods connected by a point mass. (see [6] for details of the model.) The zero null controllability for this model was proved in [7] for the case when coefficients ρ_j, a_j, b, c_j are constants and recently in [1] when coefficients ρ_j, a_j, b, c_j are space dependent functions. The method of both papers based on the analysis of eigenvalues and eigenfunctions and therefore can not be applied to the case of time dependent coefficients.

The n-dimensional generalization for linear parabolic equations with time independent coefficients was studied by J.L. Rousseau with co-authors in [14]. Exact controllability of similar problem for linear 1-d hyperbolic equations in case of one point mass attached was proved by S. Hansen and E.Zuazua in [8] and for several mass attached case by S. Avdonin and J. Edwards in [2].

The proof of the Theorem 1.1 is based on the implicit function theorem and null-controllability result for the linearized system (1.1) - (1.4). The null-controllability of the linearized system follows from the observability estimate. Observability estimate is obtained by Carleman estimate with boundary. The weight function is similar to one from work [4].

Notations. Let $\Omega_+ = (0, b), \Omega_- = (a, 0)$, $i = \sqrt{-1}$ and $D = (D_0, D_1)$, $D_0 = \frac{1}{i}\partial_{x_0}, D_1 = \frac{1}{i}\partial_{x_1}, \alpha = (\alpha_0, \alpha_1), \alpha_0 \geq 0, \alpha_1 \geq 0, |\alpha| = 2\alpha_0 + \alpha_1, \partial^\alpha = \partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}$. For any function $\tilde{\rho}$ we introduce the space $L_\rho^2(X) = \{u | \|u\|_{L_\rho^2(X)} = \sqrt{\int_X |\tilde{\rho}|u^2 dx}\}$, by Fu we denote the Fourier transform of function u in variable x_0 : $Fu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi_0 x_0} u(x_0) dx_0$. Let $\xi = (\xi_0, \xi_1), \zeta = (\xi_0, \tilde{s}), x = (x_0, x_1), \zeta^* = (\xi_0^*, \tilde{s}^*), M(\xi_0, \tilde{s}) = (\tilde{s}^4 + \xi_0^2)^{\frac{1}{4}}, \mathbb{M} = \{(\xi_0, \tilde{s}); M(\xi_0, \tilde{s}) = 1\}$. We introduce the conic neighborhood of the point ζ^* :

$$\mathcal{O}(\zeta^*, \delta) = \{(\xi_0, \tilde{s}) \in \mathbb{R}^2 \setminus \{0\} | (\xi_0/M^2(\xi_0, \tilde{s}), \tilde{s}/M(\xi_0, \tilde{s})) - (\xi_0^*, \tilde{s}^*)| \leq \delta\},$$

and the Sobolev spaces

$$H^{1,2}(Q_\pm) = \{u | u, \partial_{x_0} u, \partial_{x_1} u, \partial_{x_1}^2 u \in L^2(Q_\pm)\},$$

$H^{1,2,\tilde{s}}(Q_\pm)$ the space $H^{1,2}(Q_\pm)$ equipped with the norm

$$\|u\|_{H^{1,2,\tilde{s}}(Q_\pm)} = \sqrt{\|u\|_{H^{1,2}(Q_\pm)}^2 + \tilde{s}^2 \|u\|_{L^2(Q_\pm)}^2}.$$

For any function u we set $[u] = \lim_{x_1 \rightarrow +0} u(x_0, x_1) - \lim_{x_1 \rightarrow -0} u(x_0, x_1)$. For the symbol $M(\xi_0, \tilde{s})$ we introduce the pseudodifferential operator by formula

$$M(D_0, \tilde{s})u = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^1} M(\xi_0, \tilde{s}) e^{i\xi_0 x_0} Fud\xi_0.$$

2. OBSERVABILITY ESTIMATE.

In this section we prove the observability estimate for the following system:

$$(2.1) \quad -\tilde{\rho}_1 \partial_{x_0} v_1 - \tilde{a}_1 \partial_{x_1}^2 v_1 + \tilde{b}_1 \partial_{x_1} v_1 + \tilde{c}_1 v_1 = \tilde{f}_1 \quad \text{in } Q_+,$$

$$(2.2) \quad -\tilde{\rho}_2 \partial_{x_0} v_2 - \tilde{a}_2 \partial_{x_1}^2 v_2 + \tilde{b}_2 \partial_{x_1} v_2 + \tilde{c}_2 v_2 = \tilde{f}_2 \quad \text{in } Q_-.$$

On the interface $[0, T] \times \{0\}$ functions v_1, v_2 are connected through the boundary conditions

$$(2.3) \quad v_1(x_0, 0) - v_2(x_0, 0) = (\partial_{x_1} v_1 - \partial_{x_1} v_2 + M \partial_{x_0} v_1)(x_0, 0) - \tilde{r}(x_0) = 0 \quad \text{on } [0, T],$$

$$(2.4) \quad v_1(x_0, b) = v_2(x_0, a) = 0 \quad \forall x_0 \in [0, T].$$

We make the following standard assumptions:

$$(2.5) \quad \tilde{\rho}_1, \tilde{a}_1, \tilde{b}_1 \in C^1(\bar{Q}_+), \quad \tilde{\rho}_2, \tilde{a}_2, \tilde{b}_2 \in C^1(\bar{Q}_-), \quad \tilde{c}_1 \in L^\infty(0, T; L^2(\Omega_+)), \quad \tilde{c}_2 \in L^\infty(0, T; L^2(\Omega_-)),$$

there exist a positive constant α_0 such that

$$(2.6) \quad \tilde{\rho}_1(x) \geq \alpha_0 > 0, \quad \tilde{a}_1(x) \geq \alpha_0 > 0 \quad \forall x \in Q_+, \quad \tilde{\rho}_2(x) \geq \alpha_0 > 0, \quad \tilde{a}_2(x) \geq \alpha_0 > 0 \quad \forall x \in Q_-.$$

We have

Proposition 2.1. *Let $f_1 \in L^2(Q_+)$, $f_2 \in L^2(Q_-)$, $\tilde{r} \in L^2(0, T)$ and (2.5), (2.6) holds true. Then there exists function $\eta(x_1) \in C^2(\bar{\Omega})$, $\eta(x_1) < 0$ on $\bar{\Omega}$ and a constant C_1 independent of $v = (v_1, v_2)$ such that*

$$\begin{aligned} & \sum_{|\alpha| \leq 1} \|((T - x_0)^{-3})^{\frac{(3-2|\alpha|)}{2}} \partial^\alpha v_2 e^{\psi^*}\|_{L^2(Q_-)} + \sum_{|\alpha| \leq 1} \|((T - x_0)^{-3})^{\frac{5-2|\alpha|}{2}} \partial^\alpha v_1 e^{\psi^*}\|_{L^2(Q_+)} \\ & + \|((T - x_0)^{-3})^{\frac{3}{2}} \partial_{x_1}^+ v e^{\psi^*}\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{1}{2}} \partial_{x_1}^- v e^{\psi^*}\|_{L^2(0, T)} \\ & + \|((T - x_0)^{-3})^{\frac{1}{2}} \partial_{x_0} v e^{\psi^*}\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{5}{2}} v e^{\psi^*}\|_{L^2(0, T)} \\ & \leq C_1 (\|((T - x_0)^{-3} f_1 e^{\psi^*}\|_{L^2(Q_+)} + \|((T - x_0)^{-3} f_2 e^{\psi^*}\|_{L^2(Q_-)} \\ (2.7) \quad & + \|((T - x_0)^{-\frac{3}{2}} \tilde{r} e^{\psi^*})\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{3}{2}} \partial_{x_1} v e^{\psi^*}(\cdot, b)\|_{L^2(0, T)}), \end{aligned}$$

where $\psi^*(x) = \eta(x_1)/(T - x_0)^3$.

Proof. Making the change of variables $x_0 \rightarrow T - x_0$ and setting $w_j(x) = v_j(T - x_0, x_1)$ we have

$$(2.8) \quad \rho_1^* \partial_{x_0} w_1 - \partial_{x_1}^2 w_1 + b_1^* \partial_{x_1} w_1 + c_1^* w_1 = f_1^* \quad \text{in } Q_+,$$

$$(2.9) \quad \rho_2^* \partial_{x_0} w_2 - \partial_{x_1}^2 w_2 + b_2^* \partial_{x_1} w_2 + c_2^* w_2 = f_2^* \quad \text{in } Q_-,$$

$$(2.10) \quad w_1(x_0, 0) - w_2(x_0, 0) = (\partial_{x_1} w_1 - \partial_{x_1} w_2 - M \partial_{x_0} w_1)(x_0, 0) - r^*(x_0) = 0 \quad \text{on } [0, T],$$

$$(2.11) \quad w_1(x_0, b) = w_2(x_0, a) = 0 \quad \text{on } [0, T],$$

where $\rho_j^*(x) = \tilde{\rho}_j(T - x_0, x_1)/a_j(T - x_0, x_1)$, $b_j^*(x) = \tilde{b}_j(T - x_0, x_1)/a_j(T - x_0, x_1)$, $c_j^*(x) = \tilde{c}_j(T - x_0, x_1)/a_j(T - x_0, x_1)$, $f_j^*(x) = \tilde{f}_j(T - x_0, x_1)/a_j(T - x_0, x_1)$, $r^*(x) = \tilde{r}(T - x_0, x_1)$ and $j \in \{1, 2\}$.

Our next step is to construct of variables in domain Q_+ such that the equation (2.8) keeps the same form after change of variables but the new coefficient ρ_1^* satisfies

$$\rho_1^* = \rho_2^* \quad \text{on } [0, T] \times \{0\}.$$

Let $F(x) : C^{1,2}(\bar{Q}_+, \bar{Q}_+)$ be the diffeomorphism of Q_+ on Q_+ such that $F = (F_1, F_2)$ and $F_1(x) = x_0$ on Q_+ . In order to construct the function F_2 consider a function $q(x_0) \in C^1[0, T]$, $q(x_0) > C > 0$ on $[0, T]$. Set $\kappa_0 = \frac{b}{\|q\|_{C^0[0,T]} + 40}$. Let $\eta_1(x_1) \in C^\infty[0, b]$, $\eta_1(x_1) = x_1$ on $[0, \kappa_0]$, $\frac{d\eta_1}{dx_1} \geq 0$ on $[0, b]$, $\eta_1 = b/5$ for $x_1 \in [\frac{4b}{5}, b]$ and $\frac{d\eta_1}{dx_1} > 0$ on $[\kappa_0, \frac{b}{\|q\|_{C^0[0,T]} + 20}]$. Let

$$\eta_2(x_1) = \begin{cases} 0 & \text{for } x_1 \in [0, \frac{b}{\|q\|_{C^0[0,T]} + 20}], \\ \frac{b(x_1 - b_1)^3}{(b - b_1)^3} & \text{for } x_1 \in [b_1, b]. \end{cases} \quad F_2(x) = q(x_0)\eta_1(x_1) + \eta_2(x).$$

Then on $[0, T] \times [0, \tilde{b}]$ we have

$$F^{-1}(\tilde{x}) = (\tilde{x}_0, \tilde{x}_1/q(\tilde{x}_0)) \quad \text{and} \quad DF(x_0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & q(x_0) \end{pmatrix}.$$

Denote $\tilde{x} = F(x)$ and $\tilde{w}_1(\tilde{x}) = w_1(F^{-1}(\tilde{x}))$. Then $w_1(x) = \tilde{w}_1(F(x))$. Therefore on Q_+

$$\partial_{x_0} w_1 = \partial_{\tilde{x}_0} \tilde{w}_1 \partial_{x_0} F_1(x) + \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_0} F_2(x) = \partial_{\tilde{x}_0} \tilde{w}_1 + \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_0} F_2(x).$$

In particular

$$\partial_{x_0} w_1 = \partial_{\tilde{x}_0} \tilde{w}_1 \quad \text{on } [0, T] \times \{0\}.$$

On Q_+ we have

$$\partial_{x_1} w_1 = \partial_{\tilde{x}_0} \tilde{w}_1 \partial_{x_1} F_1(x) + \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_1} F_2(x) = \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_1} F_2(x)$$

and

$$\begin{aligned} \partial_{x_1}^2 w_1 &= \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_1}^2 F_2(x) + \partial_{\tilde{x}_0 \tilde{x}_1}^2 \tilde{w}_1 \partial_{x_1} F_1(x) \partial_{x_1} F_2(x) + \partial_{\tilde{x}_1}^2 \tilde{w}_1 (\partial_{x_1} F_2(x))^2 = \\ &\quad \partial_{\tilde{x}_1} \tilde{w}_1 \partial_{x_1}^2 F_2(x) + \partial_{\tilde{x}_1}^2 \tilde{w}_1 (\partial_{x_1} F_2(x))^2. \end{aligned}$$

Therefore function \tilde{w}_1 satisfies the parabolic equation

$$\rho_0 \partial_{\tilde{x}_0} \tilde{w}_1 - \alpha(\tilde{x}) \partial_{\tilde{x}_1}^2 \tilde{w}_1 + \beta(\tilde{x}) \partial_{\tilde{x}_1} \tilde{w}_1 + \tilde{c} \tilde{w}_1 = f_0,$$

where $f_0 = f_1^* \circ F^{-1}$, $\rho_0 = \rho_1^* \circ F^{-1}$ and $\beta(\tilde{x}) = (b_1^* \partial_{x_1} F_2 + \partial_{x_0} F_2 - \partial_{x_1}^2 F_2(x)) \circ F^{-1}$, $\alpha(\tilde{x}) = (\partial_{x_1} F_1)^2 \circ F^{-1}$, $\tilde{c} = c_1^* \circ F^{-1}$. After division of the new equation by α we have

$$\rho_1^* \partial_{\tilde{x}_0} \tilde{w}_1 - \partial_{\tilde{x}_1}^2 \tilde{w}_1 + b_1^*(\tilde{x}) \partial_{\tilde{x}_1} \tilde{w}_1 + c_1^* \tilde{w}_1 = f_1^*$$

with $\rho_1^* = \rho_0/\alpha$, $b_1^* = \beta/\alpha$, $c_1^* = \tilde{c}/\alpha$ and $f_1^* = f_0/\alpha$. Observe that on $[0, T] \times \{0\}$

$$\partial_{x_1}^2 w_1 = \partial_{\tilde{x}_1}^2 \tilde{w}_1 q^2(x_0).$$

So

$$\rho_1^*(\tilde{x}_0) = \rho_1^*(\tilde{x}_0)/q^2(\tilde{x}_0).$$

Then taking $q^2(\tilde{x}_0) = \rho_2^*(x_0, 0)/\rho_2^*(x_0, 0)$ we obtain that the function ρ given by formula

$$\rho(x) = \begin{cases} \rho_1^* & \text{for } x \in Q_+, \\ \rho_2^* & \text{for } x \in Q_- \end{cases}$$

is continuous on \bar{Q} . Equations (2.10) are transformed to

$$(2.12) \quad \tilde{w}_1(x_0, 0) - w_2(\tilde{x}_0, 0) = (q(x_0)\partial_{x_1}\tilde{w}_1 - \partial_{x_1}w_2 - M\partial_{x_0}\tilde{w}_1)(x_0, 0) - r^*(x_0) = 0 \quad \text{on } [0, T],$$

Hence instead of proving the observability estimate to system (2.1) - (2.4) it suffices to prove the observability estimate for the following system:

$$(2.13) \quad P(x, D)u = \rho\partial_{x_0}u - \partial_{x_1}^2u + b(x)\partial_{x_1}u + c(x)u = f \quad \text{in } Q \setminus [0, T] \times \{0\},$$

$$(2.14) \quad u(\cdot, a) = u(\cdot, b) = 0 \quad \text{on } [0, T],$$

$$(2.15) \quad [u](x_0, \cdot) = -\partial_{x_1}^-u(x_0, 0) + \mu(x_0)\partial_{x_1}^+u(x_0, 0) - M\partial_{x_0}u(x_0, 0) - r = 0 \quad \text{on } [0, T],$$

where

$$\begin{aligned} u(x) &= \begin{cases} \tilde{w}_1 & \text{for } x \in Q_+, \\ w_2 & \text{for } x \in Q_-, \end{cases} & b(x) &= \begin{cases} b_1^* & \text{for } x \in Q_+, \\ b_2^* & \text{for } x \in Q_-, \end{cases} \\ c(x) &= \begin{cases} c_1^* & \text{for } x \in Q_+, \\ c_2^* & \text{for } x \in Q_-, \end{cases} & f(x) &= \begin{cases} f_1^* & \text{for } x \in Q_+, \\ f_2^* & \text{for } x \in Q_-. \end{cases} \end{aligned}$$

Therefore the coefficients of equation (2.13) have the following regularity:

$$(2.16) \quad \rho \in C^1(\bar{Q}_+) \cap C^1(\bar{Q}_-) \cap C^0(\bar{Q}), \quad \rho(x) > \beta > 0 \quad \text{on } Q,$$

$$(2.17) \quad \mu \in C^1[0, T], \quad \mu(x_0) > \beta > 0 \quad \text{and} \quad b \in L^\infty(Q), \quad c \in L^\infty(0, T; L^2(\Omega)).$$

We set

$$(2.18) \quad \tilde{\varphi}(x_0) = \begin{cases} \frac{1}{x_0^3} & \text{for } x_0 \in [0, \frac{T}{4}] \\ \frac{1}{(T-x_0)^3} & \text{for } x_0 \in [\frac{3T}{4}, T] \end{cases}, \quad \varphi_*(x) = \begin{cases} \varphi_2 & \text{on } Q_+, \\ \varphi_1 & \text{on } Q_-, \end{cases}$$

where

$$(2.19) \quad \varphi_j(x) = \frac{e^{\lambda\psi_j(x_1)} - e^{10000\lambda c_0}}{x_0^3(T-x_0)^3}, \quad c_0 = \max\{b, -a\}, \quad j \in \{1, 2\},$$

where λ is a large positive parameter, $\tilde{\varphi} \in C^2[\frac{T}{8}, \frac{8T}{9}]$, and strictly positive on $[\frac{T}{8}, \frac{8T}{9}]$ and

$$(2.20) \quad \psi_1(x_1) = (x_1 + 10 + c_0)^2 \quad \text{and} \quad \psi_2(x_1) = \psi_1(0)e^{(x_1+10+c_0)^2 - (10+c_0)^2}.$$

By (2.18) - (2.20) the following is true:

$$(2.21)$$

$$\varphi_2(x) > \varphi_1(x) \quad \text{on } Q_+, \quad \varphi_1(x) > \varphi_2(x) \quad \text{on } Q_-, \quad \varphi_1(x_0, 0) = \varphi_2(x_0, 0) \quad \forall x_0 \in [0, T].$$

We introduce the Hilbert space

$$\|f\|_Y = \sqrt{\|f\|_{L^2(Q_-)}^2 + \|s\tilde{\varphi}f\|_{L^2(Q_+)}^2},$$

the operator

$$\mathbf{P}(x, D)u = (\rho\partial_{x_0}u - \partial_{x_1}^2u, -\partial_{x_1}^-u(\cdot, 0) + \mu\partial_{x_1}^+u(\cdot, 0) - M\partial_{x_0}u(\cdot, 0)) : \mathcal{X} \rightarrow Y \times L^2[0, T]$$

and the Banach space

$$\mathcal{X} = \{u | u \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-), \mathbf{P}(x, D)u \in (L^2(Q_+) \cap L^2(Q_-)) \times L^2(0, T)\},$$

$$[u](\cdot, 0) = 0, \quad u(\cdot, 0) \in H_0^1(0, T), \quad u(\cdot, a) = u(\cdot, b) = 0. \}$$

Denote $\mathcal{B}v = (\partial_{x_1}^+ v, \partial_{x_1}^- v, v)(\cdot, 0)$, and $\mathcal{Z}(0, T) = L_{(s\tilde{\varphi})^3}^2(0, T) \times L_{s\tilde{\varphi}}^2(0, T) \times H^{1, \tilde{s}}(0, T) \cap L_{(s\tilde{\varphi})^5}^2(0, T)$. We have

Proposition 2.2. *Let $u \in \mathcal{X}$ and coefficients ρ, μ, b, c satisfy (2.16), (2.17) and parameter λ fixed sufficiently large. There exists $s_0 > 1$ and positive constant C_2 such that for all $s \geq s_0$ the following estimate holds true*

$$\begin{aligned} \sum_{|\alpha| \leq 1} \| (s\tilde{\varphi})^{\frac{(3-2|\alpha|)}{2}} \partial^\alpha u e^{s\varphi_*} \|_{L^2(Q_-)} + \sum_{|\alpha| \leq 1} \| (s\tilde{\varphi})^{\frac{5-2|\alpha|}{2}} \partial^\alpha u e^{s\varphi_*} \|_{L^2(Q_+)} + \|\mathcal{B}(ue^{s\varphi_*})\|_{\mathcal{Z}(0, T)} \\ (2.22) \quad \leq C_2 (\| (fe^{s\varphi_*}, re^{s\varphi_*}) \|_{Y \times L_{s\tilde{\varphi}}^2(0, T)} + \| (s\tilde{\varphi})^{\frac{3}{2}} \partial_{x_1} ue^{s\varphi_*}(\cdot, b) \|_{L^2(0, T)}), \end{aligned}$$

where C_2 is independent of s .

Proof. Without loss of generality using the standard arguments (see e.g. [9]) we can prove an estimate (2.22) under assumption $b = c \equiv 0$. First, by an argument based on the partition of unity (e.g., Lemma 8.3.1 in [9]), it suffices to prove the inequality (2.22) locally, by assuming that

$$(2.23) \quad \text{supp } u \subset B(x^*, \delta),$$

where $B(x^*, \delta)$ is the ball in \mathbb{R}^2 of the radius $\delta > 0$ centered at some point x^* .

Let $\tilde{\theta} \in C_0^\infty(\frac{1}{2}, 2)$ be a nonnegative function such that

$$(2.24) \quad \sum_{\ell=-\infty}^{\infty} \tilde{\theta}(2^{-\ell}t) = 1 \quad \text{for all } t \in \mathbb{R}^1.$$

(For existence of such a function $\tilde{\theta}$ see e.g. [15].)

Set $u_\ell(x) = u(x)\kappa_\ell(x_0)$ where

$$(2.25) \quad \kappa_\ell(x_0) = \tilde{\theta}\left(2^{-\ell}2^{\frac{1}{\theta(x_0)^{\frac{1}{4}}}}\right),$$

where

$$(2.26) \quad \begin{aligned} \theta \in C^\infty[0, T], \quad \theta|_{[0, T/4]} = x_0, \quad \theta|_{[3T/4, T]} = T - x_0, \\ \partial_{x_0}\theta(x_0) < 0 \text{ on } (0, \frac{T}{2}), \quad \partial_{x_0}\theta(x_0) > 0 \text{ on } (\frac{T}{2}, T), \quad \partial_{x_0}^2\theta(\frac{T}{2}) < 0. \end{aligned}$$

Observe that it suffices to prove the Carleman estimate (2.22) for the function u_ℓ instead of u provided that the constant C_1 and the function s_0 are independent of ℓ . Observe that if $G \subset \mathbb{R}^m$ is a bounded domain and $q \in L^2(G)$, then there exist an independent constants C_3 and C_4 (see e.g. [15]) such that

$$(2.27) \quad C_3 \sum_{\ell=-\infty}^{\infty} \|\kappa_\ell q\|_{L^2(G)}^2 \leq \|q\|_{L^2(G)}^2 \leq C_4 \sum_{\ell=-\infty}^{\infty} \|\kappa_\ell q\|_{L^2(G)}^2.$$

Denote the norm on the left-hand side of (2.22) as $\|\cdot\|_*$. Suppose that the estimate (2.22) is true for any function u_ℓ with constants C_1 and s_0 independent of ℓ . By (2.27) for some constant C_5 independent of s we have

$$(2.28) \quad \|ue^{s\varphi_*}\|_* = \left\| \sum_{\ell=-\infty}^{+\infty} u_\ell e^{s\varphi_*} \right\|_* \leq \sum_{\ell=-\infty}^{+\infty} \|u_\ell e^{s\varphi_*}\|_* \leq C_5 \sum_{\ell=-\infty}^{\infty} (\|\kappa_\ell \mathbf{P}(x, D) ue^{s\varphi_*}\|_{Y \times L^2_{s\tilde{\varphi}}(0, T)}^2 + \|e^{s\varphi_*} [\kappa_\ell, \mathbf{P}(x, D)] u\|_{Y \times L^2_{s\tilde{\varphi}}(0, T)}^2 + \|(s\tilde{\varphi})^{\frac{3}{2}} \kappa_\ell \partial_{x_1} ue^{s\varphi_*}(\cdot, b)\|_{L^2(0, T)}^2)^{\frac{1}{2}}.$$

By (2.27) we obtain from (2.28):

$$(2.29) \quad \|ue^{s\varphi_*}\|_* \leq C_6 (\|\mathbf{P}(x, D) ue^{s\varphi_*}\|_{Y \times L^2_{s\tilde{\varphi}}(0, T)}^2 + \sum_{\ell=-\infty}^{\infty} \|e^{s\varphi_*} [\kappa_\ell, \mathbf{P}(x, D)] u\|_{Y \times L^2_{s\tilde{\varphi}}(0, T)}^2 + \|(s\tilde{\varphi})^{\frac{3}{2}} \partial_{x_1} ue^{s\varphi_*}(\cdot, b)\|_{L^2(0, T)}^2)^{\frac{1}{2}}.$$

Using (2.25) and (2.26) we estimate the norm of the commutator $[\kappa_\ell, \mathbf{P}(x, D)]$ we obtain

$$(2.30) \quad \begin{aligned} \sum_{\ell=-\infty}^{\infty} \|e^{s\varphi_*} [\kappa_\ell, \mathbf{P}(x, D)] u\|_{Y \times L^2_{s\tilde{\varphi}}(0, T)}^2 &\leq C_7 \sum_{\ell=-\infty}^{\infty} (\|\partial_{x_0} \kappa_\ell u(\cdot, 0) e^{s\varphi_*}\|_{L^2_{s\tilde{\varphi}}(0, T)}^2 + \|\partial_{x_0} \kappa_\ell u e^{s\varphi_*}\|_Y^2) \\ &\leq C_8 \sum_{\ell=-\infty}^{\infty} (\|\tilde{\varphi}^{\frac{5}{12}} \chi_{\text{supp } \kappa_\ell} u e^{s\varphi_*}\|_Y^2 + \|\tilde{\varphi}^{\frac{5}{12}} \chi_{\text{supp } \kappa_\ell} u(\cdot, 0) e^{s\varphi_*}\|_{L^2_{s\tilde{\varphi}}(0, T)}^2) \\ &\leq C_9 (\|\tilde{\varphi}^{\frac{5}{12}} u e^{s\varphi_*}\|_Y^2 + \|\tilde{\varphi}^{\frac{5}{12}} u(\cdot, 0) e^{s\varphi_*}\|_{L^2_{s\tilde{\varphi}}(0, T)}^2). \end{aligned}$$

From (2.29), and (2.30) we obtain (2.22).

Now, without loss of generality we assume that

$$(2.31) \quad \text{supp } u \subset B(x^*, \delta/2) \cap \text{supp } \kappa_\ell,$$

where $B(x^*, \delta)$ is the ball of the radius $\delta > 0$ centered at some point x^* . If x^* does not belong to the set $[0, T] \times \{0\}$ the estimate (2.22) is proved in [10]. More specifically if $\text{supp } u \cap ([0, T] \times \{0\}) = \emptyset$ there exists a constant $C_{10} = C_{10}(\delta, x^*)$ and $s_0 = s_0(\delta, x^*)$ such that

$$(2.32) \quad \begin{aligned} \sum_{|\alpha| \leq 1} \|(s\tilde{\varphi})^{\frac{(3-2|\alpha|)}{2}} \partial^\alpha u e^{s\varphi_*}\|_{L^2(Q_-)} + \sum_{|\alpha| \leq 1} \|(s\tilde{\varphi})^{\frac{5-|\alpha|}{2}} \partial^\alpha u e^{s\varphi_*}\|_{L^2(Q_+)} \\ \leq C_{10} (\|(P(x, D) u) e^{s\varphi_*}\|_Y + \|(s\tilde{\varphi})^{\frac{3}{2}} \partial_{x_1} u e^{s\varphi_*}(\cdot, b)\|_{L^2(0, T)}), \end{aligned}$$

where C_{10} is independent of s . Therefore we have to consider the case

$$(2.33) \quad x^* = (x_0^*, 0), \quad \text{supp } u \subset B(x^*, \delta/2) \cap \text{supp } \kappa_\ell, \quad B(x^*, \delta) \cap ([0, T] \times \{a, b\}) = \emptyset.$$

For any function $\varphi \in \{\varphi_1, \varphi_2\}$ we introduce the operator

$$P_\varphi(x, D, \tilde{s}) = i\rho(x)D_0 + (D_1 + |\tilde{s}|i\phi_0(x, x^*))^2, \quad \phi_0(x, x^*) = \partial_{x_1} \varphi(x)/\tilde{\varphi}(x^*), \quad \tilde{s} = s\tilde{\varphi}(x^*).$$

For any $\xi_0 \in \mathbb{R}^1 \setminus \{0\}$ and $x \in Q$ we choose $\sqrt{i\rho(x)\xi_0}$ such that

$$(2.34) \quad \text{Im} \sqrt{i\rho(x)\xi_0} > 0.$$

By (2.16) such choice is possible. We define symbol $p_\varphi(x, \xi, \tilde{s})$ by formula

$$(2.35) \quad p_\varphi(x, \xi, \tilde{s}) = i\rho(x)\xi_0 + (\xi_1 + i|\tilde{s}|\phi_0(x, x^*))^2,$$

The zeros of the polynomial $p_\varphi(x, \xi, \tilde{s})$ with respect to variable ξ_1 for $M(\xi_0, \tilde{s}) \geq 1$, and $x \in B(x^*, \delta) \cap \text{supp } \kappa_\ell$ are

$$(2.36) \quad \Gamma_\varphi^\pm(x, \xi_0, \tilde{s}) = (-i|\tilde{s}|\tilde{\mu}_\ell\varphi_0\kappa(\xi_0, \tilde{s}) + \alpha^\pm(x, \xi_0, \tilde{s})),$$

where

$$(2.37) \quad \tilde{\mu}_\ell(x) = \eta_*(x) \sum_{k=\ell-20}^{\ell+20} \kappa_\ell(x_0), \quad \eta_* \in C_0^\infty(B(x^*, 2\delta)), \quad \eta_*|_{B(x^*, \delta)} = 1,$$

the function κ_ℓ is given by (2.25),

$$(2.38) \quad \alpha^\pm(x, \xi_0, \tilde{s}) = \pm\tilde{\mu}_\ell(x)\kappa(\nu, \xi_0, \tilde{s})\sqrt{i\rho\xi_0}.$$

Next we construct the function $\kappa(\xi_0, \tilde{s}) = \kappa(\nu, \xi_0, \tilde{s})$. Let χ_ν be a $C_0^\infty(\mathbb{M})$ function such that χ_ν is identically equal 1 in some conic neighborhood of the $(\xi_0^*, \tilde{s}^*) \in \mathbb{M}$ and $\text{supp } \chi_\nu(\xi_0, \tilde{s}) \subset \mathcal{O}(\zeta^*, \delta_1)$. Assume that

$$(2.39) \quad \kappa(\nu, \xi_0, \tilde{s})|_{\text{supp } \chi_\nu} = 1, \quad \text{supp } \kappa(\nu, \xi_0, \tilde{s}) \subset \mathcal{O}(\zeta^*, 2\delta_1), \quad 1 \geq \kappa(\nu, \xi_0, \tilde{s}) \geq 0 \quad \text{on } \mathbb{M}.$$

We extend the function χ_ν on \mathbb{R}^2 as follows : $\chi_\nu(\xi_0/M^2(\xi_0, \tilde{s}), \tilde{s}/M(\xi_0, \tilde{s}))$ for $M(\xi_0, \tilde{s}) > 1$ and $\chi_\nu(\xi_0/M^2(\xi_0, \tilde{s}), \tilde{s}/M(\xi_0, \tilde{s}))\kappa^*(M(\xi_0, \tilde{s}))$ for $M(\xi_0, \tilde{s}) < 1$, where $\kappa^*(t) \in C^\infty(\mathbb{R}^1)$, $\kappa^*(t) \geq 0$, $\kappa^*(t) = 1$ for $t \geq 1$ and $\kappa^*(t) = 0$ for $t \in [0, 1/2]$. In the similar way we extend the function $\kappa(\nu, \xi_0, \tilde{s})$ on \mathbb{R}^2 . Denote by $\tilde{\chi}_\nu(x, D_0, \tilde{s})$ the pseudodifferential operator with the symbol $\eta_\ell(x)\chi_\nu(\xi_0, \tilde{s})$ and

$$(2.40) \quad \eta_\ell(x) = \eta_{**}(x) \sum_{k=-10}^{10} \kappa_{\ell+k}(x_0),$$

where $\eta_{**} \in C_0^\infty(B(x^*, \delta))$, $\eta_{**}|_{B(x^*, \frac{3}{4}\delta)} = 1$. We set

$$v_{\nu, \varphi} = \tilde{\chi}_\nu(x, D_0, \tilde{s})v_\varphi \quad \text{and} \quad v_\varphi = ue^{s\varphi},$$

$$r_{\nu, \varphi} = \tilde{\chi}_\nu(x, D_0, \tilde{s})r_\varphi \quad \text{and} \quad r_\varphi = re^{s\varphi}.$$

Let \mathcal{O} be a domain in \mathbb{R}^1 .

Definition. We say that the symbol $a(x_0, \xi_0, \tilde{s}) \in C^{\tilde{k}}(\overline{\mathcal{O}} \times \mathbb{R}^2)$ belongs to the class $C_{cl}^{\tilde{k}} S^{\kappa, \tilde{s}}(\mathcal{O})$ if

A) There exists a compact set $K \subset \subset \mathcal{O}$ such that $a(x_0, \xi_0, \tilde{s})|_{\mathcal{O} \setminus K} = 0$;

B) For any $\beta = (\beta_0, \beta_1)$ there exists a constant C_β

$$\left\| \partial_{\xi_0}^{\beta_0} \partial_{\tilde{s}}^{\beta_1} a(\cdot, \xi_0, \tilde{s}) \right\|_{C^{\tilde{k}}(\overline{\mathcal{O}})} \leq C_\beta (\tilde{s}^2 + |\xi_0|)^{\frac{\kappa - |\beta|}{2}},$$

where $|\beta| = 2\beta_0 + \beta_1$ and $M(\xi_0, \tilde{s}) \geq 1$;

C) For any $N \in \mathbb{N}$ the symbol $a(x_0, \xi_0, \tilde{s})$ can be represented as

$$a(x_0, \xi_0, \tilde{s}) = \sum_{j=1}^N a_j(x_0, \xi_0, \tilde{s}) + R_N(x_0, \xi_0, \tilde{s})$$

where the functions a_j have the following properties: for any $\tilde{\lambda} > 1$ and for all $(x_0, \xi_0, \tilde{s}) \in \{(x_0, \xi_0, \tilde{s}) | x_0 \in K, M(\xi_0, \tilde{s}) > 1\}$

$$a_j(x_0, \tilde{\lambda}^2 \xi_0, \tilde{\lambda} \tilde{s}) = \tilde{\lambda}^{\kappa-j} a_j(x_0, \xi_0, \tilde{s});$$

for any multiindex β and any (ξ_0, \tilde{s}) satisfying $M(\xi_0, \tilde{s}) \geq 1$ there exist a constant C_β such that

$$\left\| \partial_{\xi_0}^{\beta_0} \partial_{\tilde{s}}^{\beta_1} a_j(\cdot, \xi_0, \tilde{s}) \right\|_{C^{\tilde{k}}(\bar{\mathcal{O}})} \leq C_\beta (\tilde{s}^2 + |\xi_0|)^{\frac{\kappa-j-|\beta|}{2}}$$

where the term R_N satisfies the estimate

$$\|R_N(\cdot, \xi_0, \tilde{s})\|_{C^{\tilde{k}}(\bar{\mathcal{O}})} \leq C_N (\tilde{s}^2 + |\xi_0|)^{\frac{\kappa-N}{2}} \quad \forall (\xi_0, \tilde{s}) \text{ satisfying } M(\xi_0, \tilde{s}) \geq 1.$$

For the symbol a , we introduce the semi-norm

$$\begin{aligned} \pi_{C^{\tilde{k}}(\mathcal{O})}(a) &= \sum_{j=1}^{\hat{N}} \sup_{|\beta| \leq \hat{N}} \sup_{|(\xi_0, \tilde{s})| \geq 1} \left\| \frac{\partial^{\beta_0}}{\partial \xi_0^{\beta_0}} \frac{\partial^{\beta_1}}{\partial \tilde{s}^{\beta_1}} a_j(\cdot, \xi_0, \tilde{s}) \right\|_{C^{\tilde{k}}(\bar{\mathcal{O}})} / (1 + |(\xi_0, \tilde{s})|)^{\kappa-j-|\beta|} \\ &\quad + \sup_{|(\xi_0, \tilde{s})| \leq 1} \|a(\cdot, \xi_0, \tilde{s})\|_{C^{\tilde{k}}(\bar{\mathcal{O}})}. \end{aligned}$$

Obviously for any $\tilde{k} \in \{0, 1\}$

$$(2.41) \quad \pi_{C^{\tilde{k}}(B(0, \delta(x^*)))}(\chi_\nu) \leq C_{11} \tilde{\varphi}^{\frac{5\tilde{k}}{12}}(x^*).$$

Obviously the pseudodifferential operators with the symbols Γ_φ^\pm belongs to the class $C_{cl}^{\tilde{k}} S^{1,s}(B(0, \delta(x^*)))$ for any $\tilde{k} \in \{0, 1\}$ and

$$(2.42) \quad \pi_{C^{\tilde{k}}(B(0, \delta(x^*)))}(\Gamma_\varphi^\pm) \leq C_{12} \tilde{\varphi}^{\frac{5\tilde{k}}{12}}(x^*).$$

By (2.42) and Lemma 8.1 of [11]

$$(2.43) \quad \|\Gamma_\varphi^\pm(\cdot, 0, D_0, \tilde{s}) v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0, T)} \leq C_{13} \|v_{\nu, \varphi}\|_{H^{\frac{1}{2}, \tilde{s}}(0, T)}.$$

In some cases, we can represent the operator $P_\varphi(x, D, \tilde{s})$ as a product of two first order pseudodifferential operators.

Proposition 2.3. (see e.g. [11]) Let $v \in \mathcal{X}$, $\text{supp } v \subset B(x^*, \delta) \cap \text{supp } \kappa_\ell$, $x^* \in \text{supp } \kappa_\ell$ and $P_\varphi(x, D, \tilde{s}) \chi_\nu v \in L^2(Q_+) \cap L^2(Q_-)$. Assume that $\xi_0^* \neq 0$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$. Then we

can factorize the operator $P_\varphi(x, D, \tilde{s})$ into the product of two first order pseudodifferential operators:

$$(2.44) \quad P_\varphi(x, D, \tilde{s})v_{\nu, \varphi} = (D_1 - \Gamma_\varphi^-(x, D_0, \tilde{s}))(D_1 - \Gamma_\varphi^+(x, D_0, \tilde{s}))v_{\nu, \varphi} + T_{+, \varphi}v_{\nu, \varphi} = \\ (D_1 - \Gamma_\varphi^+(x, D_0, \tilde{s}))(D_1 - \Gamma_\varphi^-(x, D_0, \tilde{s}))v_{\nu, \varphi} + T_{-, \varphi}v_{\nu, \varphi},$$

Operators $T_{\pm, \varphi} : H^{\frac{1}{2}, 1, \tilde{s}}([0, b] \times \mathbb{R}^1) \rightarrow L^2(0, b; L^2(\mathbb{R}^1)) \cap H^{\frac{1}{2}, 1, \tilde{s}}([a, 0] \times \mathbb{R}^1) \rightarrow L^2(a, 0; L^2(\mathbb{R}^1))$ satisfy estimates

$$(2.45) \quad \|T_{\pm, \varphi}v_{\nu, \varphi}\|_{L^2(0, b; L^2(\mathbb{R}^1))} \leq C_{14}\tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\nu, \varphi}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_+)},$$

and

$$(2.46) \quad \|T_{\pm, \varphi}v_{\nu, \varphi}\|_{L^2(a, 0; L^2(\mathbb{R}^1))} \leq C_{15}\tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\nu, \varphi}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_-)}.$$

Denote by $V_{\nu, \varphi}^\pm = (D_1 - \Gamma_\varphi^\pm(x, D_0, \tilde{s}))v_{\nu, \varphi}$ the function with domain in \bar{Q}_+ and by $U_{\nu, \varphi}^\pm = (D_1 - \Gamma_\varphi^\pm(x, D_0, \tilde{s}))v_{\nu, \varphi}$ the function with domain in \bar{Q}_- .

Let us consider the initial value problems

$$(2.47) \quad (D_1 - \Gamma_\varphi^-(x, D_0, \tilde{s}))V_{\nu, \varphi}^+ = -T_{+, \varphi}v_{\nu, \varphi} + P_\varphi(x, D, \tilde{s})v_{\nu, \varphi} \quad x \in [0, b] \times \mathbb{R}^1.$$

and

$$(2.48) \quad (D_1 - \Gamma_\varphi^-(x, D_0, \tilde{s}))V_{\nu, \varphi}^+ = -T_{+, \varphi}v_{\nu, \varphi} + P_\varphi(x, D, \tilde{s})v_{\nu, \varphi} \quad x \in [0, b] \times \mathbb{R}^1.$$

For solutions of these problems, we can prove an a priori estimate.

Proposition 2.4. (see e.g. [11]) Let $\xi_0^* \neq 0$, $\text{supp } \chi_\nu \in \mathcal{O}(\zeta^*, \delta_1)$. There exists a constant $C_{16} > 0$ such that

$$(2.49) \quad \|V_{\nu, \varphi}^+(\cdot, 0)\|_{H^{\frac{1}{4}}(0, T) \cap L_s^2(0, T)} + \|V_{\nu, \varphi}^+\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_+)} \\ \leq C_{16}(\tilde{\varphi}^{\frac{5}{12}}(x^*)\|v\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_+)} + \|P_\varphi(x, D, \tilde{s})v_{\nu, \varphi}\|_{L^2(Q_+)}).$$

and

$$(2.50) \quad \|V_{\nu, \varphi}^+\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_-)} \leq C_{17}(\tilde{\varphi}^{\frac{5}{12}}(x^*)\|v\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_-)} \\ + \|P_\varphi(x, D, \tilde{s})v_{\nu, \varphi}\|_{L^2(Q_+)} + \|V_{\nu, \varphi}^+(\cdot, 0)\|_{H^{\frac{1}{4}, \tilde{s}}[0, T] \cap L_s^2(0, T)}).$$

Consider the initial value problem:

$$(2.51) \quad (D_1 - r(x, D_0, \tilde{s}))W = p \quad x \in [-\delta, 0] \times \mathbb{R}^1, \quad W|_{x_1=-\delta} = 0.$$

We have

Proposition 2.5. (see e.g. [11]) Let $\xi_0^* \neq 0$, $\text{supp } \chi_\nu \in \mathcal{O}(\zeta^*, \delta_1)$, $W \in H^{\frac{1}{2}, 1}(\mathbb{R}^1 \times [-\delta, 0])$, $p \in L^2(\mathbb{R}^1 \times [-\delta, 0])$. Let for each $x_1 \in [-\delta, 0]$ symbol $r(x_0, x_1, \xi_0, \tilde{s}) \in C_{cl}^1 S^{1, \tilde{s}}((0, T))$ for all $x_1 \in (-\delta, 0)$ and there exist a constant C_{18} such that

$$r(x, \xi_0, \tilde{s}) \geq C_{18} M(\xi_0, \tilde{s}) \quad (x, \xi_0, \tilde{s}) \in (B(x^*, \frac{3}{4}\delta) \cap \text{supp } \sum_{k=-8}^8 \kappa_{k+\ell}) \times \mathbb{R}^2$$

and

$$\pi_{C^1(0, T)}(r(\cdot, x_1, \cdot, \cdot)) \leq C_{19} \tilde{\varphi}^{\frac{5}{12}}(x^*) \quad \forall x_1 \in [-\delta, 0].$$

Then here exists a constant $C_{20} > 0$ such that

$$\|W(\cdot, 0)\|_{H^{\frac{1}{4}}[0, T] \cap L^2_{|\tilde{s}|}(0, T)} + \|W\|_{H^{\frac{1}{2}, 1, \tilde{s}}((- \delta, 0) \times \mathbb{R}^1)} \leq C_{20} \|p\|_{L^2((- \delta, 0) \times \mathbb{R}^1)}.$$

Now we obtain couple subelliptic estimates for the operator $P_\varphi(x, D, \tilde{s})$ on domains Q_\pm .

Proposition 2.6. (see e.g. [11]) Let parameter λ be large enough and fixed, $w \in \mathcal{X}$, $\text{supp } w \subset B(x^*, \delta) \cap \text{supp } \kappa_\ell$ and $P_\varphi(x, D, \tilde{s}) \chi_\nu w \in L^2(Q_\pm)$. Then there exist positive constants $\delta(x^*), C_{21}, C_{22}$ independent of \tilde{s} such that for all and $\tilde{s} \geq s_0$ we have

$$(2.52) \quad \begin{aligned} & C_{21} \int_{Q_+} (|\tilde{s}| |\partial_{x_1} \tilde{\chi}_\nu w|^2 + |\tilde{s}|^3 |\tilde{\chi}_\nu w|^2) dx - \text{Re} \int_{\mathbb{R}^1} \partial_{x_1}^+ \chi_\nu w \overline{\rho \partial_{x_0} \tilde{\chi}_\nu w}|_{x_1=0} dx_0 \\ & + \int_{\mathbb{R}^1} (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ \tilde{\chi}_\nu w|^2 + |\tilde{s}|^3 \varphi_0^3(x^*, x^*) |\tilde{\chi}_\nu w|^2)|_{x_1=0} dx_0 \\ & \leq \|P_\varphi(x, D, \tilde{s}) \tilde{\chi}_\nu w\|_{L^2(Q_+)}^2 + \epsilon_1(\delta) \|(\tilde{s}^{\frac{1}{2}} \partial_{x_1}^+ (\tilde{\chi}_\nu w), \tilde{s}^{\frac{3}{2}} \tilde{\chi}_\nu w)(\cdot, 0)\|_{L^2(\mathbb{R}^1) \times L^2(\mathbb{R}^1)}^2, \end{aligned}$$

where $\epsilon_1(\delta) \rightarrow +0$ as $\delta \rightarrow +0$ and

$$(2.53) \quad \begin{aligned} & C_{21} \int_{Q_-} (|\tilde{s}| |\partial_{x_1} \tilde{\chi}_\nu w|^2 + |\tilde{s}|^3 |\tilde{\chi}_\nu w|^2) dx \\ & \leq \|P_\varphi(x, D, \tilde{s}) \tilde{\chi}_\nu w\|_{L^2(Q_-)}^2 + C_{22} \left(\left| \int_{\mathbb{R}^1} \partial_{x_1}^- \tilde{\chi}_\nu w \overline{\rho \partial_{x_0} \tilde{\chi}_\nu w}|_{x_1=0} dx_0 \right| \right. \\ & \quad \left. + \int_{\mathbb{R}^1} (|\tilde{s}| \varphi_0 |\partial_{x_1}^- (\tilde{\chi}_\nu w)|^2 + |\tilde{s}|^3 \varphi_0^3 |\tilde{\chi}_\nu w|^2)|_{x_1=0} dx_0 \right). \end{aligned}$$

We use the following proposition proved in ([11]):

Proposition 2.7. Let $-\infty < \alpha < \tilde{a} < \tilde{b} < \beta < +\infty$, $p \in \mathbb{N}_+$ and $\text{supp } \mathbf{v} \subset [\tilde{a}, \tilde{b}]$. Then there exists an independent constant C_{23} such that

$$(2.54) \quad \|M^p(D_0, \tilde{s}) \mathbf{v}\|_{L^2([-R, R] \setminus [\alpha, \beta])} \leq \frac{C_{23}}{(\min\{\tilde{a} - \alpha, \beta - \tilde{b}\})^p} \|\mathbf{v}\|_{L^2(\mathbb{R}^1)}.$$

We apply the Proposition 2.7 in order to estimate the $H^{\frac{1}{2}, \tilde{s}} \cap L^2_{\tilde{s}^2}$ norm of the function $(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi(\cdot, 0)$. Observe that by (2.26), (2.25) for all sufficiently large ℓ

$$(2.55) \quad \text{supp } v_\varphi(\cdot, 0) \subset [(\ell + 2)^{-4}, (\ell - 2)^{-4}] \cup [T - (\ell - 2)^{-4}, T - (\ell + 2)^{-4}]$$

and

$$(2.56) \quad \text{supp } (1 - \eta_\ell(\cdot, 0)) \subset [0, T] \setminus [(\ell + 11)^{-4}, (\ell - 11)^{-4}] \cup [T - (\ell - 11)^{-4}, T - (\ell + 11)^{-4}].$$

Therefore, by (2.55) and (2.56) for all sufficiently large ℓ

$$(2.57) \quad \begin{aligned} & \| (1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi(\cdot, 0) \|_{H^{\frac{1}{2}, \tilde{s}}(-R, R) \cap L^2_{\tilde{s}^2}(-R, R)} \\ & \leq C_{24}\ell^5 \|v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)} \leq C_{25}\tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)}. \end{aligned}$$

Here in the last inequality we used the fact that $\ell^5 \leq C_{26}\theta^{\frac{5}{4}}(x^*) \leq C_{27}\varphi^{\frac{5}{12}}(x^*)$. By arguments, same as in Lemma 8.5 of [11] we obtain

$$(2.58) \quad |\tilde{s}| \| (1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi(\cdot, 0) \|_{H^{\frac{1}{2}, \tilde{s}}(\mathbb{R}^1 \setminus [-R, R]) \cap L^2_{\tilde{s}^2}(\mathbb{R}^1 \setminus [-R, R])} \leq C_{28} \|v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)}.$$

By (2.57) and (2.58)

$$(2.59) \quad \| (1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi(\cdot, 0) \|_{H^{\frac{1}{2}, \tilde{s}}(\mathbb{R}^1) \cap L^2_{\tilde{s}^2}(\mathbb{R}^1)} \leq C_{29}\tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)}.$$

We introduce three sets

$$\mathcal{Z}_{\varphi, \pm}(\ell) = \{(x, \xi_0, \tilde{s}) \in \text{supp } \kappa_\ell \times \{0\} \times \mathbb{M} \mid \pm(|\tilde{s}| \phi_0(x_0, 0, x^*) - \text{Im} \sqrt{i\rho(x_0, 0)\xi_0}) > 0\}$$

and

$$\mathcal{Z}_{\varphi, 0}(\ell) = \{(x, \xi_0, \tilde{s}) \in \text{supp } \kappa_\ell \times \{0\} \times \mathbb{M} \mid |\tilde{s}| \phi_0(x_0, 0, x^*) = \text{Im} \sqrt{i\rho(x_0, 0)\xi_0} = 0\}.$$

If $\zeta^* \in \mathcal{Z}_{\varphi, 0}(\ell)$ or $\zeta^* \in \mathcal{Z}_{\varphi, +}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$ we have

$$(2.60) \quad \begin{aligned} & \|\partial_{x_0} v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0, T)} \leq \|\partial_{x_0} \eta_\ell v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0, T)} + \|\xi_0 \chi_\nu F v(\cdot, 0)\|_{L^2(\mathbb{R}^1)} \\ & \leq C_{30}(\tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)} + |\tilde{s}|^2 \|\chi_\nu F v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)}) \\ & \leq C_{31}(\tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi\|_{L^2(0, T)} + |\tilde{s}|^2 \|v_{\nu, \varphi}(\cdot, 0)\|_{L^2(\mathbb{R}^1)} + |\tilde{s}|^2 \|(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)}) \\ & \leq C_{32}(|\tilde{s}| \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)} + |\tilde{s}|^2 \|v_{\nu, \varphi}(\cdot, 0)\|_{L^2(\mathbb{R}^1)}). \end{aligned}$$

Here to get the last inequality we used (2.59).

We will use the following proposition:

Proposition 2.8. ([11]) Let $w \in L^2(Q)$ and $\text{supp } w \subset \text{supp } \tilde{\mu}_\ell(x)$, where function $\tilde{\mu}_\ell$ defined by (2.37). Then there exists a constant C_{33} such that

$$(2.61) \quad \|(1 - \eta_\ell)\chi_\nu(D, \tilde{s})w\|_{H^{\frac{1}{2}, 1, \tilde{s}}(\mathbb{R}^2)} \leq C_{33}\tilde{\varphi}^{\frac{5}{12}}(x^*) \|w\|_{L^2(\mathbb{R}^2)}.$$

If $\zeta^* \in \mathcal{Z}_{\varphi,0}(\ell)$ or $\zeta^* \in \mathcal{Z}_{\varphi,+}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$ using inequality (2.61) we have

$$\begin{aligned}
\|v_{\nu,\varphi}\|_{H^{\frac{1}{2},0}(Q_\pm)} &\leq \|\chi_\nu(D_0, \tilde{s})v_\varphi\|_{H^{\frac{1}{2},0}(Q_\pm)} + \|(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi\|_{H^{\frac{1}{2},0}(Q_\pm)} = \\
&\leq C_{34}(\|\sqrt{|\xi_0|}\chi_\nu(\xi_0, \tilde{s})Fv_\varphi\|_{L^2(\mathbb{R}^1 \times I_\pm)} + \|(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_{\nu,\varphi}\|_{H^{\frac{1}{2},0}(Q_\pm)}) \\
&\leq C_{35}\|\tilde{s}\chi_\nu(\xi_0, \tilde{s})Fv_\varphi\|_{L^2(\mathbb{R}^1 \times I_\pm)} + \|(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi\|_{H^{\frac{1}{2},0}(Q_\pm)} \\
&\leq C_{36}(|\tilde{s}|\|v_{\nu,\varphi}\|_{L^2(Q_\pm)} + \|(1 - \eta_\ell)\chi_\nu(D_0, \tilde{s})v_\varphi\|_{H^{\frac{1}{2},0}(Q_\pm) \cap L_{\tilde{s}}^2(Q_\pm)}) \\
(2.62) \quad &\leq C_{37}(|\tilde{s}|\|v_{\nu,\varphi}\|_{L^2(Q_\pm)} + \tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_\varphi\|_{L^2(Q_\pm)}).
\end{aligned}$$

We consider three cases

Case 1. Let $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi,+}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1(\zeta^*))$. We observe that by (2.16) and (2.34)

$$\text{Im} \sqrt{i\rho(x_0^*, 0)\xi_0^*} = \frac{1}{\sqrt{2}}\sqrt{\rho(x_0^*, 0)}\sqrt{|\xi_0^*|}.$$

Therefore there exist positive δ_1^0 such that for any $\delta_1 \in (0, \delta_1^0)$ one can find positive $\delta_2 \in (0, \frac{1}{4})$ such that

$$(2.63) \quad \rho(x)|\xi_0| < (2 - \delta_2)\tilde{s}^2\phi_0^2(x, x^*) \quad \forall (x, \xi_0, \tilde{s}) \in B(x^*, \delta) \times \mathcal{O}(\zeta^*, \delta_1(\zeta^*)).$$

We start from the estimate of the following boundary integral

$$\begin{aligned}
|\text{Re} \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho \partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 | &\leq |\text{Re} \int_{\mathbb{R}^n} \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho(x^*) \partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 | \\
(2.64) \quad &+ |\text{Re} \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu,\varphi} \overline{(\rho(x^*) - \rho(x)) \partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 | = \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

We estimate integrals \mathcal{I}_j separately. Short computations, (2.63), (2.25), (2.40) and (2.59) imply

$$\begin{aligned}
\mathcal{I}_2 &= \sup_{x \in \text{supp } v_{\nu,\varphi}} |\rho(x^*) - \rho(x)| \int_{\mathbb{R}^n} |\partial_{x_1}^+ v_{\nu,\varphi} \overline{\partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 \\
&\leq \sup_{x \in \text{supp } v_{\nu,\varphi}} |\rho(x^*) - \rho(x)| \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} (\|\xi_0 \chi_\nu(\xi_0, \tilde{s})Fv_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)} + \|\partial_{x_0} \eta_\ell \chi_\nu v_\varphi\|_{L^2(\mathbb{R}^1)}) \\
&\leq C_{38} \sup_{x \in \text{supp } v_{\nu,\varphi}} |\rho(x^*) - \rho(x)| \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} (\tilde{s}^2 \|v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} \\
&\quad + \tilde{s}^2 \|(1 - \eta_\ell)\chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} + \|\partial_{x_0} \eta_\ell \chi_\nu v_\varphi\|_{L^2(\mathbb{R}^1)}) \\
&\leq C_{39} \sup_{x \in \text{supp } v_{\nu,\varphi}} |\rho(x^*) - \rho(x)| \left(\int_{\mathbb{R}^1} (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu,\varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu,\varphi}|^2) |_{x_1=0} dx_0 \right. \\
&\quad \left. + |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 \right).
\end{aligned}$$

Since function ρ is continuous function taking parameter δ in (2.31) sufficiently small we obtain

$$(2.65) \quad \mathcal{I}_2 = \frac{\delta_2}{6} \left(\int_{\mathbb{R}^1} (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu,\varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu,\varphi}|^2) |_{x_1=0} dx_0 + C_{40} |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 \right).$$

Next we estimate \mathcal{I}_1

$$\begin{aligned}\mathcal{I}_1 &\leq \left| \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho(x^*) \partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 \right| \leq \left| \int_{\mathbb{R}^1} |F \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho(x^*) \xi_0 \chi_\nu(\xi_0, \tilde{s}) F v_{\nu,\varphi}}|_{x_1=0} d\xi_0 \right| \\ &\quad + \left| \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho(x^*) \partial_{x_0} \eta_\ell \chi_\nu(D_0, \tilde{s}) v_\varphi}|_{x_1=0} dx_0 \right|.\end{aligned}$$

By (2.63)

$$\begin{aligned}\mathcal{I}_1 &\leq (2 - \delta_2) \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} |\tilde{s}| \phi_0(x^*, x^*) \|\chi_\nu(\xi_0, \tilde{s}) F v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)} \\ &\quad + \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)} \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} \\ &= (2 - \delta_2) \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} |\tilde{s}| \phi_0(x^*, x^*) \|v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} + \\ &(2 - \delta_2) \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} |\tilde{s}|^2 \phi_0^2(x^*, x^*) \|(1 - \eta_\ell) \chi_\nu(D_0, \tilde{s}) v_\varphi\|_{L^2(\mathbb{R}^1)} \\ &\quad + C_{41} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)} \|\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)\|_{L^2(0,T)} \\ &\leq (1 - \delta_2/5) \int_0^T (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu,\varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu,\varphi}|^2)(x_0, 0) dx_0 \\ &\quad + C_{42} |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2.\end{aligned}$$

So by (2.64)-(2.66)

$$\begin{aligned}(2.66) \quad &\left| \operatorname{Re} \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu,\varphi} \overline{\rho \partial_{x_0} v_{\nu,\varphi}}|_{x_1=0} dx_0 \right| \\ &\leq (1 - \delta_2/5) \int_0^T (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu,\varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu,\varphi}|^2)(x_0, 0) dx_0 \\ &\quad + C_{43} |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2.\end{aligned}$$

By (2.66) and inequality (2.52) of Proposition 2.6

$$\begin{aligned}(2.67) \quad &\int_0^T (|\tilde{s}| |\partial_{x_1}^+ v_{\nu,\varphi}|^2 + |\tilde{s}|^3 |v_{\nu,\varphi}|^2)(x_0, 0) dx_0 + |\tilde{s}| \|v_{\nu,\varphi}\|_{H^{0,1,\tilde{s}}(Q_+)}^2 \\ &\leq C_{44} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_{L^2(Q_+)}^2 + |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2).\end{aligned}$$

Equation

$$\mathcal{R}u = -\partial_{x_1}^- u(x_0, 0) + \mu \partial_{x_1}^+ u(x_0, 0) = M \partial_{x_0} u(x_0, 0) + r(x_0)$$

in terms of functions $v_{\nu,\varphi}$ and $r_{\nu,\varphi}$ one can write down on $[0, T]$ as

$$\begin{aligned}(2.68) \quad &\mathcal{R}v_{\nu,\varphi} + [\chi_\nu, \mathcal{R}]v_\varphi = M \partial_{x_0} v_{\nu,\varphi}(x_0, 0) \\ &- M |\tilde{s}| \partial_{x_0} \varphi(x_0, 0) v_{\nu,\varphi}(x_0, 0) - M |\tilde{s}| [\chi_\nu, \partial_{x_0} \varphi(x_0, 0)] v_\varphi + r_{\nu,\varphi}.\end{aligned}$$

By (2.68) and (2.67)

$$\begin{aligned}(2.69) \quad &\int_0^T (|\partial_{x_1}^- v_{\nu,\varphi}|^2 + |\tilde{s}|^3 |v_{\nu,\varphi}|^2)(x_0, 0) dx_0 \leq C_{45} (|\tilde{s}| \|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_{L^2(Q_+)}^2 \\ &+ |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + \|r_{\nu,\varphi}\|_{L^2(0,T)}^2).\end{aligned}$$

Applying Proposition 2.6 to function $v_{\nu,\varphi}$ on Q_- and using (2.69) to estimate the boundary integrals in (2.53) we have

$$(2.70) \quad \begin{aligned} |\tilde{s}| \|v_{\nu,\varphi}\|_{H^{0,1,\tilde{s}}(Q_-)}^2 &\leq C_{46} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_Y^2 + |\tilde{s}|^3 \varphi^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 \\ &\quad + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}| \|r_{\nu,\varphi}\|_{L^2(0,T)}^2). \end{aligned}$$

Therefore (2.67) and (2.70) imply

$$(2.71) \quad \begin{aligned} |\tilde{s}| \|v_{\nu,\varphi}\|_{H^{0,1,\tilde{s}}(Q_-)}^2 + |\tilde{s}|^3 \|v_{\nu,\varphi}\|_{H^{0,1,\tilde{s}}(Q_+)}^2 + \|\mathcal{B}v_{\nu,\varphi}\|_{\mathcal{Z}(0,T)}^2 &\leq C_{47} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_Y^2 \\ &\quad + |\tilde{s}|^3 \varphi^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}| \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}| \|r_{\nu,\varphi}\|_{L^2(0,T)}^2). \end{aligned}$$

By (2.71) and (2.62) we have

$$(2.72) \quad \begin{aligned} |\tilde{s}| \|v_{\nu,\varphi}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_-)}^2 + |\tilde{s}|^3 \|v_{\nu,\varphi}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_+)}^2 + \|\mathcal{B}v_{\nu,\varphi}\|_{\mathcal{Z}(0,T)}^2 &\leq C_{48} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_Y^2 \\ &\quad + |\tilde{s}|^3 \varphi^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}| \|r_{\nu,\varphi}\|_{L^2(0,T)}^2). \end{aligned}$$

Case 2. Let $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi,-}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$. Then $\xi_0^* \neq 0$ and decomposition (2.44) holds true. By Proposition 2.4 and (2.33) we have

$$(2.73) \quad \begin{aligned} \|V_{\nu,\varphi}^+(\cdot, 0)\|_{H^{\frac{1}{4}}(0,T) \cap L_{\tilde{s}}^2(0,T)} + \|V_{\nu,\varphi}^+\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_+)} \\ \leq C_{49} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_{L^2(Q_+)} + \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_+)}). \end{aligned}$$

By (2.36) for any $x \in B(x, \delta) \cap \text{supp } \eta_\ell$ and $(\xi, \tilde{s}) \in \text{supp } \chi_\nu$ we have $\text{Im } \Gamma_\varphi^+(x, \xi, \tilde{s}) = -|\tilde{s}| \phi_0(x, x^*) + \frac{1}{\sqrt{2}} \sqrt{\rho(x)|\xi_0|}$. Since $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi,-}(\ell)$ there exist a positive $\delta_3 = \delta_3(x^*, \zeta^*)$ such that

$$(2.74) \quad |\tilde{s}^*| \phi_0(x^*, x^*) - \frac{1}{\sqrt{2}} \sqrt{\rho(x^*) |\xi_0^*|} < -\delta_3 < 0.$$

Therefore there exists positive $\delta_4(\delta_3)$ such that

$$(2.75) \quad \text{Im } \Gamma_\varphi^\pm(x, \xi, \tilde{s}) > \delta_3/2 > 0 \quad (x, \xi, \tilde{s}) \in B(x, \delta_4) \cap \text{supp } \eta_\ell \times \text{supp } \kappa(\nu, \cdot).$$

By Proposition 2.5 and (2.75) we have

$$(2.76) \quad \|U_{\nu,\varphi}^-(\cdot, 0)\|_{H^{\frac{1}{4}}(0,T) \cap L_{\tilde{s}}^2(0,T)} \leq C_{50} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_{L^2(Q_-)} + \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_-)}).$$

The equality (2.68) we replace $\partial_{x_1}^+ v_{\nu,\varphi}(\cdot, 0)$ by $i(\Gamma_\varphi^-(x, D_0, \tilde{s}) v_{\nu,\varphi}(\cdot, 0) + V_{\nu,\varphi}^+(\cdot, 0))$ and we replace $\partial_{x_1}^- v_{\nu,\varphi}(\cdot, 0)$ by $i(\Gamma_\varphi^+(x, D_0, \tilde{s}) v_{\nu,\varphi}(\cdot, 0) + U_{\nu,\varphi}^-(\cdot, 0))$.

By (2.73) and (2.76) there exist a function q such that on $[0, T]$ we have

$$(2.77) \quad \begin{aligned} M \partial_{x_0} v_{\nu,\varphi}(x_0, 0) &= i(\Gamma_\varphi^-(x, D_0, \tilde{s}) v_{\nu,\varphi} - \mu \Gamma_\varphi^+(x, D_0, \tilde{s}) v_{\nu,\varphi})(x_0, 0) \\ &\quad + M |\tilde{s}| \partial_{x_0} \varphi(x_0, 0) v_{\nu,\varphi}(x_0, 0) + q, \end{aligned}$$

where

$$(2.78) \quad \begin{aligned} \sqrt{|\tilde{s}|} \|q\|_{L^2(0,T)} &\leq C_{51} (\|P_\varphi(x, D, \tilde{s}) v_{\nu,\varphi}\|_{L^2(Q)} + \sqrt{|\tilde{s}|} \|r_{\nu,\varphi}\|_{L^2(0,T)} \\ &\quad + |\tilde{s}|^{\frac{4}{3}} \|v_\varphi(\cdot, 0)\|_{L^2(0,T)} + \sqrt{|\tilde{s}|} \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)} + \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2},1,\tilde{s}}(Q)}). \end{aligned}$$

Inequalities (2.74), (2.59) and short computations imply

$$\begin{aligned}
& \|\partial_{x_0} v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0,T)} \geq \|\eta_\ell \partial_{x_0} \chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} - \|\partial_{x_0} \eta_\ell \chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} \geq \\
& \|\partial_{x_0} \chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} - \|(1 - \eta_\ell) \partial_{x_0} \chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} - \|\partial_{x_0} \eta_\ell \chi_\nu v_\varphi(\cdot, 0)\|_{L^2(0,T)} \geq \\
(2.79) \quad & C_{52} |\tilde{s}|^2 \|\chi_\nu v_\varphi(\cdot, 0)\|_{L^2(\mathbb{R}^1)} - C_{53} |\tilde{s}| \varphi^{\frac{5}{12}}(x^*) \|v_\varphi\|_{L^2(0,T)}.
\end{aligned}$$

By (2.79), (2.43) and (2.78) we have

$$\begin{aligned}
& \sqrt{|\tilde{s}|} \|v_{\nu, \varphi}(\cdot, 0)\|_{H^{1,\tilde{s}}(0,T)} + |\tilde{s}|^{\frac{5}{2}} \|v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0,T)} \leq C_{54} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_{L^2(Q)} \\
(2.80) \quad & + \sqrt{|\tilde{s}|} \|r_{\nu, \varphi}\|_{L^2(0,T)} + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)} + \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}).
\end{aligned}$$

So by (2.80) for any $\delta_3 > 0$

$$\begin{aligned}
& |\tilde{s}|^2 |\operatorname{Re} \int_{\mathbb{R}^1} \partial_{x_1}^+ v_{\nu, \varphi} \overline{\rho \partial_{x_0} v_{\nu, \varphi}}|_{x_1=0} dx_0 | \\
& \leq \delta_3 \int_0^T (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu, \varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu, \varphi}|^2)(x_0, 0) dx_0 \\
& \quad + C_{55}(\delta_3) |\tilde{s}| \|\partial_{x_0} v_{\nu, \varphi}(\cdot, 0)\|_{L^2(0,T)}^2 \\
& \leq \delta_3 \int_0^T (|\tilde{s}| \phi_0(x^*, x^*) |\partial_{x_1}^+ v_{\nu, \varphi}|^2 + |\tilde{s}|^3 \phi_0^3(x^*, x^*) |v_{\nu, \varphi}|^2)(x_0, 0) dx_0 \\
& \quad + C_{56}(\delta_3) (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_{L^2(Q)}^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 \\
(2.81) \quad & \quad + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2).
\end{aligned}$$

By Proposition 2.6 and estimate (4.11)

$$\begin{aligned}
& \int_0^T (|\tilde{s}|^3 |\partial_{x_1}^+ v_{\nu, \varphi}|^2 + |\tilde{s}|^5 |v_{\nu, \varphi}|^2)(x_0, 0) dx_0 + |\tilde{s}|^3 \|v_{\nu, \varphi}\|_{H^{0,1,\tilde{s}}(Q_+)}^2 \leq C_{57} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 \\
& \quad + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 \\
(2.82) \quad & \quad + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2).
\end{aligned}$$

Hence

$$\begin{aligned}
& \int_0^T (|\tilde{s}| |\partial_{x_1}^- v_{\nu, \varphi}|^2 + |\tilde{s}|^5 |v_{\nu, \varphi}|^2)(x_0, 0) dx_0 \leq C_{58} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0,T)}^2 \\
(2.83) \quad & \quad + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2).
\end{aligned}$$

By (2.83) and inequality (2.53) of Proposition 2.6

$$\begin{aligned}
& |\tilde{s}| \|v_{\nu, \varphi}\|_{H^{0,1,\tilde{s}}(Q_-)}^2 \leq C_{59} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0,T)}^2 \\
(2.84) \quad & \quad + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0,T)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2).
\end{aligned}$$

By (2.73), (2.50) and (2.82) the following estimate is true

$$(2.85) \quad \begin{aligned} & |\tilde{s}|^3 \|\alpha^+(x, D_0, \tilde{s}) v_{\nu, \varphi}\|_{L^2(Q_+)}^2 + |\tilde{s}| \|\alpha^+(x, D_0, \tilde{s}) v_{\nu, \varphi}\|_{L^2(Q_-)}^2 \\ & \leq C_{60} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 \\ & \quad + |s|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2). \end{aligned}$$

Using the Gårding inequality proved in we obtain from (2.85)

$$(2.86) \quad \begin{aligned} & |\tilde{s}|^3 \|\alpha^+(x, D_0, \tilde{s}) v_{\nu, \varphi}\|_{H^{\frac{1}{2}, 0}(Q_+)}^2 \leq C_{61} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0, T)}^2 \\ & \quad + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2). \end{aligned}$$

The inequality (2.84), (2.86) and (2.82) we have

$$(2.87) \quad \begin{aligned} & |\tilde{s}| \|v_{\nu, \varphi}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_-)}^2 + |\tilde{s}|^3 \|v_{\nu, \varphi}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q_+)}^2 + \|\mathcal{B} v_{\nu, \varphi}\|_{Z(0, T)}^2 \leq C_{62} (\|P_\varphi(x, D, \tilde{s}) v_{\nu, \varphi}\|_Y^2 \\ & \quad + |\tilde{s}|^3 \varphi^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}| \|r_{\nu, \varphi}\|_{L^2(0, T)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2). \end{aligned}$$

Case 3. Let $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi, 0}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$. Consider two subcases:

Subcase 1. Let $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi_1, 0}(\ell)$. Then by (2.18) and (2.21) $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi_2, +}(\ell)$. Therefore estimate (2.67) is true:

$$(2.88) \quad \begin{aligned} & \int_0^T (|\tilde{s}| |\partial_{x_1}^+ v_{\nu, \varphi_2}|^2 + |\tilde{s}|^3 |v_{\nu, \varphi_2}|^2)(x_0, 0) dx_0 + |\tilde{s}| \|v_{\nu, \varphi_2}\|_{H^{0, 1, \tilde{s}}(Q_+)}^2 \\ & \leq C_{63} (\|P_{\varphi_2}(x, D, \tilde{s}) v_{\nu, \varphi_2}\|_{L^2(Q_+)}^2 + |\tilde{s}| \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2). \end{aligned}$$

Since by (2.21) on the segment $[0, T]$ $\varphi_1(x_0, 0) = \varphi_2(x_0, 0)$ the inequality (2.88) implies

$$(2.89) \quad \begin{aligned} & \int_0^T (|\tilde{s}|^3 |\partial_{x_1}^+ v_{\nu, \varphi_1}|^2 + |\tilde{s}|^5 |v_{\nu, \varphi_1}|^2)(x_0, 0) dx_0 + |\tilde{s}|^3 \|v_{\nu, \varphi_2}\|_{H^{0, 1, \tilde{s}}(Q_+)}^2 \\ & \leq C_{64} (|\tilde{s}|^2 \|P_{\varphi_2}(x, D, \tilde{s}) v_{\nu, \varphi_2}\|_{L^2(Q_+)}^2 + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_\varphi(\cdot, 0)\|_{L^2(0, T)}^2). \end{aligned}$$

Observe that we have equality (2.68) with function φ_1 . Using this equality and (2.60) we estimate $\partial_{x_1}^- v_{\nu, \varphi_1}(\cdot, 0)$:

$$(2.90) \quad \begin{aligned} & \|\partial_{x_1}^- v_{\nu, \varphi_1}(\cdot, 0)\|_{L^2(0, T)} \leq \|\partial_{x_1}^+ v_{\nu, \varphi_1}(\cdot, 0)\|_{L^2(0, T)} + M \|\partial_{x_0} v_{\nu, \varphi_*}(\cdot, 0)\|_{L^2(0, T)} \\ & \quad + M \|s \partial_{x_0} \varphi_* v_{\nu, \varphi_*}(\cdot, 0)\|_{L^2(0, T)} + M |s| \|[\chi_\nu, \partial_{x_0} \varphi_*(x_0, 0)] v_{\varphi_*}\|_{L^2(0, T)} + \|\tilde{r}_{\nu, \varphi_*}\|_{L^2(0, T)} \\ & \quad + C_{65} (|\tilde{s}| \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} + |\tilde{s}|^2 \|v_{\nu, \varphi_*}(\cdot, 0)\|_{L^2(\mathbb{R}^1)}). \end{aligned}$$

Applying the Proposition 2.6 and using (2.90) and (2.60) to estimate the boundary integrals in the right hand side of (2.53) we have

$$(2.91) \quad |\tilde{s}| \|v_{\nu, \varphi_1}\|_{H^{0, 1, \tilde{s}}(Q_-)}^2 \leq C_{66} (\|P_{\varphi_*}(x, D, \tilde{s}) v_{\nu, \varphi_*}\|_Y^2 + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}| \|r_{\nu, \varphi_*}\|_{L^2(0, T)}^2).$$

Combining the estimates (2.69) (2.60) and (2.91) we have (2.87). Now we consider

Subcase 2. Let $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi_2,0}(\ell)$ and $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*, \delta_1)$. Then $(x^*, \xi_0^*, \tilde{s}^*) \in \mathcal{Z}_{\varphi_1,-}(\ell)$ and inequality (2.74) holds true for some positive δ_3 . Therefore there exists positive $\delta_4(\delta_3)$ such that

$$(2.92) \quad \text{Im } \Gamma_{\varphi_1}^\pm(x, \xi, \tilde{s}) > \delta_3/2 > 0 \quad (x, \xi, \tilde{s}) \in B(x^*, \delta_4) \cap \text{supp } \eta_\ell \times \text{supp } \kappa(\nu, \cdot).$$

By Proposition 2.4 and (2.92) we have

$$(2.93) \quad \|U_{\nu,\varphi_1}^-(\cdot, 0)\|_{H^{\frac{1}{4}}(0,T) \cap L_{\tilde{s}}^2(0,T)} \leq C_{67}(\|P_{\varphi_1}(x, D, \tilde{s})v_{\nu,\varphi_1}\|_{L^2(Q_-)} + \tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\varphi_1}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_-)}).$$

By Proposition 2.4 we have

$$(2.94) \quad \|V_{\nu,\varphi_2}^+(\cdot, 0)\|_{H^{\frac{1}{4}}(0,T) \cap L_{\tilde{s}}^2(0,T)} \leq C_{68}(\|P_{\varphi_2}(x, D, \tilde{s})v_{\nu,\varphi_2}\|_{L^2(Q_+)} + \tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\varphi_2}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q_+)}).$$

The equality (2.15) for the function u can be written as

$$(2.95) \quad -\partial_{x_1}^- v_{\varphi_1}(x_0, 0) + \mu \partial_{x_1}^+ v_{\varphi_2}(x_0, 0) + |s|(\partial_{x_1} \varphi_1 - \mu \partial_{x_1} \varphi_2)v_{\varphi_*} = \\ M \partial_{x_0} v_{\varphi_*}(x_0, 0) - M|s| \partial_{x_0} \varphi_*(x_0, 0)v_{\varphi_*}(x_0, 0) + r_{\varphi_*}.$$

We apply to both sides of equation (2.95) the operator $\chi_\nu(x, D_0, \tilde{s})$:

$$(2.96) \quad -\partial_{x_1}^- v_{\nu,\varphi_1}(x_0, 0) + \mu \partial_{x_1}^+ v_{\nu,\varphi_2}(x_0, 0) + |s|(\partial_{x_1} \varphi_1 - \mu \partial_{x_1} \varphi_2)v_{\nu,\varphi_*}(x_0, 0) \\ -M \partial_{x_0} v_{\nu,\varphi_*}(x_0, 0) + M|s| \partial_{x_0} \varphi_*(x_0, 0)v_{\nu,\varphi_*}(x_0, 0) - r_{\nu,\varphi_*}(x_0) \\ -[\chi_\nu, \partial_{x_1}^-]v_{\varphi_1}(x_0, 0) + [\chi_\nu, \mu \partial_{x_1}^+]v_{\varphi_2}(x_0, 0) + |s|[\chi_\nu, (\partial_{x_1} \varphi_1 - \mu \partial_{x_1} \varphi_2)]v_{\varphi_*}(x_0, 0) \\ +M[\chi_\nu, \partial_{x_0}]v_{\varphi_*}(x_0, 0) + M|s|[\chi_\nu, \partial_{x_0} \varphi_*(x_0, 0)]v_{\varphi_*}(x_0, 0).$$

In the equality (2.96) we replace $\partial_{x_1}^+ v_{\nu,\varphi_2}(\cdot, 0)$ by $i(\Gamma_{\varphi_2}^-(x, D_0, \tilde{s})v_{\nu,\varphi_2}(\cdot, 0) + V_{\nu,\varphi_2}^+(\cdot, 0))$ and we replace $\partial_{x_1}^- v_{\nu,\varphi_1}(\cdot, 0)$ by $i(\Gamma_{\varphi_1}^+(x, D_0, \tilde{s})v_{\nu,\varphi_1}(\cdot, 0) + U_{\nu,\varphi_1}^-(\cdot, 0))$:

$$\begin{aligned} M \partial_{x_0} v_{\nu,\varphi_*}(x_0, 0) &= i(\mu \Gamma_{\varphi_2}^-(x, D_0, \tilde{s})v_{\nu,\varphi_2} - \Gamma_{\varphi_1}^+(x, D_0, \tilde{s})v_{\nu,\varphi_1})(x_0, 0) \\ &\quad + i(\mu V_{\nu,\varphi_2}^+(x_0, 0) - U_{\nu,\varphi_1}^-(x_0, 0)) \\ &\quad - |s|(\partial_{x_1} \varphi_1 - \mu \partial_{x_1} \varphi_2)v_{\nu,\varphi_*}(x_0, 0) - M|s| \partial_{x_0} \varphi_*(x_0, 0)v_{\nu,\varphi_*}(x_0, 0) + r_{\nu,\varphi_*}(x_0) \\ &\quad + [\chi_\nu, \partial_{x_1}^-]v_{\varphi_1}(x_0, 0) - [\chi_\nu, \mu \partial_{x_1}^+]v_{\varphi_2}(x_0, 0) - |s|[\chi_\nu, (\partial_{x_1} \varphi_1 + \mu \partial_{x_1} \varphi_2)]v_{\varphi_*}(x_0, 0) \\ (2.97) \quad &\quad - M[\chi_\nu, \partial_{x_0}]v_{\varphi_*}(x_0, 0) - M|s|[\chi_\nu, \partial_{x_0} \varphi_*(x_0, 0)]v_{\varphi_*}(x_0, 0). \end{aligned}$$

Observe that estimate (2.79) holds true. By (2.79), (2.93), (2.94) from (2.97) we have

$$(2.98) \quad \begin{aligned} \sqrt{|\tilde{s}|}\|v_{\nu,\varphi_*}(\cdot, 0)\|_{H^{1,\tilde{s}}(0,T)} + |\tilde{s}|^{\frac{5}{2}}\|v_{\nu,\varphi_*}(\cdot, 0)\|_{L^2(0,T)} &\leq C_{69}(\sqrt{|\tilde{s}|}\|P_{\varphi_*}(x, D, \tilde{s})v_{\nu,\varphi_*}\|_{L^2(Q)} \\ &\quad + \tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\varphi_*}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q)} + \sqrt{|\tilde{s}|}\|r_{\nu,\varphi_*}\|_{L^2(0,T)} + |\tilde{s}|^{\frac{3}{2}}\tilde{\varphi}^{\frac{5}{12}}(x^*)\|v_{\varphi_*}(\cdot, 0)\|_{L^2(0,T)}). \end{aligned}$$

. By (2.98) and Proposition 2.6 we obtain

$$(2.99) \quad \begin{aligned} \int_0^T (|\tilde{s}|^3 |\partial_{x_1}^+ v_{\nu,\varphi_2}|^2 + |\tilde{s}|^5 |v_{\nu,\varphi_2}|^2)(x_0, 0)dx_0 + |\tilde{s}|^3 \|v_{\nu,\varphi_2}\|_{H^{0,1,\tilde{s}}(Q_+)}^2 \\ \leq C_{70}(\|P_{\varphi_*}(x, D, \tilde{s})v_{\nu,\varphi_*}\|_Y^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*)\|v_{\varphi_*}\|_{H^{\frac{1}{2},1,\tilde{s}}(Q)}^2 + |\tilde{s}|\|\tilde{r}_{\nu,\varphi_*}\|_{L^2(0,T)}^2 \\ + |\tilde{s}|^2 \|\partial_{x_1}^+ v_{\varphi_*}(\cdot, 0)\|_{L^2(0,T)}^2 + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*)\|v_{\varphi_*}(\cdot, 0)\|_{L^2(0,T)}^2). \end{aligned}$$

Hence

$$(2.100) \quad \int_0^T (|\tilde{s}| |\partial_{x_1}^- v_{\nu, \varphi_2}|^2 + |\tilde{s}|^3 |v_{\nu, \varphi_2}|^2)(x_0, 0) dx_0 \leq C_{71} (\|P_{\varphi_*}(x, D, \tilde{s}) v_{\nu, \varphi_*}\|_{L^2(Q)}^2 \\ + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2 + |\tilde{s}| \|r_{\nu, \varphi_*}\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_{\varphi}(\cdot, 0)\|_{L^2(0, T)}^2 \\ + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2).$$

By Proposition 2.6 and estimate (2.100)

$$(2.101) \quad |\tilde{s}| \|v_{\nu, \varphi_2}\|_{H^{0, 1, \tilde{s}}(Q_-)}^2 \leq C_{72} (\|P_{\varphi_*}(x, D, \tilde{s}) v_{\nu, \varphi_*}\|_{L^2(Q)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2 \\ + |\tilde{s}| \|r_{\nu, \varphi_*}\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2).$$

The estimates (2.69), (2.101) and (2.99) imply

$$(2.102) \quad \int_0^T (|\tilde{s}|^3 |\partial_{x_1}^+ v_{\nu, \varphi_2}|^2 + |\tilde{s}|^5 |v_{\nu, \varphi_2}|^2)(x_0, 0) dx_0 + |\tilde{s}|^3 \|v_{\nu, \varphi_2}\|_{H^{0, 1, \tilde{s}}(Q_+)}^2 \\ + |\tilde{s}| \|v_{\nu, \varphi_2}\|_{H^{0, 1, \tilde{s}}(Q_-)}^2 \leq C_{73} (\|P_{\varphi_*}(x, D, \tilde{s}) v_{\nu, \varphi_*}\|_{L^2(Q)}^2 + \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}^2 \\ + |\tilde{s}| \|r_{\nu, \varphi_*}\|_{L^2(0, T)}^2 + |\tilde{s}|^2 \|\partial_{x_1}^+ v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2 + |\tilde{s}|^3 \tilde{\varphi}^{\frac{5}{6}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)}^2).$$

Now we observe that in all three cases the estimate (2.102) is true. In order to finish the proof of the Proposition 2.2 let us take the covering of the surface $\mathbb{M} = \{(\xi_0, \tilde{s}) | M(\xi_0, \tilde{s}) = 1\}$ by conical neighborhoods $\mathcal{O}(\zeta^*, \delta_1(\zeta^*))$. From this covering we take the finite subcovering $\cup_{\nu=1}^N \mathcal{O}(\zeta_\nu^*, \delta_1(\zeta_\nu^*))$. Let us show that such a subcovering can be taken independently of parameter ℓ for all $\ell \geq \ell_0$. Indeed by (2.55) for any x_ℓ^* from $\text{supp } \kappa_\ell$ then

$$(2.103) \quad \text{supp}_{y \in \text{supp } \kappa_\ell} |\phi_0(y, x_\ell^*) - 1| \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

We set

$$\mathcal{Z}_\pm = \{(x, \xi_0, \tilde{s}) \in \{(0, 0), (T, 0)\} \times \mathbb{M} | \quad \pm (|\tilde{s}| - \text{Im} \sqrt{i\rho(x_0, 0)\xi_0}) > 0, \quad x_0 \in \{0, T\}\}$$

and

$$\mathcal{Z}_0 = \{(x, \xi_0, \tilde{s}) \in \{(0, 0), (T, 0)\} \times \mathbb{M} | \quad |\tilde{s}| = \text{Im} \sqrt{i\rho(x_0, 0)\xi_0} = 0, \quad x_0 \in \{0, T\}\}.$$

By (2.103)

$$(2.104) \quad \text{dist}(\mathcal{Z}_\pm, \mathcal{Z}_{\varphi, \pm}(\ell)) + \text{dist}(\mathcal{Z}_0, \mathcal{Z}_{\varphi, \pm}(\ell)) \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Without loss of generality one may assume that $x^* = (0, 0)$ or $x^* = (T, 0)$ in (2.31). Let $x^* = (T, 0)$. We construct the covering of the set \mathbb{M} in the following way: if $(\xi_0^*, \tilde{s}^*) \in \mathcal{M}_\pm = \{(\xi_0, \tilde{s}) \in \mathbb{M} | \pm (|\tilde{s}| - \text{Im} \sqrt{i\rho(T, 0)\xi_0}) > 0\}$ we consider covering of this point by the ball of centered at (ξ_0^*, \tilde{s}^*) of sufficiently small radius $\delta(\xi_0^*, \tilde{s}^*)$ such that $\overline{\mathbb{M} \cap B((\xi_0^*, \tilde{s}^*), \delta(\xi_0^*, \tilde{s}^*))} \subset \mathcal{M}_\pm$. If $(\xi_0^*, \tilde{s}^*) \in \mathcal{M}_0 = \{(\xi_0, \tilde{s}) \in \mathbb{M} | |\tilde{s}| - \text{Im} \sqrt{i\rho(T, 0)\xi_0} = 0\}$ we consider covering of this point by the ball of centered at (ξ_0^*, \tilde{s}^*) of sufficiently small radius $\delta(\xi_0^*, \tilde{s}^*)$. From this covering we take the finite subcovering $\cup_{\nu=1}^N \mathcal{O}(\zeta_\nu^*, \delta_1(\zeta_\nu^*))$ and let χ_ν be the partition of unity subjected to this finite subcovering. Hence $\sum_{\nu=1}^N \chi_\nu(\xi_0, \tilde{s}) \equiv 1$ for all (ξ_0, \tilde{s}) such that

$M(\xi_0, \tilde{s}) \geq 1$. By (2.104) there exists ℓ_1 such that for all $\ell \geq \ell_1$ if $(\xi_0^*, \tilde{s}^*) \in \mathcal{M}_\pm$ and $\text{supp } \chi_\nu \in \mathbb{M} \cap B((\xi_0^*, \tilde{s}^*), \delta(\xi_0^*, \tilde{s}^*))$ then

$$\text{supp } \kappa_\ell \times \{0\} \times \text{supp } \chi_\nu \subset \mathcal{Z}_{\varphi, \pm}(\ell).$$

On the other hand if $(\xi_0^*, \tilde{s}^*) \in \mathcal{M}_0$ and $\text{supp } \chi_\nu \in \mathbb{M} \cap B((\xi_0^*, \tilde{s}^*), \delta(\xi_0^*, \tilde{s}^*))$ then for all sufficiently large ℓ there exists $\zeta^*(\ell) = (\xi_0^*(\ell), \tilde{s}^*(\ell))$ such that $\text{supp } \chi_\nu \subset \mathcal{O}(\zeta^*(\ell), \delta_3)$ where $\delta_3 \rightarrow +0$ as $\delta(\xi_0^*, \tilde{s}^*) \rightarrow +0$.

For index $\ell \in \{1, \dots, \ell_0\}$ individual subcovering of the set \mathbb{M} and the corresponding partition of unity χ_ν . Let $\chi_0(\xi_0, \tilde{s}) \in C_0^\infty(\mathbb{R}^2)$ be a nonnegative function which is identically equal one if $M(\xi_0, \tilde{s}) \leq 1$. Then by (2.72) we have

$$\begin{aligned} \|\mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(0,T)} + \sqrt{|\tilde{s}|} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} &\leq C_{74} \sum_{\nu=0}^N (\|\eta_\ell \chi_\nu \mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(\mathbb{R}^1)} + \sqrt{|\tilde{s}|} \|\eta_\ell \chi_\nu v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}) \\ &+ \|(1 - \eta_\ell) \chi_\nu \mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(\mathbb{R}^1)} + \sqrt{|\tilde{s}|} \|(1 - \eta_\ell) \chi_\nu v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} \leq C_{75} (\tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} \\ &+ \sum_{\nu=0}^N \|\mathbf{P}(x, D, \tilde{s}) v_{\nu, \varphi_*}\|_{Y \times L_s^2(0, T)} + |\tilde{s}|^{\frac{3}{2}} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} + |\tilde{s}| \|\partial_{x_1}^+ v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} \\ (2.105) \quad &+ \|(1 - \eta_\ell) \chi_\nu \mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(\mathbb{R}^1)} + \sqrt{|\tilde{s}|} \|(1 - \eta_\ell) \chi_\nu v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}). \end{aligned}$$

By (2.58) and (2.61) there exist a constant C_{76} independent of \tilde{s}, ℓ and ν such that

$$\begin{aligned} \sum_{\nu=0}^N (\|(1 - \eta_\ell) \chi_\nu \mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(\mathbb{R}^1)} + \sqrt{|\tilde{s}|} \|(1 - \eta_\ell) \chi_\nu v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)}) \\ (2.106) \quad \leq C_{76} \left(|\tilde{s}|^{\frac{3}{2}} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} + \sqrt{\tilde{\varphi}(x^*)} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} \right). \end{aligned}$$

Using inequality (2.106) in order to estimate the last terms in (2.105) we obtain

$$\begin{aligned} \|\mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(0,T)} + \sqrt{|\tilde{s}|} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} &\leq C_{77} \left(\sqrt{\tilde{\varphi}(x^*)} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} + \|\mathbf{P}(x, D, \tilde{s}) v_{\varphi_*}\|_{Y \times L_s^2(0, T)} \right. \\ &\quad \left. + |\tilde{s}|^{\frac{3}{2}} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} + \sum_{\nu=0}^N \|[\tilde{\chi}_\nu, \mathbf{P}(x, D, \tilde{s})] v_{\varphi_*}\|_{Y \times L_s^2(0, T)} \right) \\ &\leq C_{78} \left(\sqrt{\tilde{\varphi}(x^*)} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} + \|\mathbf{P}(x, D, \tilde{s}) v_{\varphi_*}\|_{Y \times L_{s\tilde{\varphi}}^2(0, T)} + |\tilde{s}|^{\frac{3}{2}} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} \right). \end{aligned}$$

Hence there exists $s_0 > 1$ such that for all $s \geq s_0$ we see

$$\begin{aligned} (2.107) \quad &\|\mathcal{B}v_{\varphi_*}\|_{\mathcal{Z}(0,T)} + \sqrt{|\tilde{s}|} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} \\ &\leq C_{79} \left(\sqrt{\tilde{\varphi}(x^*)} \|v_{\varphi_*}\|_{H^{\frac{1}{2}, 1, \tilde{s}}(Q)} + \|\mathbf{P}(x, D, \tilde{s}) v_{\varphi_*}\|_{Y \times L_{s\tilde{\varphi}}^2(0, T)} + |\tilde{s}|^{\frac{3}{2}} \tilde{\varphi}^{\frac{5}{12}}(x^*) \|v_{\varphi_*}(\cdot, 0)\|_{L^2(0, T)} \right). \end{aligned}$$

Proof of Proposition 2.2 is complete. ■

We set

$$(2.108) \quad \psi^*(x) = \begin{cases} \hat{s} \varphi_* \circ F^{-1}(x) & \text{for } x_0 \in [\frac{T}{2}, T], \\ \hat{s} \varphi_* \circ F^{-1}(\frac{T}{2}, x_1) & \text{for } x_0 \in [0, \frac{T}{2}], \end{cases}$$

where function φ_* defined by (2.18) and diffeomorphism F is constructed in the beginning of the proof of this proposition, parameter $\hat{s} > s_0$.

From (4.1) and (2.22) we obtain (2.109). Proof of Proposition is complete. ■

Corollary 1. *Let $f_1 \in L^2(Q_+)$, $f_2 \in L^2(Q_-)$, $\tilde{r} \in L^2(0, T)$ and all conditions of Proposition 2 holds true. Then there exists function $\eta(x_1) \in C^2(\bar{\Omega})$, $\eta(x_1) < 0$ on $\bar{\Omega}$ and a constant C_{80} independent of $v = (v_1, v_2)$ such that*

$$\begin{aligned} & \sum_{|\alpha| \leq 1} \|((T - x_0)^{-3})^{\frac{(3-2|\alpha|)}{2}} \partial^\alpha v_2 e^{\psi^*}\|_{L^2(Q_-)} + \sum_{|\alpha| \leq 1} \|((T - x_0)^{-3})^{\frac{5-2|\alpha|}{2}} \partial^\alpha v_1 e^{\psi^*}\|_{L^2(Q_+)} \\ & + \|((T - x_0)^{-3})^{\frac{3}{2}} \partial_{x_1}^+ v e^{\psi^*}\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{1}{2}} \partial_{x_1}^- v e^{\psi^*}\|_{L^2(0, T)} \\ & + \|((T - x_0)^{-3})^{\frac{1}{2}} \partial_{x_0} v e^{\psi^*}\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{5}{2}} v e^{\psi^*}\|_{L^2(0, T)} \\ & \leq C_{80} (\|((T - x_0)^{-3} f_1 e^{\psi^*}\|_{L^2(Q_+)} + \|f_2 e^{\psi^*}\|_{L^2(Q_-)} \\ (2.109) \quad & + \|(T - x_0)^{-\frac{3}{2}} \tilde{r} e^{\psi^*}\|_{L^2(0, T)} + \|((T - x_0)^{-3})^{\frac{5}{2}} v e^{\psi^*}\|_{L^2(Q_\omega)}), \end{aligned}$$

where $\psi^*(x) = \eta(x_1)/(T - x_0)^3$.

3. PROOF OF THEOREM 1.1.

We consider the linearization of the null controllability problem (1.1) - (1.6):

$$(3.1) \quad L_1(x, D)z_1 = \rho_1 \partial_{x_0} z_1 - a_1 \partial_{x_1}^2 z_1 + b_1 \partial_{x_1} z_1 + c_1 z_1 = f_1 + \chi_\omega u \quad \text{in } Q_+,$$

$$(3.2) \quad L_2(x, D)z_2 = \rho_2 \partial_{x_0} z_2 - a_2 \partial_{x_1}^2 z_2 + b_2 \partial_{x_1} z_2 + c_2 z_2 = f_2 \quad \text{in } Q_-,$$

$$(3.3) \quad z_1(x_0, 0) - z_2(x_0, 0) = \partial_{x_1} z_1(x_0, 0) - \partial_{x_1} z_2(x_0, 0) - M \partial_{x_0} z_1(x_0, 0) - r(x_0) = 0 \quad \text{on } [0, T],$$

$$(3.4) \quad z_1(x_0, b) = z_2(x_0, a) = 0 \quad \text{on } [0, T],$$

$$(3.5) \quad z(0, x_1) = z_0(x_1).$$

Here

$$z = \begin{cases} z_1 & \text{for } x \in Q_+, \\ z_2 & \text{for } x \in Q_-. \end{cases}$$

We are looking for control u such that at moment T we have

$$(3.6) \quad z(T, \cdot) = 0.$$

We have

Proposition 3.1. *Suppose that assumptions (1.6)- (1.9) holds true, $\hat{s} > s_0$ where s_0 is the parameter from Proposition 2.2. Let $(T - x_0)^{\frac{15}{2}} e^{-\psi^*} f_1 \in L^2(Q_+)$, $(T - x_0)^{\frac{9}{2}} e^{-\psi^*} f_2 \in L^2(Q_-)$, $(T - x_0)^{\frac{15}{2}} e^{-\psi^*} r \in L^2(0, T)$, $z_0 \in H_0^1(\Omega)$ and function ψ^* is defined by (2.108). Then*

controllability problem (3.1)-(3.6) has solution $(z, u) \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-) \cap C^0(0, T; H^1(\Omega)) \times L^2(Q)$, $\text{supp } u \subset Q_\omega$ and satisfies the a priori estimate

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \|(T - x_0)^{3|\alpha|} \partial^\alpha z e^{-\psi^*}\|_{L^2(Q_-)} + \sum_{|\alpha| \leq 2} \|(T - x_0)^{3(|\alpha|+1)} \partial^\alpha z e^{-\psi^*}\|_{L^2(Q_+)} \\ & + \|(T - x_0)^6 z(\cdot, 0) e^{-\psi^*}\|_{H^1(0, T)} \leq C_1 (\|(T - x_0)^{\frac{15}{2}} f e^{-\psi^*}\|_{L^2(Q_+)} \\ (3.7) \quad & + \|(T - x_0)^{\frac{9}{2}} f e^{-\psi^*}\|_{L^2(Q_-)} + \|(T - x_0)^{\frac{15}{2}} r e^{-\psi^*}\|_{L^2(0, T)} + \|z_0\|_{H_0^1(\Omega)}). \end{aligned}$$

Proof. Let $\epsilon \in (0, \frac{T}{2})$ be small positive parameter. We set $\psi_\epsilon^* = \psi^*$ for $x_0 \in [0, T - \epsilon]$ and $\psi^*(x) = \psi^*(T - \epsilon, x_1)$ for $x_0 \in [T - \epsilon, T]$ where function ψ^* is constructed in (2.108). Denote $\tilde{L}(x, D) = (\tilde{L}_1(x, D), \tilde{L}_2(x, D)), \tilde{L}_j(x, D) = \frac{1}{a_j} L_j(x, D)$, $\tilde{f} = f/a_1$ on Q_+ and $\tilde{f} = f/a_2$ on Q_- . Consider the minimization problem

$$\begin{aligned} (3.8) \quad J_\epsilon(z, u) = & \|(T - x_0)^3 e^{-\psi_\epsilon^*} z\|_{L^2(Q_+)}^2 + \|e^{-\psi_\epsilon^*} z\|_{L^2(Q_-)}^2 + \|(T - x_0)^{\frac{15}{2}} e^{-\psi^*} u\|_{L^2(Q_\omega)}^2 \\ & + \frac{1}{\epsilon} \|\tilde{L}(x, D)z - \chi_\omega u - \tilde{f}\|_{L^2(Q)}^2 \rightarrow \inf, \end{aligned}$$

$$(3.9) \quad z_1(\cdot, b) = z_2(\cdot, a) = 0, \quad [z](\cdot, 0) = [\partial_{x_1} z](\cdot, 0) - M \partial_{x_0} z(\cdot, 0) + r(\cdot) = 0 \quad \text{on } [0, T],$$

$$(3.10) \quad z(0, \cdot) = z_0.$$

There exists a unique solution to the problem (3.8)-(3.10) which we denote as $(\hat{z}_\epsilon, \hat{u}_\epsilon)$. By Theorem 4.1 the functions $(\hat{z}_\epsilon, \hat{u}_\epsilon)$ belong to the space $H^{1,2}(Q_+) \cap H^{1,2}(Q_-) \times L^2(Q_\omega)$. Setting $m(x) = 1$ for $x \in Q_-$ and $m(x) = (T - x_0)^{15}$ for $x \in Q_+$, $p_\epsilon = m(x) \frac{e^{-2\psi_\epsilon^*}}{\epsilon} (\tilde{L}(x, D) \hat{z}_\epsilon - \chi_\omega \hat{u}_\epsilon - \tilde{f}) \in L^2(Q)$ by the Fermat theorem we have

$$(3.11) \quad J'_\epsilon(\hat{z}_\epsilon, \hat{u}_\epsilon)[\delta] = 0 \quad \forall \delta \in \tilde{\mathcal{X}},$$

where

$$\tilde{\mathcal{X}} = \{\delta = (\delta_1, \delta_2) \mid \tilde{L}(x, D)\delta_1 = 0 \quad \text{in } Q \setminus [0, T] \times \{0\}, \quad \delta_1(0, \cdot) = 0, \quad \delta_1(\cdot, 0) = \delta_2(\cdot, 0) \quad \text{on } [0, T],$$

$$\begin{aligned} \delta_1(\cdot, b) = \delta_2(\cdot, a) = 0, \quad & ([\partial_{x_1} \delta_1] - M \partial_{x_0} \delta_1)(\cdot, 0) = 0, \\ & (\delta_1, (T - x_0)^{-\frac{15}{2}} e^{-\psi^*} \delta_2) \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-) \times L^2(Q_\omega) \} \end{aligned}$$

equipped with the norm

$$\|p\|_{\tilde{\mathcal{X}}} = (\|(p_1, (T - x_0)^{\frac{15}{2}} e^{-\psi^*} p_2)\|_{L^2(Q) \times L^2(Q_\omega)}^2 + \|p_1\|_{H^{1,2}(Q_+)}^2 + \|p_1\|_{H^{1,2}(Q_-)}^2)^{\frac{1}{2}}.$$

From (3.11) we have

$$(3.12) \quad \tilde{L}^*(x, D)p_\epsilon = -e^{-2\psi_\epsilon^*} \hat{z}_\epsilon \quad \text{in } Q \setminus [0, T] \times \{0\},$$

$$(3.13) \quad p_{\epsilon,1}(\cdot, b) = p_{\epsilon,2}(\cdot, a) = 0, \quad [p_\epsilon](\cdot, 0) = ([\partial_{x_1} p_\epsilon] - M \partial_{x_0} p_\epsilon)(\cdot, 0) = 0 \quad \text{on } [0, T],$$

$$(3.14) \quad p_\epsilon = e^{-2\psi^*} (T - x_0)^{15} \hat{u}_\epsilon \quad \text{on } Q_\omega, \quad p_\epsilon(T, \cdot) = 0.$$

By Theorem 4.1 $p_\epsilon \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-)$. By Corollary 1 the following estimate is true:

$$(3.15) \quad \begin{aligned} & \| (T - x_0)^{\frac{15}{2}} e^{\psi^*} p_\epsilon \|_{L^2(Q_+)} + \| (T - x_0)^{\frac{9}{2}} e^{\psi^*} p_\epsilon \|_{L^2(Q_-)} \\ & + \| (T - x_0)^{\frac{15}{2}} e^{\psi^*} p_\epsilon(\cdot, 0) \|_{L^2(0,T)} \\ & \leq C_2 (\| e^{-\psi^*} (T - x_0)^{\frac{15}{2}} \hat{u}_\epsilon \|_{L^2(Q_\omega)} + \| (T - x_0)^3 e^{\psi^* - 2\psi_\epsilon^*} \hat{z}_\epsilon \|_{L^2(Q_+)} + \| e^{\psi^* - 2\psi_\epsilon^*} \hat{z}_\epsilon \|_{L^2(Q_-)}). \end{aligned}$$

Taking the scalar product in $L^2(Q)$ of equation (3.12) with function \hat{z}_ϵ and integrating by parts we have

$$2J_\epsilon(\hat{z}_\epsilon, \hat{u}_\epsilon) = -(p_\epsilon, \tilde{f})_{L^2(Q)} - (\frac{p_\epsilon}{a}(0, \cdot), z_0)_{L^2(\Omega)} + (r, p_\epsilon(\cdot, 0))_{L^2(0,T)},$$

where $a(x) = a_1(x)$ on Q_+ and $a(x) = a_2(x)$ on Q_- . Hence by (3.15) we obtain

$$(3.16) \quad \begin{aligned} J_\epsilon(\hat{z}_\epsilon, \hat{u}_\epsilon) & \leq C_3 (\| (T - x_0)^{\frac{15}{2}} f e^{-\psi^*} \|_{L^2(Q_+)} \\ & + \| (T - x_0)^{\frac{9}{2}} f e^{-\psi^*} \|_{L^2(Q_-)} + \| r e^{-\psi^*} (T - x_0)^{\frac{15}{2}} \|_{L^2(0,T)}^2 + \| z_0 \|_{L^2(\Omega)}^2). \end{aligned}$$

From the sequence $(\hat{z}_\epsilon, \hat{u}_\epsilon)$ one can take a subsequence $(e^{-\psi_{\epsilon_j}^*} \hat{z}_{\epsilon_j}, \hat{u}_{\epsilon_j})$ which converges weakly to $(e^{-\psi^*} z, u)$ in the space $L_m^2(Q) \times L_{(T-x_0)^{15} e^{-2\psi^*}}^2(Q_\omega)$. From (3.16) we have

$$(3.17) \quad \begin{aligned} & \| (T - x_0)^3 e^{-\psi^*} z \|_{L^2(Q_+)}^2 + \| e^{-\psi^*} z \|_{L^2(Q_-)}^2 \leq C_4 (\| (T - x_0)^{\frac{15}{2}} f e^{-\psi^*} \|_{L^2(Q_+)} \\ & + \| (T - x_0)^{\frac{9}{2}} f e^{-\psi^*} \|_{L^2(Q_-)} + \| r e^{-\psi^*} (T - x_0)^{\frac{15}{2}} \|_{L^2(0,T)} + \| z_0 \|_{L^2(\Omega)}). \end{aligned}$$

By Theorem 4.1 function z belongs to the space $H^{1,2}(Q_+) \cap H^{1,2}(Q_-)$. From (3.17) and (4.1)

$$(3.18) \quad \begin{aligned} & \sum_{|\alpha| \leq 1} \| (T - x_0)^{3|\alpha|} \partial^\alpha z e^{-\psi^*} \|_{L^2(Q_-)} + \sum_{|\alpha| \leq 1} \| (T - x_0)^{3(|\alpha|+1)} \partial^\alpha z e^{-\psi^*} \|_{L^2(Q_+)} \\ & \leq C_5 (\| (T - x_0)^{\frac{15}{2}} f e^{-\psi^*} \|_{L^2(Q_+)} \\ & + \| (T - x_0)^{\frac{9}{2}} f e^{-\psi^*} \|_{L^2(Q_-)} + \| (T - x_0)^{\frac{15}{2}} r e^{-\psi^*} \|_{L^2(0,T)} + \| z_0 \|_{L^2(\Omega)}). \end{aligned}$$

The pair (z, u) satisfies equations (3.9), (3.10). Estimate (3.7) follows from (3.18) and (4.1). Proof of proposition is complete. ■

Proof of Theorem 1.1. We set $\mu_1 = (T - x_0)^{15} m(x) e^{-2\psi^*}$, $m(x) = (T - x_0)^{15}$ for $x \in Q_+$, $m(x) = (T - x_0)^9$ for $x \in Q_-$, $\mu_2 = (T - x_0)^{15} e^{-2\psi^*}$, $\mathbf{Y} = L_{\mu_1}^2(Q) \times L_{\mu_2}^2(0, T) \times H_0^1(\Omega)$ and $\mathbf{X} = \{w = (w_1, w_2) | L_1(x, D)w_1 \in L_{\mu_1}^2(Q_+), L_1(x, D)w_2 \in L_{\mu_1}^2(Q_-), w_1(\cdot, b) = w_2(\cdot, a) = 0, [w](\cdot, 0) = 0, ([\partial_{x_1} w] - M\partial_{x_0} w)(\cdot, 0) \in L_{\mu_2}^2(0, T), (T - x_0)^{3(|\alpha|+1)} \partial^\alpha w \in H^{1,2}(Q_+), (T - x_0)^{3|\alpha|} \partial^\alpha w \in H^{1,2}(Q_-) \forall |\alpha| \leq 2\}$. Let (\mathbf{w}, \mathbf{u}) be the pair of functions defined by Condition 1. Consider the mapping

$$F(w, u) = (G(w + \mathbf{w}) + (\chi_\omega u + \chi_\omega \mathbf{u}, 0), ([\partial_{x_1} w] - M\partial_{x_0} w)(0, \cdot), w(0, \cdot)) : \mathbf{X} \rightarrow \mathbf{Y},$$

where $G = (G_1, G_2)$. Observe that by Condition 1 $F(0, 0) = 0$. Obviously mapping $F \in C^1(\mathbf{X}, \mathbf{Y})$ and $F'(0, 0)[\delta_1, \delta_2] = (G'(\mathbf{w})[\delta_1] + (\chi_\omega \delta_2, 0), ([\partial_{x_1} \delta_1] - M\partial_{x_0} \delta_1)(\cdot, 0), \delta_1(0, \cdot))$, where $G'(\mathbf{w})[\delta_1] = (L_1(x, D)\delta_{1,1} + \partial_{\xi_1} g_1(x, \mathbf{w}, \partial_{x_1} \mathbf{w})\delta_{1,1} + \partial_{\xi_2} g_1(x, \mathbf{w}, \partial_{x_1} \mathbf{w})\partial_{x_1} \delta_{1,1}, L_2(x, D)\delta_{1,2} + \partial_{\xi_1} g_2(x, \mathbf{w}, \partial_{x_1} \mathbf{w})\delta_{1,2} + \partial_{\xi_2} g_2(x, \mathbf{w}, \partial_{x_1} \mathbf{w})\partial_{x_1} \delta_{1,2})$. Since $\mathbf{w} \in L^\infty(Q)$ by (1.10) function $\partial_{\xi_2} g_2(x, \mathbf{w}, \partial_{x_1} \mathbf{w}) \in L^\infty(Q_-)$ and function $\partial_{\xi_2} g_1(x, \mathbf{w}, \partial_{x_1} \mathbf{w}) \in L^\infty(Q_+)$ and $\partial_{\xi_2} g_1(x, \mathbf{w}, \partial_{x_1} \mathbf{w}) \in L^2(Q_+)$, $\partial_{\xi_1} g_2(x, \mathbf{w}, \partial_{x_1} \mathbf{w}) \in L^2(Q_-)$. Since $\mathbf{X} \subset L^\infty(Q)$ operator $G' \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. By Proposition 3.1 $\text{Im } F'(0, 0) = \mathbf{Y}$. Hence by the implicit function theorem the equation $F(w, u) = \mathbf{z}$

can be solved for all $\mathbf{z} \in \mathbf{X}$ from some neighborhood of 0 in the space \mathbf{Y} . Proof of the theorem is complete. ■

4. APPENDIX

Theorem 4.1. *Let $u \equiv 0, g_1 \equiv 0, g_2 \equiv 0$. Suppose that (1.7)-(1.8) holds true. Then for any $w_0 \in H_0^1(\Omega)$, $f_1 \in L^2(Q_+)$, $f_2 \in L^2(Q_-)$, $\tilde{r} \in L^2(0, T)$ there exist a unique solution to problem (1.1)-(1.5) such that*

$$(4.1) \quad \begin{aligned} & \|w_1\|_{H^{1,2}(Q_+)} + \|w_2\|_{H^{1,2}(Q_-)} + \|w_1(\cdot, 0)\|_{H^1(0, T)} \\ & \leq C_1(\|f_1\|_{L^2(Q_+)} + \|f_2\|_{L^2(Q_-)} + \|w_0\|_{H^1(\Omega)} + \|\tilde{r}\|_{L^2(0, T)}). \end{aligned}$$

Proof. Without loss of generality we may assume that $a_j \equiv 1$. First we observe that it suffices to prove Theorem 4.1 for the case $w_0 \equiv 0$. Indeed consider the function $v_* = w_0 - \eta w_0(0)$, where $\eta \in C_0^\infty(\Omega)$, $\eta(0) = 1$. Let $v_1 \in H^{1,2}(Q_+)$ and $v_2 \in H^{1,2}(Q_-)$ be solution to the following initial value problems

$$\begin{aligned} L_1(x, D)v_1 &= f_1 \quad \text{in } Q_+, \quad v_1(x_0, 0) = v_1(x_0, b) = 0, \quad v_1(0, \cdot) = v_*, \\ L_2(x, D)v_2 &= f_2 \quad \text{in } Q_-, \quad v_2(x_0, a) = v_2(x_0, 0) = 0, \quad v_2(0, \cdot) = v_*. \end{aligned}$$

Then if z_1, z_2 be solution to the problem

$$(4.2) \quad \begin{aligned} L_1(x, D)z_1 &= g_1 \quad \text{in } Q_+, \quad L_2(x, D)z_2 = g_2 \quad \text{in } Q_-, \\ z_1(0, \cdot) &= 0, \quad z_2(0, \cdot) = 0, \quad z_2(\cdot, a) = z_1(\cdot, b) = 0, \\ z_1(\cdot, 0) &= z_2(\cdot, 0), \quad \partial_{x_1}z_1 - \partial_{x_1}z_2 = M\partial_{x_0}z(\cdot, 0) + \tilde{r} \quad \text{on } [0, T], \end{aligned}$$

where $g_1 = -L_1(x, D)(\eta w_0(0))$, $g_2 = -L_2(x, D)(\eta w_0(0))$. Then the function $(w_1, w_2) = (v_1, v_2) - (\eta w_0(0), \eta w_0(0))$ be solution to problem (1.1)-(1.5). Let K be a large positive parameter. In problem (1.1)-(1.5) we move from unknown function (w_1, w_2) to the unknown function $z = (z_1, z_2) = (w_1, w_2)e^{-Kx_0}$. the function z satisfies

$$(4.3) \quad \begin{aligned} (L_1(x, D) + K)z_1 &= g_1 \quad \text{in } Q_+, \quad (L_2(x, D) + K)z_2 = g_2 \quad \text{in } Q_-, \\ z_1(0, \cdot) &= 0, \quad z_2(0, \cdot) = 0, \quad z_2(\cdot, a) = z_1(\cdot, b) = 0, \\ z_1(\cdot, 0) - z_2(\cdot, 0) &= (\partial_{x_1}z_1 - \partial_{x_1}z_2 - M\partial_{x_0}z - MKz)(\cdot, 0) + p(\cdot) = 0 \quad \text{on } [0, T], \end{aligned}$$

where $g = (g_1, g_2)$, $g_i = e^{-Kx_0}f_i$, $p = e^{-Kx_0}\tilde{r}$.

Assume that $z \in H^{1,2}(Q_+) \cap H^{1,2}(Q_-)$ be solution to problem (4.3). The there exists a constant C_2 independent of z such that

$$(4.4) \quad \|z\|_{H^{1,2}(Q_+) \cap H^{1,2}(Q_-)} + \|z(\cdot, 0)\|_{H^1(0, T)} \leq C_2(\|p\|_{L^2(0, T)} + \|g\|_{L^2(Q)}).$$

The estimate (4.4) is the standard energy estimate which can be proved by multiplying (4.3) by z and $\partial_{x_0}z$. in particular the estimate (4.4) implies that it suffices to prove the existence of solution to problem (4.3) assuming that $g \in C^0([0, T]; L^2(\Omega))$, $p \in C^0[0, T]$ and coefficients $b, c \in C^1(\bar{Q})$.

Now consider the discretization of problem (1.1)-(1.5) in time

$$(4.5) \quad \begin{aligned} & \rho(x_{0,k+1}, \cdot) \frac{(z_{k+1} - z_k)}{h} - \partial_{x_1}^2 z_{k+1} + b(x_{0,k+1}, \cdot) \partial_{x_1} z_{k+1} \\ & + (c(x_{0,k+1}, \cdot) + K) z_{k+1} = g(x_{0,k}, \cdot) \quad \text{in } \Omega, \end{aligned}$$

$$(4.6) \quad [z_k] = 0, \quad \partial_{x_1}[z_{k+1}] = M \frac{(z_{k+1} - z_k)(0)}{h} + MKz_{k+1} + p(x_{0,k}).$$

Here $z_k = (z_{k,1}, z_{k,2})$, $z_0(x_1) = 0$, $x_{0,k} = kT/N$, $h = \frac{T}{N}$, $N \in \mathbb{Z}_+$.

Multiplying equation (4.5) by $h z_{k+1}$, integrating by parts and taking sum respect to k we have

$$(4.7) \quad \begin{aligned} & \sum_{k=0}^{N-1} \left\{ \int_{\Omega} \rho(x_{0,k+1}, x_1) (z_{k+1} - z_k) z_{k+1} dx_1 + M \left(\frac{z_{k+1} - z_k}{h} \right) (0) z_{k+1}(0) \right. \\ & \quad \left. + KM z_{k+1}^2(0) + h(\|\partial_{x_1} z_{k+1}\|_{L^2(\Omega)}^2 + K \|z_{k+1}\|_{L^2(\Omega)}^2) \right\} = \\ & \sum_{k=0}^{N-1} (g(x_{0,k}, \cdot) - b(x_{0,k+1}, \cdot) \partial_{x_1} z_{k+1} - c(x_{0,k+1}, \cdot) z_{k+1}, z_{k+1})_{L^2(\Omega)} - p(x_{0,k}) z_{k+1}(0) \leq \\ & C_3 \sum_{k=0}^{N-1} \left\{ \|g(x_{0,k}, \cdot)\|_{L^2(\Omega)}^2 + \frac{M}{2} \|\partial_{x_1} z_{k+1}\|_{L^2(\Omega)}^2 + p^2(x_{0,k}) \right\} + \frac{K}{2} \sum_{k=0}^{N-1} \|z_{k+1}\|_{L^2(\Omega)}^2. \end{aligned}$$

Observe that

$$(4.8) \quad \begin{aligned} & \sum_{k=0}^{N-1} \int_{\Omega} \rho(x_{0,k+1}, x_1) (z_{k+1} - z_k) z_{k+1} dx_1 = \\ & \sum_{k=0}^{N-1} \int_{\Omega} \left(\rho(x_{0,k+1}, x_1) z_{k+1}^2 - \sqrt{\rho(x_{0,k}, x_1)} z_k \sqrt{\rho(x_{0,k+1}, x_1)} z_{k+1} \right) dx_1 \\ & + \int_{\Omega} \left(\frac{\sqrt{\rho(x_{0,k}, x_1)} - \sqrt{\rho(x_{0,k+1}, x_1)}}{h} \right) h z_k z_{k+1} \sqrt{\rho(x_{0,k+1}, x_1)} dx_1 \geq \\ & -C_4 \|\partial_{x_0} \tilde{\rho}\|_{L^\infty(Q)} \sum_{k=0}^N \|z_k\|_{L^2(\Omega)}^2. \end{aligned}$$

From (4.7), (4.8) we obtain

$$(4.9) \quad \sum_{k=0}^N (K \|z_k\|_{L^2(\Omega)}^2 + \|\partial_{x_1} z_k\|_{L^2(\Omega)}^2) \leq C_5 (\|g\|_{C^0([0,T]; L^2(\Omega))} + \|p\|_{C^0[0,T]}).$$

Multiplying equation (4.5) by $\frac{(z_{k+1} - z_k)}{h}$, integrating by parts and taking sum respect to k we have

$$\sum_{k=0}^{N_0-1} \left\{ \int_{\Omega} \rho(x_{0,k}, x_1) \left(\frac{z_{k+1} - z_k}{h} \right)^2 dx_1 + M \left(\frac{z_{k+1} - z_k}{h} \right)^2 (0) + \frac{1}{h} \|\partial_{x_1} z_{k+1}\|_{L^2(\Omega)}^2 \right\}$$

$$\begin{aligned}
& + \left(MKz_{k+1}, \frac{(z_{k+1} - z_k)}{h} \right)_{L^2(\Omega)} - \frac{1}{h} (\partial_{x_1} z_{k+1}, \partial_{x_1} z_k)_{L^2(\Omega)} \Big\} = \\
& \sum_{k=0}^{N_0-1} (g(x_{0,k}, \cdot), \frac{(z_{k+1} - z_k)}{h})_{L^2(\Omega)} + p(x_{0,k}) \frac{(z_{k+1} - z_k)}{h} \leq \\
C_6 \sum_{k=0}^{N_0-1} & \left\{ \|g(x_{0,k}, \cdot)\|_{L^2(\Omega)}^2 + \|\partial_{x_1} z_{k+1}\|_{L^2(\Omega)}^2 + p^2(x_{0,k}) + \|z_{k+1}\|_{L^2(\Omega)}^2 \right\} \\
& + \sum_{k=0}^{N_0-1} \left(\frac{\alpha}{2} \left\| \frac{z_{k+1} - z_k}{h} \right\|_{L^2(\Omega)}^2 + \frac{M}{2} \left(\frac{(z_{k+1} - z_k)}{h} \right)^2 \right).
\end{aligned}$$

This inequality and (4.9) imply

$$\begin{aligned}
\sum_{k=0}^{N-1} & \left\{ h \int_{\Omega} \left(\frac{z_{k+1} - z_k}{h} \right)^2 dx_1 + Mh \left(\frac{z_{k+1} - z_k}{h} \right)^2 (0) + \frac{1}{h} \sup_{k \in \{1, \dots, N\}} \|\partial_{x_1} z_k\|_{L^2(\Omega)}^2 \right\} \\
(4.10) \quad & \leq C_7 \sum_{k=0}^{N-1} h \{ \|g(x_{0,k}, \cdot)\|_{L^2(\Omega)}^2 + p^2(x_{0,k}) \}.
\end{aligned}$$

We define function \tilde{z}_N as follows: $\tilde{z}_N(x_0, \cdot) = z_k$ if $x_0 = x_{0,k}$, otherwise on interval $(T(k+1)/N, Tk/N)$ function \tilde{z}_N is the linear function. Then estimate (4.10) implies

$$(4.11) \quad \|\tilde{z}_N\|_{H^{1,2}(Q)} + K^{\frac{1}{4}} \|\tilde{z}_N\|_{L^2(0,T;H^1(\Omega))} + \sqrt{K} \|\tilde{z}_N\|_{L^2(Q)} \leq C_8 (\|p\|_{C^0[0,T]} + \|g\|_{C^0([0,T];L^2(\Omega))}).$$

Functions $\tilde{z}_N = (\tilde{z}_{N,1}, \tilde{z}_{N,2})$ satisfy the initial value problem

$$\begin{aligned}
(4.12) \quad & (L_1(x, D) + K) \tilde{z}_{N,1} = g_{1,N} \quad \text{in } Q_+, \quad (L_2(x, D) + K) \tilde{z}_{N,2} = g_{2,N} \quad \text{in } Q_-, \\
& \tilde{z}_{N,1}(0, \cdot) = 0, \quad \tilde{z}_{N,2}(0, \cdot) = 0, \quad \tilde{z}_{N,2}(\cdot, a) = \tilde{z}_{N,1}(\cdot, b) = 0,
\end{aligned}$$

$$\tilde{z}_{N,1}(\cdot, 0) - \tilde{z}_{N,2}(\cdot, 0) = (\partial_{x_1} \tilde{z}_{N,1} - \partial_{x_1} \tilde{z}_{N,2} - M \partial_{x_0} \tilde{z}_{N,1}(\cdot, 0) - KM \tilde{z}_{N,1}(\cdot, 0) + p_N(\cdot)) = 0 \text{ on } [0, T].$$

Let sequence \tilde{z}_N after possibly taking a subsequence converges to the function z weakly in $H^{1,2}(Q_+) \cap H^{1,2}(Q_-)$. Observe that

$$(g_{1,N}, g_{2,N}, p_N) \rightarrow 0 \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega_+)) \times L^2(0, T; H^{-1}(\Omega_-)) \times H^{-1}(0, T).$$

Passing to the limit in (4.12) we obtain that function z solution to problem (4.3). Estimate (4.1) follows from (4.4). Proof of theorem is complete. ■

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