

Three-partition Hodge integrals and the topological vertex

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Abstract

A conjecture on the relation between the cubic Hodge integrals and the topological vertex in topological string theory is resolved. A central role is played by the notion of generalized shift symmetries in a fermionic realization of the two-dimensional quantum torus algebra. These algebraic relations of operators in the fermionic Fock space are used to convert generating functions of the cubic Hodge integrals and the topological vertex to each other. As a byproduct, the generating function of the cubic Hodge integrals at special values of the parameters \vec{w} therein is shown to be a tau function of the generalized KdV (aka Gelfand-Dickey) hierarchies.

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1 Introduction

Let $\mathcal{M}_{g,n}$ denote the moduli space of connected complex algebraic curves of genus g with n marked points and $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford compactification. Marked points on an algebraic curve C of genus g are referred to as z_1, \dots, z_n . The Hodge integrals are integrals on $\overline{\mathcal{M}}_{g,n}$ of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_1^{j_1} \cdots \lambda_g^{j_g},$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of the bundle \mathbb{L}_i of the cotangent space of C at the i -th marked point z_i , and $\lambda_j = c_j(\mathbb{E})$ is the j -th Chern class of the Hodge bundle \mathbb{E} over $\overline{\mathcal{M}}_{g,n}$. These Hodge integrals are building blocks of the localization computation of Gromov-Witten invariants [1]. The localization formula expresses those invariants as a sum of weighted graphs [2]. Integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\prod_{i=1}^s \Lambda_g^\vee(u_i)}{\prod_{j=1}^n (1 - z_j \psi_j)},$$

where $\Lambda_g^\vee(u)$ is the characteristic polynomial

$$\Lambda_g^\vee(u) = \sum_{r=0}^g (-1)^r \lambda_r u^{g-r} = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g,$$

arise as the vertex weights in the localization computation of Gromov-Witten invariants of an s -dimensional target space.

Li, Liu, Liu and Zhou [3] reformulated the topological vertex in topological string theory [4] on the basis of formal relative Gromov-Witten theory of non-compact Calabi-Yau threefolds. The topological vertex is a diagrammatic method to compute all-genus A-model topological string amplitudes on smooth non-compact toric Calabi-Yau threefolds. It is somewhat confusing that the vertex weight itself is called the topological vertex. All building blocks of the vertex weight are genuinely combinatorial objects such as partitions, the Littlewood-Richardson numbers and the Schur functions.

The geometric approach of Li et al. stems from their proof of the Mariño-Vafa conjecture and its generalization [5, 6, 7].¹⁾ The Mariño-Vafa conjecture [8] is a

¹⁾The Mariño-Vafa conjecture was also resolved by Okounkov and Pandharipande using a quite different method.

variation of the so called ELSV formula [9]. The ELSV formula expresses the Hurwitz numbers of ramified coverings of \mathbb{P}^1 in terms of linear ($s = 1$) Hodge integrals. The Mariño-Vafa conjecture claims a similar relation between certain open Gromov-Witten invariants and cubic ($s = 3$) Hodge integrals. Actually, the open Gromov-Witten invariants considered therein are related to a special case of the topological vertex.

The approach to the full (i.e., three-partition) topological vertex, just like the proof of the Mariño-Vafa conjecture, starts from a generating function of cubic Hodge integrals. The cubic Hodge integrals are labelled by a triple $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ of partitions and depend on a triple $\vec{w} = (w_1, w_2, w_3)$ of parameters. Li et al. employ various skillful ideas to derive a combinatorial expression of the topological vertex from the generating function. This expression, however, takes a form apparently different from the topological vertex of Aganagic et al. Showing several pieces of evidence, Li et al. conjecture that the two expressions of the topological vertex are equivalent.

In this paper, we prove the equivalence of the two different expressions of the topological vertex. Our strategy is to construct a generating function of the topological vertex of Aganagic et al. (modified to depend on the parameters \vec{w} of the cubic Hodge integrals), and to show that it coincides with (an exponentiated version of) the generating function of Li et al. To this end, we make use of tools developed in a series of our work on the melting crystal models [10, 11] and topological string theory [12, 13, 14]. A central role is played by the notion of generalized shift symmetries [15] in a fermionic realization of the two-dimensional quantum torus algebra. We start from a fermionic representation of the generating function of the topological vertex. The generalized shift symmetries enable us to convert this fermionic representation to the generating function of cubic Hodge integrals of Li et al. This implies the equivalence of the two expressions of the topological vertex.

As a byproduct, we find a new link between the topological vertex and integrable hierarchies. Just after the first proposal [4], particular generating functions of the topological vertex were pointed out to be tau functions of integrable hierarchies such as the KP and 2D Toda hierarchies [16, 17]. We show that the generating function of the cubic Hodge integrals at special values of the parameters \vec{w} is a tau function of the generalized KdV (aka Gelfand-Dickey) hierarchies. Such a link with the generalized KdV hierarchies seems to be unknown in the past literature.

Let us explain these backgrounds and results in more detail.

1.1 Partitions, representations of symmetric group and the Schur functions

Our notations for partitions, representations of symmetric groups and the Schur functions are mostly borrowed from Macdonald's book [18]. Let us recall these notions and basic facts.

A partition is a sequence of non-negative integers

$$\mu = (\mu_i)_{i=1}^{\infty} = (\mu_1, \mu_2, \dots)$$

satisfying $\mu_i \geq \mu_{i+1}$ for all $i \geq 1$. The number of the non-zero μ_i is the length of μ , denoted by $l(\mu)$. The sum of the non-zero μ_i is the weight of μ , denoted by $|\mu| = \mu_1 + \mu_2 + \dots$. Partitions have another expression which indicates the number of times each non-negative integers occurs in a partition

$$\mu = (1^{m_1} 2^{m_2} \dots),$$

where $m_i = m_i(\mu)$ denotes the number of i occurs in μ . The automorphism group of μ , denoted by $\text{Aut}(\mu)$, consists of possible permutations among the non-zero μ_i 's that leave μ . The number of elements in $\text{Aut}(\mu)$ is

$$|\text{Aut}(\mu)| = \prod_{i=1}^{\infty} m_i(\mu)!.$$

Note that partitions are identified with the Young diagrams. The size of the Young diagram μ is $|\mu|$, which is the total number of boxes of the diagram, and $l(\mu)$ is the height of the diagram. The conjugate (or transpose) of μ is the partition ${}^t\mu$ whose diagram is the transpose of the diagram μ .

If $|\mu| = d$, we say that μ is a partition of d . Each partition of d corresponds to a conjugacy class of the d -th symmetric group S_d . Let $C(\mu)$ denote the conjugacy class determined by a partition μ of d . It is determined by the cycle type μ of a representative $\sigma \in S_d$ of $C(\mu)$ as

$$\sigma = (1, \dots, \mu_1)(\mu_1 + 1, \dots, \mu_1 + \mu_2) \cdots (\mu_1 + \dots + \mu_{l-1}, \dots, d),$$

where $l = l(\mu)$ and (j_1, \dots, j_m) means the cyclic permutation sending $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_m \rightarrow j_1$. The number of elements in $C(\mu)$ is

$$|C(\mu)| = \frac{d!}{z_{\mu}}, \quad z_{\mu} = \prod_{i=1}^{\infty} m_i(\mu)! i^{m_i(\mu)}.$$

Each partition μ of d determines the irreducible representation (ρ_μ, V_μ) of S_d . χ_μ denotes the character $\text{Tr}_{V_\mu} \rho_\mu$. The value of χ_μ on the conjugacy class $C(\nu) \subset S_d$, denoted by $\chi_\mu(\nu) = \chi_\mu(C(\nu))$, can be computed by the Frobenius formula

$$s_\mu(\mathbf{x}) = \sum_{|\nu|=d} \frac{\chi_\mu(\nu)}{z_\nu} p_\nu, \quad \mathbf{x} = (x_i)_{i=1}^\infty = (x_1, x_2, \dots), \quad (1.1)$$

where p_ν 's are the monomials

$$p_\nu = p_{\nu_1} p_{\nu_2} \dots$$

of the power sums

$$p_k = \sum_{i=1}^{\infty} x_i^k, \quad k = 1, 2, \dots,$$

and $s_\mu(\mathbf{x})$ denotes the Schur function. (1.1) has the inversion formula

$$p_\mu = \sum_{|\nu|=d} \chi_\nu(\mu) s_\nu(\mathbf{x}). \quad (1.2)$$

Let us recall the Jacobi-Trudi formula

$$s_\mu(\mathbf{x}) = \det (h_{\mu_i - i + j}(\mathbf{x}))_{i,j=1}^n, \quad (1.3)$$

where the length of $\mu = (\mu_i)_{i=1}^\infty$ is not larger than n and $h_m(\mathbf{x})$'s are the complete symmetric functions defined by the generating function

$$\sum_{m=0}^{\infty} h_m(\mathbf{x}) z^m = \prod_{i=1}^{\infty} (1 - x_i z)^{-1}.$$

Likewise, the skew Schur functions have a determinant formula similar to (1.3). Let $\nu = (\nu_i)_{i=1}^\infty$ be a partition such that $\mu \supset \nu$, i.e., $\mu_i \geq \nu_i$ for all i . The determinant formula reads

$$s_{\mu/\nu}(\mathbf{x}) = \det (h_{\mu_i - \nu_j - i + j}(\mathbf{x}))_{i,j=1}^n, \quad (1.4)$$

where μ/ν represents the skew diagram obtained by removing the Young diagram ν from the larger one μ .

The Littlewood-Richardson numbers $c_{\mu\nu}^\eta$ are non-negative integers which are determined by the relation

$$s_\mu(\mathbf{x})s_\nu(\mathbf{x}) = \sum_{\eta \in \mathcal{P}} c_{\mu\nu}^\eta s_\eta(\mathbf{x}), \quad (1.5)$$

where the sum with respect to η ranges over the set \mathcal{P} of all partitions, and is a finite sum because $c_{\mu\nu}^\eta = 0$ unless $|\eta| = |\mu| + |\nu|$, $\eta \supset \mu$ and $\eta \supset \nu$. The skew Schur functions can be expressed in a linear combination of the Schur functions weighted by the Littlewood-Richardson numbers as

$$s_{\mu/\nu}(\mathbf{x}) = \sum_{\eta \in \mathcal{P}} c_{\nu\eta}^\mu s_\eta(\mathbf{x}). \quad (1.6)$$

1.2 Three-partition Hodge integrals

The three-partition Hodge integrals are Hodge integrals that depend on a triple of partitions in a specific manner. For a triple of partitions $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) \in \mathcal{P}^3 = \mathcal{P} \times \mathcal{P} \times \mathcal{P}$, we use the notations

$$l(\vec{\mu}) = \sum_{a=1}^3 l(\mu^{(a)}), \quad \text{Aut}(\vec{\mu}) = \prod_{a=1}^3 \text{Aut}(\mu^{(a)}).$$

$l(\vec{\mu})$ marked points on an algebraic curve of genus g are referred to as $z_1^{(1)}, \dots, z_{l(\mu^{(1)})}^{(1)}$, $z_1^{(2)}, \dots, z_{l(\mu^{(2)})}^{(2)}$ and $z_1^{(3)}, \dots, z_{l(\mu^{(3)})}^{(3)}$. Cotangent lines of curves at the marked point $z_i^{(a)}$ are glued together and form the complex line bundle $\mathbb{L}_i^{(a)}$ over $\overline{\mathcal{M}}_{g, l(\vec{\mu})}$.

Let $\vec{w} = (w_1, w_2, w_3)$ be a triple of variables which satisfy the condition $w_1 + w_2 + w_3 = 0$. The indices of those variables are understood to be cyclic as $w_{i+3} \equiv w_i$. For $\vec{\mu} \in \mathcal{P}_+^3 = \mathcal{P}^3 - \{(\emptyset, \emptyset, \emptyset)\}$, the three-partition Hodge integrals are defined by

$$\begin{aligned} G_{g, \vec{\mu}}(\vec{w}) &= \frac{(\sqrt{-1})^{l(\vec{\mu})}}{|\text{Aut}(\vec{\mu})|} \prod_{a=1}^3 \prod_{i=1}^{l(\mu^{(a)})} \left\{ w_{a+1} \prod_{j=1}^{\mu_i^{(a)}} \left(1 + \frac{\mu_i^{(a)} w_{a+1}}{j w_a} \right) \right\} \\ &\times \int_{\overline{\mathcal{M}}_{g, l(\vec{\mu})}} \prod_{a=1}^3 \frac{\Lambda_g^\vee(w_a) w_a^{l(\mu^{(a)})-1}}{\prod_{i=1}^{l(\mu^{(a)})} (w_a - \mu_i^{(a)} \psi_i^{(a)})}, \end{aligned} \quad (1.7)$$

where $\psi_i^{(a)} = c_1(\mathbb{L}_i^{(a)})$ are the ψ -classes associated with the marked points $z_i^{(a)}$.

We introduce generating functions of these three-partition Hodge integrals. Let $p^{(1)} = (p_k^{(1)})_{k=1}^\infty$, $p^{(2)} = (p_k^{(2)})_{k=1}^\infty$ and $p^{(3)} = (p_k^{(3)})_{k=1}^\infty$ be formal variables, denoted collectively by $\vec{p} = (p^{(1)}, p^{(2)}, p^{(3)})$. We put $p_\mu^{(a)} = \prod_{i=1}^{l(\mu)} p_{\mu_i}^{(a)}$ for a non-zero partition $\mu = (\mu_i)_{i=1}^\infty$, and $p_\emptyset^{(a)} = 1$ for the zero partition \emptyset . The generating functions are defined by

$$G_{\vec{\mu}}(\lambda; \vec{w}) = \sum_{g=0}^{\infty} \lambda^{2g-2+l(\vec{\mu})} G_{g, \vec{\mu}}(\vec{w}), \quad (1.8)$$

$$G(\lambda; \vec{p}; \vec{w}) = \sum_{\vec{\mu} \in \mathcal{P}_+^3} G_{\vec{\mu}}(\lambda; \vec{w}) \prod_{a=1}^3 p_{\mu^{(a)}}^{(a)}, \quad (1.9)$$

where λ amounts to the string coupling constant.

By degree counting of the RHS of (1.7), three-partition Hodge integrals turn out to be homogeneous of degree 0 with respect to \vec{w} :

$$G_{g, \vec{\mu}}(t\vec{w}) = G_{g, \vec{\mu}}(\vec{w}), \quad t\vec{w} = (tw_1, tw_2, tw_3), \quad t \neq 0.$$

This implies that

$$G_{\vec{\mu}}(\lambda; t\vec{w}) = G_{\vec{\mu}}(\lambda; \vec{w}), \quad G(\lambda; \vec{p}; t\vec{w}) = G(\lambda; \vec{p}; \vec{w}).$$

Therefore, \vec{w} can be chosen as

$$\vec{w} = (1, \tau, -1 - \tau) \quad (1.10)$$

where τ is a new variable. For \vec{w} of the form (1.10), the three-partition Hodge integrals are expressed in a shortened form as $G_{g, \vec{\mu}}(\tau) = G_{g, \vec{\mu}}(1, \tau, -1 - \tau)$. The same abbreviation is also used for the generating functions (1.8) and (1.9) as

$$G_{\vec{\mu}}(\lambda; \tau) = G_{\vec{\mu}}(\lambda; 1, \tau, -1 - \tau), \quad G(\lambda; \vec{p}; \tau) = G(\lambda; \vec{p}; 1, \tau, -1 - \tau).$$

The generating function (1.9) can be expanded in an infinite series of the Schur functions by the inversion formula (1.2). The disconnected version of (1.9) turns out to take the specific form [3]

$$\exp(G(\lambda; \vec{p}; \vec{w})) = \sum_{\vec{\mu} \in \mathcal{P}^3} \tilde{\mathcal{C}}_{\vec{\mu}}(\lambda) e^{\frac{\sqrt{-1}}{2} \lambda \sum_{a=1}^3 \kappa(\mu^{(a)}) w_{a+1}/w_a} \prod_{a=1}^3 s_{\mu^{(a)}}^{(a)}, \quad (1.11)$$

where

$$\kappa(\mu^{(a)}) = \sum_{i=1}^{\infty} \mu_i^{(a)} (\mu_i^{(a)} - 2i + 1)$$

denotes the second Casimir invariant of $\mu^{(a)}$ and

$$s_{\mu^{(a)}}^{(a)} = \sum_{\nu \in \mathcal{P}} \frac{\chi_{\mu^{(a)}}(\nu)}{z_{\nu}} p_{\nu}^{(a)}.$$

$s_{\mu^{(a)}}^{(a)}$ is the Schur function $s_{\mu^{(a)}}$ obtained by substituting $p_{\nu} = p_{\nu}^{(a)}$.

(1.11) shows that $\exp G(\lambda; \vec{p}; \vec{w})$ depends on \vec{w} in a particular form. It is a consequence of the *invariance theorem* [3] for a generating function of formal relative Gromov-Witten invariants of \mathbb{C}^3 . These invariants count stable maps from possibly disconnected curves to \mathbb{C}^3 . In the localization computation of these invariants, the fixed points are labelled by a triple of partitions. The contribution from each fixed point takes the form of three-partition Hodge integrals multiplied by the double Hurwitz numbers of \mathbb{P}^1 . Thus, the generating function of the formal relative Gromov-Witten invariants is a sum of these products over the sets of partitions. Though these terms depend on \vec{w} , the invariance theorem implies that the generating function itself does not depend on \vec{w} in total. Accordingly, \vec{w} -dependence of each term should cancel out after the summation. With the aid of a combinatorial expression [7] of the double Hurwitz numbers, the cancellation eventually yields the expansion (1.11).

Three-partition Hodge integrals at special values of τ reduce to two-partition Hodge integrals since the ψ -classes can be renumbered in terms of two partitions at such values of τ . For instance, at $\tau = 1$,

$$\begin{aligned} G_{g,(\mu^{(1)}, \mu^{(2)}, \mu^{(3)})}(1) &= (-1)^{|\mu^{(1)}| - l(\mu^{(1)})} \frac{z_{\mu^{(1)} \cup \mu^{(2)}}}{z_{\mu^{(1)}} z_{\mu^{(2)}}} G_{g,(\emptyset, \mu^{(1)} \cup \mu^{(2)}, \mu^{(3)})}(1) \\ &+ \delta_{g,0} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \delta_{(\mu^{(1)}, \mu^{(2)}, \mu^{(3)}), (\emptyset, (m), (2m))}, \end{aligned} \tag{1.12}$$

where $\mu^{(1)} \cup \mu^{(2)}$ denotes the partition obtained by the union of the parts of $\mu^{(1)}$ and $\mu^{(2)}$ which are rearranged in descending order. The second term in the RHS is an anomalous contribution from the unstable cases. In terms of the generating functions

(1.9), the reduction formula (1.12) takes the form

$$\begin{aligned} & \exp(G(\lambda; (p^{(1)}, p^{(2)}, p^{(3)}); 1)) \\ &= \exp(G(\lambda; (0, p^+, p^{(3)}); 1)) \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{2m}^{(3)}\right), \end{aligned} \quad (1.13)$$

where $p^+ = (p_k^+)_{k=1}^{\infty}$, $p_k^+ = (-1)^{k+1} p_k^{(1)} + p_k^{(2)}$.

An explicit expression of the coefficients $\tilde{\mathcal{C}}_{\vec{\mu}}(\lambda)$ can be read out from (1.13) by plugging (1.11) into (1.13):

$$\begin{aligned} & e^{-\sqrt{-1}\lambda(-\kappa(\mu^{(1)})/2 + \kappa(\mu^{(2)}) + \kappa(\mu^{(3)})/4)} \tilde{\mathcal{C}}_{(\mu^{(1)}, \mu^{(2)}, \mu^{(3)})}(\lambda) \\ &= \sum_{\nu^1, \nu^3, \nu^+, \eta^1, \eta^3 \in \mathcal{P}} c_{t_{\eta^1} \nu^1}^{\mu^{(1)}} c_{t_{\nu^1} \mu^{(2)}}^{\nu^+} c_{\eta^3 \nu^3}^{\mu^{(3)}} e^{-\sqrt{-1}\lambda(\kappa(\nu^+) + \kappa(\nu^3)/4)} \tilde{\mathcal{C}}_{(\emptyset, \nu^+, \nu^3)}(\lambda) \\ & \quad \times \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_{\xi}}, \end{aligned} \quad (1.14)$$

where $2\xi = (2\xi_i)_{i=1}^{\infty}$ for $\xi = (\xi_i)_{i=1}^{\infty}$, and $c_{t_{\eta^1} \nu^1}^{\mu^{(1)}}$, $c_{t_{\nu^1} \mu^{(2)}}^{\nu^+}$ and $c_{\eta^3 \nu^3}^{\mu^{(3)}}$ are the Littlewood-Richardson numbers (1.5). Furthermore, the *cut-and-join equations* for the two-partition Hodge integrals [7] imply that

$$\tilde{\mathcal{C}}_{(\emptyset, \nu^+, \nu^3)}(\lambda) = q^{-\kappa(\nu^3)/2} s_{\nu^+}(q^{-\rho}) s_{t_{\nu^3}}(q^{-\nu^+ - \rho}), \quad (1.15)$$

where $q = e^{-\sqrt{-1}\lambda}$, and $s_{\nu^+}(q^{-\rho})$ and $s_{t_{\nu^3}}(q^{-\nu^+ - \rho})$ are the special values of the infinite-variate Schur functions $s_{\nu^+}(\mathbf{x})$ and $s_{t_{\nu^3}}(\mathbf{x})$ at $q^{-\rho} = (q^{i-1/2})_{i=1}^{\infty}$ and $q^{-\nu^+ - \rho} = (q^{-\nu_i^+ + i - 1/2})_{i=1}^{\infty}$. As a consequence of (1.14) and (1.15), $\tilde{\mathcal{C}}_{\vec{\mu}}(\lambda)$ can be expressed in a closed form [3]:

$$\begin{aligned} & \tilde{\mathcal{C}}_{\vec{\mu}}(\lambda) = q^{\kappa(\mu^{(1)})/2 - \kappa(\mu^{(2)}) - \kappa(\mu^{(3)})/4} \\ & \quad \times \sum_{\nu^1, \nu^3, \nu^+, \eta^1, \eta^3 \in \mathcal{P}} c_{t_{\eta^1} \nu^1}^{\mu^{(1)}} c_{t_{\nu^1} \mu^{(2)}}^{\nu^+} c_{\eta^3 \nu^3}^{\mu^{(3)}} q^{\kappa(\nu^+) + \kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+ - \rho}) \\ & \quad \times \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_{\xi}}. \end{aligned} \quad (1.16)$$

1.3 Topological vertex and three-partition Hodge integrals

The toric data of a non-compact toric Calabi-Yau threefold are encoded in the associated fan of rational cones of dimension ≤ 3 on \mathbb{R}^3 . A plane section of this fan yields

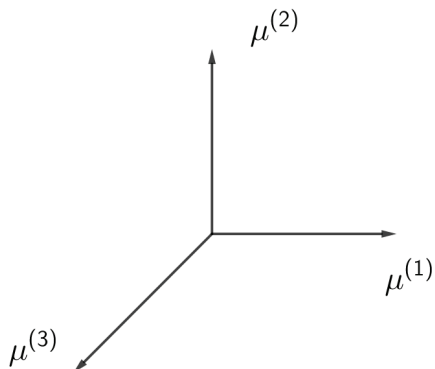


Figure 1: A diagrammatic representation of the topological vertex (1.17).

a triangulated polyhedron. Its dual graph is trivalent, and referred to as the “web” or “toric” diagram. Each vertex of this trivalent graph is given a vertex weight called the topological vertex. According to the proposal of Aganagic et al. [4], the topological string amplitudes on the Calabi-Yau threefold can be obtained by gluing these vertex weights along the edges of the graph.

Let q be a parameter in the range $0 < |q| < 1$. The vertex weight at each vertex is labelled by a triple of partitions $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) \in \mathcal{P}^3$, and defined as²⁾

$$\mathcal{C}_{\vec{\mu}}(q) = q^{\kappa(\mu^{(1)})/2} s_{\mu^{(2)}}(q^{-\rho}) \sum_{\eta \in \mathcal{P}} s_{\mu^{(1)}/\eta}(q^{-\mu^{(2)}-\rho}) s_{\mu^{(3)}/\eta}(q^{-\mu^{(2)}-\rho}). \quad (1.17)$$

$\mu^{(1)}, \mu^{(2)}$ and $\mu^{(3)}$ are assigned to the three legs of the trivalent vertex numbered in a counterclockwise direction. See Figure 1. $s_{\mu^{(1)}/\eta}(q^{-\mu^{(2)}-\rho})$ and $s_{\mu^{(3)}/\eta}(q^{-\mu^{(2)}-\rho})$ are the special values of the infinite-variate skew Schur functions $s_{\mu^{(1)}/\eta}(\mathbf{x})$ and $s_{\mu^{(3)}/\eta}(\mathbf{x})$ at $q^{-\mu^{(2)}-\rho} = (q^{-\mu_i^{(2)}+i-1/2})_{i=1}^{\infty}$ and $q^{-\mu^{(2)}-\rho} = (q^{-\mu_i^{(2)}+i-1/2})_{i=1}^{\infty}$.

We introduce a generating function of the topological vertex. Let $\mathbf{x} = (x_i)_{i=1}^{\infty}$, $\mathbf{y} = (y_i)_{i=1}^{\infty}$ and $\mathbf{z} = (z_i)_{i=1}^{\infty}$ be three sets of infinitely many variables. Define the generating function as

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \vec{w}) = \sum_{\vec{\mu} \in \mathcal{P}^3} \mathcal{C}_{\vec{\mu}}(q) q^{-\frac{1}{2} \sum_{a=1}^3 \kappa(\mu^{(a)})(1+w_{a+1}/w_a)} s_{\mu^{(1)}}(\mathbf{x}) s_{\mu^{(2)}}(\mathbf{y}) s_{\mu^{(3)}}(\mathbf{z}). \quad (1.18)$$

²⁾ We follow a definition used in the recent literature [19, 20]. This definition differs from the earlier one [4, 16] in that q is replaced by q^{-1} and an overall factor of the form $q^{\sum_{a=1}^3 \kappa(\mu^{(a)})/2}$ is multiplied.

In the case where \vec{w} takes the form (1.10), we use the abbreviated notation

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) = \mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1, \tau, -1 - \tau).$$

Our goal is to show the equivalence between the aforementioned generating functions of the topological vertex and three-partition Hodge integrals. More precisely, we prove the following:

Theorem 1. *Let $q = e^{-\sqrt{-1}\lambda}$. The generating function (1.18) coincides with the exponential of the generating function (1.9) as*

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \vec{w}) = \exp(G(\lambda; \vec{p}; \vec{w})), \quad (1.19)$$

where the variables \mathbf{x} , \mathbf{y} and \mathbf{z} are related to $\vec{p} = (p^{(1)}, p^{(2)}, p^{(3)})$ as

$$p_k^{(1)} = p_k(\mathbf{x}), \quad p_k^{(2)} = p_k(\mathbf{y}), \quad p_k^{(3)} = p_k(\mathbf{z}), \quad k = 1, 2, \dots. \quad (1.20)$$

In other words, the coefficients $\tilde{\mathcal{C}}_{\vec{\mu}}(\lambda)$ of (1.11) are connected with the topological vertex as

$$\tilde{\mathcal{C}}_{\vec{\mu}}(\lambda) = q^{-\frac{1}{2}\sum_{a=1}^3 \kappa(\mu^{(a)})} \mathcal{C}_{\vec{\mu}}(q), \quad \forall \vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) \in \mathcal{P}^3. \quad (1.21)$$

An immediate corollary of this result is cyclic symmetry of the topological vertex, which means that it is invariant under a cyclic permutation of three partitions assigned to the three legs, i.e.,

$$\mathcal{C}_{(\mu^{(1)}, \mu^{(2)}, \mu^{(3)})}(q) = \mathcal{C}_{(\mu^{(2)}, \mu^{(3)}, \mu^{(1)})}(q), \quad \forall \vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) \in \mathcal{P}^3. \quad (1.22)$$

Actually, three-partition Hodge integral is easily found to be invariant under simultaneous cyclic permutations of $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ and $\vec{w} = (w_1, w_2, w_3)$ as

$$G_{g, \mu^{(1)}, \mu^{(2)}, \mu^{(3)}}(w_1, w_2, w_3) = G_{g, \mu^{(2)}, \mu^{(3)}, \mu^{(1)}}(w_2, w_3, w_1).$$

This fact implies the cyclic symmetry of $\mathcal{C}_{\vec{\mu}}(q)$ by (1.21).

1.4 Integrable structure at positive integer values of τ

The reduction formula (1.13) of three-partition Hodge integrals can be extended to the case where τ takes any positive integer value $N = 1, 2, \dots$. Moreover, this leads to an unexpected relation with the generalized KdV hierarchies.

The generalized KdV hierarchies are reduced systems of the KP hierarchy (see Dickey's book [21] for details of these integrable hierarchies). The KP hierarchy in the Lax formalism is the system

$$\frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \dots,$$

of Lax equations for the pseudo-differential operator

$$L = \partial_x + \sum_{n=1}^{\infty} u_{n+1} \partial_x^{-n}, \quad \partial_x = \frac{\partial}{\partial x},$$

where the coefficients u_n are functions of the time variables $\mathbf{t} = (t_1, t_2, \dots)$, t_1 is identified with x , and B_k is the differential operator part

$$B_k = (L^k)_+ = \partial_x^k + b_{k2} \partial_x^{k-2} + \dots + b_{kk}$$

of the k -th power of L . If the $N + 1$ -st power $\mathcal{L} = L^{N+1}$ of L is a differential operator, i.e.,

$$\mathcal{L} = B_{N+1} = \partial_x^{N+1} + b_2 \partial_x^{N-1} + \dots + b_{N+1},$$

L does not depend on $t_{m(N+1)}$, $m = 1, 2, \dots$:

$$\frac{\partial L}{\partial t_{m(N+1)}} = 0, \quad m = 1, 2, \dots$$

This condition conversely characterizes the condition $L^{N+1} = B_{N+1}$. The remaining Lax equations can be reduced to the Lax equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad k = 1, 2, \dots,$$

for \mathcal{L} . These Lax equations comprises the N -th generalized KdV hierarchy. The KdV hierarchy amounts to the case where $N = 1$.

This reduction procedure can be reformulated in terms of the tau function $\mathcal{T} = \mathcal{T}(\mathbf{t})$. The tau function yields the dressing operator

$$W = 1 + \sum_{n=1}^{\infty} w_n \partial_x^{-n}$$

via the generating function

$$w(\mathbf{t}, z) = 1 + \sum_{n=1}^{\infty} w_n z^{-n} = \frac{\mathcal{T}(t_1 - z^{-1}, \dots, t_k - z^{-k}/k, \dots)}{\mathcal{T}(t_1, \dots, t_k, \dots)}.$$

The dressing operator, in turn, defines the Lax operator as

$$L = W \partial_x W^{-1}.$$

Consequently, if the tau function satisfies the equations

$$\frac{\partial^2 \log \mathcal{T}}{\partial t_k \partial t_{m(N+1)}} = 0, \quad k, m = 1, 2, \dots, \quad (1.23)$$

the coefficients of W and L do not depend on $t_{m(N+1)}$, $m = 1, 2, \dots$. This is a sufficient condition for the KP hierarchy to reduce to the N -th generalized KdV hierarchy. Note that this reduction condition is weaker than the commonly used condition

$$\frac{\partial \mathcal{T}}{\partial t_{m(N+1)}} = 0, \quad m = 1, 2, \dots$$

It is the weaker condition (1.23) that is relevant to the generating function (1.11) of three-partition Hodge integrals.

Let $\mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t})$ denote the function of $\mathbf{t} = (t_k)_{k=1}^{\infty}$ obtained from the generating function (1.11) by substituting $t_k = p_k^{(3)}/k$, $k = 1, 2, \dots$:

$$\mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t}) = \exp(G(\lambda; \vec{p}; N)). \quad (1.24)$$

$\lambda, N, p^{(1)}$ and $p^{(2)}$ are treated as parameters.

Theorem 2. *Let N be a positive integer. $\mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t})$ is a tau function of the KP hierarchy that satisfies the condition (1.23). The associated Lax operator L satisfies the reduction condition $L^{N+1} = B_{N+1}$ for the N -th generalized KdV hierarchy.*

Organization of the article

We start Section 2 with a brief review on a two-dimensional charged free fermion system. Various operators on the fermionic Fock space, including a realization of the two-dimensional quantum torus algebra, are introduced. These operators are used as fundamental tools in the discussions through the article. In Proposition 2.2, we give a fermionic representation of the generating function of the topological vertex.

In Section 3, we explain the generalized shift symmetries. These symmetries act on a set of basis elements $V_m^{(k)}$ of the quantum torus algebra so as to shift the indices k, m in a certain way. The shift symmetries are formulated by special values of the vertex operators $\Gamma_{\pm}(x)$ and $\Gamma'_{\pm}(x)$ and the exponential operator $q^{K/2} = \exp(\frac{K}{2} \log q)$, as

$$V_m^{(k)} \mathbb{L}_{\emptyset} = (-1)^k \mathbb{L}_{\emptyset} V_{m-k}^{(k)}, \quad V_m^{(-k)} \mathbb{L}'_{\emptyset} = \mathbb{L}'_{\emptyset} V_{m-k}^{(-k)}, \quad q^{K/2} V_m^{(k)} q^{-K/2} = V_m^{(k-m)},$$

where $\mathbb{L}_{\emptyset} = \Gamma_+(q^{-\rho})\Gamma_-(q^{-\rho})$ and $\mathbb{L}'_{\emptyset} = \Gamma'_+(q^{-\rho})\Gamma'_-(q^{-\rho})$ are the products of multivariate vertex operators $\Gamma_{\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma_{\pm}(x_i)$ and $\Gamma'_{\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma'_{\pm}(x_i)$ specialized to $q^{-\rho}$. These shift symmetries can be generalized by replacing \mathbb{L}_{\emptyset} and \mathbb{L}'_{\emptyset} with

$$\mathbb{L}_{\alpha} = s_{\alpha}(q^{-\rho})\Gamma_-(q^{-\alpha-\rho})\Gamma_+(q^{-\alpha-\rho}), \quad \mathbb{L}'_{\alpha} = s_{\alpha}(q^{-\rho})\Gamma'_-(q^{-\alpha-\rho})\Gamma'_+(q^{-\alpha-\rho})$$

and modifying $q^{K/2}$ in a certain manner. The generalized shift symmetries are summarized in Propositions 3.1 and 3.2.

The fermionic representation of the generating function in Proposition 2.2 involves a weighted sum of operators of the form

$$\sum_{\alpha \in \mathcal{P}} \mathbb{G}_{\alpha}(\tau) s_{\alpha},$$

where $\mathbb{G}_{\alpha}(\tau) = q^{-\kappa(\alpha)/2} q^{-\tau K/2} \mathbb{L}_{\alpha} q^{\tau K/2(1+\tau)}$ is a particular linear combination of the generalized shift symmetries. The generalized shift symmetries imply that, at certain values of τ , the foregoing sum is factorized into a triple product of operators. The factorization formula at $\tau = 1$ is given in Theorem 3.

In Section 4, we prove Theorem 1 using the results obtained in the preceding section. The aforementioned fermionic representation of the generating function of the topological vertex can be converted into a new representation by Theorem 3. The resulting representation is presented in Proposition 4.2. With the aid of the new representation of the generating function, we can reconsider its Schur function expansion and eventually find out the relation (1.21). Theorem 1 is thus proved.

In Section 5, we prove Theorem 2. The foregoing weighted sum of operators can be factorized in the cases where $\tau = 1/N$, $N = 2, 3, \dots$, as well. The factorized expression is presented in Theorem 4. This formula yields in Proposition 5.1 a new representation of the generating functions of the topological vertex at positive integral values of τ . Theorem 1 and Proposition 5.1 lead to a reduction formula of the generating functions of three-partition Hodge integrals in Proposition 5.2. This formula gives a generalization of (1.12) at all positive integral values of τ . We then consider integrable hierarchies underlying the generating functions of two-partition Hodge integrals. By combining these considerations, we eventually obtain Theorem 2.

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2 Fermionic representations of generating functions

2.1 Fermionic Fock space and operators

Let $\psi_n, \psi_n^*, n \in \mathbb{Z}$, denote the Fourier modes of two-dimensional charged free fermion fields

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi_n^* z^{-n}.$$

They satisfy the anti-commutation relations

$$\psi_m \psi_n^* + \psi_n^* \psi_m = \delta_{m+n,0}, \quad \psi_m \psi_n + \psi_n \psi_m = 0, \quad \psi_m^* \psi_n^* + \psi_n^* \psi_m^* = 0.$$

The associated Fock space and its dual space are decomposed into the charge- p sector for $p \in \mathbb{Z}$. Only the charge-0 sector is relevant to the subsequent computations. An orthonormal basis of the charge-0 sector is given by the ground states

$$\begin{aligned} \langle 0| &= \langle -\infty| \dots \psi_{-i+1}^* \dots \psi_{-1}^* \psi_0^*, \\ |0\rangle &= \psi_0 \psi_1 \dots \psi_{i-1} \dots |-\infty\rangle \end{aligned}$$

and the excited states

$$\langle \lambda| = \langle -\infty| \dots \psi_{\lambda_i-i+1}^* \dots \psi_{\lambda_2-1}^* \psi_{\lambda_1}^*,$$

$$|\lambda\rangle = \psi_{-\lambda_1} \psi_{-\lambda_2+1} \cdots \psi_{-\lambda_i+i-1} \cdots |-\infty\rangle$$

labelled by partitions. The normal ordered product is prescribed by

$$:\psi_m \psi_n^* : = \psi_m \psi_n^* - \langle 0 | \psi_m \psi_n^* | 0 \rangle.$$

Fundamental tools in our computations are the following operators on the Fock space which preserve the charge.

(i) The zero modes

$$L_0 = \sum_{n \in \mathbb{Z}} n : \psi_{-n} \psi_n^* :, \quad W_0 = \sum_{n \in \mathbb{Z}} n^2 : \psi_{-n} \psi_n^* :$$

of the Virasoro and W_3 algebras and the Fourier modes

$$J_m = \sum_{n \in \mathbb{Z}} : \psi_{m-n} \psi_n^* :, \quad m \in \mathbb{Z}$$

of the $U(1)$ current $J(z) = \sum_{m \in \mathbb{Z}} J_m z^{-m-1} =: \psi(z) \psi^*(z) :$.

(ii) The fermionic realization

$$K = W_0 - L_0 + J_0/4 = \sum_{n \in \mathbb{Z}} (n - 1/2)^2 : \psi_{-n} \psi_n^* :$$

of the so called “cut-and-join operator” [22].

(iii) The basis elements

$$V_m^{(k)} = q^{-k(m+1)/2} \sum_{n \in \mathbb{Z}} q^{kn} : \psi_{m-n} \psi_n^* : - \frac{q^{k/2}}{1 - q^k} \delta_{m,0}, \quad k \in \mathbb{Z}_{\neq 0}, \quad m \in \mathbb{Z}$$

and

$$V_m^{(0)} = J_m, \quad m \in \mathbb{Z}$$

of a fermionic realization³⁾ of the quantum torus algebra, which satisfy the commutation relations

$$[V_m^{(k)}, V_n^{(l)}] = \left(q^{-\frac{kn-lm}{2}} - q^{\frac{kn-lm}{2}} \right) V_{m+n}^{(k+l)} + m \delta_{k+l,0} \delta_{m+n,0}, \quad k, l, m, n \in \mathbb{Z}.$$

³⁾ The normalization of $V_m^{(k)}$ for $k \neq 0$ differs from [10, 11] in that an overall factor $q^{-k/2}$ is multiplied and that a constant term $q^{k/2}(1 - q^k)^{-1} \delta_{m,0}$ is subtracted.

(iv) The vertex operators

$$\Gamma_{\pm}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} J_{\pm k}\right), \quad \Gamma'_{\pm}(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{\pm k}\right)$$

and the multi-variable extensions

$$\Gamma_{\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma_{\pm}(x_i), \quad \Gamma'_{\pm}(\mathbf{x}) = \prod_{i=1}^{\infty} \Gamma'_{\pm}(x_i).$$

The matrix elements of those operators are well-known. $J_0 = V_0^{(0)}$, L_0 and W_0 are diagonal to the basis $\{|\lambda\rangle\}_{\lambda \in \mathcal{P}}$ in the charge-0 sector as

$$\langle \lambda | J_0 | \mu \rangle = 0, \quad \langle \lambda | L_0 | \mu \rangle = \delta_{\lambda\mu} |\lambda|, \quad \langle \lambda | W_0 | \mu \rangle = \delta_{\lambda\mu} (\kappa(\lambda) + |\lambda|).$$

Consequently, we have

$$\langle \lambda | K | \mu \rangle = \delta_{\lambda\mu} \kappa(\lambda). \quad (2.1)$$

Other zero modes of the quantum torus algebra are likewise diagonal to the basis $\{|\lambda\rangle\}_{\lambda \in \mathcal{P}}$. Their matrix elements are of the form

$$\langle \lambda | V_0^{(k)} | \mu \rangle = \delta_{\lambda\mu} \varphi_k(\mu), \quad k \in \mathbb{Z}_{\neq 0},$$

where

$$\varphi_k(\mu) = \begin{cases} -\sum_{i=1}^{\infty} q^{k(-\mu_i + i - 1/2)} & \text{for } k \geq 1, \\ \sum_{i=1}^{\infty} q^{-k(-\mu_i + i - 1/2)} & \text{for } k \leq -1. \end{cases}$$

It is also well known that the matrix elements of $\Gamma_{\pm}(\mathbf{x})$ and $\Gamma'_{\pm}(\mathbf{x})$ are skew Schur functions

$$\langle \lambda | \Gamma_{-}(\mathbf{x}) | \mu \rangle = \langle \mu | \Gamma_{+}(\mathbf{x}) | \lambda \rangle = s_{\lambda/\mu}(\mathbf{x}), \quad (2.2)$$

$$\langle \lambda | \Gamma'_{-}(\mathbf{x}) | \mu \rangle = \langle \mu | \Gamma'_{+}(\mathbf{x}) | \lambda \rangle = s_{\lambda/\mu}(\mathbf{x}). \quad (2.3)$$

2.2 Fermionic representation of generating function (1.18)

Let us provide a fermionic representation of the generating function (1.18). We first introduce a set of operator-valued functions labeled by partitions as

$$\mathbb{G}_\alpha(\tau) = q^{-\frac{\kappa(\alpha)}{2\tau}} s_\alpha(q^{-\rho}) q^{-\frac{\tau}{2}K} \Gamma_- (q^{-\alpha-\rho}) \Gamma_+ (q^{-\tau\alpha-\rho}) q^{\frac{\tau}{2(1+\tau)}K}, \quad \alpha \in \mathcal{P}, \quad \tau \neq 0, -1, \quad (2.4)$$

where $q^{-\tau K/2}$ and $q^{\tau K/2(1+\tau)}$ are the exponential operators $\exp(-\frac{\tau K}{2} \log q)$, $\exp(\frac{\tau K}{2(1+\tau)} \log q)$, and $\Gamma_- (q^{-\alpha-\rho})$ and $\Gamma_+ (q^{-\tau\alpha-\rho})$ denote respectively the multi-variate operators $\Gamma_- (\mathbf{x})$ and $\Gamma_+ (\mathbf{x})$ specialized at $q^{-\alpha-\rho}$ and $q^{-\tau\alpha-\rho}$. The matrix element of (2.4) yields the topological vertex (1.17) as follows:

Proposition 2.1.

$$\mathcal{C}_{\vec{\mu}}(q) = q^{\frac{1+\tau}{2}\kappa(\mu^{(1)}) - \frac{1}{2\tau}\kappa(\mu^{(2)}) + \frac{\tau}{2(1+\tau)}\kappa(\mu^{(3)})} \langle \mu^{(1)} | \mathbb{G}_{\tau\mu^{(2)}}(\tau) | \mu^{(3)} \rangle, \quad (2.5)$$

where $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$.

Proof. We compute the matrix element in the RHS of (2.5) by interposing the identity $\sum_{\eta \in \mathcal{P}} |\eta\rangle \langle \eta| = 1$ in the charge-0 sector as follows:

$$\begin{aligned} & \langle \mu^{(1)} | \mathbb{G}_{\tau\mu^{(2)}}(\tau) | \mu^{(3)} \rangle \\ &= q^{-\frac{\kappa(\mu^{(2)})}{2\tau}} s_{\tau\mu^{(2)}}(q^{-\rho}) \langle \mu^{(1)} | q^{-\frac{\tau}{2}K} \Gamma_- (q^{-\tau\mu^{(2)}-\rho}) \left(\sum_{\eta \in \mathcal{P}} |\eta\rangle \langle \eta| \right) \Gamma_+ (q^{-\mu^{(2)}-\rho}) q^{\frac{\tau}{2(1+\tau)}K} | \mu^{(3)} \rangle \\ &= q^{\frac{\kappa(\mu^{(2)})}{2\tau}} s_{\tau\mu^{(2)}}(q^{-\rho}) \sum_{\eta \in \mathcal{P}} \langle \mu^{(1)} | q^{-\frac{\tau}{2}K} \Gamma_- (q^{-\tau\mu^{(2)}-\rho}) |\eta\rangle \langle \eta| \Gamma_+ (q^{-\mu^{(2)}-\rho}) q^{\frac{\tau}{2(1+\tau)}K} | \mu^{(3)} \rangle. \end{aligned} \quad (2.6)$$

By (2.1) and (2.2), the matrix elements in the RHS can be expressed as

$$\begin{aligned} \langle \mu^{(1)} | q^{-\frac{\tau}{2}K} \Gamma_- (q^{-\tau\mu^{(2)}-\rho}) |\eta\rangle &= q^{-\frac{\tau}{2}\kappa(\mu^{(1)})} s_{\mu^{(1)}/\eta}(q^{-\tau\mu^{(2)}-\rho}), \\ \langle \eta | \Gamma_+ (q^{-\mu^{(2)}-\rho}) q^{\frac{\tau}{2(1+\tau)}K} | \mu^{(3)} \rangle &= q^{-\frac{\tau}{2(1+\tau)}\kappa(\mu^{(3)})} s_{\tau\mu^{(3)}/\eta}(q^{-\mu^{(2)}-\rho}). \end{aligned}$$

By plugging those expressions into (2.6), we find

$$\langle \mu^{(1)} | \mathbb{G}_{\tau\mu^{(2)}}(\tau) | \mu^{(3)} \rangle$$

$$\begin{aligned}
&= q^{-\frac{\tau}{2}\kappa(\mu^{(1)})+\frac{1}{2\tau}\kappa(\mu^{(2)})-\frac{\tau}{2(1+\tau)}\kappa(\mu^{(3)})} s_{\mathbf{t}_{\mu^{(2)}}}(q^{-\rho}) \sum_{\eta \in \mathcal{P}} s_{\mu^{(1)}/\eta}(q^{-\mathbf{t}_{\mu^{(2)}}-\rho}) s_{\mathbf{t}_{\mu^{(3)}/\eta}}(q^{-\mu^{(2)}-\rho}) \\
&= q^{-\frac{1+\tau}{2}\kappa(\mu^{(1)})+\frac{1}{2\tau}\kappa(\mu^{(2)})-\frac{\tau}{2(1+\tau)}\kappa(\mu^{(3)})} \mathcal{C}_{\vec{\mu}}(q).
\end{aligned}$$

Thus we obtain (2.5). \square

The generating function (1.18) has a fermionic expression of the following form due to (2.5):

Proposition 2.2.

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) = \langle 0 | \Gamma_+(\mathbf{x}) \left(\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(\tau) s_{\mathbf{t}_\alpha}(\mathbf{y}) \right) \Gamma'_-(\mathbf{z}) | 0 \rangle. \quad (2.7)$$

Proof. Using (2.5), three sums in the RHS of (1.18) for $\vec{w} = (1, \tau, -1 - \tau)$ become

$$\begin{aligned}
\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) &= \sum_{\vec{\mu} \in \mathcal{P}^3} \langle \mu^{(1)} | \mathbb{G}_{\mathbf{t}_{\mu^{(2)}}}(\tau) | \mathbf{t}_{\mu^{(3)}} \rangle s_{\mu^{(1)}}(\mathbf{x}) s_{\mu^{(2)}}(\mathbf{y}) s_{\mu^{(3)}}(\mathbf{z}) \\
&= \left(\sum_{\mu^{(1)} \in \mathcal{P}} s_{\mu^{(1)}}(\mathbf{x}) \langle \mu^{(1)} | \right) \sum_{\mu^{(2)} \in \mathcal{P}} \mathbb{G}_{\mathbf{t}_{\mu^{(2)}}}(\tau) s_{\mu^{(2)}}(\mathbf{y}) \left(\sum_{\mu^{(3)} \in \mathcal{P}} | \mathbf{t}_{\mu^{(3)}} \rangle s_{\mu^{(3)}}(\mathbf{z}) \right) \\
&= \langle 0 | \Gamma_+(\mathbf{x}) \left(\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(\tau) s_{\mathbf{t}_\alpha}(\mathbf{y}) \right) \Gamma'_-(\mathbf{z}) | 0 \rangle,
\end{aligned}$$

where (2.2) and (2.3) are used in the last line. \square

3 Generalized shift symmetry and factorization formula

The shift symmetries [10, 11] act on the basis elements of the quantum torus algebra. There are three different types of shift symmetries:

(i) For $k \geq 1$ and $m \in \mathbb{Z}$,

$$V_m^{(k)} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) = (-1)^k \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) V_{m-k}^{(k)}. \quad (3.1)$$

(ii) For $k \geq 1$ and $m \in \mathbb{Z}$,

$$V_m^{(-k)} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) = \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) V_{m-k}^{(-k)}. \quad (3.2)$$

(iii) For $k, m \in \mathbb{Z}$,

$$q^{K/2} V_m^{(k)} q^{-K/2} = V_m^{(k-m)}. \quad (3.3)$$

These symmetries together with their combinations are used to clarify the integrable structures of various melting crystal models [10, 11] and open topological string amplitudes [12], [13]. Likewise, quantum curves of open topological strings on the closed topological vertex and the strip geometries are derived [14] using the symmetries. An extension of the symmetries is investigated in the companion paper [15]. The generalized shift symmetries obtained therein provides a powerful tool to prove Theorem 1.

3.1 Generalized shift symmetries on quantum torus algebra

The skew Young diagram α/β is called a ribbon iff there exist positive integers r and s , $r > s$, such that partitions $\alpha = (\alpha_i)_{i=1}^\infty$ and $\beta = (\beta_i)_{i=1}^\infty$ correlate to each other through the condition

$$\{\alpha_i - i\}_{i=1}^N \cup \{\beta_r - r\} = \{\beta_i - i\}_{i=1}^N \cup \{\alpha_s - s\}, \quad \forall N > r,$$

where $\{\alpha_i - i\}_{i=1}^N$ and $\{\beta_i - i\}_{i=1}^N$ are finite subsets of \mathbb{Z} consisting of N distinct integers. See Figure 2. The semi-infinite subsets $\{\alpha_i - i\}_{i=1}^\infty$ and $\{\beta_i - i\}_{i=1}^\infty$ are referred to the Maya diagrams of α and β . The length of a ribbon α/β is given by $|\alpha| - |\beta|$ and the height, denoted by $\text{ht}(\alpha/\beta)$, is given by $r - s$.

The set of partitions of which subtraction from α leaves a ribbon of length k is denoted by

$$\mathcal{R}_{k,\alpha}^{(-)} = \left\{ \beta \in \mathcal{P} \mid \alpha/\beta \text{ is a ribbon of length } k \right\}.$$

Likewise, the set of partitions which leave a ribbon of length k by subtraction of α is

$$\mathcal{R}_{k,\alpha}^{(+)} = \left\{ \beta \in \mathcal{P} \mid \beta/\alpha \text{ is a ribbon of length } k \right\}.$$

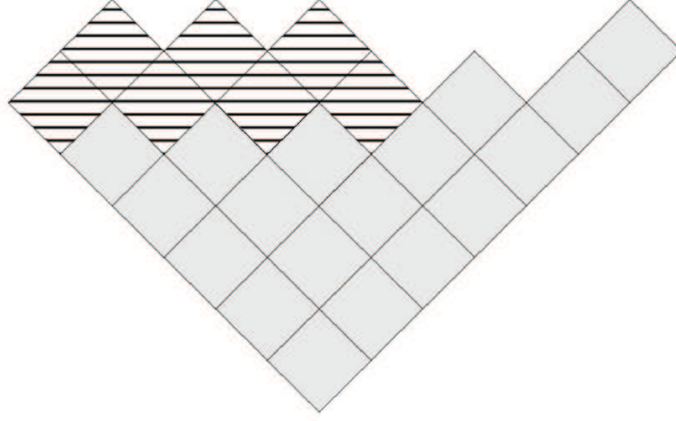


Figure 2: The Young diagrams $\alpha = (7, 5, 4, 4, 3, 2)$ and $\beta = (7, 5, 3, 2, 1)$ are laid to overlap. The skew Young diagram α/β is indicated with hatched boxes and is a ribbon of length 7 and height 3.

We describe these sets in a single symbol indexed by $\alpha \in \mathcal{P}$ and $k \in \mathbb{Z}$, $k \neq 0$, as

$$\mathcal{R}_{k,\alpha} = \begin{cases} \mathcal{R}_{k,\alpha}^{(-)} & \text{for } k \geq 1, \\ \mathcal{R}_{-k,\alpha}^{(+)} & \text{for } k \leq -1. \end{cases}$$

That means that $\mathcal{R}_{k,\alpha}$ consists of partitions which can be obtained from α by removing a ribbon of length k for $k \geq 1$ or by adding a ribbon of length $-k$ for $k \leq -1$.

For $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{R}_{k,\alpha}$, we introduce their relative sign by

$$\text{sgn}(\alpha, \beta) = \begin{cases} (-1)^{\text{ht}(\alpha/\beta)} & \text{for } k \geq 1, \\ (-1)^{\text{ht}(\beta/\alpha)} & \text{for } k \leq -1. \end{cases}$$

Since $\beta \in \mathcal{R}_{k,\alpha}$ implies $\alpha \in \mathcal{R}_{-k,\beta}$, the relative sign is symmetric under the exchange of partitions

$$\text{sgn}(\alpha, \beta) = \text{sgn}(\beta, \alpha).$$

3.1.1 Generalized shift symmetries

Let us explain the extension of the shift symmetries (i), (ii) and (iii) which is derived in [15]. For $\alpha \in \mathcal{P}$, we introduce an operator of the form

$$\mathbb{L}_\alpha = s_\alpha(q^{-\rho})\Gamma_- (q^{-\alpha-\rho}) \Gamma_+(q^{-\iota\alpha-\rho}). \quad (3.4)$$

Note that (3.1) is written with the aid of \mathbb{L}_\emptyset as

$$V_m^{(k)} \mathbb{L}_\emptyset = (-1)^k \mathbb{L}_\emptyset V_{m-k}^{(k)}, \quad k \geq 1, \quad m \in \mathbb{Z}.$$

We generalize the shift symmetry (i) by replacing \mathbb{L}_\emptyset in the LHS with \mathbb{L}_α . This yields an extension of the following form.

Proposition 3.1. [15] *For $k \in \mathbb{Z}$, $k \neq 0$ and $m \in \mathbb{Z}$, we have*

$$V_m^{(k)} \mathbb{L}_\alpha = (-1)^k \mathbb{L}_\alpha V_{m-k}^{(k)} + q^{kJ_0} \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta) q^{-m(\kappa(\alpha) - \kappa(\beta))/2k} \mathbb{L}_\beta. \quad (3.5)$$

As regards the Heisenberg sub-algebra generated by $V_m^{(0)}$, we have

$$V_m^{(0)} \mathbb{L}_\alpha = \mathbb{L}_\alpha V_m^{(0)} + \varphi_{-m}(\alpha) \mathbb{L}_\alpha. \quad (3.6)$$

Remark 3.1. (3.5) of $\alpha = \emptyset$ with $k \geq 1$ reproduces (3.1). Otherwise, there is no counterpart in (i). For instance, $\mathcal{R}_{-l,\emptyset} = \mathcal{R}_{l,\emptyset}^{(+)}$ ($l \geq 1$) consists of hooks $(1^{l-j}j^1)$ of weight l . Thus, rewriting $k = -l$, (3.5) of $\alpha = \emptyset$ with $k \leq -1$ reads

$$V_m^{(-l)} \mathbb{L}_\emptyset = (-1)^l \mathbb{L}_\emptyset V_{m+l}^{(-l)} + q^{-lJ_0} \sum_{j=1}^l (-1)^{l-j} q^{m(l+1-2j)/2} \mathbb{L}_{(1^{l-j}j^1)}, \quad l \geq 1.$$

The shift symmetry (ii) can be generalized in a manner similar to the case of (i) as described in Proposition 3.1. With $\alpha \in \mathcal{P}$, we associate an operator of the form

$$\mathbb{L}'_\alpha = s_\alpha(q^{-\rho})\Gamma'_- (q^{-\alpha-\rho}) \Gamma'_+(q^{-\iota\alpha-\rho}). \quad (3.7)$$

Note that (3.2) is written with the aid of \mathbb{L}'_\emptyset as

$$V_m^{(-k)} \mathbb{L}'_\emptyset = (-1)^k \mathbb{L}'_\emptyset V_{m-k}^{(-k)}, \quad k \geq 1, \quad m \in \mathbb{Z}.$$

The symmetry (ii) can be generalized by replacing \mathbb{L}'_\emptyset in the LHS with \mathbb{L}'_α . We then obtain an extension of the following form.

Proposition 3.2. [15] For $k \in \mathbb{Z}$, $k \neq 0$ and $m \in \mathbb{Z}$, we have

$$V_m^{(-k)} \mathbb{L}'_\alpha = \mathbb{L}'_\alpha V_{m-k}^{(-k)} + (-1)^{m+1} q^{-kJ_0} \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta) q^{-m(\kappa(\alpha) - \kappa(\beta))/2k} \mathbb{L}'_\beta. \quad (3.8)$$

As regards the generators of the Heisenberg sub-algebra, we have

$$V_m^{(0)} \mathbb{L}'_\alpha = \mathbb{L}'_\alpha V_m^{(0)} + (-1)^{m+1} \varphi_{-m}(\alpha) \mathbb{L}'_\alpha. \quad (3.9)$$

Remark 3.2. (3.8) of $\alpha = \emptyset$ with $k \geq 1$ reproduces (3.2). Otherwise, there is no counterpart in (ii).

The shift symmetry (iii) can be generalized as follows:

Proposition 3.3. For $k, m \in \mathbb{Z}$, we have

$$q^{\gamma K} V_m^{(k)} q^{-\gamma K} = V_m^{(k-2m\gamma)}, \quad (3.10)$$

where γ is chosen to be a rational number which satisfies $2m\gamma \in \mathbb{Z}$.

Proof. The commutation relations

$$[K, \psi_n] = \left(n + \frac{1}{2}\right)^2 \psi_n, \quad [K, \psi_n^*] = -\left(n - \frac{1}{2}\right)^2 \psi_n^*$$

are integrated to give

$$q^{\gamma K} \psi_n q^{-\gamma K} = q^{\gamma(n+1/2)^2} \psi_n, \quad q^{\gamma K} \psi_n^* q^{-\gamma K} = q^{-\gamma(n-1/2)^2} \psi_n^*. \quad (3.11)$$

Consider the case of $m \neq 0$. Using (3.11), the LHS of (3.10) is computed as

$$\begin{aligned} q^{\gamma K} V_m^{(k)} q^{-\gamma K} &= q^{-k(m+1)/2} \sum_n q^{kn} q^{\gamma K} : \psi_{m-n} \psi_n^* : q^{-\gamma K} \\ &= q^{-k(m+1)/2} \sum_n q^{kn + \gamma(n-n+1/2)^2 - \gamma(n-1/2)^2} : \psi_{m-n} \psi_n^* : \\ &= q^{-(k-2m\gamma)(m+1)/2} \sum_n q^{(k-2m\gamma)n} : \psi_{m-n} \psi_n^* : \\ &= V_m^{(k-2m\gamma)}. \end{aligned}$$

In the case of $m = 0$, since K commutes with $V_0^{(k)}$, we have $q^{\gamma K} V_0^{(k)} q^{-\gamma K} = V_0^{(k)}$. Thus, we obtain (3.10). \square

3.1.2 Particular combinations of generalized shift symmetries

Combining (3.5) and (3.10), we obtain an operator-valued identity.

Proposition 3.4. *For $m \in \mathbb{Z}$, $m \neq 0$, at values of τ satisfying $m\tau \in \mathbb{Z}$ and $m\tau \neq 0$, we have the operator-valued identity*

$$J_{-m}\mathbb{G}_\alpha(\tau) = \mathbb{G}_\alpha(\tau)(-1)^{m\tau}J_{-m(1+\tau)} + q^{m\tau J_0} \sum_{\beta \in \mathcal{R}_{m\tau, \alpha}} \text{sgn}(\alpha, \beta)\mathbb{G}_\beta(\tau). \quad (3.12)$$

Proof. Using \mathbb{L}_α , we express $\mathbb{G}_\alpha(\tau)$ in (2.4) as

$$\mathbb{G}_\alpha(\tau) = q^{-\kappa(\alpha)/2\tau} q^{-\tau K/2} \mathbb{L}_\alpha q^{\tau K/2(1+\tau)}.$$

The symmetry (3.10) with $\gamma = \tau/2$ reads

$$q^{\tau K/2} J_{-m} q^{-\tau K/2} = V_{-m}^{(m\tau)},$$

which converts $J_{-m}\mathbb{G}_\alpha(\tau)$ to

$$\begin{aligned} J_{-m}\mathbb{G}_\alpha(\tau) &= q^{-\kappa(\alpha)/2\tau} J_{-m} q^{-\tau K/2} \mathbb{L}_\alpha q^{\tau K/2(1+\tau)} \\ &= q^{-\kappa(\alpha)/2\tau} q^{-\tau K/2} V_{-m}^{(m\tau)} \mathbb{L}_\alpha q^{\tau K/2(1+\tau)}. \end{aligned} \quad (3.13)$$

Using (3.5), the RHS of (3.13) is further converted to

$$\begin{aligned} &\text{RHS of (3.13)} \\ &= q^{-\kappa(\alpha)/2\tau} q^{-\tau K/2} \left\{ (-1)^{m\tau} \mathbb{L}_\alpha V_{-m(1+\tau)}^{(m\tau)} + q^{m\tau J_0} \sum_{\beta \in \mathcal{R}_{m\tau, \alpha}} \text{sgn}(\alpha, \beta) q^{\frac{\kappa(\alpha) - \kappa(\beta)}{2\tau}} \mathbb{L}_\beta \right\} q^{\tau K/2(1+\tau)} \\ &= \mathbb{G}_\alpha(\tau)(-1)^{m\tau} q^{-\tau K/2(1+\tau)} V_{-m(1+\tau)}^{(m\tau)} q^{\tau K/2(1+\tau)} + q^{m\tau J_0} \sum_{\beta \in \mathcal{R}_{m\tau, \alpha}} \text{sgn}(\alpha, \beta) \mathbb{G}_\beta(\tau). \end{aligned} \quad (3.14)$$

Here, the symmetry (3.10) gives

$$q^{-\tau K/2(1+\tau)} V_{-m(1+\tau)}^{(m\tau)} q^{\tau K/2(1+\tau)} = V_{-m(1+\tau)}^{(0)} = J_{-m(1+\tau)}.$$

Thus, the last line in (3.14) becomes

$$\text{RHS of (3.14)} = \mathbb{G}_\alpha(\tau)(-1)^{m\tau} J_{-m(1+\tau)} + q^{m\tau J_0} \sum_{\beta \in \mathcal{R}_{m\tau, \alpha}} \text{sgn}(\alpha, \beta) \mathbb{G}_\beta(\tau).$$

We therefore obtain (3.12). \square

Operator-valued identity analogous to (3.12) can be obtained if we focus on (3.8) instead of (3.5). With $\alpha \in \mathcal{P}$, let us introduce an operator-valued function of the form

$$\mathbb{G}'_{\alpha}(\tau) = q^{-\kappa(\alpha)/2\tau} q^{\tau K/2} \mathbb{L}'_{\alpha} q^{-\tau K/2(1+\tau)}, \quad \tau \neq 0, -1. \quad (3.15)$$

We note that the topological vertex (1.17) can be written in terms of the matrix element of (3.15) as

$$\mathcal{C}_{\vec{\mu}}(q) = q^{\frac{1+\tau}{2}\kappa(\mu^{(1)}) - \frac{1}{2\tau}\kappa(\mu^{(2)}) + \frac{\tau}{2(1+\tau)}\kappa(\mu^{(3)})} \langle \mu^{(1)} | \mathbb{G}'_{\mu^{(2)}}(\tau) | \mu^{(3)} \rangle.$$

Proposition 3.5. *For $m \in \mathbb{Z}$, $m \neq 0$, at values of τ satisfying $m\tau \in \mathbb{Z}$, $m\tau \neq 0$, we have the operator-valued identity*

$$J_{-m} \mathbb{G}'_{\alpha}(\tau) = \mathbb{G}'_{\alpha}(\tau) J_{-m(1+\tau)} + (-1)^{m+1} q^{-m\tau J_0} \sum_{\beta \in \mathcal{R}_{m\tau, \alpha}} \text{sgn}(\alpha, \beta) \mathbb{G}'_{\beta}(\tau). \quad (3.16)$$

Proof. Using (3.8) instead of (3.5), the same computations as in the proof of Proposition 3.4 give rise to (3.16). \square

3.2 Factorization formulas at $\tau = 1$

We consider generating functions of the operator-valued functions (2.4) and (3.15). These generating functions can be factorized at $\tau = 1$. Let us introduce infinite-variate functions

$$s_{\alpha}[\mathbf{t}] = \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) | \alpha \rangle, \quad \mathbf{t} = (t_k)_{k=1}^{\infty}.$$

It is well known that the functions $s_{\alpha}[\mathbf{t}]$ are converted to the Schur functions $s_{\alpha}(\mathbf{x})$ by the Miwa transformation

$$t_k = \frac{1}{k} \sum_{i=1}^{\infty} x_i^k, \quad k = 1, 2, \dots,$$

and the Schur functions $s_{\iota_\alpha}(\mathbf{x})$ by another version of the transformation

$$t_k = -\frac{1}{k} \sum_{i=1}^{\infty} (-x_i)^k, \quad k = 1, 2, \dots$$

We note that the partial derivatives of $s_\alpha[\mathbf{t}]$ with respect to \mathbf{t} become as follows.

Proposition 3.6.

$$\partial_{t_m} s_\alpha[\mathbf{t}] = \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(-)}} \text{sgn}(\alpha, \beta) s_\beta[\mathbf{t}], \quad m = 1, 2, \dots \quad (3.17)$$

Proof. We rewrite the partial derivatives as

$$\partial_{t_m} s_\alpha[\mathbf{t}] = \partial_{t_m} \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) | \alpha \rangle = \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) J_m | \alpha \rangle, \quad (3.18)$$

which can be computed by use of a version of Murnaghan-Nakayama's rule:

$$J_k | \alpha \rangle = \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta) | \beta \rangle, \quad k \in \mathbb{Z}_{\neq 0}.$$

Applying the rule into the RHS of (3.18), we see

$$\text{RHS of (3.18)} = \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(-)}} \text{sgn}(\alpha, \beta) \langle 0 | \exp \left(\sum_{k=1}^{\infty} t_k J_k \right) | \beta \rangle = \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(-)}} \text{sgn}(\alpha, \beta) s_\beta[\mathbf{t}].$$

Thus we obtain (3.17). □

3.2.1 Operator-valued generating functions

We define operator-valued generating functions by

$$\mathbb{G}[\mathbf{t}; \tau] = \sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(\tau) s_\alpha[\mathbf{t}], \quad (3.19)$$

$$\mathbb{G}'[\mathbf{t}; \tau] = \sum_{\alpha \in \mathcal{P}} \mathbb{G}'_\alpha(\tau) s_\alpha[\mathbf{t}]. \quad (3.20)$$

Proposition 3.7. *Partial derivatives of (3.19) and (3.20) with respect to $\mathbf{t} = (t_k)_{k=1}^\infty$ are expressed in the following forms.*

$$\partial_{t_m} \mathbb{G}[\mathbf{t}; \tau] = \sum_{\alpha \in \mathcal{P}} \left(\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \operatorname{sgn}(\alpha, \beta) \mathbb{G}_\beta(\tau) \right) s_\alpha[\mathbf{t}], \quad (3.21)$$

$$\partial_{t_m} \mathbb{G}'[\mathbf{t}; \tau] = \sum_{\alpha \in \mathcal{P}} \left(\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \operatorname{sgn}(\alpha, \beta) \mathbb{G}'_\beta(\tau) \right) s_\alpha[\mathbf{t}], \quad (3.22)$$

where $m = 1, 2, \dots$.

Proof. By (3.17), the partial derivatives of $\mathbb{G}[\mathbf{t}; \tau]$ read

$$\begin{aligned} \partial_{t_m} \mathbb{G}[\mathbf{t}; \tau] &= \sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(\tau) \partial_{t_m} s_\alpha[\mathbf{t}] \\ &= \sum_{\alpha \in \mathcal{P}} \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(-)}} \operatorname{sgn}(\alpha, \beta) \mathbb{G}_\alpha(\tau) s_\beta[\mathbf{t}] = \sum_{\alpha \in \mathcal{P}} \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \operatorname{sgn}(\beta, \alpha) \mathbb{G}_\beta(\tau) s_\alpha[\mathbf{t}], \end{aligned} \quad (3.23)$$

where the roles of α and β are exchanged in the last line. Similarly, the partial derivatives of $\mathbb{G}'[\mathbf{t}; \tau]$ can be expressed as

$$\partial_{t_m} \mathbb{G}'[\mathbf{t}; \tau] = \sum_{\alpha \in \mathcal{P}} \mathbb{G}'_\alpha(\tau) \partial_{t_m} s_\alpha[\mathbf{t}] = \sum_{\alpha \in \mathcal{P}} \sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \operatorname{sgn}(\beta, \alpha) \mathbb{G}'_\beta(\tau) s_\alpha[\mathbf{t}]. \quad (3.24)$$

Taking account of the symmetry $\operatorname{sgn}(\beta, \alpha) = \operatorname{sgn}(\alpha, \beta)$, we find that (3.23) and (3.24) yield (3.21) and (3.22) respectively. \square

3.2.2 Factorization formulas at $\tau = 1$

The operator-valued generating functions $\mathbb{G}[\mathbf{t}; \tau]$ and $\mathbb{G}'[\mathbf{t}; \tau]$ at $\tau = 1$ can be factorized into a triple product of operators.

Theorem 3. (factorization formulas at $\tau = 1$) *$\mathbb{G}[\mathbf{t}; \tau]$ and $\mathbb{G}'[\mathbf{t}; \tau]$ at $\tau = 1$ satisfy the identities*

$$\mathbb{G}[\mathbf{t}; 1] = \exp \left(\sum_{k=1}^{\infty} t_k q^{kJ_0} J_k \right) \mathbb{G}_\emptyset(1) \exp \left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{kJ_0} J_{2k} \right), \quad (3.25)$$

$$\mathbb{G}'[\mathbf{t}; 1] = \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{-kJ_0} J_k\right) \mathbb{G}'_{\emptyset}(1) \exp\left(\sum_{k=1}^{\infty} (-1)^k t_k q^{-kJ_0} J_{2k}\right), \quad (3.26)$$

where $\mathbb{G}_{\emptyset}(1)$ and $\mathbb{G}'_{\emptyset}(1)$ are operators of $\mathbb{G}_{\emptyset}(\tau)$ and $\mathbb{G}'_{\emptyset}(\tau)$ specialized to $\tau = 1$.

Proof. Let us derive differential equations for $\mathbb{G}[\mathbf{t}; 1]$ and $\mathbb{G}'[\mathbf{t}; 1]$. We first consider the case of $\mathbb{G}[\mathbf{t}; 1]$. Partial derivatives of $\mathbb{G}[\mathbf{t}; 1]$ with respect to \mathbf{t} are of the form (3.21). Note here that (3.12) now reads

$$J_{-k} \mathbb{G}_{\alpha}(1) = \mathbb{G}_{\alpha}(1) (-1)^k J_{-2k} + q^{kJ_0} \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta) \mathbb{G}_{\beta}(1), \quad k \in \mathbb{Z}_{\neq 0}. \quad (3.27)$$

We rewrite (3.27) for the cases of $k \leq -1$. By letting $k = -m$, these become

$$\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \text{sgn}(\alpha, \beta) \mathbb{G}_{\beta}(1) = q^{mJ_0} J_m \mathbb{G}_{\alpha}(1) + \mathbb{G}_{\alpha}(1) (-1)^{m+1} q^{mJ_0} J_{2m}, \quad (3.28)$$

where $m = 1, 2, \dots$. By plugging (3.28) into the double sum, the RHS of (3.21) turns out to be

$$\begin{aligned} \text{RHS of (3.21) at } \tau=1 &= \sum_{\alpha \in \mathcal{P}} \left(q^{mJ_0} J_m \mathbb{G}_{\alpha}(1) + \mathbb{G}_{\alpha}(1) (-1)^{m+1} q^{mJ_0} J_{2m} \right) s_{\alpha}[\mathbf{t}] \\ &= q^{mJ_0} J_m \sum_{\alpha \in \mathcal{P}} \mathbb{G}_{\alpha}(1) s_{\alpha}[\mathbf{t}] + \sum_{\alpha \in \mathcal{P}} \mathbb{G}_{\alpha}(1) s_{\alpha}[\mathbf{t}] (-1)^{m+1} q^{mJ_0} J_{2m} \\ &= q^{mJ_0} J_m \mathbb{G}[\mathbf{t}; 1] + \mathbb{G}[\mathbf{t}; 1] (-1)^{m+1} q^{mJ_0} J_{2m}. \end{aligned}$$

We thus find that $\mathbb{G}[\mathbf{t}; 1]$ satisfies the first-order differential equations

$$\partial_{t_m} \mathbb{G}[\mathbf{t}; 1] = q^{mJ_0} J_m \mathbb{G}[\mathbf{t}; 1] + \mathbb{G}[\mathbf{t}; 1] (-1)^{m+1} q^{mJ_0} J_{2m} \quad (3.29)$$

for $m = 1, 2, \dots$.

By the uniqueness of solutions of the initial value problem, (3.29) can be solved as

$$\mathbb{G}[\mathbf{t}; 1] = \exp\left(\sum_{k=1}^{\infty} t_k q^{kJ_0} J_k\right) \mathbb{G}[\mathbf{0}; 1] \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{kJ_0} J_{2k}\right), \quad (3.30)$$

where $\mathbb{G}[\mathbf{0}; 1]$ denotes an operator at the initial time. Since $\mathbb{G}[\mathbf{0}; 1] = \mathbb{G}_{\emptyset}(1)$, (3.30) yields (3.25).

(3.26) is likewise obtained by using Proposition 3.5 in place of Proposition 3.4. Partial derivatives of $\mathbb{G}'[\mathbf{t}; 1]$ with respect to \mathbf{t} are of the form (3.22). Specialized to $\tau = 1$, the double sum therein can be computed by Proposition 3.5. (3.16) therein reads

$$J_{-k}\mathbb{G}'_{\alpha}(1) = \mathbb{G}'_{\alpha}(1)J_{-2k} + (-1)^{k+1}q^{-kJ_0} \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta)\mathbb{G}'_{\beta}(1), \quad k \in \mathbb{Z}_{\neq 0}. \quad (3.31)$$

We rewrite (3.31) for the cases of $k \leq -1$. By letting $k = -m$, these become

$$\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \text{sgn}(\alpha, \beta)\mathbb{G}'_{\beta}(1) = (-1)^{m+1} \left(q^{-mJ_0} J_m \mathbb{G}'_{\alpha}(1) - \mathbb{G}'_{\alpha}(1) q^{-mJ_0} J_{2m} \right), \quad (3.32)$$

where $m = 1, 2, \dots$. By plugging (3.32) into the double sum, the RHS of (3.22) is converted to

$$\begin{aligned} \text{RHS of (3.22) at } \tau=1 &= (-1)^{m+1} \sum_{\alpha \in \mathcal{P}} \left(q^{-mJ_0} J_m \mathbb{G}'_{\alpha}(1) - \mathbb{G}'_{\alpha}(1) q^{-mJ_0} J_{2m} \right) \\ &= (-1)^{m+1} q^{-mJ_0} J_m \mathbb{G}'[\mathbf{t}; 1] + \mathbb{G}'[\mathbf{t}; 1] (-1)^m q^{-mJ_0} J_{2m}. \end{aligned}$$

We thus find that $\mathbb{G}'[\mathbf{t}; 1]$ satisfies the first-order differential equations

$$\partial_{t_m} \mathbb{G}'[\mathbf{t}; 1] = (-1)^{m+1} q^{-mJ_0} J_m \mathbb{G}'[\mathbf{t}; 1] + \mathbb{G}'[\mathbf{t}; 1] (-1)^m q^{-mJ_0} J_{2m} \quad (3.33)$$

for $m = 1, 2, \dots$.

General solutions of the differential equations (3.33) are of the form

$$\mathbb{G}'[\mathbf{t}; 1] = \exp \left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{-kJ_0} J_k \right) \mathbb{G}'[\mathbf{0}; 1] \exp \left(\sum_{k=1}^{\infty} (-1)^k t_k q^{-kJ_0} J_{2k} \right), \quad (3.34)$$

where $\mathbb{G}'[\mathbf{0}; 1]$ denotes an operator at the initial time. In the current case, we have $\mathbb{G}'[\mathbf{0}; 1] = \mathbb{G}'_{\emptyset}(1)$. Therefore, (3.34) yields (3.26). \square

4 Proof of Theorem 1

4.1 Yet another representation at $\tau = 1$

In the case of $\tau = 1$, another expression of generating function of the topological vertex can be derived from Theorem 3. We conveniently start with an operator identity of the following form.

Proposition 4.1.

$$\mathbb{G}_\emptyset(1)\Gamma'_-(\mathbf{z}) = \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}p_{2m}(\mathbf{z})}{m}J_{-m}\right)\mathbb{G}_\emptyset(1)\Gamma_-(\mathbf{z}), \quad (4.1)$$

Proof. Let us rewrite $\Gamma'_-(\mathbf{z})$ as

$$\begin{aligned} \Gamma'_-(\mathbf{z}) &= \exp\left(\sum_{m=1}^{\infty} \frac{p_{2m-1}(\mathbf{z})}{2m-1}J_{-2m+1} - \sum_{m=1}^{\infty} \frac{p_{2m}(\mathbf{z})}{2m}J_{-2m}\right) \\ &= \exp\left(-\sum_{m=1}^{\infty} \frac{p_{2m}(\mathbf{z})}{m}J_{-2m}\right)\Gamma_-(\mathbf{z}). \end{aligned} \quad (4.2)$$

(3.27) implies the relation

$$J_{-m}\mathbb{G}_\emptyset(1) = (-1)^m\mathbb{G}_\emptyset(1)J_{-2m}, \quad m = 1, 2, \dots, . \quad (4.3)$$

Combining (4.2) with (4.3), we see

$$\begin{aligned} \mathbb{G}_\emptyset(1)\Gamma'_-(\mathbf{z}) &= \mathbb{G}_\emptyset(1)\exp\left(-\sum_{m=1}^{\infty} \frac{p_{2m}(\mathbf{z})}{m}J_{-2m}\right)\Gamma_-(\mathbf{z}) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}p_{2m}(\mathbf{z})}{m}J_{-m}\right)\mathbb{G}_\emptyset(1)\Gamma_-(\mathbf{z}). \end{aligned}$$

Thus we obtain (4.1). □

Proposition 4.2. *The generating function (1.18) at $\tau = 1$ has a fermionic representation of the form*

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) = \langle 0|\Gamma_+(\mathbf{x})\Gamma'_+(\mathbf{y})\mathbb{G}_\emptyset(1)\Gamma_-(\mathbf{z})|0\rangle \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}p_m(\mathbf{x})p_{2m}(\mathbf{z})\right). \quad (4.4)$$

Proof. We adopt the fermionic representation (2.7) for a description of the generating function (1.18). Letting $\tau = 1$, we rewrite (2.7) by using Theorem 3. Actually, substituting $t_k = (-1)^{k+1}p_k(\mathbf{y})/k$ in (3.25), we find

$$\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(1) s_{t_\alpha}(\mathbf{y}) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k(\mathbf{y})}{k} q^{kJ_0} J_k\right) \mathbb{G}_\emptyset(1) \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y})}{k} q^{kJ_0} J_{2k}\right).$$

The sum of operators in (2.7) can be thus converted into a triple product of operators as

$$\begin{aligned} \mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) &= \langle 0 | \Gamma_+(\mathbf{x}) \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k(\mathbf{y})}{k} J_k\right) \mathbb{G}_\emptyset(1) \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y})}{k} J_{2k}\right) \Gamma'_-(\mathbf{z}) | 0 \rangle \\ &= \langle 0 | \Gamma_+(\mathbf{x}) \Gamma'_+(\mathbf{y}) \mathbb{G}_\emptyset(1) \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y})}{k} J_{2k}\right) \Gamma'_-(\mathbf{z}) | 0 \rangle. \end{aligned} \quad (4.5)$$

We further rewrite the RHS of (4.5). The commutation relations

$$\Gamma'_-(\mathbf{z}) J_{2k} = J_{2k} \Gamma'_-(\mathbf{z}) + \Gamma'_-(\mathbf{z}) p_{2k}(\mathbf{z}),$$

and the annihilation property $J_{2k}|0\rangle = 0$ for $k \geq 1$ yield

$$\exp\left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y})}{k} J_{2k}\right) \Gamma'_-(\mathbf{z}) | 0 \rangle = \Gamma'_-(\mathbf{z}) | 0 \rangle \exp\left(-\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y}) p_{2k}(\mathbf{z})}{k}\right). \quad (4.6)$$

By (4.6), the RHS of (4.5) can be converted to

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) = \langle 0 | \Gamma_+(\mathbf{x}) \Gamma'_+(\mathbf{y}) \mathbb{G}_\emptyset(1) \Gamma'_-(\mathbf{z}) | 0 \rangle \exp\left(-\sum_{k=1}^{\infty} \frac{p_k(\mathbf{y}) p_{2k}(\mathbf{z})}{k}\right). \quad (4.7)$$

The bra vector in (4.7) can be expressed with the aid of (4.1) as follows:

$$\begin{aligned} &\langle 0 | \Gamma_+(\mathbf{x}) \Gamma'_+(\mathbf{y}) \mathbb{G}_\emptyset(1) \Gamma'_-(\mathbf{z}) \\ &= \langle 0 | \Gamma_+(\mathbf{x}) \Gamma'_+(\mathbf{y}) \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1} p_{2m}(\mathbf{z})}{m} J_{-m}\right) \mathbb{G}_\emptyset(1) \Gamma'_-(\mathbf{z}) \\ &= \langle 0 | \Gamma_+(\mathbf{x}) \Gamma'_+(\mathbf{y}) \mathbb{G}_\emptyset(1) \Gamma'_-(\mathbf{z}) \exp\left\{\sum_{m=1}^{\infty} \frac{1}{m} \left(p_m(\mathbf{y}) p_{2m}(\mathbf{z}) + (-1)^{m+1} p_m(\mathbf{x}) p_{2m}(\mathbf{z})\right)\right\}, \end{aligned} \quad (4.8)$$

where the negative modes J_{-m} have been moved to the left by the commutation relations

$$\begin{aligned}\Gamma_+(\mathbf{x})J_{-m} &= J_{-m}\Gamma_+(\mathbf{x}) + \Gamma_+(\mathbf{x})p_m(\mathbf{x}), \\ \Gamma'_+(\mathbf{y})J_{-m} &= J_{-m}\Gamma'_+(\mathbf{y}) + \Gamma'_+(\mathbf{y})(-1)^{m+1}p_m(\mathbf{y}),\end{aligned}$$

together with $\langle 0|J_{-m} = 0$ at the end. Designating the last part of (4.8) as the bra vector and its pairing with the ket vector $|0\rangle$ being substituted for the matrix element in (4.7), we eventually obtain (4.4). \square

The matrix element in the RHS of (4.4) has an expansion in terms of the Schur functions and the skew Schur functions.

Proposition 4.3. *The matrix element in the fermionic expression (4.4) has a Schur function expansion of the form*

$$\begin{aligned}\langle 0|\Gamma_+(\mathbf{x})\Gamma'_+(\mathbf{y})\mathbb{G}_\emptyset(1)\Gamma_-(\mathbf{z})|0\rangle \\ = \sum_{\nu^1, \nu^3, \nu^+ \in \mathcal{P}} s_{\nu^1}(\mathbf{x})s_{\nu^+ / \iota_{\nu^1}}(\mathbf{y})s_{\iota_{\nu^3}}(\mathbf{z})q^{\kappa(\nu^+) + \kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho})s_{\nu^3}(q^{-\nu^+ - \rho}).\end{aligned}\quad (4.9)$$

Proof. By inserting the identity $\sum_{\nu \in \mathcal{P}} |\nu\rangle\langle \nu| = 1$ between all pairs of adjacent operators in $\Gamma_+(\mathbf{x})\Gamma'_+(\mathbf{y})\mathbb{G}_\emptyset(1)\Gamma_-(\mathbf{z})$, the LHS of (4.9) reads

$$\begin{aligned}\langle 0|\Gamma_+(\mathbf{x})\Gamma'_+(\mathbf{y})\mathbb{G}_\emptyset\Gamma_-(\mathbf{z})|0\rangle \\ = \langle 0|\Gamma_+(\mathbf{x}) \left(\sum_{\nu^1 \in \mathcal{P}} |\nu^1\rangle\langle \nu^1| \right) \Gamma'_+(\mathbf{y}) \left(\sum_{\nu^+ \in \mathcal{P}} |\iota_{\nu^+}\rangle\langle \iota_{\nu^+}| \right) \mathbb{G}_\emptyset(1) \left(\sum_{\nu^3 \in \mathcal{P}} |\iota_{\nu^3}\rangle\langle \iota_{\nu^3}| \right) \Gamma_-(\mathbf{z})|0\rangle \\ = \sum_{\nu^1, \nu^+, \nu^3 \in \mathcal{P}} s_{\nu^1}(\mathbf{x})s_{\nu^+ / \iota_{\nu^1}}(\mathbf{y})s_{\iota_{\nu^3}}(\mathbf{z})\langle \iota_{\nu^+} | \mathbb{G}_\emptyset(1) | \iota_{\nu^3} \rangle.\end{aligned}\quad (4.10)$$

Note that (2.2) and (2.3) have been used in the last line.

The matrix element of $\mathbb{G}_\emptyset(1) = q^{-K/2}\Gamma_-(q^\rho)\Gamma_+(q^\rho)q^{K/4}$ can be expressed in terms of special values of the Schur function as

$$\langle \iota_{\nu^+} | \mathbb{G}_\emptyset(1) | \iota_{\nu^3} \rangle = q^{\kappa(\nu^+) + \kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho})s_{\nu^3}(q^{-\nu^+ - \rho}).\quad (4.11)$$

Plugging (4.11) into the RHS of (4.10), we obtain (4.9).

Let us derive (4.11). It is convenient to use the formula [12]

$$q^{K/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})|{}^t\alpha\rangle = \Gamma'_-(q^{-\alpha-\rho})|0\rangle s_{t\alpha}(q^{-\rho}), \quad \forall \alpha \in \mathcal{P}. \quad (4.12)$$

By taking the transpose and using (4.12), we can rewrite the LHS of (4.11) as

$$\begin{aligned} \langle {}^t\nu^+ | \mathbb{G}_\emptyset(1) | {}^t\nu^3 \rangle &= q^{\kappa(\nu^+)/2} \langle {}^t\nu^+ | \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})q^{K/2} | {}^t\nu^3 \rangle q^{\kappa(\nu^3)/4} \\ &= q^{\kappa(\nu^+)/2} \left(s_{t\nu^+}(q^{-\rho}) \langle 0 | \Gamma'_+(q^{-\nu^+-\rho}) \right) | {}^t\nu^3 \rangle q^{\kappa(\nu^3)/4} \\ &= q^{\kappa(\nu^+)+\kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+-\rho}). \end{aligned}$$

Note that the identity $s_{t\nu^+}(q^{-\rho}) = q^{\kappa(\nu^+)/2} s_{\nu^+}(q^{-\rho})$ is used in the last line. Thus, we obtain (4.11). \square

Remark 4.1. $\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho})$ is a self-adjoint operator, that is, satisfies

$$\langle {}^t\beta | \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) | {}^t\alpha \rangle = \langle {}^t\alpha | \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) | {}^t\beta \rangle, \quad \forall \alpha, \beta \in \mathcal{P}. \quad (4.13)$$

The matrix elements in (4.13) can be expressed in terms of special values of the Schur function by (4.12). Actually, the LHS of (4.13) becomes

$$\begin{aligned} \langle {}^t\beta | \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) | {}^t\alpha \rangle &= q^{\kappa(\beta)/2} \langle {}^t\beta | q^{K/2}\Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) | {}^t\alpha \rangle \\ &= q^{\kappa(\beta)/2} \langle {}^t\beta | q^{K/2}\Gamma'_-(q^{-\alpha-\rho})|0\rangle s_{t\alpha}(q^{-\rho}) = q^{\frac{\kappa(\alpha)+\kappa(\beta)}{2}} s_\alpha(q^{-\rho}) s_\beta(q^{-\alpha-\rho}), \end{aligned}$$

whereas the RHS of (4.13) reads

$$\langle {}^t\alpha | \Gamma_-(q^{-\rho})\Gamma_+(q^{-\rho}) | {}^t\beta \rangle = q^{\frac{\kappa(\alpha)+\kappa(\beta)}{2}} s_\alpha(q^{-\beta-\rho}) s_\beta(q^{-\rho}).$$

Thus, (4.13) implies the identity

$$s_\alpha(q^{-\rho}) s_\beta(q^{-\alpha-\rho}) = s_\alpha(q^{-\beta-\rho}) s_\beta(q^{-\rho}), \quad \forall \alpha, \beta \in \mathcal{P}. \quad (4.14)$$

The identity (4.14) is exploited in [12] to simplify open topological string amplitudes including the case of open topological string on closed vertex.

Remark 4.2. From (2.5), the matrix element of $\mathbb{G}_\emptyset(1)$ can be written in terms of the two-legged topological vertex as

$$\langle {}^t\nu^+ | \mathbb{G}_\emptyset(1) | {}^t\nu^3 \rangle = q^{\kappa(\nu^+) - \kappa(\nu^3)/4} \mathcal{C}_{({}^t\nu^+, \emptyset, \nu^3)}(q).$$

On the other hand, (1.17) and (4.11) yield another expression

$$\langle {}^t\nu^+ | \mathbb{G}_\emptyset(1) | {}^t\nu^3 \rangle = q^{\kappa(\nu^+) - \kappa(\nu^3)/4} \mathcal{C}_{(\nu^3, {}^t\nu^+, \emptyset)}(q).$$

Furthermore, by (1.17) and (4.14), we find

$$\begin{aligned} \mathcal{C}_{(\nu^3, {}^t\nu^+, \emptyset)}(q) &= q^{\kappa(\nu^3)/2} s_{\nu^3}(q^{-\nu^+ - \rho}) s_{\nu^+}(q^{-\rho}) \\ &= q^{\kappa(\nu^3)/2} s_{\nu^3}(q^{-\rho}) s_{\nu^+}(q^{-\nu^3 - \rho}) = \mathcal{C}_{(\emptyset, \nu^3, {}^t\nu^+)}(q). \end{aligned}$$

These identities imply the cyclic symmetry of the two-legged topological vertex:

$$\mathcal{C}_{({}^t\nu^+, \emptyset, \nu^3)}(q) = \mathcal{C}_{(\nu^3, {}^t\nu^+, \emptyset)}(q) = \mathcal{C}_{(\emptyset, \nu^3, {}^t\nu^+)}(q).$$

4.2 Proof of Theorem 1

We note that the exponentiation of quadratic form of the power sums in (4.4) has a Schur function expansion of the following form:

Proposition 4.4.

$$\exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m(\mathbf{x}) p_{2m}(\mathbf{z}) \right) = \sum_{\eta^1, \eta^3 \in \mathcal{P}} s_{\eta^1}(\mathbf{x}) s_{\eta^3}(\mathbf{z}) \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_\xi}. \quad (4.15)$$

Proof. We rewrite the LHS of (4.15) into

$$\begin{aligned} & \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m(\mathbf{x}) p_{2m}(\mathbf{z}) \right) \\ &= \prod_{k=1}^{\infty} \exp \left(\frac{(-1)^{k-1}}{k} p_k(\mathbf{x}) p_{2k}(\mathbf{z}) \right) = \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{(-1)^{(k-1)m_k}}{m_k! k^{m_k}} (p_k(\mathbf{x}) p_{2k}(\mathbf{z}))^{m_k} \\ &= \sum_{m_1, m_2, \dots} \frac{(-1)^{\sum_{k=1}^{\infty} (k-1)m_k}}{\prod_{k=1}^{\infty} m_k! k^{m_k}} \prod_{k=1}^{\infty} (p_k(\mathbf{x}) p_{2k}(\mathbf{z}))^{m_k} = \sum_{\xi \in \mathcal{P}} \frac{(-1)^{|\xi| - l(\xi)}}{z_\xi} p_\xi(\mathbf{x}) p_{2\xi}(\mathbf{z}), \end{aligned} \quad (4.16)$$

where the summation over non-negative integers m_k 's is replaced with the summation over partitions ξ by arranging $\xi = (1^{m_1} 2^{m_2} \dots)$. Plugging the expansions

$$(-1)^{|\xi|-l(\xi)} p_\xi(\mathbf{x}) = \sum_{\eta^1 \in \mathcal{P}} \chi_{\eta^1}(\xi) s_{t_{\eta^1}}(\mathbf{x}), \quad p_{2\xi}(\mathbf{z}) = \sum_{\eta^3 \in \mathcal{P}} \chi_{\eta^3}(2\xi) s_{\eta^3}(\mathbf{z})$$

into the RHS of (4.16), we obtain (4.15). \square

We can now rewrite $\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1)$ as follows.

Proposition 4.5.

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) = \sum_{\vec{\mu} \in \mathcal{P}^3} \tilde{\mathcal{C}}_{\vec{\mu}}(\lambda) q^{-\kappa(\mu^{(1)})/2 + \kappa(\mu^{(2)}) + \kappa(\mu^{(3)})/4} s_{\mu^{(1)}}(\mathbf{x}) s_{\mu^{(2)}}(\mathbf{y}) s_{\mu^{(3)}}(\mathbf{z}), \quad (4.17)$$

where $\tilde{\mathcal{C}}_{\vec{\mu}}(\lambda)$ is given by (1.16) with $q = e^{-\sqrt{-1}\lambda}$.

Theorem 1 is an immediate consequence of Proposition 4.5. By (1.18), $\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1)$ is a generating function of $\mathcal{C}_{\vec{\mu}}(q)$ of the form

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) = \sum_{\vec{\mu} \in \mathcal{P}^3} \mathcal{C}_{\vec{\mu}}(q) q^{-\kappa(\mu^{(1)}) + \kappa(\mu^{(2)})/2 - \kappa(\mu^{(3)})/4} s_{\mu^{(1)}}(\mathbf{x}) s_{\mu^{(2)}}(\mathbf{y}) s_{\mu^{(3)}}(\mathbf{z}).$$

Equating the coefficients of the Schur function products in this expression with those of (4.17), we obtain (1.21). This also implies that (1.19) holds.

Proof of Proposition 4.5. We derive an expansion of the fermionic representation (4.4) in the basis of the Schur functions. By (4.9) and (4.15), $\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1)$ can be expressed as

$$\begin{aligned} & \mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) \\ &= \sum_{\nu^1, \nu^3, \nu^+ \in \mathcal{P}} s_{\nu^1}(\mathbf{x}) s_{\nu^+ / t_{\nu^1}}(\mathbf{y}) s_{t_{\nu^3}}(\mathbf{z}) q^{\kappa(\nu^+) + \kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+ - \rho}) \\ & \quad \times \sum_{\eta^1, \eta^3 \in \mathcal{P}} s_{t_{\eta^1}}(\mathbf{x}) s_{\eta^3}(\mathbf{z}) \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_\xi} \\ &= \sum_{\nu^1, \nu^3, \nu^+, \eta^1, \eta^3 \in \mathcal{P}} s_{t_{\eta^1}}(\mathbf{x}) s_{\nu^1}(\mathbf{x}) s_{\nu^+ / t_{\nu^1}}(\mathbf{y}) s_{\eta^3}(\mathbf{z}) s_{t_{\nu^3}}(\mathbf{z}) \end{aligned}$$

$$\times q^{\kappa(\nu^+)+\kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+-\rho}) \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_\xi}. \quad (4.18)$$

The products of the Schur functions and the skew Schur function can be expressed in terms of the Littlewood-Richardson numbers as

$$\begin{aligned} s_{t\eta^1}(\mathbf{x}) s_{\nu^1}(\mathbf{x}) &= \sum_{\mu^{(1)} \in \mathcal{P}} c_{t\eta^1 \nu^1}^{\mu^{(1)}} s_{\mu^{(1)}}(\mathbf{x}), \\ s_{\nu^+ / t\nu^1}(\mathbf{y}) &= \sum_{\mu^{(2)} \in \mathcal{P}} c_{t\nu^1 \mu^{(2)}}^{\nu^+} s_{\mu^{(2)}}(\mathbf{y}), \\ s_{\eta^3}(\mathbf{z}) s_{t\nu^3}(\mathbf{z}) &= \sum_{\mu^{(3)} \in \mathcal{P}} c_{\eta^3 t\nu^3}^{\mu^{(3)}} s_{\mu^{(3)}}(\mathbf{z}). \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; 1) \\ &= \sum_{\nu^1, \nu^3, \nu^+, \eta^1, \eta^3 \in \mathcal{P}} \left(\sum_{\mu^{(1)} \in \mathcal{P}} c_{t\eta^1 \nu^1}^{\mu^{(1)}} s_{\mu^{(1)}}(\mathbf{x}) \right) \left(\sum_{\mu^{(2)} \in \mathcal{P}} c_{t\nu^1 \mu^{(2)}}^{\nu^+} s_{\mu^{(2)}}(\mathbf{y}) \right) \left(\sum_{\mu^{(3)} \in \mathcal{P}} c_{\eta^3 t\nu^3}^{\mu^{(3)}} s_{\mu^{(3)}}(\mathbf{z}) \right) \\ &\quad \times q^{\kappa(\nu^+)+\kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+-\rho}) \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_\xi} \\ &= \sum_{(\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) \in \mathcal{P}^3} s_{\mu^{(1)}}(\mathbf{x}) s_{\mu^{(2)}}(\mathbf{y}) s_{\mu^{(3)}}(\mathbf{z}) \\ &\quad \times \sum_{\nu^1, \nu^3, \nu^+, \eta^1, \eta^3 \in \mathcal{P}} c_{t\eta^1 \nu^1}^{\mu^{(1)}} c_{t\nu^1 \mu^{(2)}}^{\nu^+} c_{\eta^3 t\nu^3}^{\mu^{(3)}} q^{\kappa(\nu^+)+\kappa(\nu^3)/4} s_{\nu^+}(q^{-\rho}) s_{\nu^3}(q^{-\nu^+-\rho}) \sum_{\xi \in \mathcal{P}} \frac{\chi_{\eta^1}(\xi) \chi_{\eta^3}(2\xi)}{z_\xi}. \end{aligned}$$

Thus we obtain (4.17). \square

5 Proof of Theorem 2

5.1 Factorization formulas at $\tau = 1/N$

Just like $\mathbb{G}[\mathbf{t}; 1]$ and $\mathbb{G}'[\mathbf{t}; 1]$, the operator-valued generating functions (3.19) and (3.20) specialized to $\tau = 1/N$ ($N = 1, 2, \dots$) can be factorized to a triple product of operators.

Theorem 4. (factorization formulas at $\tau = 1/N$) *Let N be a positive integer. The operator-valued generating functions $\mathbb{G}[\mathbf{t}; \tau]$ and $\mathbb{G}'[\mathbf{t}; \tau]$ satisfy the following identities at $\tau = 1/N$:*

$$\mathbb{G}[\mathbf{t}; 1/N] = \exp\left(\sum_{k=1}^{\infty} t_k q^{kJ_0} J_{kN}\right) \mathbb{G}_{\emptyset}(1/N) \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{kJ_0} J_{k(N+1)}\right), \quad (5.1)$$

$$\mathbb{G}'[\mathbf{t}; 1/N] = \exp\left(\sum_{k=1}^{\infty} (-1)^{kN+1} t_k q^{-kJ_0} J_{kN}\right) \mathbb{G}'_{\emptyset}(1/N) \exp\left(\sum_{k=1}^{\infty} (-1)^{kN} t_k q^{-kJ_0} J_{k(N+1)}\right). \quad (5.2)$$

Proof. The proof is mostly parallel to the proof of Theorem 3. We derive differential equations that the operator-valued generating functions satisfy at $\tau = 1/N$. We describe the case of $\mathbb{G}[\mathbf{t}; 1/N]$ in detail. Partial derivatives of $\mathbb{G}[\mathbf{t}; 1/N]$ with respect to \mathbf{t} can be expressed in the form of (3.21). The double sum in (3.21) can be computed by Proposition 3.4. By specializing τ to $\tau = 1/N$, (3.12) reads

$$J_{-kN} \mathbb{G}_{\alpha}(1/N) = \mathbb{G}_{\alpha}(1/N) (-1)^k J_{-k(N+1)} + q^{kJ_0} \sum_{\beta \in \mathcal{R}_{k,\alpha}} \text{sgn}(\alpha, \beta) \mathbb{G}_{\beta}(1/N) \quad (5.3)$$

for $k \in \mathbb{Z}, k \neq 0$. By letting $k = -m$, we have

$$\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \text{sgn}(\alpha, \beta) \mathbb{G}_{\beta}(1/N) = q^{mJ_0} J_{mN} \mathbb{G}_{\alpha}(1/N) + \mathbb{G}_{\alpha}(1/N) (-1)^{m+1} q^{mJ_0} J_{m(N+1)} \quad (5.4)$$

for $m = 1, 2, \dots$. The RHS of (3.21) thereby becomes

$$\begin{aligned} & \text{RHS of (3.21) at } \tau = 1/N \\ &= \sum_{\alpha \in \mathcal{P}} \left(q^{mJ_0} J_{mN} \mathbb{G}_{\alpha}(1/N) + \mathbb{G}_{\alpha}(1/N) (-1)^{m+1} q^{mJ_0} J_{m(N+1)} \right) s_{\alpha}[\mathbf{t}] \\ &= q^{mJ_0} J_{mN} \mathbb{G}[\mathbf{t}; 1/N] + \mathbb{G}[\mathbf{t}; 1/N] (-1)^{m+1} q^{mJ_0} J_{m(N+1)}. \end{aligned}$$

We thus find that $\mathbb{G}[\mathbf{t}; 1/N]$ satisfies the first-order differential equations

$$\partial_{t_m} \mathbb{G}[\mathbf{t}; 1/N] = q^{mJ_0} J_{mN} \mathbb{G}[\mathbf{t}; 1/N] + \mathbb{G}[\mathbf{t}; 1/N] (-1)^{m+1} q^{mJ_0} J_{m(N+1)} \quad (5.5)$$

for $m = 1, 2, \dots$. General solutions of the differential equations (5.5) are given by

$$\mathbb{G}[\mathbf{t}; 1/N] = \exp\left(\sum_{k=1}^{\infty} t_k q^{kJ_0} J_{kN}\right) \mathbb{G}[\mathbf{0}; 1/N] \exp\left(\sum_{k=1}^{\infty} (-1)^{k+1} t_k q^{kJ_0} J_{k(N+1)}\right). \quad (5.6)$$

Since $\mathbb{G}[\mathbf{0}; 1/N] = \mathbb{G}_{\emptyset}(1/N)$, (5.6) yields (5.1).

(5.2) can be likewise derived from Proposition 3.5. Partial derivatives of $\mathbb{G}'[\mathbf{t}; 1]$ with respect to \mathbf{t} take the form of (3.22). The double sum therein can be computed using Proposition 3.5. By letting $\tau = 1/N$, (3.16) yields

$$\sum_{\beta \in \mathcal{R}_{m,\alpha}^{(+)}} \text{sgn}(\alpha, \beta) \mathbb{G}'_{\beta}(1/N) = (-1)^{mN} \left(-q^{-mJ_0} J_{mN} \mathbb{G}'_{\alpha}(1/N) + \mathbb{G}'_{\alpha}(1/N) q^{-mJ_0} J_{m(N+1)} \right) \quad (5.7)$$

for $m = 1, 2, \dots$. By plugging (5.7) into the RHS of (3.22), we eventually find out that $\mathbb{G}'[\mathbf{t}; 1/N]$ satisfies the first-order differential equations

$$\begin{aligned} \partial_{t_m} \mathbb{G}'[\mathbf{t}; 1/N] \\ = (-1)^{mN+1} q^{-mJ_0} J_{mN} \mathbb{G}'[\mathbf{t}; 1/N] + \mathbb{G}'[\mathbf{t}; 1/N] (-1)^{mN} q^{-mJ_0} J_{m(N+1)} \end{aligned} \quad (5.8)$$

for $m = 1, 2, \dots$. General solutions of (5.8) are given by

$$\begin{aligned} \mathbb{G}'[\mathbf{t}; 1/N] \\ = \exp\left(\sum_{k=1}^{\infty} (-1)^{kN+1} t_k q^{-kJ_0} J_{kN}\right) \mathbb{G}'[\mathbf{0}; 1/N] \exp\left(\sum_{k=1}^{\infty} (-1)^{kN} t_k q^{-kJ_0} J_{k(N+1)}\right). \end{aligned} \quad (5.9)$$

Since $\mathbb{G}'[\mathbf{0}; 1/N] = \mathbb{G}'_{\emptyset}(1/N)$, (5.9) yields the formula (5.2). \square

5.2 Representation of generating function at $\tau = N$

The foregoing factorization formula gives a new fermionic representation of the generating function of the topological vertex.

Proposition 5.1. *Let N be a positive integer. The generating function (1.18) at $\tau = N$ has a fermionic representation of the form*

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; N) = \langle 0 | \exp\left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{x})}{k} J_{kN}\right) \Gamma'_+(\mathbf{y}) \mathbb{G}_{\emptyset}(1/N) \Gamma_-(\mathbf{z}) | 0 \rangle$$

$$\times \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m(\mathbf{x}) p_{m(N+1)}(\mathbf{z}) \right). \quad (5.10)$$

Proof. The cyclic symmetry (1.22) can be embodied in the language of the generating functions (1.18) as

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) = \mathcal{W}\left(q; \mathbf{z}, \mathbf{x}, \mathbf{y}; \frac{-1}{1+\tau}\right).$$

In view of (2.7), this relation leads to the expression

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) = \langle 0 | \Gamma_+(z) \left(\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha \left(\frac{-1}{1+\tau} \right) s_{t_\alpha}(\mathbf{x}) \right) \Gamma'_-(\mathbf{y}) | 0 \rangle. \quad (5.11)$$

We can replace the VEV $\langle 0 | \dots | 0 \rangle$ in the RHS of (5.11) with the transpose. Actually, since the operator-valued functions $\mathbb{G}_\alpha(-1/1+\tau)$ satisfy

$$\langle \beta | \mathbb{G}_\alpha \left(\frac{-1}{1+\tau} \right) | \gamma \rangle = \langle \gamma | \mathbb{G}_{t_\alpha}(1/\tau) | \beta \rangle, \quad \forall \beta, \gamma \in \mathcal{P},$$

we can rewrite (5.11) as

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; \tau) = \langle 0 | \Gamma'_+(\mathbf{y}) \left(\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(1/\tau) s_\alpha(\mathbf{x}) \right) \Gamma_-(z) | 0 \rangle.$$

In particular, by specialization to $\tau = N$, we find

$$\mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; N) = \langle 0 | \Gamma'_+(\mathbf{y}) \left(\sum_{\alpha \in \mathcal{P}} \mathbb{G}_\alpha(1/N) s_\alpha(\mathbf{x}) \right) \Gamma_-(z) | 0 \rangle. \quad (5.12)$$

By substituting $t_k = p_k(\mathbf{x})/k$, (5.1) converts the sum of operators in (5.12) into a triple product of operators as

$$\begin{aligned} & \mathcal{W}(q; \mathbf{x}, \mathbf{y}, \mathbf{z}; N) \\ &= \langle 0 | \Gamma'_+(\mathbf{y}) \exp \left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{x})}{k} J_{kN} \right) \mathbb{G}_\emptyset(1/N) \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k(\mathbf{x})}{k} J_{k(N+1)} \right) \Gamma_-(z) | 0 \rangle \\ &= \langle 0 | \exp \left(\sum_{k=1}^{\infty} \frac{p_k(\mathbf{x})}{k} J_{kN} \right) \Gamma'_+(\mathbf{y}) \mathbb{G}_\emptyset(1/N) \Gamma_-(z) | 0 \rangle \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m(\mathbf{x}) p_{m(N+1)}(\mathbf{z}) \right). \end{aligned}$$

We thus obtain (5.10). \square

5.3 Proof of Theorem 2

We first derive a reduction formula of the generating functions of three-partition Hodge integrals at positive integral values of τ . This formula gives a generalization of (1.12) at all positive integral values of τ . We then consider integrable hierarchies that capture a certain aspect of integrable structure underlying the generating functions of two-partition Hodge integrals. By combining these considerations, we eventually obtain Theorem 2.

Let N be a positive integer.

Proposition 5.2. *The generating function (1.9) satisfies the relation*

$$G(\lambda; (p^{(1)}, p^{(2)}, p^{(3)}); N) = G(\lambda; (0, p^+, p^{(3)}); N) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)}, \quad (5.13)$$

where $p^+ = (p_k^+)_{k=1}^{\infty}$ is a linear combination of $p^{(1)}$ and $p^{(2)}$ given by

$$p_k^+ = \begin{cases} (-1)^{k+1} N p_{k/N}^{(1)} + p_k^{(2)}, & k \equiv 0 \pmod{N}, \\ p_k^{(2)}, & k \not\equiv 0 \pmod{N}. \end{cases} \quad (5.14)$$

Proof. By Theorem 1 and Proposition 5.1, we can rewrite the exponentiated generating function as

$$\begin{aligned} & \exp(G(\lambda; \vec{p}; N)) \\ &= \langle 0 | \exp \left(\sum_{k=1}^{\infty} \frac{p_k^{(1)}}{k} J_{kN} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k^{(2)}}{k} J_k \right) \mathbb{G}_{\emptyset}(1/N) \exp \left(\sum_{k=1}^{\infty} \frac{p_k^{(3)}}{k} J_{-k} \right) | 0 \rangle \\ & \quad \times \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)} \right) \\ &= \langle 0 | \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k^+}{k} J_k \right) \mathbb{G}_{\emptyset}(1/N) \exp \left(\sum_{k=1}^{\infty} \frac{p_k^{(3)}}{k} J_{-k} \right) | 0 \rangle \\ & \quad \times \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)} \right). \end{aligned} \quad (5.15)$$

By letting $\vec{p} = (0, p^{(2)}, p^{(3)})$, (5.15) takes the simplified form

$$\begin{aligned} & \exp(G(\lambda; (0, p^{(2)}, p^{(3)}); N)) \\ &= \langle 0 | \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k^{(2)}}{k} J_k\right) \mathbb{G}_{\emptyset}(1/N) \exp\left(\sum_{k=1}^{\infty} \frac{p_k^{(3)}}{k} J_{-k}\right) | 0 \rangle. \end{aligned}$$

We can replace the VEV $\langle 0 | \dots | 0 \rangle$ in the last part of (5.15) with this expression. This yields the relation

$$\begin{aligned} & \exp(G(\lambda; (p^{(1)}, p^{(2)}, p^{(3)}); N)) \\ &= \exp(G(\lambda; (0, p^+, p^{(3)}); N)) \exp\left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)}\right), \end{aligned}$$

hence (5.13). □

Proposition 5.3. $\mathcal{T}(\lambda, N, 0, p^{(2)}, \mathbf{t})$ is a tau function of the KP hierarchy with respect to the time variables \mathbf{t} , and satisfies the equations

$$\frac{\partial^2 \log \mathcal{T}(\lambda, N, 0, p^{(2)}, \mathbf{t})}{\partial t_k \partial t_{m(N+1)}} = 0, \quad k, m = 1, 2, \dots \quad (5.16)$$

Proof. We start from the fermionic expression

$$\mathcal{T}(\lambda, N, 0, p^{(2)}, \mathbf{t}) = \langle 0 | \mathbf{g}(\lambda, N; p^{(2)}) \exp\left(\sum_{k=1}^{\infty} t_k J_{-k}\right) | 0 \rangle, \quad (5.17)$$

where $\mathbf{g}(\lambda, N; p^{(2)})$ is an element of $GL(\infty)$ of the form

$$\mathbf{g}(\lambda, N; p^{(2)}) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} p_k^{(2)}}{k} J_k\right) \mathbb{G}_{\emptyset}(1/N).$$

(5.3) contains the special subset

$$\mathbb{G}_{\emptyset}(1/N) J_{-m(N+1)} = (-1)^m J_{-mN} \mathbb{G}_{\emptyset}(1/N), \quad m = 1, 2, \dots,$$

which yields the algebraic relation

$$\begin{aligned} & \mathbf{g}(\lambda, N; p^{(2)}) \exp \left(\sum_{m=1}^{\infty} t_{m(N+1)} J_{-m(N+1)} \right) \\ = & \exp \left(\sum_{m=1}^{\infty} (-1)^m t_{m(N+1)} J_{-mN} \right) \mathbf{g}(\lambda, N; p^{(2)}) \exp \left(- \sum_{m=1}^{\infty} (-1)^{m(N+1)} p_{mN}^{(2)} t_{m(N+1)} \right) \end{aligned}$$

satisfied by $\mathbf{g}(\lambda, N; p^{(2)})$. We can use this algebraic relation to rewrite (5.17) as

$$\begin{aligned} \mathcal{T}(\lambda, N, 0, p^{(2)}, \mathbf{t}) &= \langle 0 | \mathbf{g}(\lambda, N; p^{(2)}) \exp \left(\sum_{r=1}^N \sum_{m=1}^{\infty} t_{m(N+1)-r} J_{-m(N+1)+r} \right) | 0 \rangle \\ &\quad \times \exp \left(- \sum_{m=1}^{\infty} (-1)^{m(N+1)} p_{mN}^{(2)} t_{m(N+1)} \right). \end{aligned}$$

This implies that $\mathcal{T}(\lambda, N, 0, p^{(2)}, \mathbf{t})$ satisfies (5.16). \square

Proof of Theorem 2. By (5.13), $\mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t})$ can be factorized as

$$\begin{aligned} \mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t}) &= \exp(G(\lambda; 0, p^+, p^{(3)}; N)) \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)} \right) \\ &= \mathcal{T}(\lambda, N, 0, p^+, \mathbf{t}) \exp \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} p_m^{(1)} p_{m(N+1)}^{(3)} \right). \end{aligned}$$

This relation and (5.16) imply that $\mathcal{T}(\lambda, N, p^{(1)}, p^{(2)}, \mathbf{t})$ satisfies (1.23). \square

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