

On Bethe Ansatz for a Supersymmetric Vertex Model with $\mathcal{U}_q[\mathfrak{osp}(2|2)^{(2)}]$ symmetry

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Abstract

The Algebraic Bethe ansatz for a supersymmetric nineteen vertex-model constructed from a three-dimensional representation of the twisted quantum affine Lie superalgebra $\mathcal{U}_q[\mathfrak{osp}(2|2)^{(2)}]$ is presented in detail. The eigenvalues and eigenvectors of the row-to-row transfer matrix are calculated and the corresponding Bethe Ansatz equations are obtained and analyzed numerically.

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1 Introduction

One-dimensional quantum spin chain Hamiltonians and classical statistical systems in two spatial dimensions on a lattice (vertex models), share a common mathematical structure responsible by our understanding of these integrable models [1, 2]. If the Boltzmann weights underlying the vertex models are obtained from solutions of the Yang-Baxter (YB) equation the commutativity of the associated transfer matrices immediately follow, leading to their integrability.

The Bethe Ansatz (BA) is the powerful method in the analysis of integrable quantum models. There are several versions: Coordinate BA [3], Algebraic BA [4], Analytical BA [5], etc. developed for diagonalization of the corresponding Hamiltonian.

The simplest version is the Coordinate BA which we can obtain the eigenfunctions and the spectrum of the Hamiltonian from its eigenvalue problem. It is really simple and clear for the two-state models like the six-vertex models but becomes awkward for models with a higher number of states.

The Algebraic BA, also proverbial as Quantum Inverse Scattering method, is an elegant and important generalization of the Coordinate BA. It is based on the idea of constructing eigenfunctions of the Hamiltonian via creation and annihilation operators acting on a reference state. Here we use the fact the YB equation can be recast in the form of commutation relations for the matrix elements of the monodromy matrix which play the role of creation and annihilation operators. From this monodromy matrix we get the transfer matrix which, by construction, commutes with the Hamiltonian. Thus, constructing eigenfunctions of the transfer matrix determines the eigenfunctions of the Hamiltonian.

Imposing appropriate boundary conditions the BA method leads to a system of equations, the BA equations, which are useful in the thermodynamic limit. The energy of the ground state and its excitations, velocity of sound, etc., may be calculated in this limit. Moreover, in recent years we witnessed another very fruitful connection between the BA method and conformal field theory. Using the Algebraic BA, Korepin [6] found various representations of correlators in integrable models. Moreover Babujian and Flume [7] developed a method from the Algebraic BA which reveals a link to the Gaudin model and render in the quasiclassical limit solutions of the Knizhnik-Zamolodchikov equations for the $SU(2)$ Wess-Zumino-Novikov-Witten conformal theory.

Integrable quantum systems containing Fermi fields have been attracting increasing interest due to their potential applications in condensed matter physics. The prototypical examples of such systems are the supersymmetric generalizations of the Hubbard and t - J models [8], which play an important role in condensed matter physics, and also the search for solutions of the graded *Yang-Baxter equations* [] which gave origin to important algebraic construction as the supersymmetric Hopf algebras and quantum groups [9]. More recently, the integrability of supersymmetric models also proved to be important in superstring theory, more specifically in the AdS/CFT correspondence [10, 11, 12]. They lead to a generalization of the YB equation associated with the introduction of

the a Z_2 - grading [13] in the YB equation.

In the context of the Algebraic BA , the version presented here is based on the Tarosov approach [14]

The paper is organized as follows: In Section 2 we present the models. In Section 3 he main steps of the algebraic BA are developed in detail in order to solve the eigenvalue problem for the row to row transfer matrix with periodic boundary conditions, where its eigenvectors and eigenvalues are presented to be fixed by the roots of the Bethe equations. In section 4 we made a numecal analysis with the Bethe equation, and Section 5 is for our closing remarks.

2 The model

The most powerful and beautiful method to analyze these integrable quantum systems probably is the *Algebraic* BA [4]. This technique allows one to diagonalize the transfer matrix of a given integrable quantum system in an analytical way. The ABA was originally applied to systems with periodic boundary conditions but after the work of Sklyanin [15], integrable models with non-periodic boundaries could also be handled.

In this work we will study another graded three-state model with periodic boundary conditions. The R -matrix associated with this model is constructed from a three-dimensional free boson representation V of the twisted quantum affine Lie superalgebra $U_q[\text{osp}(2|2)^{(2)}] \simeq U_q[C(2)^{(2)}]$. We would like to emphasize that vertex-models described by Lie superalgebras – and, in particular, by twisted Lie superalgebras – are usually the most complex ones, which is due, of course, to the high complexity of such Lie superalgebras [16, 17, 18, 19, 20, 21].

Let $W=V\oplus U$ be a Z_2 -graded vector space where V and U denote its even and odd parts, respectively. In a Z_2 -graded vector space we associate a gradation $p(i)$ to each element ϵ_i of a given basis of V . In the present case, we shall consider only a three-dimensional representation of the twisted quantum affine Lie superalgebra $U_q[\text{osp}(2|2)^{(2)}]$ with a basis $E = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and the grading $p(1) = 0$, $p(2) = 1$ and $p(3) = 0$. Multiplication rules in the graded vector space W differ from the ordinary ones by the appearance of additional signs. For example, the graded tensor product of two homogeneous even elements $A \in \text{End}(V)$ and $B \in \text{End}(V)$ turns out to be defined by the formula,

$$A \otimes^g B = \sum_{i,j,k,l=1}^d (-1)^{p(i)p(k)+p(j)p(l)} A_{ij} B_{kl} (e_{ij} \otimes e_{kl}), \quad (1)$$

where d (in the present case, $d = 3$) is the dimension of the vector space V and e_{ij} are the Weyl matrices (e_{ij} is a matrix in which all elements are null, except that element on the $[i, j]$ position, which equals 1). In the same fashion, the graded permutation operator P^g is defined by

$$P^g = \sum_{i,j=1}^3 (-1)^{p(i)p(j)} (e_{ij} \otimes e_{ji}). \quad (2)$$

and the graded transposition A^{t^g} of a matrix $A \in \text{End}(V)$ as well as its inverse graded transposition, A^{τ^g} , are defined, respectively, by

$$A^{t^g} = \sum_{i,j=1}^3 (-1)^{p(i)p(j)+p(i)} A_{ji} e_{ij}, \quad A^{\tau^g} = \sum_{i,j=1}^3 (-1)^{p(i)p(j)+p(j)} A_{ji} e_{ij}, \quad (3)$$

so that $A^{t^g} \tau^g = A^{\tau^g} t^g = A$. Finally, the graded trace of a matrix $A \in \text{End}(V)$ is given by

$$\text{tr}^g(A) = \sum_{i=1}^3 (-1)^{p(i)} A_{ii} e_{ii}. \quad (4)$$

The YB equation \square ,

$$\mathcal{R}_{12}(x) \mathcal{R}_{13}(xy) \mathcal{R}_{23}(y) = \mathcal{R}_{23}(y) \mathcal{R}_{13}(xy) \mathcal{R}_{12}(x), \quad (5)$$

is written in the same way as in in the non-graded case: it is only necessary to employ graded operations instead of the usual operations

The R -matrix, solution of the graded YB equation (5), associated with the Yang-Zhang vertex-model [22] can be written, up to a normalizing factor and employing a different notation, as follows:

$$\mathcal{R}(x) = \begin{pmatrix} r_1(x) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r_2(x) & 0 & r_5(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3(x) & 0 & r_6(x) & 0 & r_7(x) & 0 & 0 \\ 0 & s_5(x) & 0 & r_2(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_6(x) & 0 & r_4(x) & 0 & r_6(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2(x) & 0 & r_5(x) & 0 \\ 0 & 0 & s_7(x) & 0 & s_6(x) & 0 & r_3(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_5(x) & 0 & r_2(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1(x) \end{pmatrix}, \quad (6)$$

where the amplitudes $r_i(x)$ and $s_i(x)$ are given respectively by

$$r_1(x) = q^2 x - 1, \quad (7)$$

$$r_2(x) = q(x-1), \quad (8)$$

$$r_3(x) = q(q+x)(x-1)/(qx+1), \quad (9)$$

$$r_4(x) = q(x-1) - (q+1)(q^2-1)x/(qx+1), \quad (10)$$

$$r_5(x) = q^2 - 1, \quad (11)$$

$$r_6(x) = -q^{1/2}(q^2-1)(x-1)/(qx+1), \quad (12)$$

$$r_7(x) = (q-1)(q+1)^2/(qx+1), \quad (13)$$

$$s_5(x) = (q^2-1)x = xr_5(x), \quad (14)$$

$$s_6(x) = -q^{1/2}(q^2-1)x(x-1)/(qx+1) = xr_6(x), \quad (15)$$

$$s_7(x) = (q-1)(q+1)^2 x^2/(qx+1) = x^2 r_7(x). \quad (16)$$

This R -matrix has the following properties or symmetries [27]:

$$\text{regularity:} \quad \mathcal{R}_{12}(1) = f(1)^{1/2} P_{12}^g, \quad (17)$$

$$\text{unitarity:} \quad \mathcal{R}_{12}(x) = f(x) \mathcal{R}_{21}^{-1}(x^{-1}), \quad (18)$$

$$\text{super PT:} \quad \mathcal{R}_{12}(x) = \mathcal{R}_{21}^{t_1^g \tau_2^g}(x), \quad (19)$$

$$\text{crossing:} \quad \mathcal{R}_{12}(x) = g(x) \left[V_1 \mathcal{R}_{12}^{t_2^g}(\eta^{-1} x^{-1}) V_1^{-1} \right], \quad (20)$$

where

$$f(x) = r_1(x) r_1\left(\frac{1}{x}\right) = (q^2 x - 1) \left(\frac{q^2}{x} - 1\right), \quad g(x) = -\frac{qx(x-1)}{(qx+1)}. \quad (21)$$

Here, t_1^g and t_2^g mean graded partial transpositions in the first and second vector spaces, respectively; τ_1^2 and τ_2^2 the corresponding inverse operations. Besides, $\eta = -q$ is the *crossing parameter* while

$$M = V^{t^g} V = \text{diag}(1/q, 1, q) \quad (22)$$

is the *crossing matrix*.

Besides \mathcal{R} we have to consider matrices $R = P^g \mathcal{R}$ which satisfy

$$R_{12}(x) R_{23}(xy) R_{12}(y) = R_{23}(y) R_{12}(xy) R_{23}(x) \quad (23)$$

Because only R_{12} and R_{23} are involved, Eq.(23) written in components looks the same as in the non graded case.

3 The Algebraic Bethe Ansatz

In the previous section we have presented the model through its R -matrix

The main problem now is the diagonalization of the transfer matrix of the lattice system. To do this we recall the formulation of the Algebraic Bethe ansatz [14].

Let us consider a regular lattice with L columns and L' rows. A physical state on this lattice is defined by the assignment of a *state variable* to each lattice edge. If one takes the horizontal direction as space and the vertical one as time, the transfer matrix plays the role of a discrete evolution operator acting on the Hilbert space $\mathcal{H}^{(N)}$ spanned by the *row states* which are defined by the set of vertical link variables on the same row. Thus, the transfer matrix elements can be understood as the transition probability of the one row state to project on the consecutive one after a unit of time.

The standard row-to-row monodromy matrix for an L - tensor space

$$V^{(1)} \otimes V^{(2)} \otimes \dots \otimes V^{(L)} \quad (24)$$

$$T_0(x) = \mathcal{R}_{0L}(x) \mathcal{R}_{0N-1}(x) \dots \mathcal{R}_{01}(x) \quad (25)$$

A quantum integrable system is characterized by monodromy matrix $T_0(x)$ satisfying the equation

$$R_{00'}(x/w) \left[T_0(x) \stackrel{g}{\otimes} T_{0'}(w) \right] = \left[T_{0'}(w) \stackrel{g}{\otimes} T_0(x) \right] R_{00'}(x/w) \quad (26)$$

whose consistency is guaranteed by the YB equation (23). $T_0(x)$ is a matrix in the quantum space $\otimes_j^N V^{(j)}$ with matrix elements that are operators on the states of the quantum system. The space $V_{(0)}$ is called auxiliary space of $T_0(x)$.

From the auxiliary space we can see $T_0(x)$ as a matrix 3 by 3

$$T_0(x) = \begin{pmatrix} A_1(x) & B_1(x) & B_2(x) \\ C_1(x) & A_2(x) & B_3(x) \\ C_2(x) & C_3(x) & A_3(x) \end{pmatrix} \quad (27)$$

where the operators A_i, B_i, C_i are 3^L by 3^L matrices.

The transfer matrix $\tau(x)$ for periodic boundary condition is defined as the super-trace of the row-to-row monodromy

$$\tau(x) = \text{Str} T_0(x) = \sum_{i=1}^3 (-1)^{[i]} A_i(x) = A_1(x) - A_2(x) + A_3(x) \quad (28)$$

In particular, the Hamiltonians can also be derived by the well-known relation

$$H = \alpha \frac{\partial}{\partial x} (\ln \tau(x))_{x=1} \quad (29)$$

In this section we will derive the BA equations of 19-vertex models presented in Section 2 using the Algebraic BA developed by Tarasov [14]. To do this we need of the commutation relations for entries of the monodromy matrix which are derived from the fundamental relation (26). Here these commutation relations do not share a common structure. Therefore, we only write some of them in the text and recall (26) to get the remaining ones.

First of all, let us observe that for each row state one can define the magnon number operator which commutes with the transfer matrix of the models

$$[\tau(x), M] = 0, \quad M = \sum_{k=1}^L M_k, \quad M_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad (30)$$

This is the analog of the operator S_T^z used in the previous section and the relation between M and the spin total S_T^z is simply $M = L - S_T^z$. Once again, the Hilbert space can be broken down into sectors $\mathcal{H}_M^{(L)}$. In each of these sectors, the transfer matrix can be diagonalized independently, $\tau(x)\Psi_M = \Lambda_M(\{x_i\})\Psi_M, (i = 0, 1, \dots, M), x_0 = x$. We now start to diagonalize $\tau(x)$ in every sector:

3.1 Sector $M = 0$

Let us consider the highest vector of the monodromy matrix $T(u)$ in a lattice of L sites as the even (bosonic) completely unoccupied state

$$\Psi_0 \equiv |0\rangle = \otimes_{k=1}^L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_k \quad (31)$$

It is the only state in the sector with $M = 0$. Using (27) we can compute the action of the matrix elements of $\tau(x)$ on this reference state:

$$\begin{aligned} A_k(x) |0\rangle &= r_k^L(x) |0\rangle, \\ C_k(x) |0\rangle &= 0, \quad B_k(x) |0\rangle \neq \{0, |0\rangle\}, \quad k = 1, 2, 3 \end{aligned} \quad (32)$$

Therefore in the sector $M = 0$, Ψ_0 is the eigenstate of $\tau(x) = A_1(x) - A_2(x) + A_3(x)$ with eigenvalue

$$\Lambda_0(x) = r_1^L(x) - r_2^L(x) + r_3^L(x) \quad (33)$$

Here we observe that the action of the operators $B_i(x)$ on the reference state will give us new states which lie in sectors with $M \neq 0$.

3.2 Sector $M = 1$

In this sector we have the states $B_1 |0\rangle$ and $B_3 |0\rangle$. Since $B_3 |0\rangle \propto B_1 |0\rangle$, we seek eigenstate of the form

$$\Psi_1(x_1) = B_1(x_1) |0\rangle. \quad (34)$$

The action of the operator $\tau(x)$ on this state can be computed with aid of the following commutation relations

$$A_1(x)B_1(w) = z(w/x)B_1(w)A_1(x) - \frac{r_5(w/x)}{r_2(w/x)}B_1(x)A_1(w) \quad (35)$$

$$\begin{aligned} A_2(x)B_1(w) &= -\frac{z(x/w)}{\omega(x/w)}B_1(w)A_2(x) - \frac{z(x/w)}{\omega(x/w)}\frac{1}{y(w/x)}B_2(w)C_1(x) \\ &+ \frac{s_5(x/w)}{r_2(x/w)}B_1(x)A_2(w) + \frac{s_5(x/w)}{r_2(x/w)}\frac{1}{y(x/w)}B_2(x)C_1(w) \\ &+ \frac{1}{y(x/w)}B_3(x)A_1(w) \end{aligned} \quad (36)$$

$$\begin{aligned} A_3(x)B_1(w) &= \frac{r_2(x/w)}{r_3(x/w)}B_1(w)A_3(x) + \frac{1}{y(x/w)}B_3(x)A_2(w) \\ &+ \frac{r_5(x/w)}{r_3(x/w)}B_2(w)C_3(x) - \frac{s_7(x/w)}{r_3(x/w)}B_2(x)C_3(w) \end{aligned} \quad (37)$$

where we have used Tarasov's notation[14],for the ratio functions

$$\begin{aligned} z(x) &= \frac{r_1(x)}{r_2(x)}, & \omega(x) &= \frac{r_1(x)r_3(x)}{-r_3(x)r_4(x) + r_6(x)s_6(x)}, \\ y(x) &= \frac{r_3(x)}{s_6(x)}, & y(x^{-1}) &= \frac{-r_3(x)r_4(x) + r_6(x)s_6(x)}{r_7(x)s_6(x) - r_3(x)r_6(x)}, \end{aligned} \quad (38)$$

When $\tau(x)$ act on $\Psi_1(x_1)$, the corresponding eigenvalue equation has two unwanted terms:

$$\begin{aligned} \tau(x)\Psi_1(x_1) &= (A_1(x) - A_2(x) + A_3(x)) \Psi_1(x_1) \\ &= [z(x_1/x)r_1^L(x) + \frac{z(x/x_1)}{\omega(x/x)}r_2^L(x) + \frac{r_2(x/x_1)}{r_3(x/x_1)}r_3^L(x)]\Psi_1(x_1) \\ &\quad - [\frac{r_5(x_1/x)}{r_2(x_1/x)}r_1^L(x_1) + \frac{s_5(x/x_1)}{r_2(x/x_1)}r_2^L(x_1)]B_1(x) |0\rangle \\ &\quad - \frac{1}{y(x/x)}r_1^L(x_1) - \frac{1}{y(x/x_1)}r_2^L(x_1)]B_3(x) |0\rangle \end{aligned} \quad (39)$$

From the matrix elements (7-16) we can see that $r_5(x)/r_2(x) = -s_5(x^{-1})/r_2(x^{-1})$. Therefore the unwanted terms vanish and $\Psi_1(x_1)$ is eigenstate of $\tau(x)$ with eigenvalue

$$\Lambda_1(x, x_1) = z(x_1/x)r_1^L(x) + \frac{z(x/x_1)}{\omega(x/x_1)}r_2^L(x) + \frac{r_2(x/x_1)}{r_3(x/x_1)}r_3^L(x) \quad (40)$$

provided

$$(z(x_1))^L = 1 \quad (41)$$

3.3 Sector $M = 2$

In the sector $M = 2$, we encounter two linearly independent states $B_1B_1|0\rangle$ and $B_2|0\rangle$. (The states $B_3B_3|0\rangle$, $B_1B_3|0\rangle$ and $B_3B_1|0\rangle$ also lie in the sector $M = 2$ but they are proportional to the state $B_1B_1|0\rangle$). We seek eigenstates in the form

$$\Psi_2(x_1, x_2) = B_1(x_1)B_1(x_2)|0\rangle + B_2(x_1)\Gamma(x_1, x_2)|0\rangle \quad (42)$$

where $\Gamma(x_1, x_2)$ is an operator-valued function which has to be fixed such that $\Psi_2(x_1, x_2)$ is unique state in the sector $M = 2$.

Here we observe that the operator-valued function $\Gamma(x_1, x_2)$ is.

It was demonstrated in [?] that $\Psi_2(x_1, x_2)$ is unique provided it is ordered in a normal way: In general, the operator-valued function $\Psi_n(x_1, \dots, x_n)$ is composite of normal ordered monomials. A monomial is normally ordered if in it all elements of the type $B_i(x)$ are on the left, and all elements of the type $C_j(x)$ on the right of all elements of the type $A_k(x)$. Moreover, the elements of one given type having standard ordering: $T_{i_1j_1}(x_1)T_{i_2j_2}(x_2)\dots T_{i_nj_n}(x_n)$. For a given sector $M = n$, $\Psi_n(x_1, \dots, x_n)$ is unique.

From the commutation relation

$$\begin{aligned}
B_1(x)B_1(w) &= \omega(w/x)[B_1(w)B_1(x) - \frac{1}{y(w/x)}B_2(w)A_1(x)] \\
&\quad + \frac{1}{y(x/w)}B_2(x)A_1(w)
\end{aligned} \tag{43}$$

we can see that (43) will be normally ordered if it satisfies the following swap condition

$$\Psi_2(x_2, x_1) = \omega(x_1/x_2)\Psi_2(x_1, x_2) \tag{44}$$

This condition fixes $\Gamma(x_1, x_2)$ in Eq.(42) and the eigenstate of $\tau(x)$ in the sector $M = 2$ has the form

$$\Psi_2(x_1, x_2) = B_1(x_1)B_1(x_2)|0\rangle - \frac{1}{y(x_1/x_2)}B_2(x_1)A_1(x_2)|0\rangle. \tag{45}$$

The action of transfer matrix on the states of the form (45) is more laborious. In addition to (35-37) and (43) we need appeal to (26) to derive more eight

commutation relations

$$\begin{aligned}
A_1(x)B_2(w) &= \frac{r_1(w/x)}{r_3(w/x)}B_2(w)A_1(x) - \frac{r_7(w/x)}{r_3(w/x)}B_2(x)A_1(w) \\
&\quad + \frac{r_6(w/x)}{r_3(w/x)}B_1(x)B_1(w)
\end{aligned} \tag{46}$$

$$\begin{aligned}
A_2(x)B_2(w) &= z(x/w)z(w/x)B_2(w)A_2(x) \\
&\quad + \frac{s_5(x/w)}{r_2(x/w)}[B_1(x)B_3(w) + B_3(x)B_1(w) + \frac{s_5(x/w)}{r_2(x/w)}B_2(x)A_2(w)]
\end{aligned} \tag{47}$$

$$\begin{aligned}
A_3(x)B_2(w) &= \frac{r_1(x/w)}{r_3(x/w)}B_2(w)A_3(x) - \frac{s_7(x/w)}{r_3(x/w)}B_2(x)A_3(w) \\
&\quad + \frac{1}{y(x/w)}B_3(x)B_3(w)
\end{aligned} \tag{48}$$

$$C_1(x)B_1(w) = -B_1(w)C_1(x) + \frac{s_5(x/w)}{r_2(x/w)}[A_1(w)A_2(x) - A_1(x)A_2(w)] \tag{49}$$

$$\begin{aligned}
C_3(x)B_1(w) &= -\frac{r_4(x/w)}{r_3(x/w)}B_1(w)C_3(x) - \frac{r_7(x/w)}{r_3(x/w)}B_1(x)C_3(w) \\
&\quad + \frac{1}{y(x/w)}[A_1(w)A_3(x) - A_2(x)A_2(w)] + \frac{r_6(x/w)}{r_3(x/w)}B_2(w)C_2(x)
\end{aligned} \tag{50}$$

$$B_1(x)B_2(w) = \frac{1}{z(x/w)}B_2(w)B_1(x) + \frac{s_5(x/w)}{r_1(x/w)}B_1(w)B_2(x) \tag{51}$$

$$B_1(x)B_3(w) = -B_3(w)B_1(x) - \frac{s_5(x/w)}{r_2(x/w)}B_2(w)A_2(x) + \frac{r_5(x/w)}{r_2(x/w)}B_2(x)A_2(w) \tag{52}$$

$$B_2(x)B_1(w) = \frac{1}{z(x/w)}B_1(w)B_2(x) + \frac{r_5(x/w)}{r_1(x/w)}B_2(w)B_1(x) \tag{53}$$

Here we observe that in this approach the final action of $\tau(x)$ on normally ordered states must be normal ordered. This implies in an increasing use of commutation relations needed for the diagonalization of $\tau(x)$. For example, the action of the operator $A_1(x)$ on $\Psi_2(x_1, x_2)$ has the form

$$\begin{aligned}
A_1(x)\Psi_2(x_1, x_2) &= z(x_{10})z(x_{20})r_1^L(x) \Psi_2(x_1, x_2) \\
&\quad - \frac{r_5(x_{10})}{r_2(x_{10})}z(x_{21})r_1^L(x_1) B_1(x)B_1(x_2) |0\rangle \\
&\quad - \frac{r_5(x_{20})}{r_2(x_{20})} \frac{z(x_{12})}{\omega(x_{12})} r_1^L(x_2) B_1(x)B_1(x_1) |0\rangle \\
&\quad + \left(\frac{z(x_{10})}{\omega(x_{10})} \frac{r_5(x_{20})}{r_2(x_{20})} \frac{1}{y(x_{01})} + \frac{r_7(x_{10})}{r_3(x_{10})} \frac{1}{y(x_{12})} \right) \\
&\quad \times r_1^L(x_1)r_1^L(x_2) B_2(x) |0\rangle
\end{aligned} \tag{54}$$

where $x_{ab} = x_a/x_b$, $a \neq b = 0, 1, 2$, with $x_0 = x$. Here we have used the following identities satisfied by the matrix elements of this 19-vertex model:

$$\begin{aligned} \frac{z(x_{ab})}{\omega(x_{ab})} \frac{r_5(x_{cb})}{x_2(x_{cb})} + \frac{r_6(x_{ab})}{r_3(x_{ab})} \frac{1}{y(x_{ac})} &= \frac{r_5(x_{ab})}{r_2(x_{ab})} \frac{r_5(x_{ca})}{r_2(x_{ca})} + \frac{z(x_{ac})}{\omega(x_{ac})} \frac{r_5(x_{cb})}{r_2(x_{cb})}, \\ z(x_{ab}) \frac{r_5(x_{cb})}{r_2(x_{cb})} \frac{1}{y(x_{ab})} + \frac{r_1(x_{ab})}{r_3(x_{ab})} \frac{1}{y(x_{ac})} &= z(x_{ab}) z(x_{cb}) \frac{1}{y(x_{ac})}, \\ \omega(x_{ab}) \omega(x_{ba}) &= 1, \quad (a \neq b \neq c) \end{aligned} \quad (55)$$

Similarly, for the operator $A_2(x)$ we have

$$\begin{aligned} A_2(x) \Psi_2(x_1, x_2) &= \frac{z(x_{01})}{\omega(x_{01})} \frac{z(x_{02})}{\omega(x_{02})} r_2^L(x) \Psi_2(x_1, x_2) \\ &- \frac{s_5(x_{02})}{r_2(x_{02})} z(x_{21}) r_2^L(x_2) B_1(x) B_1(x_1) |0\rangle \\ &- \frac{s_5(x_{01})}{r_2(x_{01})} \frac{z(x_{12})}{\omega(x_{12})} r_2^L(x_1) B_1(x) B_1(x_2) |0\rangle \\ &+ z(x_{21}) \frac{1}{y(x_{01})} r_1^L(x_1) B_3(x) B_1(x_2) |0\rangle \\ &+ \frac{z(x_{12})}{\omega(x_{12})} \frac{1}{y(x_{02})} r_1^L(x_2) B_3(x) B_1(x_1) |0\rangle \\ &+ \frac{s_5(x_{01})}{r_2(x_{01})} \left(\frac{s_5(x_{21})}{r_2(x_{21})} \frac{1}{y(x_{01})} + \frac{z(x_{01})}{\omega(x_{01})} \frac{1}{y(x_{02})} - \frac{s_5(x_{01})}{r_2(x_{01})} \frac{1}{y(x_{12})} \right) \\ &\times r_1^L(x_2) r_2^L(x_1) B_2(x) |0\rangle \\ &+ \frac{1}{y(x_{01})} \left(z(x_{01}) \frac{s_5(x_{02})}{r_2(x_{02})} - \frac{s_5(x_{01})}{r_2(x_{01})} \frac{s_5(x_{02})}{r_2(x_{02})} \right) r_1^L(x_1) r_2^L(x_2) B_2(x) |0\rangle \end{aligned} \quad (56)$$

In this case we have used more two identities:

$$\begin{aligned} \frac{z(x_{ab})}{\omega(x_{ab})} \frac{1}{y(x_{ac})} + \frac{s_5(x_{bc})}{r_2(x_{bc})} \frac{1}{y(x_{ab})} &= \frac{s_5(x_{ab})}{r_2(x_{ab})} \frac{1}{y(x_{bc})} + \frac{z(x_{bc})}{\omega(x_{bc})} \frac{1}{y(x_{ac})} \\ z(x_{cb}) \frac{s_5(x_{ac})}{r_2(x_{ac})} + \frac{s_5(x_{ab})}{r_2(x_{ab})} \frac{s_5(x_{bc})}{r_2(x_{bc})} &= \frac{z(x_{ab}) s_5(x_{ac})}{r_2(x_{ac})} \\ a \neq b \neq c \end{aligned} \quad (57)$$

Finally, for $A_3(x)$ we get

$$\begin{aligned} A_3(x) \Psi_2(x_1, x_2) &= \frac{r_2(x_{01})}{r_3(x_{01})} \frac{r_2(x_{02})}{r_3(x_{02})} r_3^L(x) \Psi_2(x_1, x_2) \\ &- \omega(x_{12}) \frac{1}{y(x_{01})} r_2^L(x_1) B_3(x) B_1(x_2) |0\rangle - z(x_{21}) \frac{1}{y(x_{02})} r_2^L(x_2) B_3(x) B_1(x_1) |0\rangle \\ &+ \left(\frac{s_7(x_{01})}{r_3(x_{01})} \frac{1}{y(x_{12})} - \frac{s_5(x_{01})}{r_3(x_{01})} \frac{1}{y(x_{02})} \right) r_2^L(x_1) r_2^L(x_2) B_2(x) |0\rangle \end{aligned} \quad (58)$$

Here we also have used the identities (55) and (57).

From these relations one can see that all unwanted terms of $\tau(x)\Psi_2(x_1, x_2)$ vanish. It means that $\Psi_2(x_1, x_2)$ is an eigenstate of the transfer matrix $\tau(x)$ with eigenvalue

$$\Lambda_2(x, x_1, x_2) = z(x_{10})z(x_{20})r_1^L(x) - \frac{z(x_{01})}{\omega(x_{01})} \frac{z(x_{02})}{\omega(x_{02})} r_2^L(x) + \frac{r_2(x_{01})}{r_3(x_{01})} \frac{r_2(x_{02})}{r_3(x_{02})} r_3^L(x) \quad (59)$$

provided the rapidities x_1 and x_2 satisfy the BA equations

$$(z(x_a))^L = -\frac{z(x_{ab})}{z(x_{ba})}\omega(x_{ba}), \quad a \neq b = 1, 2. \quad (60)$$

3.4 General Sector

The generalization of the above results to sectors with more than two particles proceeds through the factorization properties of the higher order phase shifts discussed in the previous section. Therefore, at this point we shall present the general result: In a generic sector $M = n$, we have $n - 1$ swap conditions

$$\Psi_n(x_1, \dots, x_{i-1}, x_{i+1}, x_i, \dots, x_n) = \omega(x_i - x_{i+1})\Psi_n(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \quad (61)$$

which yield the $n - 1$ operator-valued functions $\Gamma_i(x_1, \dots, x_n)$. The corresponding normal ordered state $\Psi_n(x_1, \dots, x_n)$ can be written with aid of a recurrence formula [?]:

$$\Psi_n(x_1, \dots, x_n) = \Phi_n(x_1, \dots, x_n) |0\rangle \quad (62)$$

where

$$\begin{aligned} \Phi_n(x_1, \dots, x_n) &= B_1(x_1)\Phi_{n-1}(x_2, \dots, x_n) \\ &- B_2(x_1) \sum_{j=2}^n \frac{1}{y(x_1/x_j)} \prod_{k=2, k \neq j}^n \mathcal{Z}(x_k/x_j) \Phi_{n-2}(x_2, \dots, \hat{x}_j, \dots, x_n) A_1(x_j) \end{aligned} \quad (63)$$

with the initial condition $\Phi_0 = 1$, $\Phi_1(x) = B_1(x)$.

The scalar function $\mathcal{Z}(x_k - x_j)$ is defined by

$$\mathcal{Z}(x_k/x_j) = \begin{cases} z(x_k/x_j) & \text{if } k > j \\ z(x_k/x_j)\omega(x_j - /x) & \text{if } k < j \end{cases} \quad (64)$$

The action of the operators $A_i(x)$, $i = 1, 2, 3$ on the operators Φ_n have the following normal ordered form

$$\begin{aligned}
A_1(x)\Phi_n(x_1, \dots, x_n) &= \prod_{k=1}^n z(x_k/x)\Phi_n(x_1, \dots, x_n)A_1(x) \\
-B_1(x) \sum_{j=1}^n \frac{x_5(x_j/x)}{x_2(x_j/x)} \prod_{k=1, k \neq j}^n \mathcal{Z}(x_k/x_j)\Phi_{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n)A_1(x_j) \\
+B_2(x) \sum_{j=2}^n \sum_{l=1}^{j-1} G_{jl}(x, x_l, x_j) \prod_{k=1, k \neq j, l}^n \mathcal{Z}(x_k/x_l)\mathcal{Z}(x_k/x_j) \\
\times \Phi_{n-2}(x_1, \dots, \hat{x}_l, \dots, \hat{x}_j, \dots, x_n)A_1(x_l)A_1(x_j)
\end{aligned} \tag{65}$$

where $G_{jl}(x, x_l, x_j)$ are scalar functions defined by

$$G_{jl}(x, x_l, x_j) = \frac{r_7(x_l/x)}{r_3(x_l/x)} \frac{1}{y(x_l/x)} + \frac{z(x_l/x)}{\omega(x_l/x)} \frac{r_5(x_j/x)}{r_2(x_j/x)} \frac{1}{y(x/x_l)} \tag{66}$$

For the action of $A_3(x)$ we have a similar expression

$$\begin{aligned}
A_3(x)\Phi_n(x_1, \dots, x_n) &= \prod_{k=1}^n \frac{r_2(x/x)}{r_3(x/x_k)}\Phi_n(x_1, \dots, x_n)A_3(x) \\
+(-1)^n B_3(x) \sum_{j=1}^n \frac{1}{y(x/x_j)} \prod_{k=1, k \neq j}^n \mathcal{Z}(x_j/x_k)\Phi_{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n)A_2(x_j) \\
+B_2(x) \sum_{j=2}^n \sum_{l=1}^{j-1} H_{jl}(x, x_l, x_j) \prod_{k=1, k \neq j, l}^n \mathcal{Z}(x_j/x_k)\mathcal{Z}(x_l/x_k) \\
\times \Phi_{n-2}(x_1, \dots, \hat{x}_l, \dots, \hat{x}_j, \dots, x_n)A_2(x_l)A_2(x_j)
\end{aligned} \tag{67}$$

where the scalar functions $H_{jl}(x, x_l, x_j)$ are given by

$$H_{jl}(x, x_l, x_j) = \frac{s_7(x/x_l)}{r_3(x/x)} \frac{1}{y(x_l/x)} - \frac{s_5(x/x_l)}{r_3(x/x_l)} \frac{1}{y(x/x_j)} \tag{68}$$

The action of the operator $A_2(x)$ is more cumbersome

$$\begin{aligned}
A_2(x)\Phi_n(x_1, \dots, x_n) &= (-1)^n \prod_{k=1}^n \frac{z(x/x)}{\omega(x/x)} \Phi_n(x_1, \dots, x_n) A_2(x) \\
&+ (-1)^n B_1(x) \sum_{j=1}^n \frac{s_5(x/x_j)}{r_2(x/x_j)} \prod_{k=1, k \neq j}^n \mathcal{Z}(x_j/x_k) \Phi_{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n) A_2(x_j) \\
&+ B_3(x) \sum_{j=1}^n \frac{1}{y(x/x_j)} \prod_{k=1, k \neq j}^n \mathcal{Z}(x_k/x_j) \Phi_{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n) A_1(x_j) \\
&+ \varepsilon^n B_2(x) \left\{ \sum_{j=2}^n \sum_{l=1}^{j-1} Y_{jl}(x, x_l, x_j) \prod_{k=1, k \neq j, l}^n \mathcal{Z}(x_k/x) \mathcal{Z}(x_j/x) \times \right. \\
&\Phi_{n-2}(x_1, \dots, \hat{x}_l, \dots, \hat{x}_j, \dots, x_n) A_1(x_l) A_2(x_j) + \\
&\sum_{j=2}^n \sum_{l=1}^{j-1} F_{jl}(x, x_l, x_j) \prod_{k=1, k \neq j, l}^n \mathcal{Z}(x_l/x_k) \mathcal{Z}(x_k/x_j) \times \\
&\left. \Phi_{n-2}(x_1, \dots, \hat{x}_l, \dots, \hat{x}_j, \dots, x_n) A_1(x_j) A_2(x_l) \right\} \tag{69}
\end{aligned}$$

where we have more two scalar functions

$$\begin{aligned}
F_{jl}(x, x_l, x_j) &= \frac{s_5(x/x_l)}{r_2(x-x_l)} \left\{ \frac{s_5(x_l/x_j)}{r_2(x_l/x_j)} \frac{1}{y(x/x_l)} + \frac{z(x/x_l)}{\omega(x/x_l)} \frac{1}{y(x/x_j)} \right. \\
&\left. - \frac{s_5(x/x_l)}{r_2(x/x_l)} \frac{1}{y(x_l/x_j)} \right\} \tag{70}
\end{aligned}$$

$$Y_{jl}(x, x_l, x_j) = \frac{1}{y(x/x_l)} \left\{ z(x/x_l) \frac{s_5(x/x_j)}{r_2(x/x_j)} - \frac{s_5(x/x)}{r_2(x/x_l)} \frac{s_5(x_l/x_j)}{r_2(x_l/x_j)} \right\} \tag{71}$$

From these relations immediately follows that $\Psi_n(x_1, \dots, x_n)$ are the eigenstates of $\tau(x)$ with eigenvalues

$$A_M(x) = r_1(x)^L \prod_{a=1}^n z(x_a/x) - (-1)^n r_2(x)^L \prod_{a=1}^n \frac{z(x/x_a)}{\omega(x/x_a)} + r_3(x)^L \prod_{a=1}^n \frac{r_2(x/x_a)}{r_3(x/x_a)} \tag{72}$$

provided their rapidities $x_i, i = 1, \dots, M$ satisfy the BA equations

$$(z(x_a))^L = (-1)^{n+1} \prod_{b \neq a=1}^n \frac{z(x_a/x_b)}{z(x_b/x_a)} \omega(x_b - x_a), \quad a = 1, 2, \dots, n \tag{73}$$

To conclude this section we remark that equations (72) and (73) reproduce the known results in the literature for the graded nineteen vertex models

4 Numerical analysis

For a small-length chain, the results above can be checked numerically. As an example, let us consider a chain with three sites, that is, let us assume that $L = 2$. In this case, both the monodromy as the transfer matrix defined at () can be explicitly constructed without difficult:

The monodromy becomes an operator with values in $\text{End}(V_0 \otimes V_1 \otimes V_2)$ and, therefore, consists in an 27-by-27 matrix. Using the graded permutation operators $P_{01}^g = P^g \otimes I_3$ and $P_{12}^g = I_3 \otimes P^g$ we have $\mathcal{R}_{01} = R \otimes I_3$ and $\mathcal{R}_{02} = P_{12}^g \mathcal{R}_{01} P_{12}^g$. Therefore for two quantum spaces the monodromy (25) is reduced to

$$T_0 = \mathcal{R}_{02} \mathcal{R}_{01} \quad (74)$$

Hence, the graded transfer matrix (28) consists in a 9-by-9 matrix.

In the framework of the ABA, on the other hand, we usually divide the spectrum of the transfer matrix into sectors, according to the *magnon number* M associated with the possible chain configurations. (We say that a spin pointing to up (1), to center (2), or to down (3) has a magnon number equal to 0, 1 or 2, respectively, and that the total magnon number of the chain is given by the sum of the magnon numbers associated with all its sites.) In this way, the reference state corresponds to the sector $M = 0$, which physically corresponds to a configuration in which all spins point up, while the n -particle states correspond to the configurations in which $M = n$, that is, they are physically formed by any combination of k spins pointing down and l spins pointing to the center, in such a way that $2k + l = n$. Therefore, for a chain of length $N = 2$, we have in total 5 sectors, corresponding to the values of M ranging from 0 to 4.

The eigenvalues (λ_i and the eigenvectors (v_i) of the transfer matrix can be numerically calculated as we give numerical values for the parameters x, q . In this example, we shall consider the (randomly generated) values, $x = 1.2970895172$ and $q = 0.3438435138$.

Here we notice that for two sites there are nine configurations of spin-1, given by

$$c_{i,j} = e_i \otimes e_j \quad i, j = 1, 2, 3 \quad (75)$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (76)$$

For this example we find the following eigenvectors their eigenvalues

$$\begin{aligned}
v_1 &= c_{1,1}, & \lambda_1 &= 0.719147295 & (n = 0) \\
v_2 &= \frac{1}{\sqrt{2}}(c_{2,1} - c_{1,2}), & \lambda_2 &= -1.1875382707 & (n = 1) \\
v_3 &= \frac{1}{\sqrt{2}}(c_{2,1} + c_{1,2}), & \lambda_3 &= 0.80021448548 & (n = 1) \\
v_4 &= -\frac{1}{\sqrt{2}}(c_{3,1} + c_{1,3}), & \lambda_4 &= 0.9231082366 & (n = 2) \\
v_5 &= -\frac{1}{\sqrt{2}}(c_{3,1} - c_{1,3}), & \lambda_5 &= -1.3365641154 & (n = 2) \\
v_6 &= \frac{1}{\sqrt{2}}(c_{3,2} - c_{2,3}), & \lambda_6 &= \lambda_2 & (n = 3) \\
v_7 &= \frac{1}{\sqrt{2}}(c_{3,2} + c_{2,3}), & \lambda_7 &= \lambda_3 & (n = 3) \\
v_8 &= c_{2,2}, & \lambda_8 &= \lambda_5 & (n = 2) \\
v_9 &= c_{3,3}, & \lambda_9 &= \lambda_1 & (n = 4) \quad (77)
\end{aligned}$$

Now we can look at the results given by the ABA
The equation (33) for N=2

$$\Lambda_0 = r_1(x)^2 - r_2(x)^2 + r_3(x)^2 \quad (78)$$

Substituting the numerical values

$$\begin{aligned}
r_1(x) &= -0.84664472310 \\
r_2(x) &= 0.1021523034 \\
r_3(x) &= 0.1159236333 \quad (79)
\end{aligned}$$

to obtain

$$\Lambda_0 = 0.7198147295 \quad (80)$$

which is equal the value λ_1 the a=eigenvalue obtained from the transfer matrix in the sector $n = 0$.

For the sector $n = 1$ we recall the equations (40) and (41). Now we solve (41) to find x_1 in order to find the eigenvalue from $\Lambda_1(x, x_1)$ from (40). For $N = 2$, the numeric solution is

$$x_1 = -2.9082997350$$

and

$$\Lambda_1(x, x_1) = 0.8002148543$$

which can be identified with the eigenvalue λ_3 of the symmetric eigenvector v_3 of the transfer matrix for the sector $n = 1$.

For the sector $n = 2$, we recall the equations (59) and (60) with $L = 2$.

The two equations supplied by (60) can be numerical solved to find x_1 and x_2 and then we can find $\Lambda_2(x, x_1, x_2)$ from (59).

The numerical results are

$$x_1 = -2.9082997360$$

$$x_2 = -2.9082997340$$

and

$$\Lambda_2(x, x_1, x_2) = 0.9231082388$$

which is the eigenvalue λ_4 of the symmetric eigenvector v_4 of the transfer matrix.

we hope these few examples should be suffice to pave the way for $L \geq 3$.

We remark however that only 5 of the 9 eigenvalues of the transfer matrix are actually distinct, which is due to the symmetry of the system regarding inversion of the spins.

In order to compute the eigenvalues of the transfer matrix in the framework of the ABA, we need to solve the BAE, since the eigenvalues given by (59) depend implicitly on the rapidities – *i.e.*, on the solutions of the BAE. Here we remark as well that for $L = 2$ is not necessary to go up to $n = 4$, as we could expect from (76). The solutions for $n = \{3, 4\}$ provide the same eigenvalues as that obtained from the cases $n = \{2, 1\}$, respectively, which is due to the above mentioned symmetry of the system regarding the inversion of the spins. This is very welcome, since the BAE are very difficult to solve, even numerically. In fact, the BAE are highly ill-conditioned: their roots are very close to each other, which requires a high accuracy in the computations; there are solutions which are not physical (for instance, in the present case when case some root equals 0, ± 1 , $\pm 1/q^2$, or when two or more roots are equal to each other etc.) and they should be discarded. Different solutions may lead to the same eigenvalue, for example those solutions differing only by a permutation of the Bethe roots, or, sometimes, roots differing only by a complex conjugation. For more details about the complexity BAE, (see [23]).

5 Conclusion

In this work we derived the periodic algebraic BA for the supersymmetric nineteen vertex model constructed from a three-dimensional free boson representation V of the twisted quantum affine Lie superalgebra $U_q[\text{osp}(2|2)^{(2)}] \simeq U_q[C(2)^{(2)}]$. Explicit results and a numerical analysis were also presented.

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