# The Facets of the Subtours Elimination Polytope

Brahim Chaourar

Department of Mathematics and Statistics, Al Imam Mohammad Ibn Saud Islamic University (IMSIU) P.O. Box 90950, Riyadh 11623, Saudi Arabia Correspondence address: P. O. Box 287574, Riyadh 11323, Saudi Arabia

#### Abstract

Let  $G = (V, E)$  be an undirected graph. The subtours elimination polytope  $P(G)$  is the set of  $x \in \mathbb{R}^E$  such that:  $0 \leq x(e) \leq 1$  for any edge  $e \in E$ ,  $x(\delta(v)) = 2$  for any vertex  $v \in V$ , and  $x(\delta(U)) \geq 2$  for any nonempty and proper subset U of V.  $P(G)$  is a relaxation of the Traveling Salesman Polytope, i.e., the convex hull of the Hamiton circuits of G. Maurras  $[13]$  and Grötschel and Padberg  $[12]$  characterize the facets of  $P(G)$  when G is a complete graph. In this paper we generalize their result by giving a minimal description of  $P(G)$  in the general case and by presenting a short proof of it.

2010 Mathematics Subject Classification: Primary 90C57, Secondary 90C27, 52B40. Key words and phrases: Traveling Salesman Problem, subtours elimination constraints, facets, locked subgraphs.

## 1 Introduction

Sets and their characteristic vectors will not be distinguished. We refer to Bondy and Murty [1], Oxley [14] and Schrijver [15] about, respectively, graphs, matroids and polyhedra terminolgy and facts.

Let  $G = (V, E)$  be an undirected graph. The subtours elimination polytope  $P(G)$  is the set of  $x \in \mathbb{R}^E$  such that:

$$
x(e) \ge 0 \text{ for any edge } e \in E
$$
 (1)

$$
x(e) \le 1 \text{ for any edge } e \in E
$$
\n<sup>(2)</sup>

 $x(\delta(v)) = 2$  for any vertex  $v \in V$  (3)

 $x(\delta(U)) \geq 2$  for any nonempty and proper subset  $U \subseteq V$  (4)

Inequalities (1) and (2) are called the trivial constraints, equalities (3) the degree constraints, and inequalities (4) the subtours elimination constraints.  $P(G)$  is a relaxation of the Traveling Salesman Polytope (TSP), i.e., the convex hull of the Hamiton circuits of G, which permits to get a lower bound for the TSP  $[7]$ . Maurras  $[13]$  and Grötschel and Padberg [12] characterize when a subtour elimination constraint is a facet of  $P(K_n)$  with  $n \geq 3$  as follows.

**Theorem 1.1.** An inequality (4) defines a facet of  $P(K_n)$  ( $n \geq 3$ ) if and only if  $2 \leq |U| \leq$  $n-2$ .

It follows a minimal description of  $P(K_n)$   $(n \geq 3)$ .

**Corollary 1.2.** A minimal description of  $P(K_n)$  ( $n \geq 3$ ) is the set of  $x \in \mathbb{R}^E$  satisfying (1), (3) and constraints (4) with  $2 \leq |U| \leq \lfloor \frac{n}{2} \rfloor$ .

A natural question is: what about any graph? In this paper we generalize this result by giving a minimal description of  $P(G)$  in the general case and by presenting a short proof of it. Another motivation is that "Facets are strongest cutting planes in an integer programming sense (see [10]) and it is thus natural to expect that such inequalities are of substantial computational value in the numerical solution of this hard combinatorial optimization problem" [11].

Since one of the motivations for this question is to optimize over  $P(G)$ , then we can suppose that G is a 2-connected graph without loops (deletion of the loops), bridges  $(P(G) = \emptyset)$ , parallel (deletion of one parallel edge) or series edges (contraction of one series edge) because any of these cases can be reduced to our case by using the indicated operation. We use the notations:  $n = |V|$ ,  $m = |E|$ , and  $n_H = |V(H)|$ ,  $m_H = |E(H)|$  for any subgraph or any subset of edges H of G. Moreover for  $y \in \mathbb{R}^E$  and  $i \in \mathbb{Z}$ , support $i(y) = \{e \in E$ such that  $y(e) = i$ . For  $k \in \mathbb{Z}_+$  and  $\mathcal{X} = \{x_1, ..., x_k\} \subset \mathbb{R}^E$ , X satisfy the intersection condition (IC) (respectively union condition (UC)) if  $\bigcap_{i=1}^{k} support_1(x_i) = \emptyset$  (respectively  $\bigcup_{i=1}^{k} support_0(x_i) = E$ , which is equivalent to the following: for any  $e \in E$ , there exists  $i \in \{1, ..., k\}$  such that  $x_i(e) \neq 1$  (respectively  $x_i(e) = 0$ ). It follows that if  $\mathcal{X} \supseteq \mathcal{X}'$ (respectively  $\mathcal{X} \subseteq \mathcal{X}'$ ) and X satisfy IC (respectively UC) then  $\mathcal{X}'$  too. IC and UC are equivalent when the  $x_i$ 's are  $\{0, 1\}$ -vectors.

Let M be a matroid defined on a finite set E.  $\mathcal{B}(M)$ ,  $M^*$ , the functions r and r<sup>\*</sup>, and  $K(M)$ are, respectively, the class of bases, the dual matroid, the rank and the dual rank functions, and the bases polytope of M. Suppose that M (and  $M^*$ ) is 2-connected. A subset  $L \subset E$  is called a locked subset of M if  $M|L$  and  $M^*|(E\backslash L)$  are 2-connected, and their corresponding ranks are at least 2, i.e.,  $min{r(L), r^*(E \setminus L)} \geq 2$ . It is not difficult to see that if L is locked then both L and  $E\setminus L$  are closed, respectively, in M and  $M^*$  (That is why we call it locked). Locked subsets were introduced to solve many combinatorial problems in matroids [2, 3, 4, 5, 6].

By analogy,  $K(G) = K(M(G))$  and a locked subgraph H of G is a subgraph for which  $E(H)$  is a locked subset of  $M(G)$ .

We can write a minimal description of  $K(G)$  as a corollary of a similar result for  $K(M)$ [6, 8, 9].

**Theorem 1.3.** A minimal description of  $K(G)$  is the set of all  $x \in \mathbb{R}^E$  satisfying the following inequalities:

$$
x(e) \ge 0 \text{ for any edge } e \in E
$$
\n<sup>(5)</sup>

$$
x(e) \le 1 \text{ for any edge } e \in E
$$
\n<sup>(6)</sup>

$$
x(E(H)) \le n_H - 1 \text{ for any locked subgraph } H \text{ of } G \tag{7}
$$

$$
x(E) = n - 1 \tag{8}
$$

For an induced subgraph H of G,  $\overline{H} = (V(E\backslash E(H)), E\backslash E(H))$  is called the complementary subgraph of H in G. Moreover, for any  $F \subseteq E$ ,  $H \cup F$  is the subgraph  $(V(H) \cup V(F), E(H) \cup$  $F$ ).

A collection C of subsets of a nonempty set X is called laminar if for all  $T, U \in \mathcal{C}$ :  $T \subset U$ or  $U \subseteq T$  or  $T \cap U = \emptyset$ . There is the following upper bound on the size of a laminar family [15]:

**Theorem 1.4.** If C is laminar and  $X \notin \mathcal{C}$ , then  $|\mathcal{C}| \leq 2|X| - 1$ .

The remainder of the paper is organized as follows: in section 2, we give a minimal description of  $P(G)$  in the general case, then we conclude in section 3.

## 2 Facets of the subtours elimination polytope

First, we need the following lemma.

**Lemma 2.1.** Let H be a 2-connected subgraph of G, and  $\{L_1, L_2\}$  be a partition of  $E(\overline{H})$ . Then H is connected if and only if  $n_H + n < n_{H\cup L_1} + n_{H\cup L_2}$ .

**Proof.**  $n_H + n < n_{H \cup L_1} + n_{H \cup L_2}$  is equivalent to:  $n_H + n_H + |V(\overline{H})| - |V(H) \cap V(\overline{H})|$  $n_H + n_{L_1} - |V(H) \cap V(L_1)| + n_H + n_{L_2} - |V(H) \cap V(L_2)|$ , i.e.,  $2n_H + |V(H)| - |V(H) \cap V(L_1)|$  $|V(\overline{H})|$  <  $2n_H + n_{L_1} + n_{L_2} - |V(H) \cap V(L_1)| - |V(H) \cap V(L_2)| = 2n_H + n_{L_1} + n_{L_2} |V(H) \cap V(\overline{H})| - |V(H) \cap V(L_1) \cap V(L_2)|$ , i.e.,  $n_{L_1} + n_{L_2} - |V(L_1) \cap V(L_2)| = |V(\overline{H})|$  $n_{L_1} + n_{L_2} - |V(H) \cap V(L_1) \cap V(L_2)|$ , i.e.,  $|V(L_1) \cap V(L_2)| > |V(H) \cap V(L_1) \cap V(L_2)| \geq 0$ , or  $|V(L_1) \cap V(L_2)| \geq 1$ , which means that H is connected.  $\Box$ 

Now we can characterize locked subgraphs by means of graphs terminology.

**Lemma 2.2.** H is a locked subgraph of G if and only if H is an induced and 2-connected subgraph such that  $3 \leq n_H \leq n-1$ , and  $\overline{H}$  is a connected subgraph.

**Proof.** It is not difficult to see that  $M(H)$  is closed and 2-connected in  $M(G)$  if and only if H is an induced and 2-connected subgraph of G. Now, suppose that  $E(H)$  is closed and 2connected, and  $E(\overline{H})$  is 2-connected in the dual matroid  $M^*(G)$ . Let  $\{L_1, L_2\}$  be a partition of  $E(\overline{H})$ . It follows that  $r^*(E(\overline{H})) < r^*(L_1) + r^*(L_2)$ , i.e.,  $|E(\overline{H}) - r(E) + r(E(H))$  $|L_1| + |L_2| - 2r(E) + r(E(H \cup L_1)) + r(E(H \cup L_2)).$  In other words,  $r(E(H)) + r(E)$  $r(E(H) \cup L_1)+r(E(H) \cup L_2)$ , which is equivalent to:  $n_H-1+n-1 < n_{H\cup L_1}-1+n_{H\cup L_2}-2$ , i.e.,  $\overline{H}$  is connected according to the previous lemma.

Let check the condition:  $min\{r(E(H)), r^*(E(\overline{H}))\} \geq 2$ . Since  $r(E(H)) = n_H - 1$  then we have  $r(E(H)) \geq 2$  if and only if  $n_H \geq 3$ . Moreover,  $r^*(E(\overline{H})) = m_{\overline{H}} + r(E(H))$  $r(E) = m_{\overline{H}} + n_H - n$  then we have  $r^*(E(\overline{H})) \ge 2$  if and only if  $n_H \ge 2 + n - m_{\overline{H}}$ , i.e.,

 $|V(H) \cap V(H)| \geq 2 + n_{\overline{H}} - m_{\overline{H}}$ . But, since G is 2-connected and H is connected, either  $|V(H) \cap V(H)| \geq 3$ , or  $|V(H) \cap V(H)| = 2$  and  $n_{\overline{H}} \leq m_{\overline{H}}$ . In both cases, the later inequality is verified and we do not need to mention it.

Furthermore,  $M(H)$  is closed and distinct from E, i.e.,  $r(E(H)) \leq r(E) - 1$ , is equivalent to:  $n_H \leq n-1$ .  $\Box$ 

We have then the following refined description of the subtours elimination polytope.

**Lemma 2.3.** Let  $v_0 \in V$ .  $P(G)$  is the set of all  $x \in \mathbb{R}^E$  satisfying  $(1)$ ,  $(2)$ , and the following constraints:

 $x(\delta(v)) = 2$  for any vertex  $v \in V \setminus \{v_0\}$  (8)

 $x(\delta(U)) > 2$  for any locked subgraph  $G(U)$  of  $G$  (9)

$$
x(E) = n \tag{10}
$$

**Proof.** It is not difficult to see that (3) is equivalent to (8) and (10).

We will prove now the equivalence of a constraint  $(9)$ . It is clear that  $G(U)$  is connected because of (9). Suppose that  $G(U)$  is not 2-connected, then there exist two subsets  $U_1$  and  $U_2$  of U such that  $G(U) = G(U_1) \oplus G(U_2)$ . Let u be the unique vertex in  $U_1 \cap U_2$ . It follows that  $x(\delta(U)) = x(\delta(U_1)) + x(\delta(U_2)) - x(\delta(u)) \ge 2 + 2 - 2 = 2$  and the constraint (4) is redundant for U. Suppose now that  $\overline{G(U)}$  is not connected. It follows that there exists a partition  $\{U_1, U_2\}$  of U such that  $x(\delta(U)) = x(\delta(U_1)) + x(\delta(U_2))$ . It follows that (4) is redundant for U. The constraints  $(8)$  and  $(10)$  imply the redundancy of a constraint  $(4)$ for  $|U| = 1$ . Suppose now that  $|U| = 2$ ,  $U = \{u, w\}$ , and let  $e = uw \in E$ . If follows that  $x(\delta(U)) = x(\delta(u)) + x(\delta(v)) - 2x(e) \geq 2 + 2 - 2 = 2$ , so constraint (4) is redundant for U. At last, since U is a proper subset of V then  $|U| \leq n - 1$ .  $\Box$ 

Let  $Q(G)$  be the set of  $x \in \mathbb{R}^E$  satisfying (1), (2), (10) and:

$$
x(H) \le n_H - 1 \text{ for any locked subgraph } H \text{ of } G \tag{11}
$$

Let  $Q'(G)$  be the set of  $x \in \mathbb{R}^E$  satisfying (1), (2), (9), (10) and:

$$
x(\delta(v)) \ge 2 \text{ for any vertex } v \in V \tag{12}
$$

It follows that:

**Lemma 2.4.** (1)  $P(G) \subseteq Q(G) \subseteq Q'(G)$ . (2) Moreover, the extreme integer points of  $Q(G)$ are exactly the Hamilton circuits of G.

**Proof.** (1) Since  $2x(E(U)) + x(\delta(U)) = \sum_{u \in U} x(\delta(u)) = 2|U|$ , then a constraint (11) with  $|V(H)| = |V| - 1$  implies a constraint (12). It is not difficult to see, similarly as in the previous lemma proof, that constraint (11) implies the following constraint:

$$
x(E(U)) \le |E(U)| - 1 \text{ for any proper subset } U \text{ of } V \tag{11}
$$

Thus, constraint (11) implies constraint (9), for a similar reason as for constraint (12), and  $Q(G) \subseteq Q'(G)$ . According to the previous lemma,  $P(G) \subseteq Q(G)$ .

(2) It is clear that Hamilton circuits of G are integer extreme points of  $Q(G)$  because they satisfy all constraints of  $Q(G)$  and  $m-1$  independent constraints of type (1) and (2). Now, let  $x \in Q(G)$  be an integer extreme point. It follows that x is a  $\{0,1\}$ -vector in  $Q'(G)$  and  $x(E) = n$ , thus x is a Hamilton circuit. □

So optimizing over  $Q(G)$  will give a lower (respectively an upper) bound for TSP (respectively MaxTSP).

We need the following lemma for the coming theorem.

**Lemma 2.5.** Let  $x \in Q(G)$  be an extreme point and  $\mathcal{L}_x$  be an independent system of tight constraints of type (11) verified by x. Then we can choose a laminar system  $L_x = \{U \subset V\}$ such that  $x(E(U)) = |U| - 1$  equivalent to  $\mathcal{L}_x$  and  $|L_x| = |\mathcal{L}_x|$ .

**Proof.** Since  $y(E(U)) + y(E(W)) = y(E(U \cap W)) + y(E(U \cup W))$  for any  $U, W \subset V$  and  $y \in \mathbb{R}^E_+$ , then U and W define tight constraints of type (11) if and only if  $U \cap W$  and  $U \cup W$ too. Applying intersections and unions to subsets of  $\mathcal{L}_x$  will transform it into  $L_x$ .  $\Box$ 

We have the following characterization of extreme points of  $Q(G)$ .

**Theorem 2.6.** (1) For any extreme point  $x \in Q(G)$ , there are  $x_i \in K(G)$ ,  $i = 1, ..., n$  such that  $\{x_1, ..., x_n\}$  satisfy IC and  $x = \frac{1}{n-1} \sum_{i=1}^n x_i$ . (2) For any  $x_i \in K(G)$ ,  $i = 1, ..., n$ , if  $\mathcal{X} = \{x_1, ..., x_n\}$  satisfy UC then  $\frac{1}{n-1} \sum_{i=1}^{n} x_i \in Q(G)$ .

**Proof.** (1) Case 1: x is integer. It follows that x is a Hamilton circuit and the  $x_i$ 's are the *n* Hamilton paths contained in  $x$ .

**Case 2:** x is not integer. Let  $A = \{e \in E \text{ such that } x(e) \geq \frac{1}{n-1}\}, E_1 = support_1(x),$  $E_x = E\$ support<sub>0</sub>(x),  $G_x = (V, A)$ , and c is the number of connected components of  $G_x$ . We can suppose that  $|E_1| \leq n-2$  because otherwise any  $T \subseteq E_1$  with  $|T| = n-1$  is a spanning tree and  $y = x - \frac{1}{n-1}T \in K(G)$ .

Claim 1:  $A \neq \emptyset$ .

By contradiction, suppose that  $x(e) < \frac{1}{n-1}$  for any edge  $e \in E$ . Thus,  $n = x(E) < \frac{m}{n-1} \le$  $n(n-1)$ 2  $\frac{1}{n-1} = \frac{n}{2}$ , a contradiction.

Claim 2:  $|A| \geq n$ .

By contradiction, suppose that  $|A| \leq n-1$ . Thus, there exists a vertex v such that  $|A \cap \delta(v)| \leq 1$  because, otherwise,  $|A| = \frac{1}{2}$  $\frac{1}{2} \sum_{v \in V} |A \cap \delta(v)| \ge \frac{1}{2} 2n = n$ , and we are done. Since  $x(\delta(v)\backslash A) + x(A \cap \delta(v)) = x(\delta(v)) \geq 2$ , because  $Q(G) \subseteq Q'(G)$ , and  $x(A \cap \delta(v)) \leq 2$  $|A \cap \delta(v)| \leq 1$ , because of constraint (2), then  $\frac{|\delta(v)|}{n-1} > x(\delta(v) \setminus A) \geq 1$ , i.e.,  $|\delta(v)| > n - 1$ , a contradiction.

Claim 3:  $G_x$  is a connected subgraph.

By contradiction, suppose that  $c \geq 2$  and let  $(V_i, E_i)$ ,  $i = 1, ..., c$ , be the connected components of  $G_x$ . For any  $i = 1, ..., c, 2 \leq x(\delta(V_i)) < \frac{|\delta(V_i)|}{n-1}$  $\frac{\partial (V_i)}{\partial n-1}$ . It follows that  $|\delta(V_i)| \geq 2n-1$ . Thus,  $|E_x \backslash A| \geq \frac{1}{2} \sum_{i=1}^c |\delta(V_i)| \geq cn - \frac{c}{2}$  $\frac{c}{2}$ , and then, according to Claim 2,  $|E_x| = |A| + |E_x \backslash A| \ge$  $(c+1)n - \frac{c}{2}$  $\frac{c}{2}$ . Since x is an extreme point of  $Q(G)$  then it should verify at least  $\ell$  tight constraints of type (11) that we can choose laminar according to the previous lemma. Thus  $\ell \ge |E_x| - |E_1| - 1 \ge (c+1)n - \frac{c}{2} - n + 2 - 1 = cn - \frac{c}{2} + 1 = \frac{c(2n-1)+2}{2} \ge \frac{2(2n-1)+2}{2} = 2n.$ But according to Theorem 1.4,  $\ell \leq 2n-1$ , a contradiction.

**Claim 4:**  $E_1$  does not contain a circuit.

 $|E_1| = x(E_1) \leq |V(E_1)| - 1 < |V(E_1)|$ . Thus,  $E_1$  is not a circuit. Actually the later inequality is true for any subset of  $E_1$  and we are done.

It follows that there exists a spanning tree  $T \supset E_1$  in  $G_x$  such that  $y = x - \frac{1}{n-1}T \in K(G)$ because of the followings.  $y(e) \geq 0$  because  $T \subseteq A$ .  $y(e) \leq x(e) \leq 1$ , for any edge  $e \in E$ . Similarly,  $y(E(H)) \le x(E(H)) \le n_H - 1$ , for any locked subgraph H of G. At last,  $y(E) = x(E) - \frac{1}{n-1}T(E) = n - \frac{n-1}{n-1} = n - 1.$ 

 $g(D) = x(D) - \frac{n-1}{n-1}$  (*D*)  $-n - \frac{n-1}{n-1} = n-1$ .<br>In this case, y can be written as a convex combination of spanning trees as follows:  $y = \sum_{j=1}^t \lambda_j T_j = \frac{1}{n-1} \sum_{j=1}^t (n-1) \lambda_j T_j = \frac{1}{n-1} \sum_{j=1}^{t'} \lambda'_j T'_j$  with  $\sum_{j=1}^{t'} \lambda'_j = \sum_{j=1}^t (n-1) \lambda_j =$  $(n-1)sum_{j=1}^{t} \lambda_j = n-1$  and  $0 \leq \lambda'_j \leq 1$ . So this sum can be partitioned into  $n-1$ convex combinations of spanning trees:  $y = \frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^{t_i} \mu_{ij} T_{ij} = \frac{1}{n-1} \sum_{i=1}^{n-1} x_i$  with  $x_i \in K(G), i = 1, ..., n-1.$ 

Now, since  $T \supset E_1$ , then  $y(e) < 1$  for any edge  $e \in E$ , i.e.,  $\frac{1}{n-1} \sum_{i=1}^{n-1} x_i(e) < 1$ . Let  $i_0 \in \{1, ..., n-1\}$  such that  $x_{i_0}(e) = Min_{1 \le i \le n-1} \{x_i(e)\}\$ . Thus,  $x_{i_0}(e) = \frac{1}{n-1} \sum_{i=1}^{n-1} x_{i_0}(e) \le$  $\frac{1}{n-1} \sum_{i=1}^{n-1} x_i(e) < 1$ , i.e.,  $x_{i_0}(e) \neq 1$  and we are done.

(2) First we prove (2) when the  $x_i$ 's are spanning trees. Let  $x = \frac{1}{n-1} \sum_{i=1}^n T_i$ . It is clear that x satisfy (1) and (10). UC for  $\mathcal T$  is equivalent to the following. For any edge  $e \in E$ , there exists  $i \in \{1, ..., n\}$  such that  $T_i(e) = 0$ . Thus,  $\sum_{i=1}^{n} T_i(e) \leq n-1$ , and then  $x(e) \leq \frac{n-1}{n-1} = 1$ . Now let H be a locked subgraph of G and  $E_i(H) = \{e \in E(H) \text{ such that } T_i(e) = 0\}$ . It follows that  $T_i(E(H)) = n_H - 1 - |E_i(H)|$ , i.e.,  $\sum_{i=1}^n T_i(E(H)) = n(n_H - 1) - \sum_{i=1}^n |E_i(H)| \le$  $n(n_H - 1) - m_H$ , because  $E(H) = \bigcup_{i=1}^n E_i(H)$  (UC), i.e.,  $m_H \le \sum_{i=1}^n E_i(H)$  $\sum$  $n_H - 1$ ) –  $m_H$ , because  $E(H) = \bigcup_{i=1}^n E_i(H)$  (UC), i.e.,  $m_H \le \sum_{i=1}^n |E_i(H)|$ . Thus,  $\sum_{i=1}^n T_i(E(H)) \le n(n_H - 1) - (n_H - 1) = (n - 1)(n_H - 1)$ , and we are done. Now let  $x = \frac{1}{n-1} \sum_{i=1}^n x_i$  with  $\mathcal{X} = \{x_1, ..., x_n\} \subset K(G)$  satisfy UC. Since  $x_i \in K(G)$ then  $x_i$  can be written as a convex combination of spanning trees:  $x_i = \sum_{j=1}^{k_i} \lambda_{i,j} T_{i,j}$ . UC for X is equivalent to the following. For any edge  $e \in E$ , there exists  $i \in \{1, ..., n\}$ such that  $x_i(e) = 0$ , i.e.,  $T_{ij}(e) = 0$  for all  $j = 1, ..., k_i$ . Thus,  $x = \frac{1}{n-1} \sum_{i=1}^{n} x_i$ 

 $\frac{1}{n-1}\sum_{i=1}^n\sum_{j=1}^{k_i}\lambda_{i,j}T_{i,j} = \sum_{j=1}^k\lambda'_{j}\frac{1}{n-1}\sum_{i=1}^nT_{i,j} = \sum_{j=1}^k\lambda'_{i,j}y_j$  (e.g.  $\lambda'_1 = Min_{i=1}^n\{\lambda_{i,1}$ and so on). But  $y_j \in Q(G)$  for all  $j = 1, ..., m$  (because for any edge  $e \in E$ , there exists a spanning tree  $T_{i,j}$  in the expression of  $y_j$  such that  $T_{ij}(e) = 0$ , i.e.,  $y_j(e) \neq 1$  and  $Q(G)$  is convex, then  $x \in Q(G)$ .  $\Box$ 

For the coming results, we need the following notations.

- For a polytope  $A$ ,  $pA$  is the set of extreme points of  $A$ .
- Let  $k \in \mathbb{Z}_+$ . For  $A \subseteq \mathbb{R}^k$ , PA denotes the smallest convex set of  $\mathbb{R}^k$  containing A.
- For  $A, B \subseteq K(G)$  and  $\lambda \in \mathbb{R}$ ,  $A + B = \{x + y \text{ such that } (x, y) \in A \times B\}$ , and  $\lambda A = \{\lambda x$ such that  $x \in A$ ,
- $\sum_{i=1}^{k} A_i = A_1 + ... + A_k$ .
- ${}^k A = \sum_{i=1}^k A_i$  with  $A_i = A$ , for  $i = 1, ..., k$ .
- Let H be a locked subgraph of G.  $F_H = \{x \in Q(G) \text{ such that } x(E(H)) = n_H 1\},\$
- $F_i = \{x \in K(G) \text{ such that } x(E(H)) = n_H 1 i\},\$

#### 2 FACETS OF THE SUBTOURS ELIMINATION POLYTOPE  $7$

- $K_i = \{x \in K(G) \text{ such that } x(E(H)) \leq n_H 1 i\}, i \in \mathbb{Q},$
- $\Omega = \{(k_1, ..., k_{n_H-1}) \in \mathbb{Z}_+^{n_H-1} \text{ such that } \sum_{i=0}^{n_H-2} k_i(n_H-1-i) = (n_H-1)(n-1) \text{ and }$  $\sum_{i=0}^{n_H-2} k_i \leq n$ ,
- $F'_{(k_0,...,k_{n_H-2})} = \sum_{i=0}^{n_H-2} k_i F_i,$
- $F'_{\lambda} = \bigcup_{(k_0, ..., k_{n_H-2}) \in \Omega} \lambda F'_{(k_0, ..., k_{n_H-2})},$
- $K' = {^n}(K(G)).$
- $\mathcal{T}(G)$  be the class of spanning trees of G,
- $\mathcal{T}_i(G) = \{T \in \mathcal{T}(G) \text{ such that } |T \cap E(H)| = n_H 1 i \},\$
- $Q'' = \bigcup_{i=0}^{n_H-1} \left( \frac{1}{n-1} \mathcal{T}_i + K_{\frac{n_H-1-i}{n-1}} \right),$

• 
$$
K'' = \frac{1}{n-1}\mathcal{T} + K(G)
$$

• For  $X \subseteq {}^{k}(\mathbb{R}^{E}), X(OC) = \{x \in X \text{ such that } x \text{ satisfy OC}\}\$  with OC $\in \{ \text{ IC}, \text{UC} \}.$ 

Note that we have removed from some notations the index  $H$  for more readability. A second corollary of Theorem 2.6 follows.

Corollary 2.7.  $PK'_{\frac{1}{n-1}}(UC) \subseteq Q(G) \subseteq PK'_{\frac{1}{n-1}}(IC)$ .

**Proof.**  $K'_{\frac{1}{n-1}}(UC) \subseteq Q(G)$  and  $pQ(G) \subseteq K'_{\frac{1}{n-1}}(IC)$  according to the Theorem 2.6. By applying  $P$  to both inclusions, we get the result.  $\Box$ 

We need the following lemma.

Lemma 2.8. If  $(k_1, ..., k_{n+1} - 2) \in \Omega$  then  $k_0 \geq n - n + 1(\geq 2)$ .

**Proof.** We have  $\sum_{i=0}^{n_H-2} k_i(n_H-1-i) = (n_H-1)(n-1)$ , i.e.,  $\sum_{i=1}^{n_H-2} k_i(n_H-1-i) =$  $(n_H-1)(n-1-k_0)$ . It follows that:  $(n_H-2)(n-k_0) \ge (n_H-2)\sum_{i=1}^{n_H-2} k_i \ge \sum_{i=1}^{n_H-2} (n_H-2)$  $1-i)k_i = (n_H-1)(n-1-k_0)$ , i.e.,  $(n_H-1)(n-k_0) - (n-k_0) \ge (n_H-1)(n-k_0) - (n_H-1)$ . Thus,  $k_0 \ge n - n_H + 1$ .  $\Box$ 

We use the following properties of the dimension.

#### Lemma 2.9.

- 1.  $Max\{dim(A), dim(B)\} \leq dim(A+B) \leq dim(A) + dim(B).$
- 2.  $dim(\lambda A) = dim(A)$  if  $\lambda \neq 0$ .
- 3.  $dim(A) = 0$  if A is a finite set.
- 4.  $dim(F_1) = m 2$  and  $dim(K(G)) = m 1$ .

#### 3 CONCLUSION 8

Now, we can state our main result.

**Theorem 2.10.** The constraints (1)-(2) and (10)-(11) give a minimal description of  $Q(G)$ .

**Proof.** It is not difficult to prove that constraints  $(1)-(2)$  and  $(10)$  are irredundant. We need only to prove that  $dim(F_H) = m - 2$ . We have  $PF'_{\frac{1}{n-1}}(UC) \subseteq F_H$  because  $F'_{\frac{1}{n-1}}(UC) \subseteq F_H$ . It follows that  $dim(F'_{\frac{1}{n-1}}(UC)) \leq dim(F_H)$ . Moreover,  $m-2 = dim(F_1) \leq dim(F'_{\frac{1}{n-1}}(UC))$ (because of Lemma 2.8) and  $dim(F_H) \le dim(Q(G)) = m-1$ . Thus,  $m-2 \le dim(F'_{\frac{1}{n-1}}(UC)) \le$  $dim(F_H) \leq m-1.$ By using the proof of Theorem 2.6, we can see that:  $F_H \subseteq \bigcup_{i=0}^{n_H-1} \left(\frac{1}{n-1}\mathcal{T}_i + F_{\frac{n_H-1-i}{n-1}}\right)$ . It follows that  $F_H \subseteq Q''$ , and, since  $dim(F_{\frac{n_H-1-i}{n-1}}) < dim(K_{\frac{n_H-1-i}{n-1}})$ , then  $dim(F_H) < dim(Q'')$ . In the other hand,  $Q'' \subseteq K''$ , i.e.,  $dim(Q'') \le dim(K'')$ . At last  $dim(K'') \le dim(\mathcal{T}(G))$  +  $dim(K(G) = m - 1$  and we are done.  $\Box$ 

And we have the following consequence for  $P(G)$ .

**Corollary 2.11.** The constraints (1), (3) and (9) with  $3 \leq |U| \leq n-2$  give a minimal description of  $P(G)$ .

## 3 Conclusion

We have characterized all facets of  $P(G)$  and given a minimal description of it. Future investigations can be improving the fractional ratio of integrity of  $P(G)$  and providing a combinatorial algorithm for optimizing on  $P(G)$ .

### References

- [1] Bondy, J. A., and Murty, U. S. R. (2008), Graph Theory, Springer.
- [2] B. Chaourar (2002), On Greedy Bases Packing in Matroids, European Journal of Combinatorics 23: 769-776.
- [3] B. Chaourar (2008), On the Kth Best Basis of a Matroid, Operations Research Letters 36 (2): 239-242.
- [4] B. Chaourar (2011), A Characterization of Uniform Matroids, ISRN Algebra, Vol. 2011, Article ID 208478, 4 pages, doi:10.5402/2011/208478.
- [5] B. Chaourar (2017), Recognizing Matroids, arXiv 1709.1025.
- [6] B. Chaourar (2018), The Facets of the Bases Polytope of a Matroid and Two Consequences, Open Journal of Discrete Mathematics 8 (1): 14-20.
- [7] G. Dantzig, R. Fulkerson, and S. Johnson (1954), Solution of a Large-scale Traveling Salesman Problem, Journal of the Operations Research Society of America 2: 393410.
- [8] E. M. Feichtner and B. Sturmfels (2005), Matroid polytopes, nested sets and Bergman fans, Portugaliae Mathematica 62 (4): 437-468.
- [9] S. Fujishige (1984), A characterization of faces of the base polyhedron associated with a submodular system, Journal of the Operations Research Society of Japan 27: 112-128.
- [10] R. Garfinkel and G.L. Nemhauser (1972), Integer programming, Wiley, New York.
- [11] M. Grötschel and M. W. Padberg (1979), On the Symmetric Traveling Salesman Problem I: Inequalities, Mathematical Programming 16: 265-280.
- [12] M. Grötschel and M. W. Padberg (1979), On the Symmetric Traveling Salesman Problem II: Lifting Theorems and Facets, Mathematical Programming 16: 281-302.
- [13] J.F. Maurras (1975), Some Results on the Convex Hull of Hamiltonian Cycles of Symetric Complete Graphs, in: Combinatorial Programming: Methods and Applications (Proceedings NATA Advanced Study Institute, Versailles, 1974; B. Roy, ed.), Reidel, Dordrecht : 179-190.
- [14] J. G. Oxley (1992), Matroid Theory, Oxford University Press, Oxford.
- [15] A. Schrijver (2004), Combinatorial Optimization: Polyhedra and Efficiency, Springer-Verlag, Berlin.