Maximum principle for stochastic optimal control problem of forward-backward stochastic difference systems

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Abstract: In this paper, we study the maximum principle for stochastic optimal control problems of forward-backward stochastic difference systems (FBSΔSs). Two types of FBSΔSs are investigated. The first one is described by a partially coupled forward-backward stochastic difference equation (FBSΔE) and the second one is described by a fully coupled FBSΔE. By adopting an appropriate representation of the product rule and an appropriate formulation of the backward stochastic difference equation (BSΔE), we deduce the adjoint difference equation. Finally, the maximum principle for this optimal control problem with the control domain being convex is established.

Keywords: backward stochastic difference equations; forward-backward stochastic difference equations; monotone condition; stochastic optimal control; maximum principle

1 Introduction

The Maximum Principle is one of the principal approaches in solving the optimal control problems. A lot of work has been done on the Maximum Principle for forward stochastic system. See, for example, Bensoussan [2], Bismut [4], Kushner [12], Peng [16]. Peng also firstly studied one kind of forward-backward stochastic control system (FBSCS) in [17] and obtained the maximum principle for this kind of control system with control domain being convex. The FBSCSs have wide applications in many fields. As the stochastic differential recursive utility, which is a generalization of a standard additive utility, can be regarded as a solution of a backward stochastic differential equation (BSDE). The recursive utility optimization problem can be described by a optimization problem for a FBSCS (see [19]). Besides, in the dynamic principal-agent problem with unobservable states and actions, the principal's problem can be formulated as a partial information optimal control problem of a FBSCS (see [22]). We refer to [8], [11], [13], [21], [24], [25] for other works on optimization problems for FBSCSs.

In this paper, we will discuss the Maximum Principle for optimal control of discrete time systems described by forward-backward stochastic difference equations (FBSΔEs). To the best of our knowledge, there are few

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results on such optimization control problems. In fact, the discrete time control systems are of great value in practice. For example, the digital control can be formulated as discrete time control problems, where the sampled data is obtained at discrete instants of time. Besides, the forward-backward stochastic difference system (FBSΔS) can be used for modeling in financial markets. For example, the solution to the backward stochastic difference equation (BSΔE) can be used to construct time-consistent nonlinear expectations (see [5], [6]) and be used for pricing in the financial markets (see [3]). However, the formulation of BSΔE is quite different from its continuous time counterpart. Many works are devoted to the study of BSΔEs (see, e.g. [3], [5], [6], [20]). Based on the driving process, there are mainly two types of formulations of BSΔEs. One is driving by a finite state process which takes values from the basis vectors (as in [5]) and the other is driving by a martingale with independent increments (as in [3]). For the latter case, the solution of the BSΔE is a triple of processes which is due to the discrete time version of the Kunita–Watanabe decomposition. In this paper, we adopt the second type of formulation to investigate the optimization problems for FBSΔSs.

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\right)$ be a probability space, and W_t be a martingale process with independent increments. Define the difference operator Δ as $\Delta V_t = V_{t+1} - V_t$. Here we consider two types of controlled FBSΔSs.

Problem 1 (partially coupled system):

The controlled system is

$$\left\{\begin{array}{rcl} \Delta X_t &=& b\left(t, X_t, u_t\right) + \sum_{i=1}^{d} \sigma_i\left(t, X_t, u_t\right) \Delta W_t^i, \\ X_0 &=& x_0, \\ \Delta Y_t &=& -f\left(t+1, X_{t+1}, Y_{t+1}, Z_{t+1}, u_{t+1}\right) + Z_t \Delta W_t + \Delta N_t, \\ Y_T &=& y_T, \end{array}\right. \tag{1.1}$$

and the cost functional is

$$J\left(u\left(\cdot\right)\right) = \mathbb{E}\left[\sum_{t=0}^{T-1} l\left(t, X_t, Y_t, Z_t, u_t\right) + h\left(X_T\right)\right]. \tag{1.2}$$

Problem 2 (fully coupled system):

The controlled system is:

$$\left\{\begin{array}{rcl} \Delta X_t &=& b\left(t, X_t, Y_t, Z_t, u_t\right) + \sum_{i=1}^{d} \sigma_i\left(t, X_t, Y_t, Z_t, u_t\right) \Delta W_t^i, \\ X_0 &=& x_0, \\ \Delta Y_t &=& -f\left(t+1, X_{t+1}, Y_{t+1}, Z_{t+1}, u_{t+1}\right) + Z_t \Delta W_t + \Delta N_t, \\ Y_T &=& y_T, \end{array}\right. \tag{1.3}$$

and the cost functional is

$$J\left(u\left(\cdot\right)\right) = \mathbb{E}\left[\sum_{t=0}^{T-1} l\left(t, X_t, Y_t, Z_t, u_t\right) + h\left(X_T\right)\right]. \tag{1.4}$$

Let $\{U_t\}_{t\in\{0,1,...,T-1\}}$ be a sequence of nonempty convex subset of $\mathbb{R}^r$. We denote the set of admissible controls $\mathcal{U}$ by $\mathcal{U} = \left\{u(\cdot) \in \mathcal{M}^2(0, T-1; \mathbb{R}^r) \,|\, u(t) \in U_t\right\}$. It can be seen that in Problem 1, b and σ do not contain the solution (Y, Z) of the backward equation. This kind of FBSΔE is called the partially coupled FBSΔE. Meanwhile, the system in Problem 2 is called the fully coupled FBSΔE.

The optimal control problem is to find the optimal control $u \in \mathcal{U}$, such that the optimal control and the corresponding state trajectory can minimize the cost functional $J(u(\cdot))$. In this paper, we assume the control domain is convex. By making the perturbation of the optimal control at a fixed time point, we obtain the maximum principle for problem 1 and 2.

To build the maximum principle, the key step is to find the adjoint variables which can be applied to deduce the variational inequality. In [14], the authors studied the maximum principle for a discrete time stochastic optimal control problem in which the state equation is only governed by a forward stochastic difference equation. By applying the Riesz representation theorem, they explicitly obtained the adjoint variables and establish the maximum principle. But to solve our problems, we need to construct the adjoint difference equations since generally the adjoint variables can not be obtained explicitly for our case. To construct the adjoint equations in our discrete time framework, the techniques which are adopted for the continuous time framework as in [16, 17] are not appliable. In this paper, we propose two techniques to deduce the adjoint difference equations. The first one is that we choose the following product rule:

$$\Delta \langle X_t, Y_t \rangle = \langle X_{t+1}, \Delta Y_t \rangle + \langle \Delta X_t, Y_t \rangle$$

where X_t (resp. Y_t) subjects to a forward (resp. backward) stochastic difference equation. The second one is that the BSΔE should be formulated as in (2.1). In other words, the generator f of the BSΔE (2.1) depends on time $t+1$. It is worth pointing out that this kind of formulation is just the formulation of the adjoint equations for stochastic optimal control problems (see [14] for the classical case). Based on these two techniques, we can deduce the adjoint difference equations. The readers may refer to Remark 3.6 for more details.

The remainder of this paper is organized as follows. In section 2, two types of the controlled FBSΔSs are formulated. We deduce the maximum principle for the partially coupled controlled FBSΔS in section 3. Finally, we establish the maximum principle for the fully coupled controlled FBSΔS in section 4.

2 Preliminaries and model formulation

Let T be a deterministic terminal time, and let $\mathcal{T} := \{0, 1, ..., T\}$. Consider a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P\right)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$. Here we define the difference operator Δ as $\Delta U_t = U_{t+1} - U_t$. Let W be a fixed $\mathbb{R}^d$-valued square integrable martingale process with independent increments, i.e. $\mathbb{E}[\Delta W_t | \mathcal{F}_t] = \mathbb{E}[\Delta W_t] = 0$ for any $t \in \{0, ..., T-1\}$. Also we suppose that $\mathbb{E}\left[\Delta W_t (\Delta W_t)^*\right] = I_d$ for any $t \in \{0, ..., T-1\}$. Here $(\cdot)^*$ denotes vector transposition. We assume that $\mathcal{F}_t$ is the completion of the σ-algebra generated by the process W up to time t.

Denote by $L^2(\mathcal{F}_t; \mathbb{R}^n)$ the set of all $\mathcal{F}_t-$measurable square integrable random variable X_t taking values in $\mathbb{R}^n$ and by $\mathcal{M}^2(0, t; \mathbb{R}^n)$ the set of all $\{\mathcal{F}_s\}_{0\le s\le t}$-adapted square integrable process X taking values in $\mathbb{R}^n$.

Moreover, we define $e_i = (0, 0, ..., 0, 1, 0, ..., 0)^* \in \mathbb{R}^n$ and mention that an inequality on a vector quantity is to hold componentwise.

Consider the following backward stochastic difference equation (BSΔE):

$$\begin{cases} \Delta Y_t &= -f(t+1, Y_{t+1}, Z_{t+1}) + Z_t \Delta W_t + \Delta N_t, \\ Y_T &= \eta, \end{cases} \tag{2.1}$$

where $\eta \in L^2(\mathcal{F}_T; \mathbb{R}^n)$, $f : \Omega \times \{1, 2, ..., T\} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \longmapsto \mathbb{R}^n$.

Assumption 2.1 *A1. The function $f(t, y, z)$ is uniformly Lipschitz continuous and independent of z at $t = T$, i.e. there exists constants $c_1, c_2 > 0$, such that for any $t \in \{1, 2, ..., T-1\}$, $y_1, y_2 \in \mathbb{R}^n$, $z_1, z_2 \in \mathbb{R}^{n\times d}$,*

$$|f(T, y_1, z_1) - f(T, y_2, z_2)| \leq c_1 |y_1 - y_2|,$$
$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_1 |y_1 - y_2| + c_2 \|z_1 - z_2\|, \ P-a.s.$$

A2. $f(t, 0, 0) \in L^2(\mathcal{F}_t; \mathbb{R}^n)$ for any $t \in \{1, 2, ..., T\}$.

Remark 2.2 *The BSΔE (2.1) is analogous to the continuous time BSDE driven by a general martingale (cf. [9]), and the solution is a triple of processes.*

Definition 2.3 *A solution to BSΔE (2.1) is a triple of processes $(Y, Z, N) \in \mathcal{M}^2(0, T; \mathbb{R}^n) \times \mathcal{M}^2(0, T-1; \mathbb{R}^{n\times d}) \times \mathcal{M}^2(0, T; \mathbb{R}^n)$ which satisfies equality (2.1) for all $t \in \{0, 1, ..., T-1\}$, and N is a martingale process strongly orthogonal to W.*

By using the Galtchouk-Kunita-Watanabe decomposition in [3], we can obtain the existence and uniqueness result of BSΔE (2.1):

Theorem 2.4 *Suppose that Assumption (2.1) holds. Then for any terminal condition $\eta \in L^2(\mathcal{F}_T; \mathbb{R}^n)$, the BS$\Delta$E (2.1) has a unique adapted solution (Y, Z, N).*

Proof. We first prove the existence and uniqueness of $(Y_{T-1}, Z_{T-1}, \Delta N_{T-1})$. Due to Assumption (2.1) and $\eta \in L^2(\mathcal{F}_T; \mathbb{R}^n)$, we get $f(T, \eta) \in L^2(\mathcal{F}_T; \mathbb{R}^n)$. Here we omit the variable Z since f is independent of Z at time T. Then we have $\mathbb{E}\left[|\mathbb{E}[\eta + f(T, \eta)|\mathcal{F}_{T-1}]|^2\right] < \infty$. Hence, $\eta + f(T, \eta) - \mathbb{E}[\eta + f(T, \eta)|\mathcal{F}_{T-1}]$ is a square integrable martingale difference. So it admits the Galtchouk-Kunita-Watanabe decomposition, which implies that there exists $Z_{T-1} \in \mathcal{F}_{T-1}$, $Z_{T-1}\Delta W_{T-1} \in L^2(\mathcal{F}_T; \mathbb{R}^n)$, $\Delta N_{T-1} \in L^2(\mathcal{F}_T; \mathbb{R}^n)$ such that $\mathbb{E}[\Delta N_{T-1}|\mathcal{F}_{T-1}] = 0$, $\mathbb{E}\left[e_i^* \Delta N_{T-1}(\Delta W_{T-1})^* |\mathcal{F}_{T-1}\right] = 0$ and

$$\eta + f(T, \eta) - \mathbb{E}[\eta + f(T, \eta)|\mathcal{F}_{T-1}] = Z_{T-1}\Delta W_{T-1} + \Delta N_{T-1}. \tag{2.2}$$

Moreover, ΔN_{T-1} is uniquely determined in this decomposition. For fixed $i \in \{1, 2, ..., n\}$, premultiply the equation by e_i^*, postmultiply the equation by $(\Delta W_{T-1})^*$ and then take the $\mathcal{F}_{T-1}$ conditional expectation. This yields that

$$\mathbb{E}\left[e_i^*(\eta + f(T, \eta))(\Delta W_{T-1})^* |\mathcal{F}_{T-1}\right] = e_i^* Z_{T-1}$$

since $\mathbb{E}\left[\Delta W_{T-1}\left(\Delta W_{T-1}\right)^{*}|\mathcal{F}_{T-1}\right]=I$. Therefore, we get the unique Z_{T-1} by

$$Z_{T-1}=\mathbb{E}\left[\left(\eta+f\left(T,\eta\right)\right)\left(\Delta W_{T-1}\right)^{*}|\mathcal{F}_{T-1}\right]$$

and

$$\mathbb{E}\left[\left\|Z_{T-1}\right\|^{2}\right]\quad\leq\quad\mathbb{E}\left[\mathbb{E}\left[\left|\eta+f\left(T,\eta\right)\right|^{2}|\mathcal{F}_{T-1}\right]\mathbb{E}\left[\left|\left(\Delta W_{T-1}\right)\right|^{2}|\mathcal{F}_{T-1}\right]\right]<\infty.$$

It leads that $Y_{T-1}=\mathbb{E}\left[\eta+f\left(T,\eta\right)|\mathcal{F}_{T-1}\right]$ and $Y_{T-1}\in L^{2}\left(\mathcal{F}_{T-1};\mathbb{R}^{n}\right)$.

Then, by similar arguments as above, we can obtain the unique solution $(Y_t,Z_t,\Delta N_t)\in L^{2}\left(\mathcal{F}_{t};\mathbb{R}^{n}\right)\times L^{2}\left(\mathcal{F}_{t};\mathbb{R}^{n\times d}\right)\times L^{2}\left(\mathcal{F}_{t};\mathbb{R}^{n}\right)$ for $t\in\{0,1,...,T-2\}$. Moreover,

$$\begin{aligned}
Z_t&=\mathbb{E}\left[\left(Y_{t+1}+f\left(t+1,Y_{t+1},Z_{t+1}\right)\right)\left(\Delta W_t\right)^{*}|\mathcal{F}_t\right],\\
Y_t&=\mathbb{E}\left[Y_{t+1}+f\left(t+1,Y_{t+1},Z_{t+1}\right)|\mathcal{F}_t\right].
\end{aligned}$$

By taking the convention $N_0=0$ and letting $N_t=N_0+\sum_{s=0}^{t-1}\Delta N_s$, we have that (2.1) holds true for all $t\in\{0,1,...,T-1\}$. Finally, since

$$\begin{aligned}
&\mathbb{E}\left[e_i^{*}N_t\left(W_t\right)^{*}|\mathcal{F}_{t-1}\right]\\
&=e_i^{*}\sum_{s=0}^{t-2}\Delta N_s\mathbb{E}\left[\left(W_t\right)^{*}|\mathcal{F}_{t-1}\right]+\mathbb{E}\left[e_i^{*}\Delta N_{t-1}\left(W_{t-1}+\Delta W_{t-1}\right)^{*}|\mathcal{F}_{t-1}\right]\\
&=e_i^{*}N_{t-1}\left(W_{t-1}\right)^{*},
\end{aligned}$$

we conclude that N is strongly orthogonal to W. ■

Now we consider the control systems (1.1)-(1.2) and (1.3)-(1.4).

Let the coefficients in system (1.1)-(1.2) be such that:

$$\begin{aligned}
b\left(\omega,t,x,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{m}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{m},\\
\sigma_i\left(\omega,t,x,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{m}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{m},\\
f\left(\omega,t,x,y,z,u\right)&:\Omega\times\{1,2,...,T\}\times\mathbb{R}^{m}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{n},\\
l\left(\omega,t,x,y,z,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{m}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R},\\
h\left(\omega,x\right)&:\Omega\times\mathbb{R}^{m}\rightarrow\mathbb{R}.
\end{aligned}$$

And the coefficients in system (1.3)-(1.4) be such that:

$$\begin{aligned}
b\left(\omega,t,x,y,z,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{n}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{n},\\
\sigma_i\left(\omega,t,x,y,z,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{n}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{n},\\
f\left(\omega,t,x,y,z,u\right)&:\Omega\times\{1,2,...,T\}\times\mathbb{R}^{n}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R}^{n},\\
l\left(\omega,t,x,y,z,u\right)&:\Omega\times\{0,1,...,T-1\}\times\mathbb{R}^{n}\times\mathbb{R}^{n}\times\mathbb{R}^{n\times d}\times\mathbb{R}^{r}\rightarrow\mathbb{R},\\
h\left(\omega,x\right)&:\Omega\times\mathbb{R}^{n}\rightarrow\mathbb{R}.
\end{aligned}$$

Remark 2.5 *The cost functional in [17] consists of three parts: the running cost functional, the terminal cost functional of X_T, the initial cost functional of Y_0. In our formulation, if we take $l\left(\omega,0,X_0,Y_0,Z_0,u_0\right)=\gamma\left(\omega,Y_0\right)$, then the cost functional (1.4) for our discrete time framework can be reduced to the cost functional in [17] formally.*

For system (1.1)-(1.2), we assume that:

Assumption 2.6 *For $\varphi = b, \sigma_i, f, l, h$, we assume that*

1. *φ is adapted map, i.e. for any $(x, y, z, u) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^r$, $\varphi(\cdot, \cdot, x, y, z, u)$ is $\{\mathcal{F}_t\}$-adapted process. Moreover, $\varphi(\cdot, t, 0, 0, 0, 0) \in L^2(\mathcal{F}_t)$.*
2. *$\forall t \in \{0, 1, ..., T\}$, $\varphi(\cdot, t, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable with respect to x, y, z, u, and $\varphi_x, \varphi_y, \varphi_{z_i}, \varphi_u$ are uniformly bounded $P - a.s.$. Also, for $t = T$, $f_{z_i} \equiv 0$, i.e. f is independent of z at time T. Here we use z_i to represent the i-th column of the matrix z.*

Let

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, A(t, \lambda; u) = \begin{pmatrix} -f \\ b \\ \sigma \end{pmatrix} (t, \lambda; u).$$

For control system (1.3)-(1.4), we additionally assume that:

Assumption 2.7 *For any $u \in \mathcal{U}$, the coefficients in (1.3) satisfy the following monotone conditions, i.e. when $t \in \{1, ..., T-1\}$,*

$$\langle A(t, \lambda_1; u) - A(t, \lambda_2; u), \lambda_1 - \lambda_2 \rangle \leq -\alpha |\lambda_1 - \lambda_2|^2, P - a.s.,$$
$$\forall \lambda_1, \lambda_2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n;$$

when $t = T$,

$$\langle -f(T, x_1, y, z, u) + f(T, x_2, y, z, u), x_1 - x_2 \rangle \leq -\alpha |x_1 - x_2|^2, P - a.s.;$$

when $t = 0$,

$$\begin{aligned} &\langle b(0, \lambda_1; u) - b(0, \lambda_2; u), y_1 - y_2 \rangle + \langle \sigma(0, \lambda_1; u) - \sigma(0, \lambda_2; u), z_1 - z_2 \rangle \\ &\leq -\alpha \left[|y_1 - y_2|^2 + \|z_1 - z_2\|^2 \right], \end{aligned}$$

where α is a given positive constant.

Besides, in the following, we formally denote $b(T, x, y, z, u) \equiv 0$, $\sigma(T, x, y, z, u) \equiv 0$, $l(T, x, y, z, u) \equiv 0$, $f(0, x, y, z, u) \equiv 0$.

3 Maximum principle for the partially coupled FBSΔE system

For any $u \in \mathcal{U}$, it is obvious that there exists a unique solution $\{X_t\}_{t=0}^T \in \mathcal{M}^2(0, T; \mathbb{R}^m)$ to the forward stochastic difference equation in the system (1.1). Then, by Theorem 2.4, the backward equation in the system (1.1) has a unique solution (Y, Z, N) where $Y = \{Y_t\}_{t=0}^T$, $Z = \{Z_t\}_{t=0}^{T-1}$ and $N = \{N_t\}_{t=0}^T$.

Suppose that $\bar{u} = \{\bar{u}_t\}_{t=0}^{T}$ is the optimal control of problem (1.1)-(1.2) and $\left(\bar{X}, \bar{Y}, \bar{Z}\right)$ is the corresponding optimal trajectory. For a fixed time $0 \leq s \leq T$, choose any $\Delta v \in L^2\left(\mathcal{F}_s; \mathbb{R}^r\right)$ such that $\bar{u}_s + \Delta v$ takes values in U_s. For any $\varepsilon \in [0,1]$, construct the perturbed admissible control

$$u_t^{\varepsilon} = (1-\delta_{ts})\,\bar{u}_t + \delta_{ts}\left(\bar{u}_s + \varepsilon\Delta v\right) = \bar{u}_t + \delta_{ts}\varepsilon\Delta v, \tag{3.1}$$

where $\delta_{ts} = 1$ for $t = s$, $\delta_{ts} = 0$ for $t \neq s$ and $t \in \{0, 1, ..., T\}$. Since U_s is a convex set, $\{u_t^{\varepsilon}\}_{t=0}^{T} \in \mathcal{U}$ is an admissible control. Let $(X^{\varepsilon}, Y^{\varepsilon}, Z^{\varepsilon}, N^{\varepsilon})$ be the solution of (1.1) corresponding to the control u^{ε}.

Set

$$\begin{aligned} \bar{\varphi}(t) &= \varphi\left(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t\right), \quad & \varphi^{\varepsilon}(t) &= \varphi\left(t, X_t^{\varepsilon}, Y_t^{\varepsilon}, Z_t^{\varepsilon}, u_t^{\varepsilon}\right), \\ \widetilde{\varphi}^{\varepsilon}(t) &= \varphi\left(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, u_t^{\varepsilon}\right), \quad & \varphi_{\mu}(t) &= \varphi_{\mu}\left(t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t\right), \end{aligned} \tag{3.2}$$

where $\varphi = b$, σ_i, g, f, l, h and $\mu = x$, y, z_i and u.

Then, we have the following estimates.

Lemma 3.1 *Under Assumption (2.6), we have*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left|X_t^{\varepsilon} - \bar{X}_t\right|^2 \leq C\varepsilon^2 \mathbb{E}\left|\Delta v\right|^2. \tag{3.3}$$

Proof. In the following, the positive constant C may change from lines to lines.

When $t = 0, ..., s$, $X_t^{\varepsilon} = \bar{X}_t$.

When $t = s+1$,

$$X_{s+1}^{\varepsilon} - \bar{X}_{s+1} = \widetilde{b}^{\varepsilon}(s) - \overline{b}(s) + \sum_{i=1}^{d}\left[\widetilde{\sigma}_i^{\varepsilon}(s) - \overline{\sigma}_i(s)\right]\Delta W_s^i.$$

Then,

$$\mathbb{E}\left|X_{s+1}^{\varepsilon} - \bar{X}_{s+1}\right|^2 \leq (d+1)\,\mathbb{E}\left[\left|\widetilde{b}^{\varepsilon}(s) - \overline{b}(s)\right|^2 + \sum_{i=1}^{d}\left|\left[\widetilde{\sigma}_i^{\varepsilon}(s) - \overline{\sigma}_i(s)\right]\Delta W_s^i\right|^2\right].$$

By the boundedness of b_u, we have

$$\mathbb{E}\left[\left|\widetilde{b}^{\varepsilon}(s) - \overline{b}(s)\right|^2\right] \leq C\mathbb{E}\left[\left|u_s^{\varepsilon} - \bar{u}_s\right|^2\right] = C\varepsilon^2\mathbb{E}\left[\left|\Delta v\right|^2\right].$$

By the boundedness of σ_{iu}, we have

$$\begin{aligned} &\mathbb{E}\left|\left[\widetilde{\sigma}_i^{\varepsilon}(s) - \overline{\sigma}_i(s)\right]\Delta W_s^i\right|^2 \\ &= \mathbb{E}\left[\left|\widetilde{\sigma}_i^{\varepsilon}(s) - \overline{\sigma}_i(s)\right|^2 \mathbb{E}\left[\left|\Delta W_s^i\right|^2 |\mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[\left|\left[\widetilde{\sigma}_i^{\varepsilon}(s) - \overline{\sigma}_i(s)\right]\right|^2\right] \\ &\leq C\varepsilon^2\mathbb{E}\left[\left|\Delta v\right|^2\right] \end{aligned}$$

which leads to

$$\mathbb{E}\left|X_{s+1}^{\varepsilon} - \bar{X}_{s+1}\right|^2 \leq C\varepsilon^2\mathbb{E}\left[\left|\Delta v\right|^2\right].$$

When $t=s+2,...,T$,

$$\begin{aligned}\mathbb{E}\left|X_t^{\varepsilon}-\bar{X}_t\right|^2 &\leq (d+1)\,\mathbb{E}\left|b\left(t-1,X_{t-1}^{\varepsilon},\bar{u}_{t-1}\right)-b\left(t-1,\bar{X}_{t-1},\bar{u}_{t-1}\right)\right|^2\\ &+\sum_{i=1}^d\left|\left[\sigma_i\left(t-1,X_{t-1}^{\varepsilon},\bar{u}_{t-1}\right)-\sigma_i\left(t-1,\bar{X}_{t-1},\bar{u}_{t-1}\right)\right]\Delta W_s^i\right|^2.\end{aligned}$$

Due to the boundedness of b_x, σ_{ix}, we obtain $\mathbb{E}\left|X_t^{\varepsilon}-\bar{X}_t\right|^2\leq C\mathbb{E}\left[\left|X_{t-1}^{\varepsilon}-\bar{X}_{t-1}\right|^2\right]$. Thus, by induction we prove the result. ■

Let $\xi=\{\xi_t\}_{t=0}^T$ be the solution to the following difference equation,

$$\begin{cases}\Delta\xi_t &= b_x(t)\,\xi_t+\delta_{ts}b_u(t)\,\varepsilon\Delta v+\sum_{i=1}^d\left[\sigma_{ix}(t)\,\xi_t+\delta_{ts}\varepsilon\sigma_{iu}(t)\,\Delta v\right]\Delta W_t^i,\\ \xi_0 &= 0,\end{cases}\tag{3.4}$$

It is easy to check that

$$\sup_{0\leq t\leq T}\mathbb{E}\left|\xi_t\right|^2\leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2,\tag{3.5}$$

and we have the following result:

Lemma 3.2 *Under Assumption 2.6, we have*

$$\sup_{0\leq t\leq T}\mathbb{E}\left|X_t^{\varepsilon}-\bar{X}_t-\xi_t\right|^2=o\left(\varepsilon^2\right).$$

Proof. When $t=0,...,s$, $X_t^{\varepsilon}=\bar{X}_t$ and $\xi_t=0$ which lead to $X_t^{\varepsilon}-\bar{X}_t-\xi_t=0$.

When $t=s+1$,

$$X_{s+1}^{\varepsilon}-\bar{X}_{s+1}-\xi_{s+1}=\left[\widetilde{b}_u(s)-b_u(s)\right]\varepsilon\Delta v+\sum_{i=1}^d\left[\widetilde{\sigma}_{iu}(s)-\sigma_{iu}(s)\right]\varepsilon\Delta v\Delta W_s^i$$

where

$$\begin{aligned}\widetilde{b}_u(s) &= \int_0^1 b_u\left(s,\bar{X}_s,\bar{u}_s+\lambda\left(u_s^{\varepsilon}-\bar{u}_s\right)\right)d\lambda,\\ \widetilde{\sigma}_{iu}(s) &= \int_0^1 \sigma_{iu}\left(s,\bar{X}_s,\bar{u}_s+\lambda\left(u_s^{\varepsilon}-\bar{u}_s\right)\right)d\lambda.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left|X_{s+1}^{\varepsilon}-\bar{X}_{s+1}-\xi_{s+1}\right|^2 &\leq (d+1)\,\mathbb{E}\left[\left|\left[\widetilde{b}_u(s)-b_u(s)\right]\varepsilon\Delta v\right|^2+\sum_{i=1}^d\left|\left[\widetilde{\sigma}_{iu}(s)-\sigma_{iu}(s)\right]\varepsilon\Delta v\Delta W_s^i\right|^2\right]\\ &\leq C\mathbb{E}\left[\left\|\widetilde{b}_u(s)-b_u(s)\right\|^2\left|\Delta v\right|^2+\sum_{i=1}^d\left\|\widetilde{\sigma}_{iu}(s)-\sigma_{iu}(s)\right\|^2\left|\Delta v\right|^2\right]\varepsilon^2.\end{aligned}$$

Since $\left\|\widetilde{b}_u(s)-b_u(s)\right\|\to 0$ and $\left\|\widetilde{\sigma}_{iu}(s)-\sigma_{iu}(s)\right\|\to 0$ as $\varepsilon\to 0$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{E}\left|X_{s+1}^{\varepsilon}-\bar{X}_{s+1}-\xi_{s+1}\right|^2=0.$$

When $t = s+2, ..., T$,

$$\begin{aligned} X_t^\varepsilon - \bar{X}_t - \xi_t &= \widetilde{b}_x(t-1)\left(X_{t-1}^\varepsilon - \bar{X}_{t-1} - \xi_{t-1}\right) + \left[\widetilde{b}_x(t-1) - b_x(t-1)\right]\xi_{t-1} \\ &+ \textstyle\sum_{i=1}^d \left\{\widetilde{\sigma}_{ix}(t-1)\left(X_{t-1}^\varepsilon - \bar{X}_{t-1} - \xi_{t-1}\right) + \left[\widetilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)\right]\xi_{t-1}\right\}\Delta W_{t-1}^i \end{aligned}$$

where

$$\begin{aligned} \widetilde{b}_x(t) &= \int_0^1 b_x\left(t, \bar{X}_t + \lambda\left(X_t^\varepsilon - \bar{X}_t\right), \bar{u}_t\right) d\lambda, \\ \widetilde{\sigma}_{ix}(t) &= \int_0^1 \sigma_{ix}\left(t, \bar{X}_t + \lambda\left(X_t^\varepsilon - \bar{X}_t\right), \bar{u}_t\right) d\lambda. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\left|X_t^\varepsilon - \bar{X}_t - \xi_t\right|^2 \leq\ & C\mathbb{E}\left\|\widetilde{b}_x(t-1)\right\|^2\left|X_{t-1}^\varepsilon - \bar{X}_{t-1} - \xi_{t-1}\right|^2 + \left\|\widetilde{b}_x(t-1) - b_x(t-1)\right\|^2\left|\xi_{t-1}\right|^2 \\ & + \textstyle\sum_{i=1}^d \left\|\widetilde{\sigma}_{ix}(t-1)\right\|^2\left|X_{t-1}^\varepsilon - \bar{X}_{t-1} - \xi_{t-1}\right|^2 + \sum_{i=1}^d \left\|\widetilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)\right\|^2\left|\xi_{t-1}\right|^2 \\ \leq\ & C\mathbb{E}\left(\left\|\widetilde{b}_x(t-1)\right\|^2 + \textstyle\sum_{i=1}^d \left\|\widetilde{\sigma}_{ix}(t-1)\right\|^2\right)\left|X_{t-1}^\varepsilon - \bar{X}_{t-1} - \xi_{t-1}\right|^2 \\ & + \left\|\widetilde{b}_x(t-1) - b_x(t-1)\right\|^2\left|\xi_{t-1}\right|^2 + \textstyle\sum_{i=1}^d \left\|\widetilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)\right\|^2\left|\xi_{t-1}\right|^2\Big]. \end{aligned}$$

$\left\|\widetilde{b}_x(t-1) - b_x(t-1)\right\| \to 0$ and $\left\|\widetilde{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)\right\| \to 0$ as $\varepsilon \to 0$. Since $\widetilde{b}_x(t-1)$ and $\widetilde{\sigma}_{ix}(t-1)$ are bounded, by the estimation (3.5), we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\mathbb{E}\left|X_t^\varepsilon - \bar{X}_t - \xi_t\right|^2 = 0.$$

This completes the proof. ■

Lemma 3.3 *Under Assumption 2.6, we have*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left|Y_t^\varepsilon - \bar{Y}_t\right|^2 \leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2 \tag{3.6}$$

$$\sup_{0 \leq t \leq T-1} \mathbb{E}\left\|Z_t^\varepsilon - \bar{Z}_t\right\|^2 \leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2. \tag{3.7}$$

Proof. It is obvious that $Y_T^\varepsilon - \bar{Y}_T = 0$ at time T.

When $t = s, ..., T-1$ (if $s = T$, skip this part), we have

$$\begin{aligned} & \mathbb{E}\left|f\left(t+1, X_{t+1}^\varepsilon, Y_{t+1}^\varepsilon, Z_{t+1}^\varepsilon, \bar{u}_{t+1}\right) - f\left(t+1, \bar{X}_{t+1}, \bar{Y}_{t+1}, \bar{Z}_{t+1}, \bar{u}_{t+1}\right)\right|^2 \\ & \leq C\mathbb{E}\left[\left|X_{t+1}^\varepsilon - \bar{X}_{t+1}\right|^2 + \left|Y_{t+1}^\varepsilon - \bar{Y}_{t+1}\right|^2 + \left\|Z_{t+1}^\varepsilon - \bar{Z}_{t+1}\right\|^2\right] \\ & \leq C\mathbb{E}\left[\left|Y_{t+1}^\varepsilon - \bar{Y}_{t+1}\right|^2 + \left\|Z_{t+1}^\varepsilon - \bar{Z}_{t+1}\right\|^2\right] + C_1\varepsilon^2\mathbb{E}\left|\Delta v\right|^2. \end{aligned}$$

It yields that

$$\mathbb{E}\left|Y_t^\varepsilon - \bar{Y}_t\right|^2 \leq C\mathbb{E}\left[\left|Y_{t+1}^\varepsilon - \bar{Y}_{t+1}\right|^2 + \left\|Z_{t+1}^\varepsilon - \bar{Z}_{t+1}\right\|^2\right] + C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.$$

Similarly, we have

$$\begin{aligned}&\mathbb{E}\left\|Z_t^{\varepsilon}-\bar{Z}_t\right\|^2\\&\leq C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right|^2+\left\|Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right\|^2\right]+C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.\end{aligned}$$

When $t=s-1$, by similar analysis,

$$\left\{\begin{array}{rcl}\mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t\right|^2&\leq&C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right|^2+\left\|Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right\|^2\right]+C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2,\\\mathbb{E}\left[\left\|Z_t^{\varepsilon}-\bar{Z}_t\right\|^2\right]&\leq&C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right|^2+\left\|Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right\|^2\right]+C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.\end{array}\right.$$

If $s=T$, it shows like

$$\left\{\begin{array}{rcl}\mathbb{E}\left|Y_{T-1}^{\varepsilon}-\bar{Y}_{T-1}\right|^2&\leq&C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2,\\\mathbb{E}\left[\left\|Z_{T-1}^{\varepsilon}-\bar{Z}_{T-1}\right\|^2\right]&\leq&C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.\end{array}\right.$$

When $t=0,...,s-2$, we have

$$\left\{\begin{array}{rcl}\mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t\right|^2&\leq&C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right|^2+\left\|Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right\|^2\right],\\\mathbb{E}\left[\left\|Z_t^{\varepsilon}-\bar{Z}_t\right\|^2\right]&\leq&C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right|^2+\left\|Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right\|^2\right].\end{array}\right.$$

Thus, there exists $C>0$, such that for any $t\in\{0,1,...,T\}$,

$$\left\{\begin{array}{rcl}\mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t\right|^2&\leq&C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2,\\\mathbb{E}\left[\left\|Z_t^{\varepsilon}-\bar{Z}_t\right\|^2\right]&\leq&C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.\end{array}\right.$$

This completes the proof. ■

Let (η,ζ,V) be the solution to the following BSΔE,

$$\left\{\begin{array}{rcl}\Delta\eta_t&=&-f_x\left(t+1\right)\xi_{t+1}-f_y\left(t+1\right)\eta_{t+1}-\delta_{(t+1)s}f_u\left(t+1\right)\varepsilon\Delta v\\&&-\sum_{i=1}^d f_{z_i}\left(t+1\right)\zeta_{t+1}e_i+\zeta_t\Delta W_t+\Delta V_t,\\\eta_T&=&0.\end{array}\right.$$

Notice that $f_x\left(T\right)=f_x\left(T,\bar{X}_T,\bar{Y}_T,\bar{u}_T\right)$ since f is independent of Z, also as $f_y\left(T\right)$, $f_u\left(T\right)$.

It is easy to check that

$$\begin{aligned}\sup_{0\leq t\leq T}\mathbb{E}\left|\eta_t\right|^2\leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2,\\\sup_{0\leq t\leq T-1}\mathbb{E}\left\|\zeta_t\right\|^2\leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.\end{aligned}$$

and we have the following result:

Lemma 3.4 *Under Assumption 2.6, we have*

$$\sup_{0\le t\le T}\mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t-\eta_t\right|^2=o\left(\varepsilon^2\right),$$
$$\sup_{0\le t\le T-1}\mathbb{E}\left\|Z_t^{\varepsilon}-\bar{Z}_t-\zeta_t\right\|^2=o\left(\varepsilon^2\right).$$

Proof. When $t=T$, $Y_T^{\varepsilon}-\bar{Y}_T-\eta_T=0$.

When $t\in\{0,1,...,T-1\}$, we have

$$\begin{aligned}
& Y_t^{\varepsilon}-\bar{Y}_t-\eta_t \\
= \ & \mathbb{E}\left[Y_{t+1}^{\varepsilon}+f^{\varepsilon}(t+1)-\bar{Y}_{t+1}-\overline{f}(t+1)-\eta_{t+1}-f_x(t+1)\xi_{t+1}\right. \\
& \left.-f_y(t+1)\eta_{t+1}-\textstyle\sum_{i=1}^d f_{z_i}(t+1)\zeta_{t+1}e_i-\delta_{(t+1)s}f_u(t+1)\varepsilon\Delta v|\mathcal{F}_t\right] \\
= \ & \mathbb{E}\left[Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}-\eta_{t+1}+\widetilde{f}_x(t+1)\left(X_{t+1}^{\varepsilon}-\bar{X}_{t+1}\right)+\widetilde{f}_y(t+1)\left(Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}\right)\right. \\
& +\textstyle\sum_{i=1}^d\widetilde{f}_{z_i}(t+1)\left(Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}\right)e_i+\delta_{(t+1)s}\widetilde{f}_u(t+1)\varepsilon\Delta v-f_x(t+1)\xi_{t+1} \\
& \left.-f_y(t+1)\eta_{t+1}-\textstyle\sum_{i=1}^d f_{z_i}(t+1)\zeta_{t+1}e_i-\delta_{(t+1)s}f_u(t+1)\varepsilon\Delta v|\mathcal{F}_t\right],
\end{aligned}$$

where

$$\widetilde{f}_{\mu}(t)=\int_0^1 f_{\mu}(t,\bar{X}_t+\lambda\left(X_t^{\varepsilon}-\bar{X}_t\right),\bar{Y}_t+\lambda\left(Y_t^{\varepsilon}-\bar{Y}_t\right),\bar{Z}_t+\lambda\left(Z_t^{\varepsilon}-\bar{Z}_t\right),\bar{u}_t+\lambda\left(u_t^{\varepsilon}-\bar{u}_t\right)d\lambda$$

for $\mu=x$, y, z_i and u. Then,

$$\begin{aligned}
& \mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t-\eta_t\right|^2 \\
& \le C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}-\eta_{t+1}\right|^2\right. \\
& +\left|\widetilde{f}_x(t+1)\left(X_{t+1}^{\varepsilon}-\bar{X}_{t+1}-\xi_{t+1}\right)\right|^2+\left|\left[\widetilde{f}_x(t+1)-f_x(t+1)\right]\xi_{t+1}\right|^2 \\
& +\left|\widetilde{f}_y(t+1)\left(Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}-\eta_{t+1}\right)\right|^2+\left|\left[\widetilde{f}_y(t+1)-f_y(t+1)\right]\eta_{t+1}\right|^2 \\
& +\sum_{i=1}^d\left|\widetilde{f}_{z_i}(t+1)\left(Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}-\zeta_{t+1}\right)e_i\right|^2+\sum_{i=1}^d\left|\left[\widetilde{f}_{z_i}(t+1)-f_{z_i}(t+1)\right]\zeta_{t+1}e_i\right|^2 \\
& \left.+\delta_{(t+1)s}\left|\left[\widetilde{f}_u(t+1)-f_u(t+1)\right]\varepsilon\Delta v\right|^2\right]
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\left\|Z_t^{\varepsilon}-\bar{Z}_t-\zeta_t\right\|^2\\
&\leq C\mathbb{E}\left[\left|Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}-\eta_{t+1}\right|^2\right.\\
&+\left|\widetilde{f}_x(t+1)\left(X_{t+1}^{\varepsilon}-\bar{X}_{t+1}-\xi_{t+1}\right)\right|^2+\left|\left[\widetilde{f}_x(t+1)-f_x(t+1)\right]\xi_{t+1}\right|^2\\
&+\left|\widetilde{f}_y(t+1)\left(Y_{t+1}^{\varepsilon}-\bar{Y}_{t+1}-\eta_{t+1}\right)\right|^2+\left|\left[\widetilde{f}_y(t+1)-f_y(t+1)\right]\eta_{t+1}\right|^2\\
&+\sum_{i=1}^{d}\left|\widetilde{f}_{z_i}(t+1)\left(Z_{t+1}^{\varepsilon}-\bar{Z}_{t+1}-\zeta_{t+1}\right)e_i\right|^2+\sum_{i=1}^{d}\left|\left[\widetilde{f}_{z_i}(t+1)-f_{z_i}(t+1)\right]\zeta_{t+1}e_i\right|^2\\
&\left.+\delta_{(t+1)s}\left|\left[\widetilde{f}_u(t+1)-f_u(t+1)\right]\varepsilon\Delta v\right|^2\right].
\end{aligned}
$$

Notice that $\widetilde{f}_x(t)-f_x(t)\to 0$, $\widetilde{f}_y(t)-f_y(t)\to 0$, $\widetilde{f}_{z_i}(t)-f_{z_i}(t)\to 0$, $\widetilde{f}_u(t)-f_u(t)\to 0$ as $\varepsilon\to 0$. We obtain that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon^2}\mathbb{E}\left|Y_t^{\varepsilon}-\bar{Y}_t-\eta_t\right|^2 &= 0,\\
\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon^2}\mathbb{E}\left\|Z_t^{\varepsilon}-\bar{Z}_t-\zeta_t\right\|^2 &= 0.
\end{aligned}
$$

This completes the proof. ■

By Lemma 3.2 and Lemma 3.4, we have

$$
\begin{aligned}
&J\left(u^{\varepsilon}(\cdot)\right)-J\left(\bar{u}(\cdot)\right)\\
=\;&\mathbb{E}\textstyle\sum_{t=0}^{T-1}\left[\langle l_x(t),\xi_t\rangle+\langle l_y(t),\eta_t\rangle+\sum_{i=1}^{d}\langle l_{z_i}(t),\zeta_te_i\rangle+\delta_{ts}\langle l_u(s),\varepsilon\Delta v\rangle\right]\\
&+\mathbb{E}\left\langle h_x\left(\bar{X}_T\right),\xi_T\right\rangle+o(\varepsilon).
\end{aligned}
$$

Introducing the following adjoint equation:

$$
\begin{cases}
\Delta p_t = -b_x^*(t+1)p_{t+1}-\sum_{i=1}^{d}\sigma_{ix}^*(t+1)q_{t+1}e_i\\
\qquad\quad +f_x^*(t+1)k_{t+1}+l_x(t+1)+q_t\Delta W_t+\Delta Q_t,\\
\Delta k_t = f_y^*(t)k_t+l_y(t)+\sum_{i=1}^{d}\left[f_{z_i}^*(t)k_t+l_{z_i}(t)\right]\Delta W_t^i,\\
p_T = -h_x\left(\bar{X}_T\right),\\
k_0 = 0,
\end{cases}
\tag{3.8}
$$

where W and Q are square integrable martingale processes and Q is strongly orthogonal to W.

Obviously the forward equation in (3.8) admits a unique solution $k\in\mathcal{M}^2(0,T;\mathbb{R}^n)$. Then, based on the solution k, according to Theorem 2.4, the backward equation in (3.8) has a unique solution $(p,q,Q)\in\mathcal{M}^2(0,T;\mathbb{R}^m)\times\mathcal{M}^2\left(0,T-1;\mathbb{R}^{m\times d}\right)\times\mathcal{M}^2(0,T;\mathbb{R}^m)$. So FBS$\Delta$E has a unique solution (p,q,Q,k).

We obtain the following maximum principle for the optimal control problem (1.1)-(1.2).

Define the Hamiltonian function

$$H(\omega,t,u,x,y,z,p,q,k)=b^*(\omega,t,x,u)\,p+\sum_{i=1}^{d}\sigma_i^*(\omega,t,x,u)\,qe_i$$
$$-f^*(\omega,t,x,y,z,u)\,k-l(\omega,t,x,y,z,u).$$

Theorem 3.5 *Suppose that Assumption (2.6) holds. Let $\bar{u}$ be an optimal control of the problem (1.1)-(1.2), $\left(\bar{X},\bar{Y},\bar{Z}\right)$ be the corresponding optimal trajectory and (p,q,k) be the solution to the adjoint equation (3.8). Then for any $t\in\{0,1,...,T\}$, for any $v\in U_t$, we have*

$$\left\langle H_u\left(t,\bar{u}_t,\bar{X}_t,\bar{Y}_t,\bar{Z}_t,p_t,q_t,k_t\right),v-\bar{u}_t\right\rangle\leq 0,\ \ P-a.s.. \tag{3.9}$$

Proof. For $t\in\{0,1,...,T-1\}$, we have

$$\begin{aligned}
&\Delta\left\langle \xi_t,p_t\right\rangle\\
=\ &\left\langle \xi_{t+1},\Delta p_t\right\rangle+\left\langle \Delta\xi_t,p_t\right\rangle\\
=\ &\left\langle \xi_{t+1},-b_x^*(t+1)\,p_{t+1}-\sum_{i=1}^{d}\sigma_{ix}^*(t+1)\,q_{t+1}e_i+f_x^*(t+1)\,k_{t+1}+l_x(t+1)\right\rangle\\
&+\left\langle \sum_{j=1}^{d}\left[\sigma_{jx}(t)\,\xi_t+\delta_{ts}\varepsilon\sigma_{ju}(t)\,\Delta v\right]\Delta W_t^j,q_t\Delta W_t\right\rangle+\left\langle b_x(t)\,\xi_t+\delta_{ts}b_u(t)\,\varepsilon\Delta v,p_t\right\rangle+\Phi_t,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\Phi_t\ =\ &\left\langle \xi_t+b_x(t)\,\xi_t+\delta_{ts}b_u(t)\,\varepsilon\Delta v,q_t\Delta W_t\right\rangle+\left\langle \sum_{i=1}^{d}\left[\sigma_{ix}(t)\,\xi_t+\delta_{ts}\varepsilon\sigma_{iu}(t)\,\Delta v\right]\Delta W_t^i,p_t\right\rangle\\
&+\left\langle \xi_t+b_x(t)\,\xi_t+\delta_{ts}b_u(t)\,\varepsilon\Delta v,\Delta Q_t\right\rangle+\left\langle \sum_{j=1}^{d}\left[\sigma_{jx}(t)\,\xi_t+\delta_{ts}\varepsilon\sigma_{ju}(t)\,\Delta v\right]\Delta W_t^j,\Delta Q_t\right\rangle.
\end{aligned}$$

It is obvious that $\mathbb{E}\left[\Phi_t\right]=0$. We have

$$\begin{aligned}
\mathbb{E}\left\langle \sum_{j=1}^{d}\sigma_{jx}(t)\,\xi_t\Delta W_t^j,q_t\Delta W_t\right\rangle=\ &\mathbb{E}\left\langle \sum_{j=1}^{d}\sigma_{jx}(t)\,\xi_t\Delta W_t^j,\sum_{i=1}^{d}q_te_i\Delta W_t^i\right\rangle\\
=\ &\mathbb{E}\left[\sum_{i=1}^{d}\left\langle \xi_t,\sum_{j=1}^{d}\sigma_{jx}^*(t)\,q_te_i\mathbb{E}\left[\Delta W_t^i\Delta W_t^j|\mathcal{F}_t\right]\right\rangle\right]\\
=\ &\mathbb{E}\left[\sum_{i=1}^{d}\left\langle \xi_t,\sigma_{ix}^*(t)\,q_te_i\right\rangle\right],
\end{aligned}$$

and

$$\mathbb{E}\left\langle \sum_{j=1}^{d}\delta_{ts}\varepsilon\sigma_{ju}(t)\,\Delta v\Delta W_t^j,q_t\Delta W_t\right\rangle=\mathbb{E}\left[\delta_{ts}\varepsilon\sum_{j=1}^{d}\left\langle \sigma_{ju}(t)\,\Delta v,q_te_j\right\rangle\right].$$

Similarly, it can be shown that for $t\in\{0,1,...,T-1\}$, we have

$$\begin{aligned}
&\Delta\left\langle \eta_t,k_t\right\rangle\\
=\ &\left\langle -f_x(t+1)\,\xi_{t+1}-f_y(t+1)\,\eta_{t+1}-\sum_{i=1}^{d}f_{z_i}(t+1)\,\zeta_{t+1}e_i-\delta_{(t+1)s}f_u(t+1)\,\varepsilon\Delta v,k_{t+1}\right\rangle\\
&+\left\langle \zeta_t\Delta W_t,\sum_{i=1}^{d}\left[f_{z_i}^*(t)\,k_t+l_{z_i}(t)\right]\Delta W_t^i\right\rangle+\left\langle \eta_t,f_y^*(t)\,k_t+l_y(t)\right\rangle+\Psi_t,
\end{aligned}$$

where

$$\begin{aligned}\Psi_t &= \left\langle \zeta_t \Delta W_t, k_t + f_y^* (t) k_t + l_y (t) \right\rangle + \left\langle \eta_t, \sum_{i=1}^d \left[f_{z_i}^* (t) k_t + l_{z_i} (t) \right] \Delta W_t^i \right\rangle \\ &\quad + \left\langle k_t + f_y^* (t) k_t + l_y (t), \Delta V_t \right\rangle + \left\langle \sum_{i=1}^d \left[f_{z_i}^* (t) k_t + l_{z_i} (t) \right] \Delta W_t^i, \Delta V_t \right\rangle .\end{aligned}$$

It is easy to check that

$$\mathbb{E} \left\langle \zeta_t \Delta W_t, \sum_{i=1}^d f_{z_i}^* (t) k_t \Delta W_t^i \right\rangle = \mathbb{E} \left[\sum_{i=1}^d \langle f_{z_i} (t) \zeta_t e_i, k_t \rangle \right],$$

$$\mathbb{E} \left\langle \zeta_t \Delta W_t, \sum_{i=1}^d l_{z_i} (t) \Delta W_t^i \right\rangle = \mathbb{E} \left[\sum_{i=1}^d \langle l_{z_i} (t), \zeta_t e_i \rangle \right].$$

Then we have

$$\begin{aligned}&\mathbb{E} \left[\Delta \left(\langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right) \right] \\ &= \mathbb{E} \left[\langle -b_x (t+1) \xi_{t+1}, p_{t+1} \rangle + \langle b_x (t) \xi_t, p_t \rangle \right. \\ &- \sum_{i=1}^d \langle \xi_{t+1}, \sigma_{ix}^* (t+1) q_{t+1} e_i \rangle + \sum_{i=1}^d \langle \xi_t, \sigma_{ix}^* (t) q_t e_i \rangle \\ &- \langle f_y (t+1) \eta_{t+1}, k_{t+1} \rangle + \langle f_y (t) \eta_t, k_t \rangle \\ &- \sum_{i=1}^d \langle f_{z_i} (t+1) \zeta_{t+1} e_i, k_{t+1} \rangle + \sum_{i=1}^d \langle f_{z_i} (t) \zeta_t e_i, k_t \rangle \\ &+ \langle l_x (t+1), \xi_{t+1} \rangle + \langle \eta_t, l_y (t) \rangle + \sum_{i=1}^d \langle l_{z_i} (t), \zeta_t e_i \rangle \\ &+ \varepsilon \langle \delta_{ts} b_u (t) \Delta v, p_t \rangle + \delta_{ts} \varepsilon \sum_{i=1}^d \langle \sigma_{iu} (t) \Delta v, q_t e_i \rangle \\ &\left. - \varepsilon \left\langle \delta_{(t+1)s} f_u (t+1) \Delta v, k_{t+1} \right\rangle \right] .\end{aligned} \tag{3.11}$$

Therefore,

$$\begin{aligned}&-\mathbb{E} \left\langle h_x \left(\bar{X}_T \right), \xi_T \right\rangle \\ =\ & \mathbb{E} \left[\langle \xi_T, p_T \rangle + \langle \eta_T, k_T \rangle - \langle \xi_0, p_0 \rangle - \langle \eta_0, k_0 \rangle \right] \\ =\ & \sum_{t=0}^{T-1} \mathbb{E} \Delta \left(\langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right) \\ =\ & \mathbb{E} \left[\langle b_x (0) \xi_0, p_0 \rangle + \sum_{i=1}^d \langle \xi_0, \sigma_{ix}^* (0) q_0 e_i \rangle + \langle f_y (0) \eta_0, k_0 \rangle + \sum_{i=1}^d \langle f_{z_i} (0) \zeta_0 e_i, k_0 \rangle \right] \\ & + \sum_{t=0}^{T-1} \mathbb{E} \left[\langle l_x (t), \xi_t \rangle + \langle l_y (t), \eta_t \rangle + \sum_{i=1}^d \langle l_{z_i} (t), \zeta_t e_i \rangle \right] \\ & + \sum_{t=0}^{T} \delta_{ts} \varepsilon \mathbb{E} \left[\langle b_u^* (t) p_t, \Delta v \rangle + \sum_{i=1}^d \langle \sigma_{iu}^* (t) q_t e_i, \Delta v \rangle - \langle f_u^* (t) k_t, \Delta v \rangle \right] .\end{aligned} \tag{3.12}$$

Since $\xi_0 = 0$ and $k_0 = 0$, we deduce

$$\begin{aligned} &\mathbb{E}\sum_{t=0}^{T-1}\left[\langle l_x(t),\xi_t\rangle + \langle l_y(t),\eta_t\rangle + \sum_{i=1}^{d}\langle l_{z_i}(t),\zeta_t e_i\rangle\right] + \mathbb{E}\left\langle h_x\left(\bar{X}_T\right),\xi_T\right\rangle \\ = \; & -\varepsilon\mathbb{E}\left[\left\langle b_u^*(s)\,p_s + \sum_{i=1}^{d}\sigma_{iu}^*(s)\,q_t e_i - f_u^*(s)\,k_s, \Delta v\right\rangle\right]. \end{aligned} \tag{3.13}$$

By $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[J\left(u^\varepsilon(\cdot)\right) - J\left(\bar{u}(\cdot)\right)\right] \geq 0$, we obtain

$$\mathbb{E}\left[\left\langle b_u^*(s)\,p_s + \sum_{i=1}^{d}\sigma_{iu}^*(s)\,q_s e_i - f_u^*(s)\,k_s - l_u(s),\Delta v\right\rangle\right] \leq 0.$$

Thus, it is easy to obtain equation (3.9) since s is taking arbitrarily. This completes the proof. ■

Remark 3.6 *In the introduction we point out that we need a reasonable representation of the product rule. When we calculate $\Delta\langle\xi_t,p_t\rangle$ in (3.10), $\Delta\langle\xi_t,p_t\rangle$ is represented as $\langle\xi_{t+1},\cdots\rangle+\cdots$. Combining the formulation of the BSΔE mentioned in the introduction, this representation will lead to the terms such as $\langle\Box_t,\Diamond_t\rangle - \langle\Box_{t+1},\Diamond_{t+1}\rangle$ in (3.11). By summing and rearranging these terms in (3.12), we obtain the dual relation (3.13).*

When $g \equiv 0$ and $f \equiv 0$, our control system (1.1)-(1.2) degenerates to the classical discrete control system which only contains a forward stochastic difference equation as in [14]. For this special case, the adjoint equation becomes

$$\begin{cases} \Delta p_t = -b_x^*(t+1)\,p_{t+1} - \sum_{i=1}^{d}\sigma_{ix}^*(t+1)\,q_{t+1}e_i + l_x(t+1) + q_t\Delta W_t + \Delta Q_t, \\ p_T = -h_x\left(\bar{X}_T\right), \end{cases} \tag{3.14}$$

and the Hamiltonian function becomes

$$H(\omega,t,u,x,p,q) = b^*(\omega,t,x,u)\,p + \sum_{i=1}^{d}\sigma_i^*(\omega,t,x,u)\,qe_i - l(\omega,t,x,u).$$

The adjoint equation has the following explicit solution

$$\begin{cases} p_{T-1} = -\mathbb{E}\left[h_x\left(\bar{X}_T\right)|\mathcal{F}_{T-1}\right], \\ q_{T-1} = -\mathbb{E}\left[h_x\left(\bar{X}_T\right)\left(\Delta W_t\right)^*|\mathcal{F}_{T-1}\right], \\ p_t = \mathbb{E}\left[\left[I + b_x^*(t+1)\right]p_{t+1} - l_x(t+1) + \sum_{i=1}^{d}\sigma_{ix}^*(t+1)\,q_{t+1}e_i|\mathcal{F}_t\right], \\ q_t = \mathbb{E}\left[\left[\left[I + b_x^*(t+1)\right]p_{t+1} - l_x(t+1) + \sum_{i=1}^{d}\sigma_{ix}^*(t+1)\,q_{t+1}e_i\right]\left(\Delta W_t\right)^*|\mathcal{F}_t\right]. \end{cases}$$

which coincides with the results in [14].

4 Maximum principle for the fully coupled FBSΔE system

In this section we suppose W to be one-dimensional driving process. Let $\bar{u} = \{\bar{u}_t\}_{t=0}^{T}$ be the optimal control for the control problem (1.3)-(1.4) and $(\bar{X}, \bar{Y}, \bar{Z})$ be the corresponding optimal trajectory. Note that the existence and uniqueness of $(\bar{X}, \bar{Y}, \bar{Z})$ is guaranteed by the results in [15]. The perturbed control u^{ε} is the same as (3.1) and we denote by $(X^{\varepsilon}, Y^{\varepsilon}, Z^{\varepsilon})$ the corresponding trajectory.

Let

$$\widehat{X}_t = X_t^{\varepsilon} - \bar{X}_t,\ \widehat{Y}_t = Y_t^{\varepsilon} - \bar{Y}_t,\ \widehat{Z}_t = Z_t^{\varepsilon} - \bar{Z}_t,\ \widehat{N}_t = N_t^{\varepsilon} - \overline{N}_t.$$

Using the similar notations (3.2) in section 3, we have

$$\begin{cases} \Delta \widehat{X}_t &= b^{\varepsilon}(t) - \overline{b}(t) + (\sigma^{\varepsilon}(t) - \overline{\sigma}(t)) \Delta W_t, \\ \Delta \widehat{Y}_t &= -f^{\varepsilon}(t+1) + \overline{f}(t+1) + \widehat{Z}_t \Delta W_t + \Delta \widehat{N}_t, \\ \widehat{X}_0 &= 0, \\ \widehat{Y}_T &= 0. \end{cases} \tag{4.1}$$

Lemma 4.1 *Under Assumption 2.6 and Assumption 2.7, we have*

$$\mathbb{E}\left(\sum_{t=0}^{T}\left|\widehat{X}_t\right|^2 + \sum_{t=0}^{T}\left|\widehat{Y}_t\right|^2 + \sum_{t=0}^{T-1}\left|\widehat{Z}_t\right|^2\right) \leq C\varepsilon^2 \mathbb{E}\left|\Delta v\right|^2. \tag{4.2}$$

Proof. By (4.1),

$$\begin{aligned} 0 = {} & \mathbb{E}\left\langle \widehat{X}_T, \widehat{Y}_T \right\rangle - \mathbb{E}\left\langle \widehat{X}_0, \widehat{Y}_0 \right\rangle \\ = {} & \mathbb{E}\textstyle\sum_{t=0}^{T-1} \Delta \left\langle \widehat{X}_t, \widehat{Y}_t \right\rangle \\ = {} & \mathbb{E}\textstyle\sum_{t=0}^{T}\left[\left\langle \widehat{X}_t, -f^{\varepsilon}(t) + \overline{f}(t)\right\rangle + \left\langle \widehat{Y}_t, b^{\varepsilon}(t) - \overline{b}(t)\right\rangle + \left\langle \widehat{Z}_t, \sigma^{\varepsilon}(t) - \overline{\sigma}(t)\right\rangle\right] \\ = {} & \mathbb{E}\textstyle\sum_{t=1}^{T-1}\left\langle A(t, \lambda_t^{\varepsilon}; u_t^{\varepsilon}) - A\left(t, \overline{\lambda}_t; u_t^{\varepsilon}\right), \widehat{\lambda}_t \right\rangle \\ & + \mathbb{E}\left\langle \widehat{X}_T, -f^{\varepsilon}(T) + \widetilde{f}^{\varepsilon}(T)\right\rangle + \left\langle \widehat{Y}_0, b^{\varepsilon}(0) - \widetilde{b}^{\varepsilon}(0)\right\rangle + \left\langle \widehat{Z}_0, \sigma^{\varepsilon}(0) - \widetilde{\sigma}^{\varepsilon}(0)\right\rangle \\ & + \mathbb{E}\textstyle\sum_{t=0}^{T}\left[\left\langle \widehat{X}_t, -\widetilde{f}^{\varepsilon}(t) + \overline{f}(t)\right\rangle + \left\langle \widehat{Y}_t, \widetilde{b}^{\varepsilon}(t) - \overline{b}(t)\right\rangle + \left\langle \widehat{Z}_t, \widetilde{\sigma}^{\varepsilon}(t) - \overline{\sigma}(t)\right\rangle\right] \\ = {} & \mathbb{E}\textstyle\sum_{t=1}^{T-1}\left\langle A(t, \lambda_t^{\varepsilon}; u_t^{\varepsilon}) - A\left(t, \overline{\lambda}_t; u_t^{\varepsilon}\right), \widehat{\lambda}_t \right\rangle \\ & + \mathbb{E}\left\langle \widehat{X}_T, -f^{\varepsilon}(T) + \widetilde{f}^{\varepsilon}(T)\right\rangle + \left\langle \widehat{Y}_0, b^{\varepsilon}(0) - \widetilde{b}^{\varepsilon}(0)\right\rangle + \left\langle \widehat{Z}_0, \sigma^{\varepsilon}(0) - \widetilde{\sigma}^{\varepsilon}(0)\right\rangle \\ & + \mathbb{E}\left[\left\langle \widehat{X}_s, -\widetilde{f}^{\varepsilon}(s) + \overline{f}(s)\right\rangle + \left\langle \widehat{Y}_s, \widetilde{b}^{\varepsilon}(s) - \overline{b}(s)\right\rangle + \left\langle \widehat{Z}_s, \widetilde{\sigma}^{\varepsilon}(s) - \overline{\sigma}(s)\right\rangle\right]. \end{aligned}$$

By the monotone condition, we obtain

$$\begin{array}{rl} & \mathbb{E}\left[\left\langle \widehat{X}_s, -\widetilde{f}^{\varepsilon}(s)+\overline{f}(s)\right\rangle + \left\langle \widehat{Y}_s, \widetilde{b}^{\varepsilon}(s)-\overline{b}(s)\right\rangle + \left\langle \widehat{Z}_s, \widetilde{\sigma}^{\varepsilon}(s)-\overline{\sigma}(s)\right\rangle\right] \\ \geq & \alpha\mathbb{E}\left[\sum_{t=0}^{T}\left|\widehat{X}_t\right|^2 + \sum_{t=0}^{T}\left|\widehat{Y}_t\right|^2 + \sum_{t=0}^{T-1}\left|\widehat{Z}_t\right|^2\right]. \end{array} \tag{4.3}$$

On the other hand,

$$\begin{aligned} & \mathbb{E}\left\langle \widehat{X}_s, -\widetilde{f}^{\varepsilon}(s)+\overline{f}(s)\right\rangle \\ & \leq \frac{\alpha}{2}\mathbb{E}\left|\widehat{X}_s\right|^2 + \frac{1}{2\alpha}\mathbb{E}\left|\overline{f}(s)-\widetilde{f}^{\varepsilon}(s)\right|^2 \\ & \leq \frac{\alpha}{2}\mathbb{E}\left|\widehat{X}_s\right|^2 + \frac{C}{2\alpha}\varepsilon^2\mathbb{E}\left|\Delta v\right|^2 \end{aligned}$$

and similarly,

$$\begin{array}{rl} & \mathbb{E}\left[\left\langle \widehat{X}_s, -\widetilde{f}^{\varepsilon}(s)+\overline{f}(s)\right\rangle + \left\langle \widehat{Y}_s, \widetilde{b}^{\varepsilon}(s)-\overline{b}(s)\right\rangle + \left\langle \widehat{Z}_s, \widetilde{\sigma}^{\varepsilon}(s)-\overline{\sigma}(s)\right\rangle\right] \\ \leq & \frac{\alpha}{2}\mathbb{E}\left[\left|\widehat{X}_s\right|^2 + \left|\widehat{Y}_s\right|^2 + \left|\widehat{Z}_s\right|^2\right] + C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2. \end{array} \tag{4.4}$$

Combining (4.3) and (4.4), we have

$$\mathbb{E}\left[\sum_{t=0}^{T}\left|\widehat{X}_t\right|^2 + \sum_{t=0}^{T}\left|\widehat{Y}_t\right|^2 + \sum_{t=0}^{T-1}\left|\widehat{Z}_t\right|^2\right] \leq C\varepsilon^2\mathbb{E}\left|\Delta v\right|^2.$$

This completes the proof. ■

Next we introduce the following variational equation:

$$\left\{\begin{array}{rcl} \Delta\xi_t & = & b_x(t)\xi_t + b_y(t)\eta_t + b_z(t)\zeta_t + \delta_{ts}b_u(t)\varepsilon\Delta v \\ & & +\left[\sigma_x(t)\xi_t + \sigma_y(t)\eta_t + \sigma_z(t)\zeta_t + \delta_{ts}\varepsilon\sigma_u(t)\Delta v\right]\Delta W_t, \\ \Delta\eta_t & = & -f_x(t+1)\xi_{t+1} - f_y(t+1)\eta_{t+1} - f_z(t+1)\zeta_{t+1} \\ & & -\delta_{(t+1)s}f_u(t+1)\varepsilon\Delta v + \zeta_t\Delta W_t + \Delta V_t, \\ \xi_0 & = & 0, \\ \eta_T & = & 0. \end{array}\right. \tag{4.5}$$

By Assumption 2.6 and Assumption 2.7, when $t \in \{1, ..., T-1\}$,

$$\begin{pmatrix} -f_x(t) & -f_y(t) & -f_z(t) \\ b_x(t) & b_y(t) & b_z(t) \\ \sigma_x(t) & \sigma_y(t) & \sigma_z(t) \end{pmatrix} \leq -\alpha I_{3n},\ P-a.s.; \tag{4.6}$$

when $t=0$,

$$\begin{pmatrix} b_y(0) & b_z(0) \\ \sigma_y(0) & \sigma_z(0) \end{pmatrix} \leq -\alpha I_{2n},\ P-a.s.; \tag{4.7}$$

when $t=T$,

$$-f_x(T) \leq -\alpha I_n,\ P-a.s.. \tag{4.8}$$

Thus, the coefficients of (4.5) satisfy the monotone condition and there exists a unique solution (ξ,η,ζ,V) to (4.5). Similar to the proof of Lemma 4.1, we have

$$\mathbb{E}\left[\sum_{t=0}^{T}|\xi_t|^2+\sum_{t=0}^{T}|\eta_t|^2+\sum_{t=0}^{T-1}|\zeta_t|^2\right] \leq C\varepsilon^2\mathbb{E}\,|\Delta v|^2. \tag{4.9}$$

Define

$$\widetilde{\varphi}_\mu(t)=\int_0^1 \varphi_\mu t,\bar{X}_t+\lambda\left(X_t^\varepsilon-\bar{X}_t\right),\bar{Y}_t+\lambda\left(Y_t^\varepsilon-\bar{Y}_t\right),\bar{Z}_t+\lambda\left(Z_t^\varepsilon-\bar{Z}_t\right),\bar{u}_t+\lambda\left(u_t^\varepsilon-\bar{u}_t\right)d\lambda$$

where $\varphi=b,\ \sigma_i,\ g,\ f,\ l,\ h$ and $\mu=x,\ y,\ z_i$ and u.

Lemma 4.2 *Under Assumption 2.6 and Assumption 2.7, we have*

$$\mathbb{E}\left[\sum_{t=0}^{T}\left|\widehat{X}_t-\xi_t\right|^2+\sum_{t=0}^{T}\left|\widehat{Y}_t-\eta_t\right|^2+\sum_{t=0}^{T-1}\left|\widehat{Z}_t-\zeta_t\right|^2\right]=o\left(\varepsilon^2\right).$$

Proof. Note that

$$\begin{aligned}&\varphi^\varepsilon(t)-\overline{\varphi}(t)\\ &=\widetilde{\varphi}_x(t)\left(X_t^\varepsilon-\bar{X}_t\right)+\widetilde{\varphi}_y(t)\left(Y_t^\varepsilon-\bar{Y}_t\right)+\widetilde{\varphi}_z(t)\left(Z_t^\varepsilon-\bar{Z}_t\right)+\delta_{ts}\widetilde{\varphi}_u(t)\,\varepsilon\Delta v.\end{aligned}$$

Set

$$\widetilde{X}_t=\widehat{X}_t-\xi_t,\ \widetilde{Y}_t=\widehat{Y}_t-\eta_t,\ \widetilde{Z}_t=\widehat{Z}_t-\zeta_t,\ \widetilde{N}_t=\widehat{N}_t-V_t.$$

Then,

$$\begin{cases}\Delta\widetilde{X}_t &= b_x(t)\widetilde{X}_t+b_y(t)\widetilde{Y}_t+b_z(t)\widetilde{Z}_t+\Lambda_1(t)\\ &\quad+\left[\sigma_x(t)\widetilde{X}_t+\sigma_y(t)\widetilde{Y}_t+\sigma_z(t)\widetilde{Z}_t+\Lambda_2(t)\right]\Delta W_t,\\ \Delta\widetilde{Y}_t &= -f_x(t+1)\widetilde{X}_{t+1}-f_y(t+1)\widetilde{Y}_{t+1}-f_z(t+1)\widetilde{Z}_{t+1}\\ &\quad-\Lambda_3(t+1)+\widetilde{Z}_t\Delta W_t+\Delta\widetilde{N}_t,\\ \widetilde{X}_0 &= 0,\\ \widetilde{Y}_T &= 0,\end{cases} \tag{4.10}$$

where

$$\begin{aligned}\Lambda_1(t) &= \left(\widetilde{b}_x(t) - b_x(t)\right)\widehat{X}_t + \left(\widetilde{b}_y(t) - b_y(t)\right)\widehat{Y}_t \\ &\quad + \left(\widetilde{b}_z(t) - b_z(t)\right)\widehat{Z}_t + \delta_{ts}\left(\widetilde{b}_u(t) - b_u(t)\right)\varepsilon\Delta v, \\ \Lambda_2(t) &= \left(\widetilde{\sigma}_x(t) - \sigma_x(t)\right)\widehat{X}_t + \left(\widetilde{\sigma}_y(t) - \sigma_y(t)\right)\widehat{Y}_t \\ &\quad + \left(\widetilde{\sigma}_z(t) - \sigma_z(t)\right)\widehat{Z}_t + \delta_{ts}\left(\widetilde{\sigma}_u(t) - \sigma_u(t)\right)\varepsilon\Delta v, \\ \Lambda_3(t) &= -\left(\widetilde{f}_x(t) - f_x(t)\right)\widehat{X}_t - \left(\widetilde{f}_y(t) - f_y(t)\right)\widehat{Y}_t \\ &\quad - \left(\widetilde{f}_z(t) - f_z(t)\right)\widehat{Z}_t - \delta_{ts}\left(\widetilde{f}_u(t) - f_u(t)\right)\varepsilon\Delta v.\end{aligned}$$

According to (4.10),

$$\begin{aligned}0 = &\ \mathbb{E}\left\langle \widetilde{X}_T, \widetilde{Y}_T\right\rangle - \mathbb{E}\left\langle \widetilde{X}_0, \widetilde{Y}_0\right\rangle \\ = &\ \mathbb{E}\textstyle\sum_{t=0}^{T-1}\Delta\left\langle \widetilde{X}_t, \widetilde{Y}_t\right\rangle \\ = &\ \mathbb{E}\textstyle\sum_{t=0}^{T}\left[\left\langle \widetilde{X}_t, -f_\lambda(t)\widetilde{\lambda}_t\right\rangle + \left\langle \widetilde{Y}_t, b_\lambda(t)\widetilde{\lambda}_t\right\rangle + \left\langle \widetilde{Z}_t, \sigma_\lambda(t)\widetilde{\lambda}_t\right\rangle\right] \\ &+\mathbb{E}\textstyle\sum_{t=0}^{T}\left[\left\langle \widetilde{X}_t, -\Lambda_3(t)\right\rangle + \left\langle \widetilde{Y}_t, \Lambda_1(t)\right\rangle + \left\langle \widetilde{Z}_t, \Lambda_2(t)\right\rangle\right]\end{aligned}$$

where

$$\begin{aligned}\widetilde{\lambda}_t &= \left(\widetilde{X}_t^*, \widetilde{Y}_t^*, \widetilde{Z}_t^*\right)^*, \\ b_\lambda(t) &= \left(b_x(t), b_y(t), b_z(t)\right), \\ \sigma_\lambda(t) &= \left(\sigma_x(t), \sigma_y(t), \sigma_z(t)\right), \\ f_\lambda(t) &= \left(f_x(t), f_y(t), f_z(t)\right).\end{aligned}$$

Combining (4.6), (4.7) and (4.8), we have

$$\begin{aligned}&\mathbb{E}\textstyle\sum_{t=0}^{T}\left[\left\langle \widetilde{X}_t, -\Lambda_3(t)\right\rangle + \left\langle \widetilde{Y}_t, \Lambda_1(t)\right\rangle + \left\langle \widetilde{Z}_t, \Lambda_2(t)\right\rangle\right] \\ \geq &\ \alpha\mathbb{E}\left[\textstyle\sum_{t=0}^{T}\left|\widetilde{X}_t\right|^2 + \sum_{t=0}^{T}\left|\widetilde{Y}_t\right|^2 + \sum_{t=0}^{T-1}\left|\widetilde{Z}_t\right|^2\right].\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}&\mathbb{E}\left\langle \widetilde{X}_t, -\Lambda_3(t)\right\rangle \\ = &\ \mathbb{E}\left\langle \widetilde{X}_t, \left(\widetilde{f}_x(t) - f_x(t)\right)\widehat{X}_t\right\rangle + \mathbb{E}\left\langle \widetilde{X}_t, \left(\widetilde{f}_y(t) - f_y(t)\right)\widehat{Y}_t\right\rangle \\ &+\mathbb{E}\left\langle \widetilde{X}_t, \left(\widetilde{f}_z(t) - f_z(t)\right)\widehat{Z}_t\right\rangle + \mathbb{E}\left\langle \widetilde{X}_t, \delta_{ts}\left(\widetilde{f}_u(t) - f_u(t)\right)\varepsilon\Delta v\right\rangle \\ \leq &\ \tfrac{\alpha}{2}\mathbb{E}\left|\widetilde{X}_t\right|^2 + \tfrac{2}{\alpha}\mathbb{E}\left[\left\|\widetilde{f}_x(t) - f_x(t)\right\|^2\left|\widehat{X}_t\right|^2 + \left\|\widetilde{f}_y(t) - f_y(t)\right\|^2\left|\widehat{Y}_t\right|^2\right] \\ &+\tfrac{2}{\alpha}\mathbb{E}\left[\left\|\widetilde{f}_z(t) - f_z(t)\right\|^2\left|\widehat{Z}_t\right|^2 + \delta_{ts}\varepsilon^2\left\|\widetilde{f}_u(t) - f_u(t)\right\|^2\left|\Delta v\right|^2\right].\end{aligned}$$

When $\varepsilon \to 0$, $\left\| \widetilde{f}_\mu (t) - f_\mu (t) \right\| \to 0$ for $\mu = x,\ y,\ z$ and u. Then, by Lemma 4.1,

$$\mathbb{E} \left\langle \widetilde{X}_t, -\Lambda_3 (t) \right\rangle \leq \frac{\alpha}{2} \mathbb{E} \left| \widetilde{X}_t \right|^2 + o\left(\varepsilon^2\right).$$

Similar results hold for the other terms in (4.11). Finally, we have

$$\mathbb{E} \left[\sum_{t=0}^{T} \left| \widetilde{X}_t \right|^2 + \sum_{t=0}^{T} \left| \widetilde{Y}_t \right|^2 + \sum_{t=0}^{T-1} \left| \widetilde{Z}_t \right|^2 \right] \leq o\left(\varepsilon^2\right).$$

This completes the proof. ■

By Lemma 4.2, we obtain

$$\begin{aligned} & J\left(u^\varepsilon (\cdot)\right) - J\left(\bar{u}(\cdot)\right) \\ = \ & \mathbb{E} \textstyle\sum_{t=0}^{T-1} \left[\langle l_x (t), \xi_t \rangle + \langle l_y (t), \eta_t \rangle + \langle l_z (t), \zeta_t \rangle + \delta_{ts} \langle l_u (s), \varepsilon \Delta v \rangle \right] + \mathbb{E} \left\langle h_x \left(\bar{X}_T \right), \xi_T \right\rangle + o(\varepsilon). \end{aligned}$$

Introduce the following adjoint equation:

$$\left\{ \begin{array}{rcl} \Delta p_t & = & -b_x^* (t+1) p_{t+1} - \sigma_x^* (t+1) q_{t+1} \\ & & + f_x^* (t+1) k_{t+1} + l_x (t+1) + q_t \Delta W_t + \Delta Q_t, \\ \Delta k_t & = & f_y^* (t) k_t - b_y^* (t) p_t - \sigma_y^* (t) q_t + l_y (t) \\ & & + \left[f_z^* (t) k_t - b_z^* (t) p_t - \sigma_z^* (t) q_t + l_z (t) \right] \Delta W_t, \\ p_T & = & -h_x \left(\bar{X}_T \right), \\ k_0 & = & 0. \end{array} \right. \tag{4.12}$$

Define the Hamiltonian function as follows:

$$\begin{aligned} H(\omega, t, u, x, y, z, p, q, k) = {} & b^* (\omega, t, x, y, z, u) p + \textstyle\sum_{i=1}^d \sigma_i^* (\omega, t, x, y, z, u) q e_i \\ & - f^* (\omega, t, x, y, z, u) k - l(\omega, t, x, y, z, u). \end{aligned}$$

Theorem 4.3 *Suppose that Assumption 2.6 and Assumption 2.7 hold. Let $\bar{u}$ be an optimal control for (1.3)-(1.3), $\left(\bar{X}, \bar{Y}, \bar{Z}\right)$ be the corresponding optimal trajectory and (p, q, k) be the solution to the adjoint equation (4.12). Then, for any $t \in \{0, 1, ..., T\}$ and any $v \in U_t$, we have*

$$\left\langle H_u \left(t, \bar{u}_t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, p_t, q_t, k_t \right), v - \bar{u}_t \right\rangle \leq 0, \ \ P-a.s.. \tag{4.13}$$

Proof. From the expression of ξ_t, p_t for $t \in \{0,1,...,T-1\}$, we have

$$\begin{aligned}\Delta\left\langle \xi_t, p_t\right\rangle = &\ \left\langle \xi_{t+1}, \Delta p_t\right\rangle + \left\langle \Delta\xi_t, p_t\right\rangle \\ = &\ \left\langle \xi_{t+1}, -b_x^*\left(t+1\right)p_{t+1} - \sigma_x^*\left(t+1\right)q_{t+1} + f_x^*\left(t+1\right)k_{t+1} + l_x\left(t+1\right)\right\rangle \\ &+ \left\langle \left[\sigma_x\left(t\right)\xi_t + \sigma_y\left(t\right)\eta_t + \sigma_z\left(t\right)\zeta_t + \delta_{ts}\varepsilon\sigma_{iu}\left(t\right)\Delta v\right]\Delta W_t, q_t\Delta W_t\right\rangle \\ &+ \left\langle b_x\left(t\right)\xi_t + b_y\left(t\right)\eta_t + b_z\left(t\right)\zeta_t + \delta_{ts}b_u\left(t\right)\varepsilon\Delta v, p_t\right\rangle + \Phi_t,\end{aligned}$$

where

$$\begin{aligned}\Phi_t = &\ \left\langle \xi_t + b_x\left(t\right)\xi_t + b_y\left(t\right)\eta_t + b_z\left(t\right)\zeta_t + \delta_{ts}b_u\left(t\right)\varepsilon\Delta v, q_t\Delta W_t\right\rangle \\ &+ \left\langle \left[\sigma_x\left(t\right)\xi_t + \sigma_y\left(t\right)\eta_t + \sigma_z\left(t\right)\zeta_t + \delta_{ts}\varepsilon\sigma_{iu}\left(t\right)\Delta v\right]\Delta W_t, p_t\right\rangle \\ &+ \left\langle \xi_t + b_x\left(t\right)\xi_t + b_y\left(t\right)\eta_t + b_z\left(t\right)\zeta_t + \delta_{ts}b_u\left(t\right)\varepsilon\Delta v, \Delta Q_t\right\rangle \\ &+ \left\langle \left[\sigma_x\left(t\right)\xi_t + \sigma_y\left(t\right)\eta_t + \sigma_z\left(t\right)\zeta_t + \delta_{ts}\varepsilon\sigma_{iu}\left(t\right)\Delta v\right]\Delta W_t, \Delta Q_t\right\rangle.\end{aligned}$$

Since W and Q are square integrable martingale processes and Q is strongly orthogonal to W, we have $\mathbb{E}\left[\Phi_t\right] = 0$. Similarly,

$$\begin{aligned}\Delta\left\langle \eta_t, k_t\right\rangle = &\ \left\langle \Delta\eta_t, k_{t+1}\right\rangle + \left\langle \eta_t, \Delta k_t\right\rangle \\ = &\ \left\langle -f_x\left(t+1\right)\xi_{t+1} - f_y\left(t+1\right)\eta_{t+1} - f_z\left(t+1\right)\zeta_{t+1} - \delta_{(t+1)s}f_u\left(t+1\right)\varepsilon\Delta v, k_{t+1}\right\rangle \\ &+ \left\langle \zeta_t\Delta W_t, \left[f_z^*\left(t\right)k_t - b_z^*\left(t\right)p_t - \sigma_z^*\left(t\right)q_t + l_z\left(t\right)\right]\Delta W_t\right\rangle \\ &+ \left\langle \eta_t, f_y^*\left(t\right)k_t - b_y^*\left(t\right)p_t - \sigma_y^*\left(t\right)q_t + l_y\left(t\right)\right\rangle + \Psi_t,\end{aligned}$$

where

$$\begin{aligned}\Psi_t = &\ \left\langle \zeta_t\Delta W_t, k_t + f_y^*\left(t\right)k_t - b_y^*\left(t\right)p_t - \sigma_y^*\left(t\right)q_t + l_y\left(t\right)\right\rangle \\ &+ \left\langle \eta_t, \left[f_z^*\left(t\right)k_t - b_z^*\left(t\right)p_t - \sigma_z^*\left(t\right)q_t + l_z\left(t\right)\right]\Delta W_t\right\rangle \\ &+ \left\langle k_t + f_y^*\left(t\right)k_t - b_y^*\left(t\right)p_t - \sigma_y^*\left(t\right)q_t + l_y\left(t\right), \Delta V_t\right\rangle \\ &+ \left\langle \left[f_z^*\left(t\right)k_t - b_z^*\left(t\right)p_t - \sigma_z^*\left(t\right)q_t + l_z\left(t\right)\right]\Delta W_t, \Delta V_t\right\rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\mathbb{E}\left\langle \left[\sigma_x\left(t\right)\xi_t + \sigma_y\left(t\right)\eta_t + \sigma_z\left(t\right)\zeta_t + \delta_{ts}\varepsilon\sigma_{iu}\left(t\right)\Delta v\right]\Delta W_t, q_t\Delta W_t\right\rangle \\ &= \mathbb{E}\left\langle \sigma_x\left(t\right)\xi_t + \sigma_y\left(t\right)\eta_t + \sigma_z\left(t\right)\zeta_t + \delta_{ts}\varepsilon\sigma_{iu}\left(t\right)\Delta v, q_t\mathbb{E}\left[\Delta W_t^2|\mathcal{F}_t\right]\right\rangle \\ &= \mathbb{E}\left[\sum_{i=1}^d \left\langle \xi_t, \sigma_{ix}^*\left(t\right)q_te_i\right\rangle\right]\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\left\langle \zeta_t \Delta W_t, \left[f_z^*(t)\, k_t - b_z^*(t)\, p_t - \sigma_z^*(t)\, q_t + l_z(t)\right] \Delta W_t\right\rangle \\
&= \mathbb{E}\left\langle \zeta_t, \left[f_z^*(t)\, k_t - b_z^*(t)\, p_t - \sigma_z^*(t)\, q_t + l_z(t)\right] \mathbb{E}\left[\Delta W_t^2 | \mathcal{F}_t\right]\right\rangle \\
&= \mathbb{E}\left\langle \sum_{j=1}^{d} \zeta_t e_j \Delta W_t^j, \sum_{i=1}^{d} f_{z_i}^*(t)\, k_t \Delta W_t^i \right\rangle \\
&= \mathbb{E}\left[\sum_{i=1}^{d} \left\langle f_{z_i}(t)\, \zeta_t e_i, k_t\right\rangle\right].
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
& \mathbb{E}\left[\Delta\left(\langle \xi_t, p_t\rangle + \langle \eta_t, k_t\rangle\right)\right] \\
= \quad & \mathbb{E}\left\langle -b_x(t+1)\, \xi_{t+1}, p_{t+1}\right\rangle + \left\langle b_x(t)\, \xi_t, p_t\right\rangle - \left\langle \xi_{t+1}, \sigma_x^*(t+1)\, q_{t+1}\right\rangle + \left\langle \xi_t, \sigma_x^*(t)\, q_t\right\rangle \\
& - \left\langle f_y(t+1)\, \eta_{t+1}, k_{t+1}\right\rangle + \left\langle f_y(t)\, \eta_t, k_t\right\rangle - \left\langle f_z(t+1)\, \zeta_{t+1}, k_{t+1}\right\rangle + \left\langle f_z(t)\, \zeta_t, k_t\right\rangle \\
& + \left\langle l_x(t+1), \xi_{t+1}\right\rangle + \left\langle l_y(t), \eta_t\right\rangle + \left\langle l_z(t), \zeta_t\right\rangle + \varepsilon \left\langle \delta_{ts} b_u(t)\, \Delta v, p_t\right\rangle + \delta_{ts}\varepsilon \left\langle \sigma_u(t)\, \Delta v, q_t\right\rangle \\
& - \varepsilon \left\langle \delta_{(t+1)s} f_u(t+1)\, \Delta v, k_{t+1}\right\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& -\mathbb{E}\left\langle h_x\left(\bar{X}_T\right), \xi_T\right\rangle \\
= \quad & \mathbb{E}\left[\langle \xi_T, p_T\rangle + \langle \eta_T, k_T\rangle - \langle \xi_0, p_0\rangle - \langle \eta_0, k_0\rangle\right] \\
= \quad & \textstyle\sum_{t=0}^{T-1} \mathbb{E}\Delta\left(\langle \xi_t, p_t\rangle + \langle \eta_t, k_t\rangle\right) \\
= \quad & \mathbb{E}\left[\langle b_x(0)\, \xi_0, p_0\rangle + \textstyle\sum_{i=1}^{d} \langle \xi_0, \sigma_x^*(0)\, q_0\rangle + \langle f_y(0)\, \eta_0, k_0\rangle + \langle f_z(0)\, \zeta_0, k_0\rangle\right] \\
& + \textstyle\sum_{t=0}^{T-1} \mathbb{E}\left[\langle l_x(t), \xi_t\rangle + \langle l_y(t), \eta_t\rangle + \langle l_z(t), \zeta_t\rangle\right] \\
& + \textstyle\sum_{t=0}^{T} \delta_{ts}\varepsilon \mathbb{E}\left[\langle b_u^*(t)\, p_t, \Delta v\rangle + \langle \sigma_u^*(t)\, q_t, \Delta v\rangle - \langle f_u^*(t)\, k_t, \Delta v\rangle\right].
\end{aligned}
$$

Notice that $\xi_0 = 0$, $k_0 = 0$. So

$$
\begin{aligned}
& \mathbb{E}\textstyle\sum_{t=0}^{T-1} \left[\langle l_x(t), \xi_t\rangle + \langle l_y(t), \eta_t\rangle + \langle l_z(t), \zeta_t\rangle\right] + \mathbb{E}\left\langle h_x\left(\bar{X}_T\right), \xi_T\right\rangle \\
= \quad & -\varepsilon \mathbb{E}\left[\langle b_u^*(s)\, p_s + \sigma_u^*(s)\, q_s - f_u^*(s)\, k_s, \Delta v\rangle\right].
\end{aligned}
$$

Since $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[J\left(u^{\varepsilon}(\cdot)\right) - J\left(\bar{u}(\cdot)\right)\right] \geq 0$, we obtain

$$
\mathbb{E}\left[\langle b_u^*(s)\, p_s + \sigma_u^*(s)\, q_s - f_u^*(s)\, k_s - l_u(s), \Delta v\rangle\right] \leq 0.
$$

Then, (4.13) holds due to that s is taking arbitrarily. This completes the proof. ■

References

[1] Bender, C., & Zhang, J. (2008). Time discretization and Markovian iteration for coupled FBSDEs. The Annals of Applied Probability, 18(1), 143-177.

[2] Bensoussan, A. (1982). Lectures on stochastic control. In Nonlinear filtering and stochastic control (pp. 1-62). Springer, Berlin, Heidelberg.

[3] Bielecki, T. R., Cialenco, I., & Chen, T. (2015). Dynamic conic finance via backward stochastic difference equations. SIAM Journal on Financial Mathematics, 6(1), 1068-1122.

[4] Bismut, J. M. (1978). An introductory approach to duality in optimal stochastic control. SIAM review, 20(1), 62-78.

[5] Cohen, S. N., & Elliott, R. J. (2010). A general theory of finite state backward stochastic difference equations. Stochastic Processes and their Applications, 120(4), 442-466.

[6] Cohen, S. N., & Elliott, R. J. (2011). Backward stochastic difference equations and nearly time-consistent nonlinear expectations. SIAM Journal on Control and Optimization, 49(1), 125-139.

[7] Delarue, F., & Menozzi, S. (2006). A forward–backward stochastic algorithm for quasi-linear PDEs. The Annals of Applied Probability, 16(1), 140-184.

[8] Dokuchaev, N., & Zhou, X. Y. (1999). Stochastic controls with terminal contingent conditions. Journal of Mathematical Analysis and Applications, 238(1), 143-165.

[9] El Karoui, N., & Huang, S. J. (1997). A general result of existence and uniqueness of backward stochastic differential equations. Pitman Research Notes in Mathematics Series, 27-38.

[10] Huang, J., & Shi, J. (2012). Maximum principle for optimal control of fully coupled forward-backward stochastic differential delayed equations*. ESAIM: Control, optimisation and calculus of variations, 18(4), 1073-1096.

[11] Huang, J., Wang, G., & Xiong, J. (2009). A maximum principle for partial information backward stochastic control problems with applications. SIAM journal on Control and Optimization, 48(4), 2106-2117.

[12] Kushner, H. J. (1972). Necessary conditions for continuous parameter stochastic optimization problems. SIAM Journal on Control, 10(3), 550-565.

[13] Lim, A. E., & Zhou, X. Y. (2001). Linear-quadratic control of backward stochastic differential equations. SIAM journal on control and optimization, 40(2), 450-474.

[14] Lin, X., & Zhang, W. (2015). A maximum principle for optimal control of discrete-time stochastic systems with multiplicative noise. IEEE Transactions on Automatic Control, 60(4), 1121-1126.

[15] Ji, S. & Liu, H. (2018). Fully coupled forward-backward stochastic difference equations, arXiv.

[16] Peng, S. (1990). A general stochastic maximum principle for optimal control problems. SIAM Journal on control and optimization, 28(4), 966-979.

[17] Peng, S. (1993). Backward stochastic differential equations and applications to optimal control. Applied Mathematics and Optimization, 27(2), 125-144.

[18] Pardoux, E., & Peng, S. (1990). Adapted solution of a backward stochastic differential equation. Systems & Control Letters, 14(1), 55-61.

[19] Schroder, M., & Skiadas, C. (1999). Optimal consumption and portfolio selection with stochastic differential utility. Journal of Economic Theory, 89(1), 68-126.

[20] Stadje, M. (2010). Extending dynamic convex risk measures from discrete time to continuous time: A convergence approach. Insurance: Mathematics and Economics, 47(3), 391-404.

[21] Wang, G., & Wu, Z. (2009). The maximum principles for stochastic recursive optimal control problems under partial information. IEEE Transactions on Automatic control, 54(6), 1230-1242.

[22] Williams, N. (2009). On dynamic principal-agent problems in continuous time. University of Wisconsin, Madison.

[23] Wu, Z. (1998). Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems. Systems Science and Mathematical sciences, 3, 249-259.

[24] Wu, Z. (2013). A general maximum principle for optimal control of forward–backward stochastic systems. Automatica, 49(5), 1473-1480.

[25] Xu, W. (1995). Stochastic maximum principle for optimal control problem of forward and backward system. The ANZIAM Journal, 37(2), 172-185.

[26] Yong, J. (2010). Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions. SIAM Journal on Control and Optimization, 48(6), 4119-4156.

[27] Xu, J., Zhang, H., & Xie, L. (2017). Solvability of general linear forward and backward stochastic difference equations. Control Conference (pp.1888-1891). IEEE.

[28] Xu, J., Zhang, H., & Xie, L. (2018). General linear forward and backward stochastic difference equations with applications. Automatica, 96, 40-50.