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SOME EXAMPLES OF CALABI-YAU PAIRS WITH MAXIMAL INTERSECTION AND NO TORIC MODEL

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Abstract. It is known that a maximal intersection log canonical Calabi-Yau surface pair is crepant birational to a toric pair. This does not hold in higher dimension: this article presents some examples of maximal intersection Calabi-Yau pairs that admit no toric model.

1. Introduction and motivation

A Calabi–Yau (CY) pair (X, D_X) consists of a normal projective variety X and a reduced sum of integral Weil divisors D_X such that $K_X + D_X \sim_{\mathbb{Z}} 0$.

The class of CY pairs arises naturally in a number of problems and comprises examples with very different birational geometry. Indeed, on the one hand, a Gorenstein Calabi–Yau variety X can be identified with the CY pair $(X, 0)$. On the other hand, if X is a Fano variety, and if D_X is an effective reduced anticanonical divisor, then (X, D_X) is also a CY pair.

Definition 1.1. (a) A pair (X, D_X) is (t,dlt) (resp. (t,lc)) if X is $\mathbb{Q}$-factorial, terminal and (X, D_X) divisorially log terminal (resp. log canonical).

(b) A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is volume preserving if $a_E(K_X + D_X) = a_E(K_Y + D_Y)$ for every geometric valuation E with centre on X and on Y.

The dual complex of a dlt pair $(Z, D_Z = \sum D_i)$ is the regular cell complex obtained by attaching an $(|I| - 1)$-dimensional cell for every irreducible component of a non-empty intersection $\bigcap_{i \in I} D_i$.

The dual complex encodes the combinatorics of the lc centres of a dlt pair and [4] show that its PL homeomorphism class is a volume preserving birational invariant.

By [3, Theorem 1.9], a (t,lc) CY pair (X, D_X) has a volume preserving (t,dlt) modification $(\widetilde{X}, D_{\widetilde{X}}) \to (X, D_X)$, and the birational map between two such modifications is volume preserving.

Abusing notation, I call dual complex the following volume preserving birational invariant of a (t,lc) CY pair (X, D_X).

Definition 1.2. $\mathcal{D}(X, D_X)$ is the PL homeomorphism class of the dual complex of a volume preserving (t,dlt) modification of (X, D_X).

As the underlying varieties of CY pairs range from CY to Fano varieties, they can have very different birational properties. However, X being Fano is not a volume preserving birational invariant of the pair (X, D_X). Following [13], I consider the following volume preserving birational invariant notion:

Definition 1.3. A (t,lc) CY pair (X, D_X) has maximal intersection if $\dim \mathcal{D}(X, D_X) = \dim X - 1$.

In other words, (X, D_X) has maximal intersection if there is a volume preserving (t,dlt) modification of (X, D_X) with a 0-dimensional log canonical centre. Maximal intersection CY pairs have some Fano-type properties; Kollár and Xu show the following:

Theorem 1.4. *Let (X, D_X) be a dlt maximal intersection CY pair, then:*

1. [13, Proposition 19] *X is rationally connected,*

2. [13, Theorem 21] *there is a volume preserving map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Z, D_Z)$ such that D_Z fully supports a big and semiample divisor.*

Remark 1.5. The expression "Fano-type" should be understood with a pinch of salt. Having maximal intersection is a degenerate condition: a general (t,lc) CY pair (X, D_X) with X Fano and D_X a reduced anticanonical section needs not have maximal intersection.

Definition 1.6. A toric pair (X, D_X) is a (t,lc) CY pair formed by a toric variety and the reduced sum of toric invariant divisors.

A toric model is a volume preserving birational map to a toric pair.

Example 1.7. A CY pair with a toric model has maximal intersection.

Remark 1.8. In dimension 2, the converse holds: maximal intersection CY surface pairs are precisely those with a toric model [6].

The characterisation of CY pairs with a toric model is an open and difficult problem. A characterisation of toric pairs was conjectured by Shokurov and is proved in [1], but it is not clear how to refine it to get information on the existence of a toric model. A motivation to better understand the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry.

The mirror conjecture extends from a duality between Calabi-Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models, i.e. non-compact Kähler manifolds endowed with a superpotential. Most known constructions of mirror partners rely on toric features such as the existence of a toric model or of a toric degeneration. In an exciting development, Gross, Hacking and Keel conjecture the following construction for mirrors of maximal intersection CY pairs.

Conjecture 1.9. [6] *Let (Y, D_Y) be a simple normal crossings maximal intersection CY pair. Assume that D_Y supports an ample divisor, let R be the ring $k[\mathrm{Pic}(Y)^\times]$, Ω the canonical volume form on U and*

$$U^{\mathrm{trop}}(\mathbb{Z}) = \Big\{ \textit{divisorial valuations } v\colon k(U) \setminus \{0\} \to \mathbb{Z} \textit{ with } v(\Omega) < 0 \Big\} \cup \{0\}.$$

Then, the free R-module V with basis $U^{\mathrm{trop}}(\mathbb{Z})$ has a natural finitely generated R-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on U.

Denote by K the torus $\mathrm{Ker}\{\mathrm{Pic}(Y) \to \mathrm{Pic}(U)\}$. The fibration

$$p\colon \mathrm{Spec}(V) \to \mathrm{Spec}(R) = T_{\mathrm{Pic}(Y)}$$

is a T_K-equivariant flat family of affine maximal intersection log CY varieties. The quotient

$$\mathrm{Spec}(V)/T_K \to T_{\mathrm{Pic}(U)}$$

only depends on U and is the mirror family of U.

Versions of Conjecture 1.9 are proved for cluster varieties in [7], but relatively few examples are known.

The goal of this note is to present examples of maximal intersection CY pairs that do not admit a toric model and for which one can hope to construct the mirror partner proposed in Conjecture 1.9 (see Section 2 for a precise statement).

2. Auxiliary results on 3-fold CY pairs

The examples in Section 3 are 3-fold maximal intersection CY pairs whose underlying varieties are birationally rigid. In particular, such pairs admit no toric model; this shows that [6]'s results on maximal intersection surface CY pairs do not extend to higher dimensions. In this section, I first recall some results on birational rigidity of Fano 3-folds. Then, I introduce the (t,dlt) modifications suited to the construction outlined in Conjecture 1.9 and discuss the singularities of the boundary D_X.

2.1. **Birational rigidity.** Let X be a terminal $\mathbb{Q}$-factorial Fano 3-fold. When X has Picard rank 1, X is a Mori fibre space, i.e. an end product of the classical MMP.

Definition 2.1. A birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ between Mori fibre spaces Y/S and Y'/S' is square if it fits into a commutative square

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & Y' \\ \downarrow & & \downarrow \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

where g is birational and the restriction $Y_\eta \overset{\varphi_\eta}{\dashrightarrow} Y'_\eta$ is biregular, where η is the function field of the base $k(S)$.

A Mori fibre space Y/S is (birationally) rigid if for every birational map $Y/S \overset{\varphi}{\dashrightarrow} Y'/S'$ to another Mori fibre space, there is a birational self map $Y/S \overset{\alpha}{\dashrightarrow} X/S$ such that $\varphi \circ \alpha$ is square.

In particular, if X is a rigid Mori fibre space, then X is non-rational and no (t,lc) CY pair (X, D_X) admits a toric model.

Non-singular quartic hypersurfaces $X_4 \subset \mathbb{P}^4$ are probably the most famous examples of birationally rigid 3-folds [9]. Some mildly singular quartic hypersurfaces are also known to be birationally rigid, in particular, we have:

Proposition 2.2. [2, 16] *Let $X_4 \subset \mathbb{P}^4$ be a quartic hypersurface with no worse than ordinary double points. If $|\operatorname{Sing}(X)| \leq 8$, then X is $\mathbb{Q}$-factorial (in particular, X is a Mori fibre space) and is birationally rigid.*

2.2. **Singularities of the boundary.** I now state some results on the singularities of the boundary of a 3-fold (t,lc) CY pair. Let (X, D_X) be a 3-fold (t,lc) CY pair and $(\widetilde{X}, D_{\widetilde{X}})$ a (t,dlt) modification. A stratum of $(\widetilde{X}, D_{\widetilde{X}})$ is an irreducible component of a non-empty intersection of components of $D_{\widetilde{X}}$. Given a stratum W, there is a divisor $\operatorname{Diff}_W D_{\widetilde{X}}$ on W such that $(W, \operatorname{Diff}_W D_{\widetilde{X}})$ is a lc CY pair and

$$K_W + \operatorname{Diff}_W D_{\widetilde{X}} \sim_{\mathbb{Q}} \left(K_{\widetilde{X}} + D_{\widetilde{X}}\right)_{|W}.$$

When $K_{\widetilde{X}} + D_{\widetilde{X}}$ is Cartier and $D_{\widetilde{X}}$ reduced, $\operatorname{Diff}_W D_{\widetilde{X}}$ is the sum of the restrictions of the components of $D_{\widetilde{X}}$ that do not contain W.

In particular, for any irreducible component S of $D_{\widetilde{X}}$, the link of $[S]$ in $\mathcal{D}(X, D_X)$ is the dual complex $\mathcal{D}(S, \operatorname{Diff}_S D_{\widetilde{X}})$. Therefore, if (X, D_X) has maximal intersection, so does $(S, \operatorname{Diff}_S D_{\widetilde{X}})$. By the results of [6], $(S, \operatorname{Diff}_S D_X)$ then has a toric model.

As X has terminal singularities, X is normal and Cohen-Macaulay. Any Cartier component S of the boundary D_X is Cohen-Macaulay and satisfies Serre's condition S_2. By [12, Proposition 16.9], $(S, \operatorname{Diff}_S D_X)$ is semi log canonical (slc). In particular, if X is Gorenstein and D_X irreducible, D_X has slc singularities.

I am particularly interested in producing examples of (t,lc) CY pairs for which the mirror partners proposed in Conjecture 1.9 (see also [8]) can be constructed; this motivates the following definition:

Definition 2.3. A (t,dlt) modification $(\widetilde{X}, D_{\widetilde{X}}) \to (X, D_X)$ is called good if $(\widetilde{X}, D_{\widetilde{X}})$ is log smooth in the sense of log geometry, that is if the components of $D_{\widetilde{X}}$ are non-singular and if $\widetilde{X}$ has only cyclic quotient singularities.

An immediate consequence of the definition is that if $(\widetilde{X}, D_{\widetilde{X}}) \xrightarrow{f} (X, D_X)$ is a good (t,dlt) modification and $D_X = \sum_i D_i$, then

$$D_{\widetilde{X}} = \sum_i f_*^{-1} D_i + E,$$

where E is reduced and f-exceptional, and the restriction of f to $f_*^{-1} D_i$ is a resolution for all i.

Normal singularities Let $p \in \operatorname{Sing}(D_i)$ be an isolated singularity lying on a single component of the boundary. The restriction $f_i\colon \widetilde{D}_i \to D_i$ is a resolution and we have:

$$K_{\widetilde{D}_i} = (K_{\widetilde{X}} + D_i)_{|\widetilde{D}_i} = (f_{|\widetilde{D}_i})^* K_{D_i} - (E)_{|\widetilde{D}_i}$$

where E is defined by $K_{\widetilde{X}} + f_*^{-1} D_{\widetilde{X}} + E = f^*(K_X + D_X)$.

We now assume that D_i is Cartier, as is the case when X is Gorenstein and D_X irreducible. Without loss of generality, assume that $\mathrm{Sing}(D_i) = p$. Then, p is canonical if $E \cap \widetilde{D}_i = \emptyset$, and elliptic otherwise. Indeed, let

$$f_i \colon \widetilde{D}_i \xrightarrow{q} \overline{D}_i \xrightarrow{\mu} D_i$$

be the factorisation through the minimal resolution of $(p \in D_i)$. Then, q is either an isomorphism or an isomorphism at the generic point of each component of $E_{|\widetilde{D}_i}$ because f is volume preserving. We have: $K_{\overline{D}_i} = \mu^* K_{D_i} - Z$, where the effective cycle $Z = q_*(E_{D_i})$ is either empty (and p is canonical) or a reduced sum of μ-exceptional curves (and p is elliptic). In the second case, $Z \sim -K_{\widetilde{D}_i}$ is the fundamental cycle of $(p \in D_i)$. If Z is irreducible, it is reduced and has genus 1; if not, every irreducible component of Z is a smooth rational curve of self-intersection -2.

When p is elliptic, Z is reduced and p is a Kodaira singularity [10, Theorem 2.9], i.e. a resolution is obtained by blowing up points of the singular fibre in a degeneration of elliptic curves; further, in Arnold's terminology, the singularity p is uni or bimodal.

Further, $p \in D_i$ is a hypersurface singularity (resp. a codimension 2 complete intersection, resp. not a complete intersection) when $-3 \leq Z^2 \leq -1$ (resp. $Z^2 = -4$, resp. $Z^2 \leq -5$) [14]. When $-1 \leq Z^2 \leq -4$, normal forms are known for $p \in D_i$: Table 1 lists normal forms of slc hypersurface singularities, while normal forms of codimension 2 complete intersections elliptic singularities are given in [19].

3. Examples of rigid maximal intersection 3-fold CY pairs

All the examples below are (t, lc) CY pairs (X, D_X) which admit no toric model. Except for Example 3.4, all underlying varieties X are birationally rigid quartic hypersurfaces by Proposition 2.2; the underlying variety in Example 3.4 is a smooth cubic 3-fold, and therefore non-rational.

3.1. **Examples with normal boundary.**

Example 3.1. Consider the CY pair (X, D_X) where X is the nonsingular quartic hypersurface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_0x_1x_2x_3 + x_4(x_0^3 + x_4^3) = 0\}$$

and D_X is its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic surface D_X has a unique singular point $p = (1{:}0{:}0{:}0{:}0)$, and using the notation of Table 1, p is locally analytically equivalent to a $T_{4,4,4}$ cusp

$$\{0\} \in \{x^4 + y^4 + z^4 + xyz = 0\}.$$

type	name	symbol	equation $f \in \mathbb{C}[x,y,z]$		$\operatorname{mult}_0 f$
terminal	smooth	A_0	x		1
canonical	du Val	A_n	$x^2+y^2+z^{n+1}$	$n \geq 1$	2
		D_n	$x^2+z(y^2+z^{n-2})$	$n \geq 4$	2
		E_6	$x^2+y^3+z^4$		2
		E_7	$x^2+y^3+yz^3$		2
		E_8	$x^2+y^3+z^5$		2
lc	simple elliptic	$X_{1,0}$	$x^2+y^4+z^4+\lambda xyz$	$\lambda^4 \neq 64$	2
		$J_{2,0}$	$x^2+y^3+z^6+\lambda xyz$	$\lambda^6 \neq 432$	2
		$T_{3,3,3}$	$x^3+y^3+z^3+\lambda xyz$	$\lambda^3 \neq -27$	3
	cusp	$T_{p,q,r}$	$x^p+y^q+z^r+xyz$	$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$	2 or 3
slc	normal crossing	A_∞	x^2+y^2		2
	pinch point	D_∞	x^2+y^2z		2
	degenerate cusp	$T_{2,\infty,\infty}$	$x^2+y^2+z^2$		2
		$T_{2,q,\infty}$	$x^2+y^2(z^2+y^{q-2})$	$q \geq 3$	2
		$T_{\infty,\infty,\infty}$	xyz		3
		$T_{p,\infty,\infty}$	$xyz+x^p$	$p \geq 3$	3
		$T_{p,q,\infty}$	$xyz+x^p+y^q$	$q \geq p \geq 3$	3

TABLE 1. Dimension 2 slc hypersurface singularities

D_X is easily seen to be rational: the projection from the triple point p is

$$D_X \dashrightarrow \mathbb{P}^2_{x_1,x_2,x_3};$$

this map is the blowup of the 12 points $\{x_1^4+x_2^4+x_3^4 = x_1x_2x_3 = 0\}$, of which 4 lie on each coordinate line $L_i = \{x_i = 0\}$, for $i = 1,2,3$.

I treat this example in detail and construct explicitly a good (t,dlt) modification of the pair (X, D_X).

Let $f\colon X_p \to X$ be the blowup of p, then X_p is non-singular, the exceptional divisor E satisfies $(E, \mathcal{O}_E(E)) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$, and if D denotes the proper transform of D_X, we have:

$$K_{X_p} + D + E = f^*(K_X + D).$$

Explicitly, the blowup $\mathcal{F} \to \mathbb{P}^4$ of $\mathbb{P}^4$ at p is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0) \cap (x_1, \ldots, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \ldots, x_4]$ and A is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & s_1 & s_2 & s_3 & s_4 \\ 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \qquad (1)$$

The equation of X_p is

$$X_p = \{u^2\big(u(s_1^4+s_2^4+s_3^4+(x_0s_1s_2s_3)\big)+s_4(x_0^3+u^3s_4^3) = 0\},$$

while $E = \{u = 0\}$ and $D = \{u(s_1^4 + s_2^4 + s_3^4) + x_0 s_1 s_2 s_3 = 0\}$. By construction, E is the projective plane with coordinates s_1, s_2, s_3. Note that $(X_p, D + E)$ is not dlt because $D \cap E = \{x_0 s_1 s_2 s_3 = 0\}$ consists of 3 concurrent lines C_1, C_2, C_3.

Consider $g_1 \colon X_1 \to X_p$ the blowup of the nonsingular curve

$$C_1 = \{u = s_1 = s_4 = 0\} \subset X_p.$$

The exceptional divisor of g_1 is a surface $E_1 \simeq \mathbb{P}(\mathcal{N}_{C_1/X_p})$, and since $C_1 \simeq \mathbb{P}^1$, the restriction sequence of normal bundles gives

$$\mathcal{N}_{C_1/X_p} \simeq \mathcal{N}_{C_1/E} \oplus (\mathcal{N}_{E/X_p})_{|C_1} \simeq \mathcal{O}_{C_1}(1) \oplus \mathcal{O}_E(-1)_{|C_1},$$

so that $E_1 = \mathbb{F}_2$. Further,

$$K_{X_1} + D + E + E_1 = g_1^*(K_{X_p} + D + E)$$

where, abusing notation, I denote by D and E the proper transforms of the divisors D and E. The "restricted pair" on E_1 is a surface CY pair $(E_1, (D + E)_{|E_1})$ by adjunction. By construction, $E \cap E_1$ is the negative section σ. The curve $\Gamma = D \cap E_1$ is irreducible, and since $(D + E)_{|E_1}$ is anticanonical, we have

$$\Gamma \sim \sigma + 4f \text{ where } f \text{ is a fibre of } \mathbb{F}_2 \to \mathbb{P}^1, \text{ and } \Gamma^2 = 6, \Gamma \cdot E_{|E_1} = 2.$$

The divisors D, E, E_1 meet in two points, the dual complex $\mathcal{D}(X_1, D + E + E_1)$ is not simplicial it is a sphere S^2 whose triangulation is given by 3 vertices on an equator. While not strictly necessary, we consider a further blowup to obtain a (t,dlt) pair with simplicial dual complex.

Denote by C_2 the proper transform of the curve

$$\{u = s_2 = s_4 = 0\}.$$

Then $C_2 \subset E \cap D$ is rational, and as above

$$\mathcal{N}_{C_2/X_1} \simeq \mathcal{N}_{C_2/E} \oplus (\mathcal{N}_{E/X_2})_{|C_2} = \mathcal{O}_{C_2}(1) \oplus \mathcal{O}_{C_2}(-2).$$

Let $g_2 \colon X_2 \to X_1$ be the blowup of C_2, then the exceptional divisor of g_2 is a Hirzebruch surface

$$E_2 \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{N}_{C_2/X_1}) \simeq \mathbb{F}_3.$$

Still denoting by D, E, E_1 the strict transforms of D, E, E_1, we have:

$$K_{X_2} + D + E + E_1 + E_2 = g_2^*(K_{X_1} + D + E + E_1).$$

The pair $(X_2, D + E + E_1 + E_2)$ is dlt; the composition

$$g_2 \circ g_1 \circ f \colon (\widetilde{X}, D_{\widetilde{X}}) = (X_2, D + E + E_1 + E_2) \to (X, D_X)$$

is a good (t,dlt) modification.

The "restrictions" of $(\widetilde{X}, D_{\widetilde{X}})$ to the component of the boundary are the following surface anticanonical pairs:

- On D: $(E + E_1 + E_2)_{|D}$ is a cycle of (-3)-curves, the morphism $D \to D_X$ is the familiar resolution of the $T_{4,4,4}$ cusp singularity;

- On E: $(D + E_1 + E_2)_E$ is the triangle of coordinate lines with self-intersections $(1, 1, 1)$;
- On E_1: $(D + E + E_2)_{E_1}$ is an anticanonical cycle with self-intersections $(5, -3, -1)$;
- On E_2: $(D + E + E_1)_{E_2}$ is an anticanonical cycle with self-intersections $(5, -3, 0)$ (as above, $E_{|E_2} \sim \sigma$ is a negative section, $E_{1|E_2} \sim f$ a fibre of $\mathbb{F}_3 \to \mathbb{P}^1$, and $D_{|E_2} \sim 4f + \sigma$).

It follows that the dual complex $\mathcal{D}(X, D_X)$ is PL homeomorphic to a tetrahedron and (X, D_X) has maximal intersection. Note that $(0 \in D_X)$ is a maximal intersection lc point, and since D_X is a rational surface, it has a toric model.

Example 3.2. Let X be the hypersurface

$$X = \{x_3(x_0^3 + x_1^3) + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\},$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$.

The quartic X has 3 ordinary double points at the intersection points

$$L \cap \{x_0^3 + x_1^3 = 0\},$$

where L is the line $\{x_2 = x_3 = x_4 = 0\}$. The singular locus of D_X is $\mathrm{Sing}(X) \cup \{p\}$, where $p = (0{:}0{:}0{:}1{:}0)$ is a $T_{3,3,4}$ cusp, i.e. locally analytically equivalent to

$$\{0\} \in \{x^3 + y^3 + z^4 + xyz = 0\}.$$

The quartic surface D_X is rational; the projection of D_X from p is

$$D_X \dashrightarrow \mathbb{P}^2_{x_0,x_1,x_2};$$

this map is defined outside of the 12 points (counted with multiplicity) defined by $\{x_2^4 = x_0^3 + x_1^3 + x_0x_1x_2 = 0\}$.

If $\widetilde{X} \xrightarrow{f} X$ is the composition of the blowups at the ordinary double points and at p, $\widetilde{X}$ is smooth and $D_{\widetilde{X}}$ is non-singular, so that f is a good (t,dlt) modification.

The minimal resolution of $p \in D_X$ is a rational curve with self intersection $C^2 = -3$. Explicitly, taking the blowup of X at p, the proper transform is a rational surface D. The exceptional curve is the preimage of a nodal cubic in $\mathbb{P}^2$ blown up at 12 points counted with multiplicities. Note that $(\widetilde{X}, D + E)$ is not dlt, but in order to obtain a (t,dlt) modification, we just need to blowup the node of $D \cap E$ which is a nonsingular point of $\widetilde{X}, D$ and E. The (t,dlt) modification of (X, D_X) in a neighbourhood of p is good and the associated dual complex is 2-dimensional.

The pair (X, D_X) has maximal intersection; but as in the previous examples, X is rigid, so that (X, D_X) can have no toric model.

Example 3.3. Let X be the nonsingular quartic hypersurface

$$X = \{x_0^3x_3 + x_1^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \subset \mathbb{P}^4$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$.

The surface D_X has a unique singular point $p = (0{:}0{:}0{:}1{:}0)$ of D_X, which is a cusp $T_{3,4,4}$, i.e. is locally analytically equivalent to

$$\{0\} \in \{x^3 + y^4 + z^4 + xyz = 0\}.$$

As in Example 3.1, X is non-singular, and finding a good (t,dlt) modification of (X, D_X) will amount to taking a minimal resolution of the singular point of D_X. Let $X_p \to X$ be the blowup of X at p; X_p is non-singular and if D denotes the proper transform of D_X, and E the exceptional divisor, $D \cap E$ consists of 2 rational curves of self intersection -3 and -4. These curves are the proper transforms of $\{x_0 = 0\}$ and of $\{x_0^2 + x_1x_2\}$ under the blow up of $\mathbb{P}^2_{x_0,x_1,x_2}$ at the points

$$\{x_1^4 + x_2^4 = x_0(x_1x_2 + x_0^2) = 0\}.$$

The dual complex consists of 3 vertices that are joined by edges and span 2 distinct faces: $\mathcal{D}(X, D_X)$ is PL homeomorphic to a sphere S^2 whose triangulation is given by 3 vertices on an equator. The CY pair (X, D_X) has maximal intersection but no toric model.

3.2. **Examples with non-normal boundary.**

Example 3.4. This example is due to R. Svaldi. Consider the cubic 3-fold

$$X = \{x_0x_1x_2 + x_1^3 + x_2^3 + x_3q + x_4q' = 0\} \subset \mathbb{P}^4$$

where q, q' are homogeneous polynomials of degree 2 in $x_0, \cdots, x_4$. If the quadrics q and q' are general and if

$$(q(1,0,0,0,0), q'(1,0,0,0,0)) \neq (0,0),$$

then X and $S = \{x_3 = 0\} \cap X$ and $T = \{x_4 = 0\} \cap X$ are nonsingular.

Let D_X be the anticanonical divisor $S+T$. The curve $C = S \cap T = \Pi \cap X$ for $\Pi = \{x_3 = x_4 = 0\}$ is a nodal cubic. It follows that both (S, C) and (T, C) are log canonical, and therefore so is (X, D_X).

Since S and T are smooth, $\mathrm{Sing}(D_X) = S \cap T = C$, and if p is the node of C, we have:

$$(p \in D_X) \sim \{0\} \in \big\{(xy + x^3 + y^3 + z)(xy + x^3 + y^3 + t) = 0\big\}$$
$$\sim \{0\} \in \big\{(xy + z)(xy + t) = 0\big\} \sim \{0\} \in \big\{(xy + z)(xy - z) = 0\big\}.$$

Thus, $p \in D_X$ is a double pinch point, i.e. p is locally analytically equivalent to $\{0\} \in \{x^2y^2 - z^2 = 0\}$.

We now construct a good (t,dlt) modification of (X, D_X). Let $f\colon X_C \to X$ be the blowup of X along C; $\mathrm{Sing}(X_C)$ is an ordinary double point.

Indeed, let $\Pi = \{x_3 = x_4 = 0\}$, then f is the restriction to X of the blowup $\mathcal{F} \to \mathbb{P}^4$, where $\mathcal{F}$ is the rank 2 toric variety $\mathrm{TV}(I, A)$, where $I = (u, x_0, x_1, x_2) \cap (x_3, x_4)$ is the irrelevant ideal of $\mathbb{C}[u, x_0, \dots, x_4]$ and A is the action of $\mathbb{C}^* \times \mathbb{C}^*$ with weights:

$$\begin{pmatrix} u & x_0 & x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

The equation of X_C is

$$\{x_0x_1x_2 + x_1^3 + x_2^3 + u(x_3q + x_4q') = 0\},$$

so that X_C has a unique singular point at

$$x_0 - 1 = u = x_1 = x_2 = x_3q(1,0,0,0,0) + x_4q'(1,0,0,0,0) = 0,$$

and this is a 3-fold ordinary double point. In addition, denoting by $E_f = \{u = 0\} \cap X_C$ the exceptional divisor, we have

$$K_{X_C} + \widetilde{S} + \widetilde{T} + E_f = K_X + S + T,$$

so that the pair $(X_C, \widetilde{S} + \widetilde{T} + E_f)$ is a (t,lc) CY pair.

The pair $(X_C, \widetilde{S} + \widetilde{T} + E_f)$ is not dlt as the boundary has multiplicity 3 along the fibre F over the node of $S \cap T$. The blowup of F is not $\mathbb{Q}$-factorial, therefore in order to obtain a good (t,dlt) modification, we consider the divisorial contraction $g\colon \widetilde{X} \to X_C$ centred along F. This is obtained by (a) blowing up the node, (b) then blowing up the proper transform of F, (c) flopping a pair of lines with normal bundle $(-1,-1)$ and (d) contracting the proper transform of the $\mathbb{P}^1 \times \mathbb{P}^1$ above the node to a point $\frac{1}{2}(1,1,1)$. The exceptional divisor of g is denoted by E_g.

The pair $(\widetilde{X}, \widetilde{S} + \widetilde{T} + \widetilde{E}_f + E_g)$ is the desired (t,dlt) modification of (X, D_X), and it has maximal intersection. The dual complex is PL homeomorphic to a tetrahedron.

Example 3.5. Let X be the quartic hypersurface

$$X = \{x_1^2x_2^2 + x_1x_2x_3l + x_3^2q + x_4f_3 = 0\} \subset \mathbb{P}^4,$$

where l (resp. q) is a general linear (resp. quadratic) form in $x_0, \cdots, x_3$, and f_3 a general homogeneous form of degree 3 in $x_0, \cdots, x_4$. Let D_X be the hyperplane section $X \cap \{x_4 = 0\}$.

As l, q and f_3 are general, X has 6 ordinary double points. Indeed, denote by $L = \{x_1 = x_3 = x = 4 = 0\}$ and $L' = \{x_2 = x_3 = x = 4 = 0\}$, then

$$\operatorname{Sing}(X) = \big\{L \cap \{f_3 = 0\}\big\} \cup \big\{L' \cap \{f_3 = 0\}\big\} = \{q_1, q_2, q_3\} \cup \{q_1', q_2', q_3'\}$$

which consists of 3 points on each of the lines. In the neighbourhood of each point q_i (resp. q_i') for $i = 1, 2, 3$, the equation of X is of the form

$$\{0\} \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of X are ordinary double points. The quartic hypersurface X is birationally rigid by Proposition 2.2.

The surface D_X is non-normal as it has multiplicity 2 along L and L'. The point $p = L \cap L'$ is locally analytically equivalent to

$$\{0\} \in \{x^2y^2 + z^2 = 0\},$$

so that $p \in D_X$ is a double pinch point. We conclude that the surface D_X has slc singularities, and hence (X, D_X) is a (t,lc) CY pair.

We construct a good (t,dlt) modification as follows.

First, since $\mathrm{Sing}(X) \cap L$ (resp. $\mathrm{Sing}(X) \cap L'$) is non-empty, the blowup of X along L (resp. along L') is not $\mathbb{Q}$-factorial. In order to remain in the (t,dlt) category, we consider the divisorial extraction $f\colon X_L \to X$ centered on L (resp. L').This is obtained by (a) blowing up the 3 nodes lying on L, (b) blowing up the proper transform of L, (c) flopping 3 pairs of lines with normal bundle $(-1,-1)$ and (d) contracting the proper transforms of the three exceptional divisors $\mathbb{P}^1 \times \mathbb{P}^1$ lying above the nodes to points $\frac{1}{2}(1,1,1)$. The exceptional divisor of f is denoted by E. Let $p\colon \widetilde{X} \to X$ denote the morphism obtained by composing the divisorial extraction centered on L with that centered on L' (in any order), and let E, E' denote the exceptional divisors of the divisorial extractions. Then

$$K_{\widetilde{X}} + \widetilde{D} + E + E' = p^*(K_X + D)$$

is a (t,dlt) modification of (X, D_X) and it has maximal intersection. The dual complex $\mathcal{D}(X, D_X)$ is is PL homeomorphic to a sphere S^2 whose triangulation is given by 3 vertices on an equator.

4. Further results on quartic 3-fold CY pairs: beyond maximal intersection

This section concentrates on (t,lc) CY pairs (X, D_X), where X is a factorial quartic hypersurface in $\mathbb{P}^4$ and D is an irreducible hyperplane section of X. I give some more detail on the possible dual complexes of such pairs.

As explained in Section 2.2, D_X is slc because (X, D_X) is lc. In order to study completely the dual complexes of such (t,lc) CY pairs, one needs a good understanding of the normal forms of slc singularities that can lie on D. In the case of a general Fano X, this step would require additional work, but here, D_X is a quartic surface in $\mathbb{P}^3$ and the study of singularities of such surfaces has a rich history. I recall some results directly relevant to the construction of degenerate CY pairs (X, D_X). The classification of singular quartic surfaces in $\mathbb{P}^3$ can be broken in three independent cases.

(a) Quartic surfaces with no worse than rational double points: the minimal resolution is a $K3$ surface. Possible configurations of canonical singularities were studied by several authors using the moduli theory of $K3$ surfaces; there are several thousands possible configurations. The pair (X, D_X) is (t,dlt) and the dual complex of (X, D_X) is reduced to a point.
(b) Non-normal quartic surfaces were classified by Urabe [17]; there are a handful of cases recalled in Theorem 4.1.
(c) Non-canonical quartic surfaces with isolated singularities. These are studied by Wall [20] and Degtyarev [5] among others; their results are recalled in Theorem 4.5.

Theorem 4.1. [17]*A non-normal quartic surface $D \subset \mathbb{P}^3$ is one of:*

1. *the cone over an irreducible plane quartic curve with a singular point of type A_1 or A_2.*
2. *a ruled surface over a smooth elliptic curve G, $D = \varphi_{\mathcal{L}}(Z)$, where:*

(a) $\mathcal{L} = \mathcal{O}_Z(C_1) \otimes \pi^* M$, *and* $Z = \mathbb{P}_G(\mathcal{O}_G \oplus N)$, *for*
-M a line bundle of degree 2 and
-N a non-trivial line bundle of degree 0.
Denoting by L_i the images by $\varphi_{\mathcal{L}}$ of the sections of Z associated to $\mathcal{O}_G \oplus N \to \mathcal{O}_G$ and $\mathcal{O}_G \oplus N \to N$, $\mathrm{Sing}(D) = L_1 \cup L_2$.

(b) $\mathcal{L} = \mathcal{O}_Z(C) \otimes \pi^* M$, *and* $Z = \mathbb{P}_G(E)$ *for a rank 2 vector bundle E that fits in a non-splitting*

$$0 \to \mathcal{O}_G \to E \to \mathcal{O}_G \to 0.$$

Denoting by L the image by $\varphi_{\mathcal{L}}$ of a section $G \to Z$, $\mathrm{Sing}\, D = L$.

3. a rational surface $D \subset \mathbb{P}^3$ which is

(a) the image of a smooth $S \subset \mathbb{P}^5$ under the projection from a line disjoint from S; D has no isolated singular point and
-$S = v_2(\mathbb{P}^2)$, where v_2 is the Veronese embedding; D is the Steiner Roman surface and is homeomorphic to $\mathbb{RP}^2$;
-$S = \varphi(\mathbb{P}^1 \times \mathbb{P}^1)$, where φ is the embedding defined by $|l_1 + 2l_2|$ for $l_{1,2}$ the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$;
-$S = \varphi(\mathbb{F}_2)$, where φ is the embedding defined by $|\sigma + f|$ for σ the negative section and f the fibre of $\mathbb{F}_2$.

(b) the image of a surface $\hat{D} \subset \mathbb{P}^4$ with canonical singularities under the projection from a point not lying on it; $\hat{D}$ is a degenerate dP4 surface which is the blowup of $\mathbb{P}^2$ in 5 points in almost general position.

(c) a rational surface embedded by a complete linear system on its normalisation $\hat{D}$; the non-normal locus of D is a line L and D may have isolated singularities outside L. The minimal resolution of the normalisation of D is a blowup of $\mathbb{P}^2$ in 9 points. The normalisation of D has at most two rational triple points lying on the inverse image of the non-normal locus; their images on D are also triple points.

Remark 4.2. D is not slc in case 1.

Corollary 4.3. *Let (X, D_X) be a (t,lc) quartic CY pair with non-normal boundary. Then, (X, D_X) has maximal intersection except in the cases described in 2.(a) and (b) of Theorem 4.1.*

Example 4.4. Consider the pair (X, D_X) where:

$$X = \{x_0^2 x_3^2 + x_1^2 x_2 x_3 + x_2^2 q(x_0, x_1) + x_4 f_3 = 0\}, D_X = X \cap \{x_4 = 0\},$$

where q is a general quadratic form in (x_1, x_2) and f_3 a general cubic in $x_0, \cdots, x_4$.

When q and f_3 are general, the quartic hypersurface X has 3 ordinary double points. Indeed, denote by $L = \{x_0 = x_1 = x_4 = 0\}$, then $\mathrm{Sing}(X)$ consists of points of intersection of L with $\{f_3 = 0\}$; there are 3 such points $\{q_1, q_2, q_3\}$ when f_3 is general. In the neighbourhood of each point q_i for $i = 1, 2, 3$, the equation of X is of the form

$$\{0\} \in \{xy + zt = 0\}$$

(and $D_X = \{t = 0\}$) so that all singular points of X are ordinary double points. The nodal quartic X is terminal and $\mathbb{Q}$-factorial because it has less than 9 ordinary double points; X is birationally rigid by [2, 16].

Taking the divisorial extraction of the line L is enough to produce a dlt modification $(\widetilde{X}, D_{\widetilde{X}} + E)$ of (X, D_X); this shows that (X, D_X) does not have maximal intersection. The dual complex has a single 1-stratum, the elliptic curve $D_{\widetilde{X}} \cap E$, which is a $(2,2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$. The quartic surface D_X is a ruled surface over an elliptic curve isomorphic to $D_{\widetilde{X}} \cap E$; it is an example of case 2.(b) in Theorem 4.1.

Theorem 4.5. [20] *A normal quartic surface $D \subset \mathbb{P}^3$ with at least one non-canonical singular point is one of:*

1. *D has a single elliptic singularity and D is rational, or*
2. *D is a cone, or*
3. *D is elliptically ruled and*

(a) *D has a double point p with tangent cone z^2, the projection away from p is the double cover of $\mathbb{P}^2$ branched over a sextic curve Γ. The curve Γ is the union of 3 conics in a pencil that also contains a double line. When this line is a common chord, D has two $T_{2,3,6}$ singularities, when this line is a common tangent, D has one singularity of type $E_{4,0}$. In the first case, D may have an additional A_1 singular point.*

(b) *D is $\{(x_0x_3 + q(x_1, x_2))^2 + f_4(x_1, x_2, x_3) = 0\}$ and $\{f_4 = 0\}$ is four concurrent lines. Depending on whether $L = \{x_3 = 0\}$ is one of these lines or not and on whether the point of concurrence lies on L, D has either two $T_{2,4,4}$ singular points or one trimodal elliptic singularity. The surface may have additional canonical points A_n for $n = 1, 2, 3$ or $2A_1$.*

Example 4.6. Let X be the nonsingular quartic hypersurface

$$X = \{x_0^2x_3^2 + x_0x_1^3 + x_3x_2^3 + x_0x_1x_2x_3 + x_4(x_0^3 + x_3^3 + x_4^3) = 0\}$$

and D_X its hyperplane section $X \cap \{x_4 = 0\}$. The surface D_X is normal,

$$\operatorname{Sing}(D_X) = \{p, p'\} = \{(1{:}0{:}0{:}0{:}0), (0{:}0{:}0{:}1{:}0)\},$$

and each singular point is simple elliptic $J_{2,0} = T_{2,3,6}$, i.e. is locally analytically equivalent to $\{0\} \in \{x^2 + y^3 + z^6 + xyz = 0\}$.

Here X is nonsingular and D_X is irreducible and normal, and as I explain below, finding a good (t,dlt) modification amounts to constructing a minimal resolution of D_X. Let $\widetilde{X} \to X$ be the composition of the weighted blowups at $p = (1{:}0{:}0{:}0{:}0)$ with weights $(0, 2, 1, 3, 1)$ and at $p' = (0{:}0{:}0{:}1{:}0)$ with weights $(3, 1, 2, 0, 1)$, and denote by E and E' the corresponding exceptional divisors. Note that $\widetilde{X}$ is terminal and $\mathbb{Q}$-factorial by [11, Theorem 3.5] and has no worse than cyclic quotient singularities. The morphism

$$(\widetilde{X}, D + E + E') \xrightarrow{f} (X, D)$$

is volume preserving and the intersection of D with each exceptional divisor is a smooth elliptic curve $C_6 \subset \mathbb{P}(1,1,2,3)$ not passing through the singular points of E and E'; f is a good (t,dlt) modification.

The dual complex $\mathcal{D}(X, D_X)$ is 1-dimensional, it has 3 vertices and 2 edges; (X, D_X) does not have maximal intersection. The quartic surface D_X is an example of case 3.(a) in Theorem 4.5.

Corollary 4.7. *Let (X, D_X) be a (t,lc) quartic CY pair. Assume that D_X is normal, has non-canonical singularities but is not a cone. Then (X, D_X) has maximal intersection except in cases 3.(a) and (b) of Theorem 4.5.*

Remark 4.8. When $\dim \mathcal{D}(X, D_X) = 1$, D_X either has two $T_{2,3,6}$ or two $T_{2,4,4}$ singularities. Indeed, as is explained in Section 2.2, singular points $p \in D$ are Kodaira singularities, and in particular are at worst bimodal. The description of cases 3.(a) and (b) of Theorem 4.5 immediately implies the result, because a surface singularity of type $E_{4,0}$ is trimodal.

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