

AN OVERVIEW OF (κ, τ) -REGULAR SETS AND THEIR APPLICATIONS

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ABSTRACT. A (κ, τ) -regular set is a vertex subset S inducing a κ -regular subgraph such that every vertex out of S has τ neighbors in S . This article is an expository overview of the main results obtained for graphs with (κ, τ) -regular sets. The graphs with classical combinatorial structures, like perfect matchings, Hamilton cycles, efficient dominating sets, etc, are characterized by (κ, τ) -sets whose determination is equivalent to the determination of those classical combinatorial structures. The characterization of graphs with these combinatorial structures are presented. The determination of (κ, τ) -regular sets in a finite number of steps is deduced and the main spectral properties of these sets are described.

Key words: perfect matching; Hamilton cycle; efficient dominating set; maximum k -regular induced subgraph; graph spectra.

1. INTRODUCTION

The graphs G considered in this work are simple and undirected with *order* (number of vertices) n and *size* (number of edges) m . The vertex set of G is denoted by $V(G)$ and its edge set by $E(G)$. An edge with *end-vertices* i and j is denoted by ij . In such a case, we say that these vertices are *adjacent*. The *neighborhood* of a vertex $u \in V(G)$ is $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and the *degree* of the vertex u is $d_G(u) = |N_G(u)|$. The *maximum (minimum) degree* of the vertices of a graph G is $\Delta(G)$ ($\delta(G)$). Several other concepts and notation will be introduced throughout the text and for further basic notation and basic concepts the reader is referred to [4] and [17].

The concept of (κ, τ) -regular set for general graphs first appeared in [6] as a particular case of the concept of (X, Y) -set introduced in [26] as a vertex subset S of a graph G such that

$$|N_G(v) \cap S| \in \begin{cases} X, & \text{if } v \in S \\ Y, & \text{otherwise,} \end{cases}$$

where X and Y are subsets of $\{0, 1, \dots, n-1\}$. The vertex subset S is (κ, τ) -regular when the subsets X and Y are the singleton subsets $X = \{k\}$ and $Y = \{\tau\}$, that is, S is (κ, τ) -regular if $\forall v \in V(G)$

$$|N_G(v) \cap S| = \begin{cases} \kappa, & \text{if } v \in S \\ \tau, & \text{otherwise.} \end{cases}$$

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When a graph G is κ -regular, for convenience, its vertex set $V(G)$ is considered a $(\kappa, 0)$ -regular set. From now on, for any graph G , it is assumed that a (κ, τ) -regular set $S \subset V(G)$ (that is, S is strictly included in $V(G)$) is such that $\tau > 0$.

Example 1.1. The Petersen graph depicted in Figure 1.1 has several (k, τ) -regular sets. For instance, $S = \begin{cases} \{1, 2, 3, 4\}, & \text{is } (0, 2)\text{-regular;} \\ \{5, 6, 7, 8, 9, 10\}, & \text{is } (1, 3)\text{-regular;} \\ \{1, 2, 5, 7, 8\}, & \text{is } (2, 1)\text{-regular.} \end{cases}$

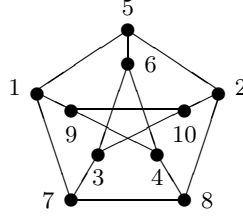


FIGURE 1.1. The Petersen graph.

The following properties are immediate.

- (1) If an arbitrary graph H has a (κ, τ) -regular set $S \subset V(H)$, then S is a $(|S| - \kappa - 1, |S| - \tau)$ -regular set for the complement graph of H .
- (2) If a p -regular graph G has a (κ, τ) -regular set $S \subset V(H)$, then $V(G) \setminus S$ is $(p - \tau, p - \kappa)$ -regular.

Now we recall that the *adjacency matrix* of a graph G of order n is the $n \times n$ matrix \mathbf{A}_G whose (i, j) -entry is equal to 1 whether $ij \in E(G)$ and is equal to 0 otherwise. This matrix \mathbf{A}_G is symmetric and then it has n real eigenvalues, herein indexed in non increasing order and denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$. These eigenvalues of \mathbf{A}_G are also called the *eigenvalues* of G . The *spectrum* of G is the multiset

$$\sigma(G) = \{\mu_1^{[m_1]}, \mu_2^{[m_2]}, \dots, \mu_q^{[m_q]}\},$$

where $\mu_1, \mu_2, \dots, \mu_q$ are the q distinct eigenvalues of G and m_j denotes the multiplicity of μ_j , for $j = 1, 2, \dots, q$. Therefore,

$$\underbrace{\lambda_1(G) = \dots = \lambda_{m_1}(G)}_{\mu_1} > \underbrace{\lambda_{m_1+1}(G) = \dots = \lambda_{m_1+m_2}(G)}_{\mu_2} > \dots > \underbrace{\lambda_{n-m_q+1}(G) = \dots = \lambda_n(G)}_{\mu_q}$$

and $\sum_{j=1}^q m_j = n$. The associated eigenspace of an eigenvalue λ of a graph G is denoted by $\mathcal{E}_G(\lambda)$.

The (κ, τ) -regular sets were first investigated in [25], in the context of regular graphs, where a subgraph of a graph G induced by a (κ, τ) -regular set was called an *eigengraph* of G . Furthermore, a necessary and sufficient condition for the existence of a (κ, τ) -regular set in a regular graph was deduced. Theorem 1.1 states a slightly different version deduced in [7] also for regular graphs. From now on, \hat{e} denotes the all-one vector with n components and the *characteristic vector* of a vertex subset S of a graph is $x(S) \in \{0, 1\}^{V(G)}$ such that its i -th component is equal to 1 if $i \in S$ and equal to 0 otherwise.

Theorem 1.1. [25, 7] *A p -regular graph G has a (κ, τ) -regular set, with $\kappa < p$, if and only if $\kappa - \tau$ is an eigenvalue of G and there exists $\mathbf{x} \in \{0, 1\}^{V(G)}$ such that $\mathbf{x} - \frac{\tau}{p+\tau-\kappa}\hat{\mathbf{e}} \in \mathcal{E}_G(\kappa - \tau)$. Furthermore, \mathbf{x} is the characteristic vector of a $(\kappa - \tau)$ -regular set.*

More generally, the next theorem states a necessary and sufficient condition for the existence of a (κ, τ) -regular set in an arbitrary graph. This version is a stronger variant than the one presented in [6].

Theorem 1.2. *A graph G has a (κ, τ) -regular set if and only if the linear system*

$$(1.1) \quad (\mathbf{A}_G - (\kappa - \tau)I_n)x = \tau\hat{\mathbf{e}},$$

where I_n is the identity matrix of order n , has a 0–1 solution. Furthermore, every 0–1 solution is the characteristic vector of a (κ, τ) -regular set.

Proof. Let $S \subset V(G)$ be a (κ, τ) -regular set. Then it is immediate that the characteristic vector of S , $x(S)$, is a solution of (1.1). Conversely, assuming that $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \{0, 1\}^{V(G)}$ is a solution of (1.1), its i -th equation, for $i = 1, \dots, n$, can be written as follows

$$\sum_{j \in N_G(i)} \mathbf{x}_j - (\kappa - \tau)\mathbf{x}_i = \tau \Leftrightarrow \sum_{j \in N_G(i): \mathbf{x}_j=1} \mathbf{x}_j = \begin{cases} \kappa, & \text{if } \mathbf{x}_i = 1 \\ \tau, & \text{otherwise.} \end{cases}$$

Therefore, the vertex set defined by \mathbf{x} as being its characteristic vector is (κ, τ) -regular. \square

Regarding the cardinality of (κ, τ) -regular sets, the next theorem extends the result obtained in [3] for $(0, \tau)$ -regular sets. The graphs with $(0, \tau)$ -regular sets were called τ -regular-stable graphs in [3].

Theorem 1.3. *Let G be graph with a (κ, τ) -regular set $S \subset V(G)$, such that $\delta(G) + \tau > \kappa$. Then*

$$(1.2) \quad \frac{n\tau}{\Delta(G) - (\kappa - \tau)} \leq |S| \leq \frac{n\tau}{\delta(G) - (\kappa - \tau)}.$$

Proof. Since $S \subset V(G)$, from the definition of a (κ, τ) -regular set, $\tau > 0$ and we obtain

$$|S|(\delta(G) - \kappa) \leq (n - |S|)\tau \leq |S|(\Delta(G) - \kappa).$$

Therefore, $|S|(\delta(G) - (\kappa - \tau)) \leq n\tau \leq |S|(\Delta(G) - (\kappa - \tau))$ and the inequalities (1.2) follow. \square

As immediate consequence of Theorem 1.3, if G is a p -regular graph with a (κ, τ) -regular set $S \subset V(G)$, then $|S| = \frac{n\tau}{p - (\kappa - \tau)}$.

2. CHARACTERIZATION OF GRAPHS WITH CLASSICAL COMBINATORIAL STRUCTURES

There are several classical combinatorial structures which can be characterized by (κ, τ) -regular sets as it is the case of perfect matchings, Hamiltonian cycles, efficient dominating sets and dominating induced matchings (also called efficient edge dominating sets). Furthermore, there are graphs with particular combinatorial structure that can be characterized by using (κ, τ) -regular sets, as it is the case of strongly regular graphs.

The next theorem which appear in [3] (see also [5]) states a necessary and sufficient condition for the existence of perfect matchings in graphs.

Theorem 2.1. [3] *A graph $G \neq K_2$ has a perfect matching if and only if its line graph $L(G)$ has a $(0, 2)$ -regular set.*

It follows a necessary and sufficient condition for Hamiltonian graphs published in [1] (for the reader convenience the proof is also presented).

Theorem 2.2. [1] *A graph G is Hamiltonian if and only if its line graph has a $(2, 4)$ -regular set inducing a connected subgraph.*

Proof. Let C be a Hamilton cycle of a graph G and let $L(C)$ be the vertex subset of the line graph $L(G)$ corresponding to the edges in C . Then it is immediate that $L(C)$ is a $(2, 4)$ -regular set of $L(G)$ inducing a connected subgraph. Conversely, assume that the line graph $L(G)$ of a graph G has a $(2, 4)$ -regular set S inducing a connected subgraph. The edges of G corresponding to S form just one cycle C in G . Furthermore, since each vertex not in S has 4 neighbors in S , the corresponding edges in G have both end-vertices in the cycle C . Therefore, C is Hamiltonian. \square

Example 2.1. The Figure 2.1 depicts a Hamiltonian graph H and its line graph $L(H)$. The graph H has a Hamiltonian cycle define by the edge set $C = \{a, b, c, d, e, f, g\}$ and this edge set corresponds in $L(H)$ to a $(2, 4)$ -regular set inducing a connected subgraph.



FIGURE 2.1. A Hamiltonian graph H and its line graph $L(H)$ with the $(2, 4)$ -regular set $S = \{a, b, c, d, e, f, g\}$.

In [23] it was proved that a graph G is Hamiltonian if and only the subdivision of G (that is, a graph obtained from G after inserting a vertex in the middle of each edge) has a $(2, 2)$ -regular set inducing a connected subgraph.

Before to proceed, it is worth recall some domination concepts. Given a graph G and a vertex $v \in V(G)$, v *dominates* itself and all its neighbors. A vertex set $S \subset V(G)$ is *dominating* if every vertex of G is dominated by at least one vertex of S . The *domination number* of a graph G , $\gamma(G)$, is the cardinality of a dominating set in G with minimum cardinality. A *dominating set* S is *efficient dominating* (also known as *independent perfect*

dominating set) if each vertex of G is dominated by precisely one vertex of S . Not every graph has an efficient dominating set (for example, C_4 has no efficient dominating sets). The problem of determining an efficient dominating set in a graph (if there exists) is called the *efficient dominating set problem*. This problem is known to be \mathcal{NP} -complete for general graphs [2]. A closely related problem is that of determining if G has an *efficient edge dominating set*, that is, a set E of edges such that every edge of G shares a vertex with precisely one edge in E (assuming that an edge shares a vertex with itself). This edge set is also known as a *dominating induced matching problem* and the problem of determining such edge set is also \mathcal{NP} -complete [19]. An instance of a dominating induced matching can be transformed into an instance of an efficient dominating set by associating to the input graph G its line graph $L(G)$.

Theorem 2.3. *A vertex subset S of a graph G is an efficient dominating set if and only if S is $(0, 1)$ -regular.*

Proof. Taking into account the definitions of a (κ, τ) -regular set and efficient dominating set, the result follows. \square

The (κ, τ) -regular sets are also related with determination of vertex subsets of maximum cardinality inducing a κ -regular subgraph, as it is highlighted by the next theorem.

Theorem 2.4. [9] *Let G be a graph of order n and let $\tau = -\lambda_n(G)$. If $S \subset V(G)$ is $(\kappa, \kappa + \tau)$ -regular, then S is a maximum cardinality vertex subset of G inducing a κ -regular subgraph.*

Example 2.2. Consider the graph G depicted in Figure 2.2 for which $\sigma(G) = \{-2^{[2]}, 0^{[3]}, 4^{[1]}\}$ and the vertex subset $S = \{1, 3, 4, 6\}$ is $(2, 4)$ -regular. Therefore, applying Theorem 2.4, we may conclude that S is a maximum cardinality vertex subset inducing a 2-regular subgraph.

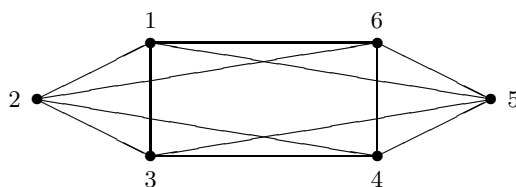


FIGURE 2.2. Graph G with the $(2, 4)$ -regular set $S = \{1, 3, 4, 6\}$ and $\sigma(G) = \{-2^{[2]}, 0^{[3]}, 4^{[1]}\}$.

A *strongly regular graph* with parameters (n, p, a, c) is a p -regular graph of order n , where each pair of vertices have a common neighbors if they are adjacent and c common neighbors otherwise. For instance, the Petersen graph depicted in Figure 1.1 is a strongly regular graph with parameters $(10, 3, 0, 1)$ and the graph depicted in Figure 2.2 is a strongly regular graph with parameters $(6, 4, 2, 4)$. A strongly regular graph G is *primitive* if G and the complement graph of G are both connected; otherwise it is called *imprimitive*. A strongly regular graph with parameters (n, p, a, c) is imprimitive if and only if $c = p$ or $c = 0$ (see [18, p. 178]). The graph depicted in

Figure 2.2 is an example of an imprimitive strongly regular graph.

As was noticed in [8], if S is a maximum stable set of a primitive strongly regular graph G with parameters (n, p, a, c) , then for all k in S

$$|S| = \sum_{j \in N_G(k)} \frac{|N_G(j) \cap S|}{c} - \frac{p-c}{c}.$$

Furthermore, if S is $(0, \tau)$ -regular, then

$$|S| = \frac{p(\tau - 1) + c}{c}.$$

The next theorem gives a necessary and sufficient condition for a regular graph be strongly regular, based on the existence of (κ, τ) -regular sets.

Theorem 2.5. [11] *A p -regular graph G is strongly regular with parameters (n, p, a, c) if and only if for every vertex $v \in V(G)$, the vertex subset $S = N_G(v)$ is (a, c) -regular in $G - v$ (the graph obtained from G after the deletion of the vertex v).*

Example 2.3. Consider the strongly regular graph G with parameters $(10, 3, 0, 1)$ depicted in Figure 1.1. Deleting the vertex 1 (for instance), the vertex subset $N_G(1) = \{5, 7, 9\}$ is $(0, 1)$ -regular in the graph $G - 1$. A similar $(0, 1)$ -regular set is determined deleting any other vertex of G .

3. DETERMINATION OF (κ, τ) -REGULAR SETS

As a consequence of the results obtained in [20], in general, the recognition of graphs with a (κ, τ) -regular set is \mathcal{NP} -complete. However, there are families of graphs for which such recognition and the determination of (κ, τ) -regular sets can be done in polynomial time, in particular for the graphs whose maximum multiplicity of the eigenvalues is small. This section is devoted to the results and algorithmic techniques developed for the determination of (κ, τ) -regular sets.

The next theorem is a variant of a theorem which appears in [12].

Theorem 3.1. *Let G be a graph with at least one edge and let $\bar{\mathbf{x}}$ be a particular solution of the linear system (1.1). The graph G has a (κ, τ) -regular set if and only if there is a $0 - 1$ vector \mathbf{x} such that*

$$(3.1) \quad \mathbf{x} = \bar{\mathbf{x}} + \hat{\mathbf{u}},$$

where $\hat{\mathbf{u}} = \mathbf{0}$ if $\kappa - \tau$ is not an eigenvalue of G and $\hat{\mathbf{u}} \in \mathcal{E}_G(\kappa - \tau)$ otherwise. Furthermore, every $0 - 1$ solution \mathbf{x} in (3.1) is the characteristic vector of a (κ, τ) -regular set $S \subset V(G)$.

Proof. It is immediate that every solution of the linear system (1.1) can be obtained from the equation (3.1). Therefore, applying Theorem 1.2, the result follows. \square

Corollary 3.2. [12] *If a graph G has a (κ, τ) -regular set $S \subseteq V(G)$ and $\bar{\mathbf{x}}$ is a particular solution of the linear system (1.1), then $|S| = \hat{\mathbf{e}}^T \bar{\mathbf{x}}$.*

Algorithm 1 for the recognition whether a graph G has a (κ, τ) regular set and its determination when it exists.

Require: \mathbf{A}_G , κ , τ , t and the $n \times t$ matrix U whose columns are linear independent vectors of $\mathcal{E}_G(\kappa - \tau)$ when $t > 0$.

Ensure: a (κ, τ) -regular set S or the conclusion that it does not exists.

- 1: **COMPUTE** a particular solution $\bar{\mathbf{x}}$ of the linear system (1.1).
 - 2: **SET** $\mathbf{x} := \bar{\mathbf{x}}$.
 - 3: **IF** $t = 0$, **THEN GOTO** Step 11
 - 4: **DETERMINE** the matrix V , with columns $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_t$, obtained from U as in Theorem 3.3 and the index subset $I = \{i_1, \dots, i_t\} \subset \{1, \dots, n\}$.
 - 5: **SET** $\Lambda := \{(\delta_{i_1}, \dots, \delta_{i_t}) : \delta_{i_j} \in \{-\bar{\mathbf{x}}_{i_j}, 1 - \bar{\mathbf{x}}_{i_j}\}, i_j \in I\}$.
 - 6: **WHILE** \mathbf{x} is not 0-1 and $\Lambda \neq \emptyset$
 - 7: **CHOOSE** $(\delta_{i_1}, \dots, \delta_{i_t}) \in \Lambda$ and **SET** $\Lambda := \Lambda \setminus \{(\delta_{i_1}, \dots, \delta_{i_t})\}$.
 - 8: **IF** $\bar{\mathbf{x}} + \sum_{j=1}^t \delta_{i_j} \hat{\mathbf{v}}_j$ is a 0-1 vector **THEN SET** $\mathbf{x} := \bar{\mathbf{x}} + \sum_{j=1}^t \delta_{i_j} \hat{\mathbf{v}}_j$.
 - 9: **END IF**
 - 10: **END WHILE**
 - 11: **IF** \mathbf{x} is 0-1 **THEN** return \mathbf{x} as the characteristic vector of S .
 - 12: **ELSE** G has no (κ, τ) -regular sets.
 - 13: **END IF**
-

Example 3.1. Let us apply Algorithm 1 repeatedly (updating Λ in each run of step 5, removing the t-tuples already determined which are 0-1 solutions) to the determination of all $(0, 2)$ -regular sets S_1 and all $(1, 3)$ -regular sets S_2 in the graph G depicted in Figure 1.1. Note that the adjacency matrix \mathbf{A}_G has three distinct eigenvalues: 3 with multiplicity 1, 1 with multiplicity 5 and -2 with multiplicity 4. Since $0 - 2 = 1 - 3 = -2$, in both cases we can consider the matrix

$$U^T = \begin{pmatrix} -1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ -3 & -1 & -1 & -1 & 2 & 0 & 2 & 0 & 2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & -1 & -1 & -2 & 2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the rows of U^T are linear independent eigenvectors belonging to $\mathcal{E}_G(\kappa - \tau)$. The matrix V obtained from the matrix U in step 4 can take the form

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & -2/3 & 1/3 \end{pmatrix},$$

and then $I = \{1, 2, 3, 4\}$. For each case, consider a particular solution $\bar{\mathbf{x}}$ of the linear system (1.1).

- (1) For $(\kappa, \tau) = (0, 2)$, using the particular solution of (1.1) $\bar{\mathbf{x}} = \frac{2}{5}\hat{e}$, by Corollary 3.2 $|S_1| = 4$ and by Theorem 3.3 $\delta_1, \delta_2, \delta_3, \delta_4 \in \{-\frac{2}{5}, \frac{3}{5}\}$. The 0 – 1 solutions \mathbf{x} in (3.2) (and consequently the characteristic vectors of the $(0, 2)$ -regular sets) are obtained for each tuple $(\delta_1, \delta_2, \delta_3, \delta_4)$ of the table (in each row, $\delta_1, \dots, \delta_4$ appear in the first 4 entries and the corresponding vertices of S_1 appear in the last 4 entries):

δ_1	δ_2	δ_3	δ_4				
$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$	4	5	7	10
$-\frac{2}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$	$-\frac{2}{5}$	3	5	8	9
$-\frac{2}{5}$	$\frac{3}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$	2	6	7	9
$\frac{3}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{2}{5}$	1	6	8	10
$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	1	2	3	4

- (2) For $(\kappa, \tau) = (1, 3)$, using the particular solution of (1.1) $\bar{\mathbf{x}}^T = (3/2, 3/2, 3/2, 3/2, 0, 0, 0, 0, 0, 0)$, by Corollary 3.2 $|S_2| = 6$ and by Theorem 3.3 $\delta_1, \delta_2, \delta_3, \delta_4 \in \{-\frac{3}{2}, -\frac{1}{2}\}$. The 0 – 1 solutions \mathbf{x} in (3.2) (and consequently the characteristic vectors of the $(1, 3)$ -regular sets) are obtained for the tuples $(\delta_1, \delta_2, \delta_3, \delta_4)$ of the table:

δ_1	δ_2	δ_3	δ_4						
$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	5	6	7	8	9	10
$-\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	2	3	4	5	7	9
$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	3	4	5	8	10
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	1	2	3	6	8	9
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	2	4	6	7	10

Note that in both cases, among the $2^4 = 16$ possible tuples $(\delta_1, \delta_2, \delta_3, \delta_4)$, we found 5 tuples producing 0 – 1 solutions.

Note that in Theorem 3.3, despite the set of possible tuples $(\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_t})$ is finite, its cardinality is the exponential number 2^t . Therefore, when the multiplicity t of the eigenvalue $\kappa - \tau$ is larger, the determination of a tuple of scalars $(\delta_{i_1}, \delta_{i_2}, \dots, \delta_{i_t})$ producing a 0 – 1 solution in (3.2) or the recognition that there is no such solution, can not be computationally effective by checking all possible tuples. Considering the inequality (see, for instance, [17])

$$\max_{\mu \in \sigma(G) \setminus \{0\}} m_G(\mu) \leq n - \gamma(G),$$

where $m_G(\mu)$ is the multiplicity of the eigenvalue μ of the graph G , it follows that the graphs with higher domination number have smaller upper bound for the maximum multiplicity of the nonzero eigenvalues of G . Thus, for those graphs the determination of (κ, τ) -regular sets with $\kappa - \tau \neq 0$ or the recognition that none of them there exist is computationally effective. Anyway, with the same purpose, the development of computationally effective techniques for particular graph families remains an open problem and it is an interesting research line.

4. SPECTRAL PROPERTIES

The presence of (κ, τ) -regular sets in graphs has deep influence on their spectrum. From Theorem 1.1 it is immediate that a regular graph with a (κ, τ) -regular set has $\kappa - \tau$ as an eigenvalue. However, in the case of non-regular graphs, the presence of a (κ, τ) -regular set does not imply that

$\kappa - \tau$ is an eigenvalue. For instance, the graph G depicted in Figure 4.1 has the $(1, 1)$ -regular set $\{b, e\}$ but 0 is not an eigenvalue of G . The graph G

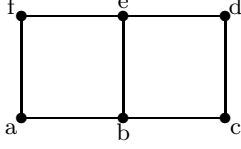


FIGURE 4.1. A graph G with a $(1, 1)$ -regular set and spectrum $\sigma(G) = \{-1 - \sqrt{2}, -1, 1 - \sqrt{2}, -1 + \sqrt{2}, 1, 1 + \sqrt{2}\}$

of Figure 4.1 has also the $(2, 1)$ -regular sets $\{a, b, e, f\}$ and $\{b, c, d, e\}$, the $(1, 2)$ -regular set $\{a, c, d, f\}$ and the $(0, 1)$ -regular sets $\{a, d\}$ and $\{c, f\}$.

The next theorem establishes a sufficient condition for arbitrary graphs with (κ, τ) -regular sets to have $\kappa - \tau$ as an eigenvalue.

Theorem 4.1. [10] *Let λ be an integer and let G be a graph with a (κ_1, τ_1) -regular set S_1 , with $\tau_1 > 0$, and a (κ_2, τ_2) -regular set S_2 , such that $S_1 \neq S_2$ and $\kappa_1 - \tau_1 = \kappa_2 - \tau_2 = \lambda$. Then $\lambda \in \sigma(G)$ and $\hat{\mathbf{u}} = \frac{\tau_2}{\tau_1} \mathbf{x} - \mathbf{y} \in \mathcal{E}_G(\lambda)$, where \mathbf{x} is the characteristic vector of S_1 and \mathbf{y} is the characteristic vector of S_2 .*

Example 4.1. Consider the graph G depicted in Figure 4.1. Since the vertex subsets $S_1 = \{a, b, e, f\}$ and $S_2 = \{b, c, d, e\}$ are both $(2, 1)$ -regular and the vertex subsets $T_1 = \{a, d\}$ and $T_2 = \{c, f\}$ are both $(0, 1)$ -regular, applying Theorem 4.1, it follows that 1 and -1 are both eigenvalues of G . Furthermore,

$$\begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{E}_G(1) \quad \text{and} \quad \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{E}_G(-1).$$

Now, it is worth mention the concepts of main and non-main eigenvalues. An eigenvalue μ of a graph G , which has an associated eigenspace $\mathcal{E}_G(\mu)$ not orthogonal to the all-one vector $\hat{\mathbf{e}}$, is said to be *main*. When $\mathcal{E}_G(\mu)$ is orthogonal to $\hat{\mathbf{e}}$ the eigenvalue μ is referred as *non-main*. The concept of main (non-main) eigenvalue was introduced in 1970 by Cvetković [14] (see also [15, 16]) and further investigated in many papers since then. For every graph G , its largest eigenvalue $\lambda_1(G)$ is a main eigenvalue. In particular, it is well known that a graph is regular if and only if it has only one main eigenvalue (see [22], where a survey on main (non-main) eigenvalues was published).

Regarding graphs with just two main eigenvalues we may use (κ, τ) -regular sets for the determination of particular families using a graph operation introduced in [11] and herein described as follows:

Consider the graph operation $H = G_1 \oplus_s^\tau G_2$, where G_1 is a κ_1 -regular graph, G_2 is a κ_2 -regular graph and H is obtained from G_1 and G_2 by

connecting each vertex of $V(G_1) = \{x_1, \dots, x_{n_1}\}$ to $\tau > 0$ vertices in $V(G_2) = \{y_1, \dots, y_{n_2}\}$, such that $B_i = N_H(x_i) \cap V(G_2), i = 1, \dots, n_1$, is a $1 - (n_2, \tau, s)$ combinatorial design [24] (that is, $\bigcup_{i=1}^{n_1} B_i = V(G_2)$, $|B_i| = \tau$, for $i = 1, \dots, n_1$ and each vertex $v \in V(G_2)$ belongs to exactly s blocks B_i). Therefore, the vertex subsets $V(G_1)$ and $V(G_2)$ are (κ_1, s) -regular and (κ_2, τ) -regular, respectively.

Theorem 4.2. [11] *Considering a κ_1 -regular graph G_1 and a κ_2 -regular graph G_2 , let $H = G_1 \oplus_s^T G_2$ be the graph obtained as above described. If μ is a main eigenvalue of H , then*

$$(4.1) \quad \mu = \frac{\kappa_1 + \kappa_2 \pm \sqrt{(\kappa_2 - \kappa_1)^2 + 4s\tau}}{2}.$$

The particular case of unicyclic graphs with just two main eigenvalues was investigated in [21].

Theorem 4.3. [21] *The graphs attained from a cycle C_n by attaching $s > 0$ pendent vertices are all unicyclic graphs with exactly two main eigenvalues.*

Note that any unicyclic graph H as referred in Theorem 4.3 is such that $H = snK_1 \oplus_s^1 C_n$ and by Theorem 4.2 its main eigenvalues are

$$\mu = 1 \pm \sqrt{1 + s}.$$

Since the largest eigenvalue of a graph is main, it follows that $\mu = 1 + \sqrt{1 + s}$ is the largest eigenvalue of H .

Theorem 4.4. [11] *Let S be a (κ, τ) -regular set of a graph G , with characteristic vector $\mathbf{x}(S)$ and let μ be an eigenvalue of G . Then μ is non-main if and only if*

$$\mu = \kappa - \tau \quad \text{or} \quad \mathbf{x}(S) \in (\mathcal{E}_G(\mu))^\perp,$$

where $(\mathcal{E}_G(\mu))^\perp$ denotes the vector space orthogonal to $\mathcal{E}_G(\mu)$.

From the above theorem we have the following corollary.

Corollary 4.5. [11] *Let $\kappa, \tau \in \mathbb{Z}^+ \cup \{0\}$, with $\tau > 0$, where \mathbb{Z}^+ denotes the set of positive integers. If $\mu = \kappa - \tau$ is a main eigenvalue of G , then G does not have a (κ, τ) -regular set.*

Now we present the following theorem.

Theorem 4.6. *Let G be a graph with a (κ, τ) -regular set $S \subset V(G)$ and let $\bar{\mathbf{x}}$ be a particular solution of (1.1). If μ is a main eigenvalue of G and $\hat{\mathbf{u}} \in \mathcal{E}_G(\mu)$ is not orthogonal to $\hat{\mathbf{e}}$, then*

$$(4.2) \quad \hat{\mathbf{u}}^T \bar{\mathbf{x}} \neq 0 \quad \text{and} \quad \mu = \tau \frac{\hat{\mathbf{u}}^T \hat{\mathbf{e}}}{\hat{\mathbf{u}}^T \bar{\mathbf{x}}} + (\kappa - \tau).$$

Proof. Since μ is a main eigenvalue of G , by Theorem 4.4, $\mu \neq \kappa - \tau$. Taking into account that $\bar{\mathbf{x}}$ is a particular solution of (1.1), multiplying both sides of (1.1) on the left by $\hat{\mathbf{u}}^T$, we obtain

$$(4.3) \quad (\mu - (\kappa - \tau)) \hat{\mathbf{u}}^T \bar{\mathbf{x}} = \tau \hat{\mathbf{u}}^T \hat{\mathbf{e}}.$$

Therefore, $\hat{\mathbf{u}}^T \bar{\mathbf{x}} \neq 0$ and from (4.3) it follows $\mu = \tau \frac{\hat{\mathbf{u}}^T \hat{\mathbf{e}}}{\hat{\mathbf{u}}^T \bar{\mathbf{x}}} + (\kappa - \tau)$. \square

A result not much different from (4.2) but more restrictive appear in [11].

Considering a graph G , a *star set* for an eigenvalue $\mu \in \sigma(G)$, with multiplicity $m_G(\mu)$, is a vertex subset X such that $|X| = m_G(\mu)$ and the graph $G - X$ does not have μ as an eigenvalue. The vertex complement subset $V(G) \setminus X$ is called a *co-star set* and the graph $G - X$ is called a *star complement* for μ . The main properties of star sets and star complements appear in [17].

Theorem 4.7. [1] *Let G be a graph and $X \subset V(G)$ a star (or co-star) set for the eigenvalue $\mu \in \sigma(G)$. If X or $V(G) \setminus X$ is (κ, τ) -regular in G , then μ is non-main if and only if $\mu = \kappa - \tau$.*

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