

Discrete Uncertainty Principle in Quaternion Setting and Application in Signal Reconstruction

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Abstract. In this paper, the uncertainty principle of discrete signals associated with Quaternion Fourier transform is investigated. It suggests how sparsity helps in the recovery of missing frequency.

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1. Introduction

The classical uncertainty principle (the continuous-time uncertainty principle) says that if a function $f(t)$ is essentially zero outside an interval of length Δ_t and its Fourier transform $\hat{f}(\omega)$ defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\omega} dt$$

is essentially zero outside an interval of length Δ_ω , then

$$\Delta_t \Delta_\omega \geq 1.$$

In mathematics, that means a function and its Fourier transform cannot both be higher concentrated. Uncertainty principle was first introduced by Werner Heisenberg in quantum mechanics [8], which plays an important role in physics and engineering over the past century. Recently, uncertainty principle was applied to signal processing by Donoho and Stark [6], Candes and Tao [5], Tropp [15], and Bandeira, Lewis, and Mixon [1]. In [6], the authors first gave the uncertainty principles of discrete 1D signals. It states that: if

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$\{x_t\}_{t=0}^{N-1}$ is a sequence of length N and $\{\hat{x}_\omega\}_{\omega=0}^{N-1}$ is the sequence of its discrete Fourier transform, which is defined by

$$\hat{x}_\omega := \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-2\pi i \frac{\omega t}{N}}, \quad \text{for } \omega = 0, 1, \dots, N-1.$$

Then

$$N_t N_\omega \geq N, \quad (1.1)$$

where $\{x_t\}$ is nonzero at N_t points and $\{\hat{x}_\omega\}$ is nonzero at N_ω points. In [5], the uniform uncertainty principle was obtained and which played an crucial role in compressed sensing. The discrete uncertainty principles with applications on sparse signal processing was investigated in [1]. To the authors' knowledge, the higher dimensional investigation was first considered in [10], inspired by Donoho and Startk's uncertainty principle [6], the authors [10] study the uncertainty principle and signal recovery for continuous quaternion-valued signals.

The quaternion Fourier transform (QFT) plays a vital role in the representation of 2D signals. It is an extension of Fourier transform (FT) to the quaternion algebra, which was first proposed by Ell [7]. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex FT. The QFT has wide range of application, such as color image analysis [4, 13], color image digital watermarking scheme [3], image pre-processing and neural computing techniques for speech recognition [2], envelope [11] and edge detectors [9] of color images.

In this paper, we study a novel discrete uncertainty principle associated with the QFT and discuss its application to signal recovery. The main contributions of this paper are summarized as follows.

1. The discrete case of uncertainty principle associated with Quaternion Fourier transform is established to give the relationship between the nonzero numbers of the discrete quaternion-valued signals and their QFTs.
2. The discrete uncertainty principle suggests how sparsity helps in the recovery of missing frequencies.

The article is organized as follows. The Quaternion algebra and Quaternion Fourier transform are reviewed in Section 2. The uncertainty principle of discrete 2D signals is obtained for two-sided discrete Quaternion Fourier transform in Section 3. In Section 4, the discussion for application of uncertainty principles in spare signal recovery is investigated.

2. Preliminaries

The quaternion algebra \mathcal{H} was first invented by W. R. Hamilton in 1843 for extending complex numbers to a 4D algebra [12]. A quaternion $q \in \mathcal{H}$ can be

written in this form

$$q = q_0 + \underline{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3, \quad q_k \in \mathbf{R}, \quad k = 0, 1, 2, 3,$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy Hamilton's multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$

$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Applying the Hamilton's multiplication rules, the multiplication of two quaternions $p = p_0 + \underline{p}$ and $q = q_0 + \underline{q}$ can be expressed by

$$pq = p_0q_0 + \underline{p} \cdot \underline{q} + p_0\underline{q} + q_0\underline{p} + \underline{p} \times \underline{q},$$

where

$$\underline{p} \cdot \underline{q} := -(p_1q_1 + p_2q_2 + p_3q_3)$$

and

$$\underline{p} \times \underline{q} := \mathbf{i}(p_3q_2 - p_2q_3) + \mathbf{j}(p_1q_3 - p_3q_1) + \mathbf{k}(p_2q_1 - p_1q_2).$$

We define the conjugation of $q \in \mathcal{H}$ by $\bar{q} := q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$. Clearly, $q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. The modulus of a quaternion q is defined by

$$|q| := \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

In this paper, we study the quaternion-valued signal $f : \mathbf{R}^2 \rightarrow \mathcal{H}$ which can be expressed by

$$f(t, s) = f_0(t, s) + \mathbf{i}f_1(t, s) + \mathbf{j}f_2(t, s) + \mathbf{k}f_3(t, s),$$

where $f_k, (k = 0, 1, 2, 3)$ are real-valued functions.

In 1997, Sangwine [14] defined the fundamental idea of a discrete Quaternion Fourier transform (DQFT) and inversion discrete Quaternion Fourier transform (IDQFT) of Quaternion-valued signals, which we recall next.

Definition 2.1. (DQFT and IDQFT) Let $\{f(t, s)\}$ ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$) be a sequence of length MN and $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$) be its two-sided discrete Quaternion Fourier transform (DQFT), which is defined by

$$\hat{f}(u, v) := \frac{1}{\sqrt{MN}} \sum_{t=0}^{M-1} \sum_{s=0}^{N-1} e^{-2\pi i \frac{ut}{M}} f(t, s) e^{-2\pi j \frac{vs}{N}}. \quad (2.1)$$

Moreover, the inverse discrete Quaternion Fourier transform (IDQFT) of $\{\hat{f}(u, v)\}$ is defined by

$$f(t, s) := \frac{1}{\sqrt{MN}} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} e^{2\pi i \frac{ut}{M}} \hat{f}(u, v) e^{2\pi j \frac{vs}{N}}. \quad (2.2)$$

As a consequence of Definition 2.1, formula (2.1) can be represented in the matrix form. Denote two $M \times N$ matrices

$$\begin{aligned} A &:= \begin{pmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & \vdots & \vdots \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{pmatrix} \\ &= (f(t,s)) \in \mathcal{H}^{M \times N} \end{aligned}$$

and

$$\begin{aligned} \hat{A} &:= \begin{pmatrix} \hat{f}(0,0) & \hat{f}(0,1) & \cdots & \hat{f}(0,N-1) \\ \hat{f}(1,0) & \hat{f}(1,1) & \cdots & \hat{f}(1,N-1) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}(M-1,0) & \hat{f}(M-1,1) & \cdots & \hat{f}(M-1,N-1) \end{pmatrix} \\ &= (\hat{f}(u,v)) \in \mathcal{H}^{M \times N}, \end{aligned}$$

then formula (2.1) can be expressed as

$$\hat{A} = V_{\mathbf{i}} A V_{\mathbf{j}},$$

where $V_{\mathbf{i}}$ and $V_{\mathbf{j}}$ are $M \times M$ and $N \times N$ Vandermonde matrices, which are defined by

$$V_{\mathbf{i}} := \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi\mathbf{i}\frac{1}{M}} & \cdots & e^{-2\pi\mathbf{i}\frac{(M-1)}{M}} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-2\pi\mathbf{i}\frac{(M-1)}{M}} & \cdots & e^{-2\pi\mathbf{i}\frac{(M-1)^2}{M}} \end{pmatrix}$$

and

$$V_{\mathbf{j}} := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi\mathbf{j}\frac{1}{N}} & \cdots & e^{-2\pi\mathbf{j}\frac{(N-1)}{N}} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-2\pi\mathbf{j}\frac{(N-1)}{N}} & \cdots & e^{-2\pi\mathbf{j}\frac{(N-1)^2}{N}} \end{pmatrix},$$

respectively. Clearly, they are non-singular matrices.

Similarly, formula (2.2) can be expressed as

$$A = V_{\mathbf{i}}^{-1} \hat{A} V_{\mathbf{j}}^{-1} = V_{-\mathbf{i}} \hat{A} V_{-\mathbf{j}}. \quad (2.3)$$

3. The Discrete Uncertainty Principle in Quaternion Setting

Uncertainty principle has a significant role in both science and engineering for most of the past century. In this section, we show that the discrete uncertainty principle of quaternion-valued signals.

Theorem 3.1 (Main Theorem). Let $N_{(t,s)}$ and $N_{(u,v)}$ be the numbers of nonzero elements of sequences $\{f(t,s)\}$ ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$) and $\{\hat{f}(u,v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$), respectively. Then we have

$$N_{(t,s)} \cdot N_{(u,v)} \geq MN. \quad (3.1)$$

By the arithmetic mean-Geometric mean inequality, which we describe next.

Corollary 3.2.

$$N_{(t,s)} + N_{(u,v)} \geq 2\sqrt{MN}.$$

In particular, when $M = N$, we have

Corollary 3.3.

$$N_{(t,s)} \cdot N_{(u,v)} \geq N^2. \quad (3.2)$$

Corollary 3.4 (1D discrete uncertainty principle). In Theorem 3.1, when $M = 1$ or $N = 1$, one has the classical discrete uncertainty principle (1.1) in [6].

Remark 3.5. Theorem 3.1 gives a lower bound on the value of the time-bandwidth product. Corollary 3.2 gives a lower bound of the sum of nonzero elements in both time and frequency spaces. Furthermore, it is easy to construct examples on the equality of (3.1) in Theorem 3.1. In this sense, the discrete-time principle is sharp.

Example (A trivial case). Let $t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$ and

$$f(t,s) = \begin{cases} 1, & \text{for } t = 0, s = 0, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence of (2.1), we have

$$\hat{f}(u,v) = \frac{1}{\sqrt{MN}} f(0,0) = \frac{1}{\sqrt{MN}},$$

when $u = 0, 1, \dots, M-1$ and $v = 0, 1, \dots, N-1$. Therefore, $N_{(t,s)} = 1$, $N_{(u,v)} = MN$, or equivalently $N_{(t,s)} \cdot N_{(u,v)} = MN$ as desired.

The next result is a non-trivial example.

Example. Suppose that $M = N$ admits the factorization $N = k \cdot l$ and

$$A = (f(t,s)) = \begin{pmatrix} E_k & E_k & \cdots & E_k \\ E_k & E_k & \cdots & E_k \\ \vdots & \vdots & \vdots & \vdots \\ E_k & E_k & \cdots & E_k \end{pmatrix}, \quad (3.3)$$

where

$$E_k := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

is a $k \times k$ matrix with entries are determined by $e_{st} = 1$ if $s = t = 1$, otherwise 0. We can prove that

$$\hat{A} = \left(\hat{f}(u, v) \right) = \frac{l}{k} \begin{pmatrix} E_l & E_l & \cdots & E_l \\ E_l & E_l & \cdots & E_l \\ \vdots & \vdots & \vdots & \vdots \\ E_l & E_l & \cdots & E_l \end{pmatrix}$$

is a $N \times N$ element block matrix with $k \times k$ sub-Matrices E_l . Here, $E_l = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ with entries are determined by $e_{u,v} := 1$ if $u = v = 1$, otherwise is 0. Equivalently, we have $N_{(t,s)} = l^2$ and $N_{(u,v)} = k^2$. Therefore,

$$N_{(t,s)} \cdot N_{(u,v)} = l^2 \cdot k^2 = N^2.$$

Proof. Let $p, q = 0, 1, \dots, l-1$. As a consequence of (3.3), we have

$$f(t, s) = \begin{cases} 1, & \text{for } t = kp, s = kq, \\ 0, & \text{otherwise.} \end{cases} \quad (3.4)$$

That is

$$\begin{aligned} A &= \begin{pmatrix} f(0,0) & f(0,k) & \cdots & f(0,(l-1)k) \\ f(k,0) & f(k,k) & \cdots & f(k,(l-1)k) \\ \vdots & \vdots & \vdots & \vdots \\ f((l-1)k,0) & f((l-1)k,k) & \cdots & f((l-1)k,(l-1)k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \end{aligned} \quad (3.5)$$

Then we have to prove that

$$\hat{f}(u, v) = \begin{cases} \frac{l}{k}, & \text{for } u = al, v = bl, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b = 0, 1, \dots, k-1$.

In fact, for $a, b = 0, 1, \dots, k-1$, $N = l \cdot k$ and as a consequence of (2.1), (3.4) and (3.5), we have

$$\begin{aligned}
\hat{f}(al, bl) &= \frac{1}{N} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} e^{-2\pi i \frac{ta}{N}} f(t, s) e^{-2\pi j \frac{sb}{N}} \\
&= \frac{1}{kl} \sum_{t=0}^{kl-1} \sum_{s=0}^{kl-1} e^{-2\pi i \frac{ta}{k}} f(t, s) e^{-2\pi j \frac{sb}{k}} \\
&= \frac{1}{kl} \sum_{p=0}^{l-1} \sum_{s=0}^{l-1} e^{-2\pi i \frac{kp a}{k}} e^{-2\pi j \frac{kqb}{k}}. \tag{3.6}
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\hat{f}(al, bl) &= \frac{1}{kl} \sum_{p=0}^{l-1} \sum_{s=0}^{l-1} e^{-2\pi i p a} e^{-2\pi j q b} \\
&= \frac{1}{kl} \sum_{p=0}^{l-1} \sum_{s=0}^{l-1} 1 = \frac{l}{k}.
\end{aligned}$$

When $u \neq al, v \neq bl$, for $a, b = 0, 1, \dots, k-1$, we have

$$\hat{f}(u, v) = \frac{1}{kl} \sum_{t=0}^{kl-1} \sum_{s=0}^{kl-1} e^{-2\pi i \frac{ut}{kl}} f(t, s) e^{-2\pi j \frac{vs}{kl}}.$$

As a consequence of (3.4) and (3.5), we obtain

$$\begin{aligned}
&\hat{f}(u, v) \\
&= f(0, 0) + f(0, k) e^{-2\pi j \frac{kv}{kl}} + \dots + f(0, (l-1)k) e^{-2\pi j \frac{kv(l-1)}{kl}} \\
&\quad + e^{-2\pi i \frac{ku}{kl}} [f(k, 0) + f(k, k) e^{-2\pi j \frac{kv}{kl}} + \dots + f(k, (l-1)k) e^{-2\pi j \frac{kv(l-1)}{kl}}] \\
&\quad + \dots \\
&\quad + e^{-2\pi i \frac{(l-1)ku}{kl}} [f(k, 0) + f((l-1)k, k) e^{-2\pi j \frac{kv}{kl}} \\
&\quad + \dots + f((l-1)k, (l-1)k) e^{-2\pi j \frac{kv(l-1)}{kl}}] \\
&= 1 + e^{-2\pi j \frac{v}{l}} + \dots + e^{-2\pi j \frac{v(l-1)}{l}} \\
&\quad + e^{-2\pi i \frac{u}{l}} [1 + e^{-2\pi j \frac{v}{l}} + \dots + e^{-2\pi j \frac{v(l-1)}{l}}] \\
&\quad + \dots \\
&\quad + e^{-2\pi i \frac{(l-1)u}{l}} [1 + e^{-2\pi j \frac{v}{l}} + \dots + e^{-2\pi j \frac{v(l-1)}{l}}] = 0.
\end{aligned}$$

This yields the desired conclusion. \square

To show the discrete uncertainty principle for quaternion-valued signal, the consecutive $m \times n$ sub-matrix stated in the following definition are sufficient.

Definition 3.6 (Consecutive Sub-Matrix of a Given Matrix). Given a $M \times N$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ \vdots & & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} \end{pmatrix},$$

define the $(2M - 1) \times (2N - 1)$ matrix Λ_A as follows

$$\Lambda_A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{M,1} & a_{M,2} & \cdots & a_{M,N} & a_{M,1} & \cdots & a_{M,N-1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,N} & & & \\ \vdots & \vdots & \ddots & \vdots & & & 0 \\ a_{M-1,1} & a_{M,2} & \cdots & a_{M-1,N} & & & \end{pmatrix},$$

then the consecutive $m \times n$ sub-matrix of A are defined by the $m \times n$ sub-matrix of Λ_A with rows in m consecutive terms or columns in n consecutive terms, where integers $m \leq M$ and $n \leq N$.

The following lemmas will be essential in proving these discrete uncertainty principle. Let $[r]$ be the smallest integer greater than or equal to r and denote $m := \sqrt{N_{(t,s)}}$.

Lemma 3.7. *Let $1 < N_{(t,s)} \leq MN$. Assume that the sequence $\{f(t, s)\}$ has $N_{(t,s)}$ nonzero elements ($t = 0, 1, \dots, M - 1, s = 0, 1, \dots, N - 1$). If $[m] = m$ and $m \leq \min\{M, N\}$, then the sequence $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M - 1, v = 0, 1, \dots, N - 1$) forms any consecutive m -matrix, i.e.,*

$$\begin{aligned} \hat{A}_{u,v} &= \left(\hat{f}(u, v) \right) \tag{3.7} \\ &= \begin{pmatrix} \hat{f}(u, v) & \hat{f}(u, v + 1) & \cdots & \hat{f}(u, v + m - 1) \\ \hat{f}(u + 1, v) & \hat{f}(u + 1, v + 1) & \cdots & \hat{f}(u + 1, v + m - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}(u + m - 1, v) & \hat{f}(u + m - 1, v + 1) & \cdots & \hat{f}(u + m - 1, v + m - 1) \end{pmatrix}, \end{aligned}$$

which has at least one nonzero element.

Proof. Without loss of generality, let $f(s, t) \neq 0$ ($s, t = 0, 1, \dots, m - 1$). Denote $m \times m$ matrix by

$$\begin{aligned} A_{0,0} &:= (f(t, s)) \\ &= \begin{pmatrix} f(0,0) & f(0,1) & \cdots & f(0, m - 1) \\ f(1,0) & f(1,1) & \cdots & f(1, m - 1) \\ \vdots & \vdots & \vdots & \vdots \\ f(m - 1, 0) & f(m - 1, 1) & \cdots & f(m - 1, m - 1). \end{pmatrix}, \end{aligned}$$

then $A_{0,0} \neq \mathbf{0}$. Here $\mathbf{0}$ is a $m \times m$ zero matrix.

Then one only needs to prove that any consecutive m -matix $\hat{A}_{u,v} \neq \mathbf{0}$. We prove it by contradiction. Suppose that there exists $\hat{A}_{u,v} = \mathbf{0}$, applying (2.3), we have

$$A_{0,0} = (V_i^u)^{-1} \hat{A}_{u,v} (V_j^v)^{-1} = \mathbf{0}.$$

Here

$$V_i^u := \frac{1}{\sqrt{M}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i \frac{u}{M}} & \cdots & e^{-2\pi i \frac{(m-1)u}{M}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-2\pi i \frac{(m-1)u}{M}} & \cdots & e^{-2\pi i \frac{(m-1)^2 u}{M}} \end{pmatrix}$$

and

$$V_j^v := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi j \frac{v}{N}} & \cdots & e^{-2\pi j \frac{(m-1)v}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-2\pi j \frac{(m-1)v}{N}} & \cdots & e^{-2\pi j \frac{(m-1)^2 v}{N}} \end{pmatrix},$$

respectively. It contradicts with $A_{0,0} \neq \mathbf{0}$. Therefore, $\hat{A}_{u,v} \neq \mathbf{0}$. This also means that the sequence $\{\hat{f}(u, v)\}$ has at least one nonzero element. It completes the proof. \square

The following example illustrates the consecutive m -matix $\hat{A}_{u,v}$.

Example. For $M = 2$, $N = 3$, let

$$A = \begin{pmatrix} f(0,0) & f(0,1) & f(0,2) \\ f(1,0) & f(1,1) & f(1,2) \end{pmatrix}.$$

We have

$$\hat{A} = \begin{pmatrix} \hat{f}(0,0) & \hat{f}(0,1) & \hat{f}(0,2) \\ \hat{f}(1,0) & \hat{f}(1,1) & \hat{f}(1,2) \end{pmatrix}.$$

As a consequence of periodic, the consecutive 2-matrices are:

$$\hat{A}_{0,0} = \begin{pmatrix} \hat{f}(0,0) & \hat{f}(0,1) \\ \hat{f}(1,0) & \hat{f}(1,1) \end{pmatrix},$$

$$\hat{A}_{0,1} = \begin{pmatrix} \hat{f}(0,1) & \hat{f}(0,2) \\ \hat{f}(1,1) & \hat{f}(1,2) \end{pmatrix}$$

and

$$\hat{A}_{0,2} = \begin{pmatrix} \hat{f}(0,2) & \hat{f}(0,0) \\ \hat{f}(1,2) & \hat{f}(0,1) \end{pmatrix}.$$

The following corollary is an immediate consequence of Lemma 3.7.

Corollary 3.8. *If the sequence $\{f(t, s)\}$ ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$) has MN nonzero elements, then the sequence $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$) has at least one nonzero element. Thus $N_{t,s} \cdot N_{u,v} \geq MN$.*

Lemma 3.9. *Let $1 < N_{(t,s)} < MN$. Assume that the sequence $\{f(t, s)\}$ ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$) has $N_{(t,s)}$ nonzero elements. If $[m] \neq m$ and $m \leq \min\{M, N\}$, then the sequence $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$) forms any consecutive m -matrix (3.9) which has at least two nonzero elements.*

Proof. To prove it by contradiction, let

$$\begin{aligned} A_{0,0} &= (f(t, s)) \\ &= \begin{pmatrix} f(0, 0) & f(0, 1) & \cdots & f(0, m-1) \\ f(1, 0) & f(1, 1) & \cdots & f(1, m-1) \\ \vdots & \vdots & \vdots & \vdots \\ f(m-1, 0) & f(m-1, 1) & \cdots & f(m-1, m-1) \end{pmatrix}. \end{aligned}$$

Without loss of generality, assume that the non-zeros number $N_{(t,s)}$ of the sequence $\{f(t, s)\}$ are all in A , and if the sequence $\{\hat{f}(u, v)\}$ in some consecutive m -matrix (3.9) only has one nonzero point, with the aid of (2.3), we have

$$\begin{aligned} A_{0,0} &= (V_{\mathbf{i}}^u)^{-1} \hat{A}_{u,v} (V_{\mathbf{j}}^v)^{-1} \\ &= V_{-\mathbf{i}}^u \hat{A}_{u,v} V_{-\mathbf{j}}^v \\ &= \frac{1}{\sqrt{MN}} \begin{pmatrix} \hat{f}(u, v) & \hat{f}(u, v) & \cdots & \hat{f}(u, v) \\ \hat{f}(u, v) & \hat{f}(u, v) & \cdots & \hat{f}(u, v) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}(u, v) & \hat{f}(u, v) & \cdots & \hat{f}(u, v) \end{pmatrix}. \end{aligned}$$

That means the sequence $\{f(t, s)\}$ has at least $m^2 = [\sqrt{N_{(t,s)}}]^2 > N_{(t,s)}$ nonzero points. This contradicts with the condition of the sequence $\{f(t, s)\}$ which has $N_{(t,s)}$ nonzero elements. It completes the proof. \square

Lemma 3.10. *Let $N \in \mathbf{N}^+$ be a positive number. Then we have*

$$[\sqrt{N}]^2 \leq 2N. \quad (3.8)$$

Proof. It is straightforward to verify the inequality (3.8) is true for $N = 1, 2, 3, 4$ and 5 . When $N \geq 6$, one obtains

$$\begin{aligned} N^2 + 1 > 6N &\Leftrightarrow N^2 - 2N + 1 > 4N \\ &\Leftrightarrow (N-1)^2 > 4N \\ &\Leftrightarrow N-1 > 2\sqrt{N} \\ &\Leftrightarrow N > 2\sqrt{N} + 1. \end{aligned}$$

Therefore, for $N \geq 6$,

$$[\sqrt{N}]^2 < (\sqrt{N} + 1)^2 = N + 2\sqrt{N} + 1 < 2N.$$

Consequently, equality (3.8) holds. It completes the proof. \square

Without loss of generality, we assume that $M \leq N$. A similar argument as in the proof of Lemma 3.7 is also true for the following lemma.

Lemma 3.11. *Let $1 < N_{(t,s)} \leq MN$. Assume that the sequence $\{f(t, s)\}$ has $N_{(t,s)}$ nonzero elements ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$). When $m > M = \min\{M, N\}$, then the sequence $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$) forms any consecutive $M \times (m-1)$ -matrix*

$$\begin{aligned} \hat{A}_{u,v} &= \left(\hat{f}(u, v) \right) \\ &= \begin{pmatrix} \hat{f}(u, v) & \hat{f}(u, v+1) & \cdots & \hat{f}(u, v+m-2) \\ \hat{f}(u+1, v) & \hat{f}(u+1, v+1) & \cdots & \hat{f}(u+1, v+m-2) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}(u+M-1, v) & \hat{f}(u+M-1, v+1) & \cdots & \hat{f}(u+M-1, v+m-2) \end{pmatrix}, \end{aligned} \quad (3.9)$$

which has at least one nonzero element.

We can now proceed to the proof of Main Theorem 3.1.

Proof of Main Theorem 3.1. • When $N_{(t,s)} = 1$ and $N_{(u,v)} = MN$, with the aid of Example 3 and Corollary 3.8, the conclusion holds.

- When $1 < N_{(t,s)} < MN$. Without loss of generality, we assume that $M \leq N$. There are two cases:

- 1) Assume that $[\sqrt{N_{(t,s)}}] \leq M$, when $[\sqrt{N_{(t,s)}}] = \sqrt{N_{(t,s)}}$, by Lemma 3.7, we have

$$\begin{aligned} N_{(t,s)} \cdot N_{(u,v)} &\geq \frac{M}{\sqrt{N_{(t,s)}}} \frac{N}{\sqrt{N_{(t,s)}}} 1N_{(t,s)} \\ &= \frac{MN}{(\sqrt{N_{(t,s)}})^2} N_{(t,s)} \\ &= MN. \end{aligned}$$

When $[\sqrt{N_{(t,s)}}] \neq \sqrt{N_{(t,s)}}$, then $[\sqrt{N_{(t,s)}}] > \sqrt{N_{(t,s)}}$. By Lemma 3.9 and (3.8), we have

$$\begin{aligned} N_{(t,s)} \cdot N_{(u,v)} &\geq \frac{M}{[\sqrt{N_{(t,s)}}]} \frac{N}{[\sqrt{N_{(t,s)}}]} 2N_{(t,s)} \\ &\geq MN. \end{aligned}$$

- 2) Assume that $[\sqrt{N_{(t,s)}}] > M$, then $\sqrt{N_{(t,s)}} \geq M$. By Lemma 3.11, we have

$$\begin{aligned} N_{(t,s)} \cdot N_{(u,v)} &\geq \frac{N}{[\sqrt{N_{(t,s)}}] - 1} 1N_{(t,s)} \\ &\geq \frac{N}{\sqrt{N_{(t,s)}}} N_{(t,s)} \\ &= N\sqrt{N_{(t,s)}} \geq MN. \end{aligned}$$

This completes the proof. \square

The following examples illustrate the discrete uncertainty principle.

Example. For $M = N = 2$, let

$$A = \begin{pmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{pmatrix}.$$

We have

$$\begin{aligned} \hat{A} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix} \begin{pmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi j} \end{pmatrix} \\ &= \begin{pmatrix} \hat{f}(0,0) & \hat{f}(0,1) \\ \hat{f}(1,0) & \hat{f}(1,1) \end{pmatrix}. \end{aligned}$$

Here

$$\begin{aligned} \hat{f}(0,0) &= \frac{1}{2} [f(0,0) + f(0,1) + f(1,0) + f(1,1)] \\ \hat{f}(0,1) &= \frac{1}{2} [f(0,0) + f(1,0) + (f(0,1) + f(1,1)) e^{-\pi j}] \\ \hat{f}(1,0) &= \frac{1}{2} [f(0,0) + f(0,1) + e^{-\pi i} (f(1,0) + f(1,1))] \\ \hat{f}(1,1) &= \frac{1}{2} [f(0,0) + e^{-\pi i} f(1,0) + f(0,1) e^{-\pi j} + e^{-\pi i} f(1,1) e^{-\pi j}]. \end{aligned}$$

We conclude that

- When $\{f(t, s)\}$ has only one nonzero point, clearly, $\{\hat{f}(u, v)\}$ has 4 nonzero points.
- When $\{f(t, s)\}$ has two or three nonzero points, $\{\hat{f}(u, v)\}$ has at least 2 nonzero points.
- When $\{f(t, s)\}$ has four nonzero points, then $\{\hat{f}(u, v)\}$ has at least 1 nonzero point.

Therefore we have $N_{(t,s)} \cdot N_{(u,v)} \geq 4$. It demonstrates the discrete uncertainty principle. Figure 1 shows an example for $\{f(t, s)\}$ with $M = N = 2$.

Example. For $M = 2, N = 3$, let

$$A = \begin{pmatrix} f(0,0) & f(0,1) & f(0,2) \\ f(1,0) & f(1,1) & f(1,2) \end{pmatrix}.$$

We have

$$\begin{aligned} &\hat{A} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix} \begin{pmatrix} f(0,0) & f(0,1) & f(0,2) \\ f(1,0) & f(1,1) & f(1,2) \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{4\pi j}{3}} \\ 1 & e^{-\frac{4\pi j}{3}} & e^{-\frac{8\pi j}{3}} \end{pmatrix} \\ &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 \\ 1 & e^{-\pi i} \end{pmatrix} \begin{pmatrix} f(0,0) & f(0,1) & f(0,2) \\ f(1,0) & f(1,1) & f(1,2) \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{4\pi j}{3}} \\ 1 & e^{-\frac{4\pi j}{3}} & e^{-\frac{8\pi j}{3}} \end{pmatrix} \\ &= \begin{pmatrix} \hat{f}(0,0) & \hat{f}(0,1) & \hat{f}(0,2) \\ \hat{f}(1,0) & \hat{f}(1,1) & \hat{f}(1,2) \end{pmatrix}. \end{aligned}$$

$N_{(t,s)}$	$f(t,s)$	$\hat{f}(u,v)$	$N_{(u,v)}$	$\hat{N}_{(t,s)} \cdot N_{(u,v)}$
1	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$	4	$1 \times 4 = 2 \times 2$
	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix}$	4	$1 \times 4 = 2 \times 2$
	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}$	4	$1 \times 4 = 2 \times 2$
	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$	4	$1 \times 4 = 2 \times 2$
3	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.5 & 0.75 \\ 0.75 & 0 \end{bmatrix}$	3	$3 \times 3 \geq 2 \times 2$
	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1.5 & 0.75 \\ 0 & 0.375 \end{bmatrix}$	3	$3 \times 3 \geq 2 \times 2$
	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1.5 & 0 \\ 0.75 & 0.375 \end{bmatrix}$	3	$3 \times 3 \geq 2 \times 2$
	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.375 \end{bmatrix}$	2	$3 \times 2 \geq 2 \times 2$

$N_{(t,s)}$	$f(t,s)$	$\hat{f}(u,v)$	$N_{(u,v)}$	$\hat{N}_{(t,s)} \cdot N_{(u,v)}$
2	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ 1 & 0.25 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0.25 & 0.25 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.625 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ 0.25 & -0.5 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
	$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ 0.25 & -0.125 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ -0.5 & -0.125 \end{bmatrix}$	4	$2 \times 4 \geq 2 \times 2$
4	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0.5 \\ 0.5 & 0.125 \end{bmatrix}$	4	$4 \times 4 \geq 2 \times 2$

FIGURE 1. The cases of $N_{(t,s)}$ and $N_{(u,v)}$ for an example of $\{f(t,s)\}$ with $M = N = 2$.

A straightforward computation gives

$$\begin{aligned}
 \hat{f}(0,0) &= \frac{1}{\sqrt{6}}[f(0,0) + f(1,0) + f(0,1) + f(1,1) + f(0,2) + f(1,2)] \\
 \hat{f}(0,1) &= \frac{1}{\sqrt{6}}[f(0,0) + f(1,0) + (f(0,1) + f(1,1))e^{-\frac{2\pi j}{3}} + (f(0,2) + f(1,2))e^{-\frac{4\pi j}{3}}] \\
 \hat{f}(0,2) &= \frac{1}{\sqrt{6}}[f(0,0) + f(1,0) + (f(0,1) + f(1,1))e^{-\frac{4\pi j}{3}} + (f(0,2) + f(1,2))e^{-\frac{2\pi j}{3}}] \\
 \hat{f}(1,0) &= \frac{1}{\sqrt{6}}[f(0,0) + f(0,1) + f(0,2) + e^{-\pi i}(f(1,0) + f(1,1) + f(1,2))] \\
 \hat{f}(1,1) &= \frac{1}{\sqrt{6}}[f(0,0) + e^{-\pi i}f(1,0) + f(0,1)e^{-\frac{2\pi j}{3}} + e^{-\pi i}f(1,1)e^{-\frac{2\pi j}{3}} \\
 &\quad + f(0,2)e^{-\frac{4\pi j}{3}} + e^{-\pi i}f(1,2)e^{-\frac{4\pi j}{3}}] \\
 \hat{f}(1,2) &= \frac{1}{\sqrt{6}}[f(0,0) + e^{-\pi i}f(1,0) + f(0,1)e^{-\frac{4\pi j}{3}} + e^{-\pi i}f(1,1)e^{-\frac{4\pi j}{3}} \\
 &\quad + f(0,2)e^{-\frac{2\pi j}{3}} + e^{-\pi i}f(1,2)e^{-\frac{2\pi j}{3}}].
 \end{aligned}$$

Clearly,

- When $\{f(t,s)\}$ has only one nonzero point, then $\{\hat{f}(u,v)\}$ has 6 nonzero points.
- When $\{f(t,s)\}$ has two or three nonzero points, then $\{\hat{f}(u,v)\}$ has at least 3 nonzero points.
- When $\{f(t,s)\}$ has four nonzero points, then $\{\hat{f}(u,v)\}$ has at least 2 nonzero points.
- When $\{f(t,s)\}$ has five nonzero points, then $\{\hat{f}(u,v)\}$ has at least 2 nonzero points.

$N_{(t,s)}$	$f(t,s)$	$\hat{f}(u,v)$	$N_{(u,v)}$	$N_{(t,s)} \cdot N_{(u,v)}$												
1	<table border="1"><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>	1	0	0	0	0	0	<table border="1"><tr><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td></tr><tr><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td></tr></table>	$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$	6	$1 \times 6 = 2 \times 3$
1	0	0														
0	0	0														
$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$														
$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$														
2	<table border="1"><tr><td>1</td><td>1</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>	1	1	0	0	0	0	<table border="1"><tr><td>$\sqrt{6}/3$</td><td>$\sqrt{6}(1-\sqrt{3}j)/12$</td><td>$\sqrt{6}(1+\sqrt{3}j)/12$</td></tr><tr><td>$\sqrt{6}/3$</td><td>$\sqrt{6}(1-\sqrt{3}j)/12$</td><td>$\sqrt{6}(1+\sqrt{3}j)/12$</td></tr></table>	$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$	$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$	6	$2 \times 6 \geq 2 \times 3$
1	1	0														
0	0	0														
$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$														
$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$														
3	<table border="1"><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>	1	1	1	0	0	0	<table border="1"><tr><td>$\sqrt{6}/2$</td><td>0</td><td>0</td></tr><tr><td>$\sqrt{6}/2$</td><td>0</td><td>0</td></tr></table>	$\sqrt{6}/2$	0	0	$\sqrt{6}/2$	0	0	4	$3 \times 4 \geq 2 \times 3$
1	1	1														
0	0	0														
$\sqrt{6}/2$	0	0														
$\sqrt{6}/2$	0	0														
4	<table border="1"><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr></table>	1	1	1	1	0	0	<table border="1"><tr><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td><td>$\sqrt{6}/6$</td></tr><tr><td>0</td><td>$-\sqrt{6}/6$</td><td>$-\sqrt{6}/6$</td></tr></table>	$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$	0	$-\sqrt{6}/6$	$-\sqrt{6}/6$	5	$4 \times 5 \geq 2 \times 3$
1	1	1														
1	0	0														
$\sqrt{6}/6$	$\sqrt{6}/6$	$\sqrt{6}/6$														
0	$-\sqrt{6}/6$	$-\sqrt{6}/6$														
5	<table border="1"><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table>	1	1	1	1	1	0	<table border="1"><tr><td>$\sqrt{6}/3$</td><td>$\sqrt{6}(1-\sqrt{3}j)/12$</td><td>$\sqrt{6}(1+\sqrt{3}j)/12$</td></tr><tr><td>$\sqrt{6}/3$</td><td>$\sqrt{6}(-1+\sqrt{3}j)/12$</td><td>$\sqrt{6}(-1-\sqrt{3}j)/12$</td></tr></table>	$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$	$\sqrt{6}/3$	$\sqrt{6}(-1+\sqrt{3}j)/12$	$\sqrt{6}(-1-\sqrt{3}j)/12$	6	$5 \times 6 \geq 2 \times 3$
1	1	1														
1	1	0														
$\sqrt{6}/3$	$\sqrt{6}(1-\sqrt{3}j)/12$	$\sqrt{6}(1+\sqrt{3}j)/12$														
$\sqrt{6}/3$	$\sqrt{6}(-1+\sqrt{3}j)/12$	$\sqrt{6}(-1-\sqrt{3}j)/12$														
6	<table border="1"><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	1	1	1	1	1	1	<table border="1"><tr><td>$\sqrt{6}$</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>	$\sqrt{6}$	0	0	0	0	0	1	$6 \times 1 = 2 \times 3$
1	1	1														
1	1	1														
$\sqrt{6}$	0	0														
0	0	0														

FIGURE 2. The cases of $N_{(t,s)}$ and $N_{(u,v)}$ for an example of $f(t,s)$ with $M = 2$ and $N = 3$.

- When $\{f(t,s)\}$ has six nonzero points, then $\{\hat{f}(u,v)\}$ has at least 1 nonzero point.

Therefore, we have $N_{(t,s)} \cdot N_{(u,v)} \geq 6$. It also demonstrates the discrete uncertainty principle. Figure 2 shows an example for $\{f(t,s)\}$ with $M = 2$, $N = 3$.

4. Uncertainty Principle for Bandlimited Signal Recovery

Donoho and Stark in [6] gave an example where the discrete-time uncertainty principle (1.1) shows something unexpected is possible. That is the recovery of a “sparse” wide-band signal from narrow-band measurements. The discrete-time uncertainty principle suggests how sparsity helps in the recovery of missing frequencies. We derive the results in the quaternionic setting.

Suppose there is an observed discrete quaternion-valued signal r , which is a combination of an ideal Ω -bandlimited signal f and noise, i.e.

$$r := P_{\Omega}f + n \quad (4.1)$$

where n denotes the noise and P_{Ω} is the operator that limits the measurements to the passband Ω of the system. Let P_{Ω} be the ideal bandpass operator

$$P_{\Omega}f := \frac{1}{\sqrt{MN}} \sum_{(u,v) \in \Omega} e^{2\pi i \frac{u}{M}} \hat{f}(u,v) e^{2\pi j \frac{v}{N}}. \quad (4.2)$$

If we apply the QDFT, (4.1) becomes

$$\hat{r} = \begin{cases} \hat{f} + \hat{n}, & (u,v) \in \Omega, \\ 0, & (u,v) \in \Omega^c. \end{cases} \quad (4.3)$$

Here, we assumed that the noise n is also bandlimited and Ω^c denotes the set of unobserved frequencies $\mathbf{R}^2 \setminus \Omega$.

Let $\Lambda := \Omega^c$ and $N_{(u,v)}$ denote its cardinality. As we see, the data $\{\hat{r}(u, v) : (u, v) \in \Lambda\}$ are not observed. The receiver's aim is to reconstruct the discrete-time signal f from the noisy observed signal r . Although it may seem that it is impossible, the uncertainty principles says recovery is possible provided that $2N_{(t,s)} \cdot N_{(u,v)} < MN$. Here $N_{(t,s)}$ and $N_{(u,v)}$ are the numbers of nonzero elements of sequences $\{f(t, s)\}$ ($t = 0, 1, \dots, M-1, s = 0, 1, \dots, N-1$) and $\{\hat{f}(u, v)\}$ ($u = 0, 1, \dots, M-1, v = 0, 1, \dots, N-1$), respectively. Donoho and Stark in [6] proved this result in the one dimensional case.

Theorem 4.1. *Suppose there is no noise in (4.1), that is $r = P_\Omega f$. If f has only $N_{(t,s)}$ nonzero elements and if*

$$2N_{(t,s)} \cdot N_{(u,v)} < MN, \quad (4.4)$$

then f can be uniquely reconstructed from r .

Proof. To prove this, we first show that f is the unique sequence satisfying the condition (4.4) that can generate the given data r . Suppose there is another sequence f_1 which also generates r , i.e., $P_\Omega f = r = P_\Omega f_1$. Let $h := f - f_1$, we have $P_\Omega h = 0$. Since f and f_1 have at most $N_{(t,s)}$ nonzero elements, clearly, h has fewer than $N'_{(t,s)} = 2N_{(t,s)}$ nonzero elements. On the other hand, $P_\Omega h = 0$, we have $\hat{h}(u, v) = 0$, for $(u, v) \in \Omega$. Therefore the DQFT of h has at most $N_{(u,v)}$ nonzero elements. Then h must be zero, for otherwise it would be a contradiction with the discrete-time uncertainty principle 3.1 (Here $N'_{(t,s)} \cdot N_{(u,v)} \leq MN$). Thus $f = f_1$. It establishes the uniqueness.

To reconstruct f from observed r , a ideal closest point algorithm could be used. Let $N_{(t,s)}$ be given and denote \mathbb{I} be the subsets τ of $\{0, 1, \dots, MN-1\}$ having $N_{(t,s)}$ elements. For a given subsets $\tau \in \mathbb{I}$, let \tilde{f}_τ be the sequence supported on τ , which is closed to generating the observed signal r , i.e.

$$\tilde{f}_\tau = \arg \min_{\tau \in \mathbb{I}} \|r - P_\Omega f'\|, \text{ s.t. } P_\tau f' = f. \quad (4.5)$$

For a fixed $\tau \in \mathbb{I}$, we merely have to find that

$$\tilde{f} = \arg \min_{\tilde{f}_\tau, \tau \in \mathbb{I}} \{\|r - P_\Omega \tilde{f}_\tau\|\}. \quad (4.6)$$

This step requires solving $\binom{N_{(t,s)}}{MN}$ sets of linear least-squares problems, each one requiring $\mathcal{O}((MN)^3)$ operations, therefore it is totally impractical for large $N_{(t,s)}$. It completes the proof and this theorem establishes uniqueness. \square

In the following, one establishes stability in the presence of noise.

Theorem 4.2. *Suppose that f has at most $N_{(t,s)}$ nonzero elements, with*

$$2N_{(t,s)} \cdot N_{(u,v)} < MN.$$

Assume that the norm of the noise is small, i.e., $\|n\| \leq \varepsilon$. If \tilde{f} has at most $N_{(t,s)}$ nonzero elements and satisfies

$$\|r - P_{\Omega}\tilde{f}\| \leq \varepsilon, \quad (4.7)$$

then

$$\|f - \tilde{f}\| \leq \frac{2\varepsilon}{\sqrt{1 - \frac{2N_{(t,s)}N_{(u,v)}}{MN}}}.$$

Proof. Let T denote the support of $f - \tilde{f}$, then the cardinality of T is at most $N'_{(t,s)} = 2N_{(t,s)}$. Denote by P_T the operator that timelimited a sequence on T . We have

$$\|f - \tilde{f}\|^2 = \|P_B(f - \tilde{f})\|^2 + \|(I - P_B)(f - \tilde{f})\|^2. \quad (4.8)$$

As a consequence of triangle inequality, the hypothesis condition $\|n\| \leq \varepsilon$ and inequality (4.7), we have

$$\begin{aligned} \|P_T(f - \tilde{f})\|^2 &= \|P_T(f) - r + r - P_T(\tilde{f})\|^2 \\ &\leq (\|P_T(f) - r\| + \|r - P_T(\tilde{f})\|)^2 \\ &= (\|n\| + \varepsilon)^2 \\ &\leq 4\varepsilon^2. \end{aligned} \quad (4.9)$$

Let $P_W = I - P_T$. Then the second term of (4.8) is

$$\begin{aligned} \|P_W(f - \tilde{f})\|^2 &= \|P_W P_T(f - \tilde{f})\|^2 \\ &\leq \|P_W P_T\|^2 \|f - \tilde{f}\|^2 \\ &\leq \frac{2N_{(t,s)}N_{(u,v)}}{MN} \|f - \tilde{f}\|^2. \end{aligned} \quad (4.10)$$

Combining (4.8)-(4.10), we obtain

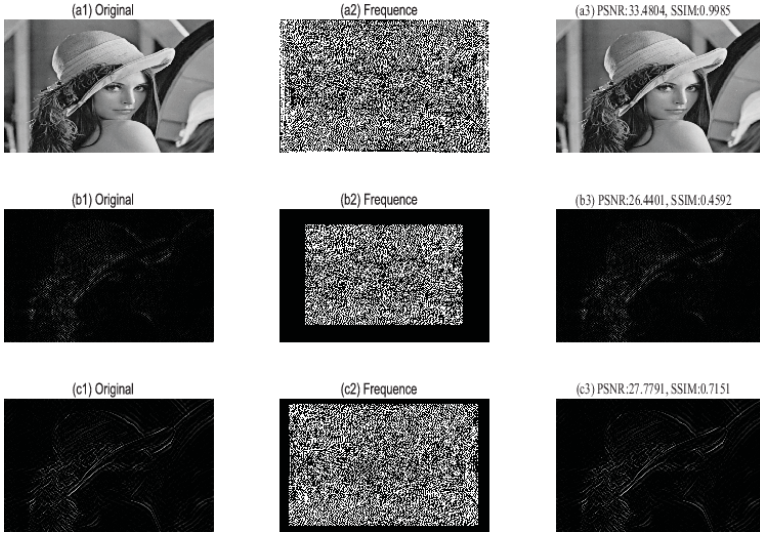
$$\|f - \tilde{f}\|^2 \leq \frac{4\varepsilon^2}{1 - \frac{2N_{(t,s)}N_{(u,v)}}{MN}}.$$

This completes the proof. \square

Example. For an image with size $M = N = 400$, we can also find the uncertainty principle in the image recovery processing in Fig. 3. For the original Lena *a1* and different bandlimited Lena *b1* and *c1* in Fig. 3, they are recovered with different numbers of $N_{(t,s)}$ and $N_{(u,v)}$. The results show that for different numbers of $N_{(t,s)}$ and $N_{(u,v)}$ the PSNR and SSIM are different. For an image with the numbers of $N_{(t,s)}$ and $N_{(u,v)}$ are smaller, the recovery results are worse. From the Table 1, we can show the recovery results in data, the bigger of PSNR and SSIM, the quality of images are better.

TABLE 1. The number of $N_{(t,s)}$ and $N_{(u,v)}$ in the Fig. 3.

Image size	$N_{(t,s)}$	$N_{(u,v)}$	Uncertainty Principle	PSNR	SSIM
(a) 400×400	160000	82057	$160000 \times 82057 \geq 400 \times 400$	33.4804	0.9985
(b) 400×400	79841	46396	$79841 \times 46396 \geq 400 \times 400$	26.4401	0.4592
(c) 400×400	79800	65990	$79800 \times 65990 \geq 400 \times 400$	27.7791	0.7151

FIGURE 3. The cases of $N_{(t,s)}$ and $N_{(u,v)}$ for an example of $\{f(t, s)\}$ with $M = N = 2$.

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