

Modified Proca Theory in Arbitrary and Two Dimensions

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Abstract: We demonstrate that the standard Stückelberg-modified Proca theory (i.e. a *massive* Abelian 1-form theory) respects the *classical* gauge and corresponding *quantum* (anti-)BRST symmetry transformations in any arbitrary dimension of spacetime within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. We further show that the Stückelberg formalism gets *modified* in the two (1+1)-dimensions of spacetime due to a couple of discrete *duality* symmetries in the theory which turn out to be responsible for the existence of the nilpotent (anti-)co-BRST symmetry transformations corresponding to the nilpotent (anti-)BRST symmetry transformations of our theory. These *nilpotent* symmetries exist *together* in the modified version of the two (1+1)-dimensional (2D) Proca theory. We provide the mathematical basis for the *modification* of the Stückelberg technique, the existence of the discrete *duality* as well as the *continuous* (anti-)co-BRST symmetry transformations in the 2D *modified* version of Proca theory.

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1 Introduction

The celebrated Proca theory (i.e., a *massive* Abelian 1-form theory) is the generalization of the Maxwell (i.e., a *massless* Abelian 1-form) gauge theory. Whereas the *latter* corresponds to a physical photon with *two* physical degrees of freedom, the *former* describes a *massive* vector boson with *three* physical degrees of freedom in the *physical* four ($3 + 1$)-dimensions of spacetime. The key signature of the Maxwell theory is the existence of the first-class constraints on it (see, e.g., [1, 2]). On the contrary, the Proca theory is endowed with the second-class constraints in the terminology of Dirac's prescription for the classification scheme of constraints [1, 2]. The well-known Stückelberg technique (see, e.g., [3] for details) converts the second-class constraints of the Proca theory into the first-class constraints thereby restoring the beautiful gauge symmetry for *even* the *massive* Abelian 1-form theory where the *gauge invariance* and *mass* co-exist *together* in a beautiful fashion. This statement is *true* in any arbitrary dimension of spacetime and it is valid for *even* the higher p -form ($p = 2, 3, \dots$) gauge theories. For instance, we have exploited *this* technique in the context of *massive* Abelian 2-form gauge theory [4, 5] where the *mass* and *gauge* symmetry co-exist *together* (due to the application of the Stückelberg formalism).

The central purpose of our present endeavor is to show that there is a mathematically-backed precise *modification* of the Stückelberg technique for the p -form ($p = 1, 2, 3, \dots$) *massive* gauge theories in $D = 2p$ dimensions of spacetime as these theories turn out to be the *massive* models of Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. To corroborate this claim, in our present endeavor, we take up the 2D Stückelberg-modified Proca theory (i.e. a *massive* Abelian 1-form theory) to establish that there is a precise *modification* of the Stückelberg technique [cf. Eq. (15) below] which leads to the existence of the (anti-)co-BRST symmetries *corresponding* to the nilpotent (anti-)BRST symmetries (that exist in any arbitrary dimension of spacetime) because there is a set of discrete *duality* symmetry transformations [cf. Eqs. (23), (29)] in our 2D theory. We also show that the *physical state* of our theory is annihilated by the first-class constraints as well as their *dual* versions when we choose the physical state to be the *harmonic* state (of the Hodge decomposed state in the quantum Hilbert space) that is required to be annihilated by the *conserved* (anti-)BRST and (anti-)co-BRST charges of our modified 2D Proca theory which has been proven to be an *interesting* model for the Hodge theory [6, 7]. We have also demonstrated that the discrete *duality* symmetry transformations [see, Eq. (29) below] connect the (anti-)BRST charges with the (anti-)co-BRST charges and vice-versa. Hence, the restrictions imposed by *these* conserved and nilpotent charges on the *physical* state are also intimately connected due to the *duality* transformations (29). These *restrictions* and their *connections* are *true* at the *tree-level* as well as any arbitrary loop-level diagrams in the perturbative computations for a given physical process.

The theoretical contents of our present endeavor are organized as follows. First of all, to set the notations and convention, we begin with the general form of the Proca theory in the next section and discuss the (anti-)BRST symmetries in any arbitrary dimension of spacetime for the Stückelberg-modified version of *this* theory. The third section is devoted to the *modification* of the *standard* Stückelberg formalism in 2D for the *massive* Abelian 1-form (i.e. Proca) theory where we provide the mathematical basis for the existence as well as validity of this *modification*. The subject matter of the fourth section deals with the

existence of the (anti-)BRST and (anti-)co-BRST symmetries *together* and their intimate relationships due to the existence of the discrete duality symmetry [cf. Eq. (29)] for the *modified* version of 2D Proca theory. Finally, in the last section, we summarize our key results and point out a few future directions for further investigations in the context of higher p -form ($p = 2, 3\dots$) theories.

2 Preliminaries: (Anti-)BRST Symmetries

We begin with the following Lagrangian density $[\mathcal{L}_{(P)}]$ for the *massive* Abelian 1-form ($A^{(1)} = dx^\mu A_\mu$) vector-boson in any arbitrary dimension of spacetime (see, e.g. [3])

$$\mathcal{L}_{(P)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \quad (1)$$

where the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (with $\mu, \nu\dots = 0, 1, 2\dots D-1$) has been derived from the 2-form $F^{(2)} = dA^{(1)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) F_{\mu\nu}$ where $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative of the differential geometry [8-10] and m is the rest mass of the vector boson. It can be readily checked that under the following *standard* Stückelberg technique, the vector bosonic field A_μ is *modified* in the following manner [3]

$$A_\mu \longrightarrow A_\mu \mp \frac{1}{m} \partial_\mu \phi, \quad (2)$$

where ϕ is a *pure* scalar field. Insertion of (2) into (1) leads to the following form of the Stueckelberg-modified Lagrangian density $[\mathcal{L}_{(S)}]$ in any arbitrary dimension of spacetime

$$\mathcal{L}_{(P)} \longrightarrow \mathcal{L}_{(S)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \mp m A_\mu \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (3)$$

which respects [i.e. $\delta_g \mathcal{L}_{(S)} = 0$] the following continuous and infinitesimal local *classical* gauge symmetry transformations (δ_g)

$$\delta_g A_\mu = \partial_\mu \chi, \quad \delta_g \phi = \pm m \chi, \quad \delta_g F_{\mu\nu} = 0, \quad (4)$$

where $\chi(x)$ is the *local* gauge transformation parameter*. The *latter* can be exploited within the purview of BRST formalism to derive the off-shell nilpotent (anti-)BRST symmetry transformations $[s_{(a)b}]$ in terms of the *fermionic* (anti-)ghost fields (\bar{C}) C as (see, e.g. [7])

$$\begin{aligned} s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= -i B, & s_{ab} B &= 0, & s_{ab} \phi &= \pm m \bar{C}, \\ s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= i B, & s_b B &= 0, & s_b \phi &= \pm m C, \end{aligned} \quad (5)$$

*The Lagrangian density (3) is endowed with the first-class constraints: $\Pi^0 \approx 0$, $\vec{\nabla} \cdot \vec{E} \mp m \Pi_\phi \approx 0$ where $\Pi^0 = -F^{00} \approx 0$ is the primary constraint and $\frac{\partial \Pi^0}{\partial t} = \vec{\nabla} \cdot \vec{E} \mp m \Pi_\phi \approx 0$ is the secondary constraint with $\Pi_\phi = \dot{\phi} \mp m A_0$ as the canonical conjugate momentum w.r.t. the ϕ field. These *first-class* constraints are present in the generator $G = \int d^{D-1}x [\dot{\chi} \Pi^0 - \chi (\vec{\nabla} \cdot \vec{E} \mp m \Pi_\phi)]$ which leads to the derivation of the *local* and infinitesimal classical gauge symmetry transformations (4) in any arbitrary dimension of spacetime.

which are respected by the following (anti-)BRST invariant Lagrangian density (\mathcal{L}_b) for the *modified* Proca theory in any arbitrary dimension of spacetime, namely,

$$\begin{aligned}
\mathcal{L}_b &= \mathcal{L}_{(S)} + s_b s_{ab} \left[\frac{i}{2} A^\mu A_\mu - \frac{i}{2} \phi^2 - \frac{1}{2} \bar{C} C \right], \\
&\equiv \mathcal{L}_{(S)} + s_b \left[-i \bar{C} \{(\partial \cdot A) \pm m \phi + \frac{1}{2} B\} \right], \\
&\equiv \mathcal{L}_{(S)} + s_{ab} \left[i C \{(\partial \cdot A) \pm m \phi + \frac{1}{2} B\} \right],
\end{aligned} \tag{6}$$

where B is the Nakanishi-Lautrup auxiliary field and the fermionic ($C^2 = 0$, $\bar{C}^2 = 0$, $C \bar{C} + \bar{C} C = 0$) (anti-)ghost fields (\bar{C}) C are invoked to maintain the *unitarity* in the theory. In its full blaze of glory, the Lagrangian density \mathcal{L}_b looks as:

$$\begin{aligned}
\mathcal{L}_b &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \mp m A_\mu \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
&\quad + B (\partial \cdot A \pm m \phi) + \frac{B^2}{2} - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C.
\end{aligned} \tag{7}$$

It is straightforward to check that $s_b \mathcal{L}_b = \partial_\mu [B \partial^\mu C]$ and $s_{ab} \mathcal{L}_b = \partial_\mu [B \partial^\mu \bar{C}]$ which render the D -dimensional action integral $S = \int d^D x \mathcal{L}_b$ *invariant* under $s_{(a)b}$.

According to the celebrated Noether theorem, the above continuous, off-shell nilpotent ($s_{(a)b}^2 = 0$), absolutely anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) and *infinitesimal* symmetry transformations lead to the following explicit expressions for the (anti-)BRST invariant ($s_{ab} J_{(ab)}^\mu = 0$, $s_b J_{(b)}^\mu = 0$) Noether conserved currents ($J_{(r)}^\mu$ with $r = ab, b$):

$$\begin{aligned}
J_{(ab)}^\mu &= -F^{\mu\nu} \partial_\nu \bar{C} + B \partial^\mu \bar{C} \pm m \bar{C} \partial^\mu \phi - m^2 \bar{C} A^\mu, \\
J_{(b)}^\mu &= -F^{\mu\nu} \partial_\nu C + B \partial^\mu C \pm m C \partial^\mu \phi - m^2 C A^\mu.
\end{aligned} \tag{8}$$

The conservation law $\partial_\mu J_{(r)}^\mu = 0$ (with $r = b, ab$) can be proven by using the following Euler-Lagrange (EL) equations of motion (EoM), namely,

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial^\nu B \pm m \partial^\nu \phi - m^2 A^\nu, & (\square + m^2) C &= 0, \\
\square \phi \mp m (\partial \cdot A) &= \pm m B, & (\square + m^2) \bar{C} &= 0.
\end{aligned} \tag{9}$$

The above conserved currents lead to the definition of conserved charges as

$$\begin{aligned}
Q_b &= \int d^{D-1} x J_{(b)}^0 = \int d^{D-1} x \left[-F^{0i} (\partial_i C) + B \dot{C} \pm m C \dot{\phi} - m^2 A_0 C \right] \\
&\equiv \int d^{D-1} x [B \dot{C} - \dot{B} C], \\
Q_{ab} &= \int d^{D-1} x J_{(ab)}^0 = \int d^{D-1} x \left[-F^{0i} (\partial_i \bar{C}) + B \dot{\bar{C}} \pm m \bar{C} \dot{\phi} - m^2 A_0 \bar{C} \right] \\
&\equiv \int d^{D-1} x [B \dot{\bar{C}} - \dot{B} \bar{C}],
\end{aligned} \tag{10}$$

where we have applied the Gauss divergence theorem to drop the total *space* derivative terms and used: $\dot{B} = \vec{\nabla} \cdot \vec{E} + m^2 A_0 \mp m \dot{\phi}$ [that arises from Eq. (9)] to derive the concise

forms of conserved[†] charges $Q_{(a)b}$. It is straightforward to note that the following *exact* forms of the charges w.r.t. $s_{(a)b}$ are *true*, namely,

$$\begin{aligned} Q_b &= \int d^{D-1} x [s_b \{i \dot{\bar{C}} C - i \bar{C} \dot{C}\}] \equiv \int d^{D-1} x [s_{ab} (i C \dot{C})], \\ Q_{ab} &= \int d^{D-1} x [s_{ab} \{i \bar{C} \dot{C} - i \dot{\bar{C}} C\}] \equiv \int d^{D-1} x [s_b (-i \bar{C} \dot{C})], \end{aligned} \quad (11)$$

which establish the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges as illustrated below

$$\begin{aligned} s_b Q_b &= -i \{Q_b, Q_b\} = 0 \implies Q_b^2 = 0 \iff s_b^2 = 0, \\ s_{ab} Q_{ab} &= -i \{Q_{ab}, Q_{ab}\} = 0 \implies Q_{ab}^2 = 0 \iff s_{ab}^2 = 0, \\ s_{ab} Q_b &= -i \{Q_b, Q_{ab}\} = 0 \implies s_{ab}^2 = 0 \iff \{Q_b, Q_{ab}\} = 0, \\ s_b Q_{ab} &= -i \{Q_{ab}, Q_b\} = 0 \implies s_b^2 = 0 \iff \{Q_{ab}, Q_b\} = 0, \end{aligned} \quad (12)$$

where the basic principle behind the *continuous* symmetry transformations and their *generators* has been used. It is interesting to point out that the off-shell nilpotency [i.e. $Q_{(a)b}^2 = 0$] of the conserved (anti-)BRST charges ($Q_{(a)b}$) is deeply related with the off-shell nilpotency [i.e. $s_{(a)b}^2 = 0$] of the (anti-)BRST transformations $s_{(a)b}$. However, we note that the absolute anticommutativity of the BRST charge *with* the anti-BRST charge is connected with the nilpotency ($s_{ab}^2 = 0$) of the anti-BRST transformations (s_{ab}). On the other hand, the absolute anticommutativity of the anti-BRST charge *with* the BRST charge is intimately connected with the off-shell nilpotency ($s_b^2 = 0$) of the BRST transformations (s_b). These statements are corroborated by a close and careful look at Eqs. (11) and (12). These observations will be very useful when we shall discuss the conserved (anti-)co-BRST charges and corresponding symmetries for the Stückelberg-modified 2D Proca theory in the fourth section.

3 Stückelberg Formalism for the 2D Proca Theory: Mathematical Basis for Its Modification

The 2D Proca theory is very *special* in the sense that the field strength tensor $F_{\mu\nu}$ has only *one* existing component which is nothing but the electric field (i.e. $F_{01} = E$). The *latter* (i.e. the electric field E) turns out to be a *pseudo-scalar* because it is a *single* object which changes sign under the parity symmetry transformation. Furthermore, as far as the mathematical aspect of the Stückelberg modification [cf. Eq. (2)] in any arbitrary dimension of spacetime is concerned, we note that the vector field A_μ (belonging to the 1-form $A^{(1)} = dx^\mu A_\mu$) is modified by a 1-form $\Phi^{(1)} = d\Phi^{(0)} = d\phi \equiv dx^\mu \partial_\mu \phi$ where the 0-form $\Phi^{(0)} = \phi$ is a *pure* scalar field and $d = dx^\mu \partial_\mu$ is the exterior derivative of the differential geometry [8-10]. A close look at Eq. (2) shows that the 0-form scalar field ϕ has

[†]The conservation law (i.e. $\dot{Q}_{(a)b} = 0$) for the concise forms of the (anti-)BRST charges $Q_{(a)b}$ can be easily proven by using the EL-EoMs: $(\square + m^2) B = 0$, $(\square + m^2) C = 0$, $(\square + m^2) \bar{C} = 0$ which emerge out from the variation of the action integral $S = \int d^{D-1} x \mathcal{L}_b$ defined w.r.t. the Lagrangian density \mathcal{L}_b .

been incorporated into the Stückelberg modification with a *mass* factor in the denominator on the dimensional ground (in the natural units: $\hbar = c = 1$). It is straightforward to note that, for the 2D Proca theory, the mass dimensions of A_μ and ϕ fields are *zero*. Hence, the modification in Eq. (2) is *correct* on the dimensional ground. As a passing remarks, it is interesting to point out that the field strength tensor $F_{\mu\nu}$ remains invariant under the modification (2) due to the well-known Stückelberg formalism.

As pointed out earlier, the two (1+1)-dimensional (2D) theory is very *special* because we have the freedom to add/subtract a pseudo-scalar field $\tilde{\phi}$ in the modification (2) with some condition. In other words, an axial-vector 1-form $\tilde{\Phi}^{(1)} = d\tilde{\Phi}^{(0)} \equiv dx^\mu \partial_\mu \tilde{\phi}$ constructed with a pseudo-scalar field $\tilde{\phi}$ (i.e. the 0-form $\tilde{\Phi}^{(0)} = \tilde{\phi}$) is at our disposal. However, the 1-form $\tilde{\Phi}^{(1)}$ is a pseudo-vector (i.e. axial-vector). To make it a *polar-vector* in 2D, we have to take a *single* Hodge duality $*$ operation on it (with the input: $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$), as follows:

$$*\tilde{\Phi}^{(1)} = *dx^\mu \partial_\mu \tilde{\phi} = \varepsilon^{\mu\nu} dx_\nu \partial_\mu \tilde{\phi} \equiv dx^\mu (-\varepsilon_{\mu\nu} \partial^\nu \tilde{\phi}), \quad (13)$$

where, at this stage, the explicit expression: $-\varepsilon_{\mu\nu} \partial^\nu \tilde{\phi}$ is a *polar-vector* in 2D modified version of Proca theory. In the mathematical language, the modification (2) can now be expressed, in terms of the 1-forms, as

$$A^{(1)} \longrightarrow A^{(1)} \mp \frac{1}{m} d\Phi^{(0)} \pm \frac{1}{m} *d\tilde{\Phi}^{(0)}, \quad (14)$$

which leads to the following explicit *modification* of the Stückelberg technique, namely,

$$A_\mu \longrightarrow A_\mu \mp \frac{1}{m} \partial_\mu \phi \mp \frac{1}{m} \varepsilon_{\mu\nu} \partial^\nu \tilde{\phi}. \quad (15)$$

It should be pointed out that the mass term (m) has to be present in the *third* term, too, in the above equation on the dimensional ground because the mass dimension of the pseudo-scalar field ($\tilde{\phi}$) is also *zero* in 2D theory. As a side remark, we note that the field-strength tensor $F_{\mu\nu}$ does *not* remain *invariant* under (15) [which was *invariant* under the *original* modification (2)]. In fact, the field-strength tensor $F_{\mu\nu}$ explicitly transforms, under the *modified* Stückelberg-technique (15), as follows:

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} \mp \frac{1}{m} [\varepsilon_{\nu\lambda} \partial_\mu \partial^\lambda \tilde{\phi} - \varepsilon_{\mu\lambda} \partial_\nu \partial^\lambda \tilde{\phi}]. \quad (16)$$

The special feature of 2D theory is the observation that *only* $F_{01} = E$ exists for the field strength tensor $F_{\mu\nu}$. This implies that the following is true due to (16), namely,

$$F_{01} \longrightarrow F_{01} \mp (\partial_1 \partial_1 \tilde{\phi} - \partial_0 \partial_0 \tilde{\phi}). \quad (17)$$

In other words, the electric field ($E = F_{01}$) of the Proca theory is modified, under the *modified* 2D Stückelberg-technique (15) (with the choice: $\varepsilon_{01} = +1 = \varepsilon^{10}$), as follows:

$$E \longrightarrow E \pm \frac{1}{m} \square \tilde{\phi}, \quad \square = \partial_0 \partial_0 - \partial_1 \partial_1. \quad (18)$$

Thus, ultimately, the 2D Proca Lagrangian density ($\mathcal{L}_{(P)}^{(2D)}$) is modified to $\mathcal{L}_{(S)}^{(2D)}$

$$\begin{aligned} \mathcal{L}_{(P)}^{(2D)} = \frac{1}{2} E^2 + \frac{m^2}{2} A_\mu A^\mu \quad \longrightarrow \quad \mathcal{L}_{(S)}^{(2D)} = \frac{1}{2} (E \pm \frac{1}{m} \square \tilde{\phi})^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \\ + \frac{m^2}{2} A_\mu A^\mu \mp m A_\mu \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \end{aligned} \quad (19)$$

where $\mathcal{L}_{(S)}^{(2D)}$ is the 2D Stückelberg-modified version of the Lagrangian density for the Proca theory (modulo some *total* spacetime derivative terms). We lay emphasis on the fact that the square of the *first-term* of (19) leads to

$$(E \pm \frac{1}{m} \square \tilde{\phi})^2 = E^2 + \frac{1}{m^2} (\square \tilde{\phi})^2 \pm \frac{2}{m} E \square \tilde{\phi}, \quad (20)$$

where the last two terms are the higher derivative terms for a 2D theory in view of the fact that $E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$ (with the inputs: $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, $\varepsilon^{10} = +1 = \varepsilon_{01}$) *also* contains a partial derivative. However, there is a *solution* to this problem if we assume, at this stage, that the pseudo-scalar field $\tilde{\phi}$ obeys the Klein-Gordon equation[‡]: $(\square + m^2) \tilde{\phi} = 0$. Thus, ultimately, the substitution $\square \tilde{\phi} = -m^2 \tilde{\phi}$ leads to the following form of the Lagrangian density from the modified Lagrangian density in Eq. (19), namely,

$$\mathcal{L}_{(S)}^{(2D)} = \frac{1}{2} (E \mp m \tilde{\phi})^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu \mp m A_\mu \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (21)$$

which has been *considered* by us in our earlier works [6, 7]. However, this is for the *first-time*, we are able to derive the Lagrangian density (21) by exploiting the theoretical beauty and strength of the 2D *modified* Stückelberg technique that has been quoted in Eq. (15).

At this crucial juncture, there are a few remarks on the theoretical beauty of the Lagrangian density (21). First of all, we find that the kinetic terms for the *pure* scalar field (ϕ) and pseudo-scalar field ($\tilde{\phi}$) carry *opposite* signs. In other words, the pseudo-scalar field ($\tilde{\phi}$) carries a *negative* kinetic term which is interesting from the point of view of the existence of the possible candidates for the dark matter and dark energy (see. e.g. [11, 12]). Furthermore, such fields (which have been christened as the “ghost” and “phantom” fields in the realm of modern-day cosmology) play important roles in the cyclic, bouncing and self-accelerated cosmological models of the Universe (see. e.g. [13-15]). Second, the signs of *all* the terms of the Lagrangian density (21), along with the *gauge-fixing* term [3] in the 't Hooft gauge (\mathcal{L}_{gf}) [cf. Eq. (7)], namely,

$$\begin{aligned} \mathcal{L}_{(S)}^{(2D)} + \mathcal{L}_{(gf)} &= \frac{1}{2} (E \mp m \tilde{\phi})^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu \\ &\mp m A_\mu \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} (\partial \cdot A \pm m \phi)^2, \end{aligned} \quad (22)$$

[‡]We shall see *later* that the *final* Lagrangian density [cf. Eqs. (22), (27)], with the replacement: $\square \tilde{\phi} = -m^2 \tilde{\phi}$, does lead to the derivation of the Klein-Gordon equation [i.e. $(\square + m^2) \tilde{\phi} = 0$] for the pseudo-scalar field [cf. Eqs. (22) and (24) below] which has been used in the derivation of (21) from (19).

are *fixed* as it respects a set of a couple of very useful discrete *duality* symmetry transformations (modulo some *total* spacetime derivative terms). These duality[§] transformations for the basic fields $(A_\mu, \phi, \tilde{\phi})$ along with E and $(\partial \cdot A)$ are listed as follows:

$$\begin{aligned} A_\mu &\rightarrow \mp i \varepsilon_{\mu\nu} A^\nu, & \phi &\rightarrow \mp i \tilde{\phi}, & \tilde{\phi} &\rightarrow \mp i \phi, \\ E &\rightarrow \pm i (\partial \cdot A), & (\partial \cdot A) &\rightarrow \pm i E. \end{aligned} \quad (23)$$

Third, we note that the Lagrangian density (22) leads to the EL-EoMs for the fields ϕ and $\tilde{\phi}$ as the Klein-Gordon equations, namely,

$$(\square + m^2) \phi = 0, \quad (\square + m^2) \tilde{\phi} = 0. \quad (24)$$

It is interesting to point out that *one* of the above equations [i.e. $(\square + m^2) \tilde{\phi} = 0$] has been used in the derivation of (22) from (19). Fourth, we note that the *mass* term for the Proca field A_μ (i.e. $\frac{m^2}{2} A_\mu A^\mu$) remains *invariant* under the discrete *duality* symmetry transformations (23). As a consequence, the fields ϕ and $\tilde{\phi}$ carry the *same* rest mass (m). Fifth, we note that the 2D *modified* Stückelberg-technique (15) remains *invariant* under the discrete *duality* symmetry[¶] transformations (23). Finally, we observe that the gauge-fixed Lagrangian density (22) respects the local and infinitesimal *gauge* and *dual-gauge* symmetry transformations δ_g and δ_{dg} , respectively, as listed below (see, e.g. [7])

$$\begin{aligned} \delta_g A_\mu &= \partial_\mu \Sigma, & \delta_g \phi &= \pm m \Sigma, & \delta_g E &= 0, & \delta_g \tilde{\phi} &= 0, \\ \delta_{dg} A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \Omega, & \delta_{dg} \tilde{\phi} &= \mp m \Omega, & \delta_{dg} E &= \square \Omega, & \delta_{dg} \phi &= 0, \end{aligned} \quad (25)$$

if we impose *exactly* the same kinds of restrictions $(\square + m^2) \Sigma = 0$, $(\square + m^2) \Omega = 0$ on the *gauge* and *dual-gauge* symmetry transformation parameters Σ and Ω because we note that the following transformations of the Lagrangian density (22) are *true*, namely,

$$\begin{aligned} \delta_g [\mathcal{L}_{(S)}^{(2D)} + \mathcal{L}_{(gf)}] &= -(\partial \cdot A \pm m \phi) [\square + m^2] \Sigma, \\ \delta_{dg} [\mathcal{L}_{(S)}^{(2D)} + \mathcal{L}_{(gf)}] &= \partial_\mu \left[m \varepsilon^{\mu\nu} (m A_\nu \Omega \pm \phi \partial_\nu \Omega) \pm m \tilde{\phi} \partial^\mu \Omega \right] \\ &+ (E \mp m \tilde{\phi}) [\square + m^2] \Omega. \end{aligned} \quad (26)$$

We shall see in the next section that, within the framework of BRST formalism, there is no *outside* restrictions on the (anti-)ghost fields which are the *generalization* of the *classical gauge* and *dual-gauge* transformation parameters Σ and Ω to the *quantum* level.

4 Comments on (Anti-)BRST and (Anti-)Co-BRST Symmetries: 2D Modified Proca Theory

The (anti-)BRST and (anti-)co-BRST symmetry invariant Lagrangian density for the 2D *modified* Proca theory is the *generalization* of the (anti-)BRST invariant Lagrangian density

[§]The root-cause behind (23) is the self-duality condition on 2D Abelian 1-form. In other words, we note that: $*A^{(1)} = *(dx^\mu A_\mu) = dx^\mu (-\varepsilon_{\mu\nu} A^\nu) \equiv dx^\mu \tilde{A}_\mu$. The transformation $A_\mu \rightarrow \mp i \varepsilon_{\mu\nu} A^\nu$ in (23) owes its origin to the self-duality condition where the dual-vector $\tilde{A}_\mu = -\varepsilon_{\mu\nu} A^\nu$.

[¶]Invariance of the *modified* Stückelberg-technique (15) under the discrete symmetry transformations (23) and/or (29) demonstrates that there is a perfect *duality* symmetry in the 2D modified version of Proca theory. In fact, we note that (15) and (23) are intertwined *together* in a subtle fashion.

(7) (valid in any arbitrary D-dimensions of spacetime) as^{||} (see, e.g. [7])

$$\begin{aligned}\mathcal{L}_{(B)} &= \mathcal{B}(E \mp m \tilde{\phi}) - \frac{1}{2} \mathcal{B}^2 \pm m E \tilde{\phi} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{m^2}{2} A_\mu A^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\ &\mp m A_\mu \partial^\mu \phi + B(\partial \cdot A \pm m \phi) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C + i m^2 \bar{C} C,\end{aligned}\quad (27)$$

where we have (i) invoked Nakanishi-Lautrup auxiliary field \mathcal{B} to linearize the kinetic term of the Lagrangian density (22), and (ii) *chosen* a specific form of the Lagrangian density. In fact, there are *four* Lagrangian densities corresponding to (22) which have been discussed in our earlier work [7]. We have taken only *one specific* form in (27) which is *different* from the choices taken in [7]. It is straightforward to note that the (anti-)BRST transformations (5) are *now* generalized with *additional* transformations for our 2D theory as:

$$\begin{aligned}s_{ab} A_\mu &= \partial_\mu \bar{C}, & s_{ab} \bar{C} &= 0, & s_{ab} C &= -i B, & s_{ab} \phi &= \pm m \bar{C}, \\ s_{ab} B &= 0, & s_{ab} E &= 0, & s_{ab} \mathcal{B} &= 0, & s_{ab} \tilde{\phi} &= 0, \\ s_b A_\mu &= \partial_\mu C, & s_b C &= 0, & s_b \bar{C} &= +i B, & s_b \phi &= \pm m C, \\ s_b B &= 0, & s_b E &= 0, & s_b \mathcal{B} &= 0, & s_b \tilde{\phi} &= 0.\end{aligned}\quad (28)$$

Once again, we note that $s_b \mathcal{L}_B = \partial_\mu [B \partial^\mu C]$, $s_{ab} \mathcal{L}_B = \partial_\mu [B \partial^\mu \bar{C}]$. Hence, the action integral $S = \int d^2x \mathcal{L}_{(B)}$ remains (anti-)BRST invariant for the physical fields which vanish-off as $x \rightarrow \pm\infty$ due to the application of Gauss's divergence theorem.

Before we derive the (anti-)co-BRST symmetry transformations from (28), it is interesting to point out that the discrete *duality* symmetry transformations (23) are generalized for the (anti-)BRST and (anti-)co-BRST *invariant* 2D Lagrangian density $\mathcal{L}_{(B)}$ as:

$$\begin{aligned}A_\mu &\rightarrow \mp i \varepsilon_{\mu\nu} A^\nu, & \phi &\rightarrow \mp i \tilde{\phi}, & \tilde{\phi} &\rightarrow \mp i \phi, & B &\rightarrow \mp i \mathcal{B}, & \mathcal{B} &\rightarrow \mp i B, \\ C &\rightarrow \pm i \bar{C} & \bar{C} &\rightarrow \pm i C, & (\partial \cdot A) &\rightarrow \pm i E & E &\rightarrow \pm i (\partial \cdot A).\end{aligned}\quad (29)$$

In other words, we find that the Lagrangian density $\mathcal{L}_{(B)}$ remains invariant under (29). As a consequence of the discrete *duality* symmetry transformations (29), we find that the nilpotent (anti-)co-BRST symmetry transformations $[s_{(a)d}]$ can be derived from the nilpotent (anti-)BRST symmetry transformations (28) as follows:

$$\begin{aligned}s_{ad} A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu C, & s_{ad} C &= 0, & s_{ad} \bar{C} &= +i \mathcal{B}, & s_{ad} \mathcal{B} &= 0, & s_{ad} \phi &= 0, \\ s_{ad} B &= 0, & s_{ad} \tilde{\phi} &= \mp m C, & s_{ad} (\partial \cdot A) &= 0, & s_{ad} E &= \square C, \\ s_d A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, & s_d \bar{C} &= 0, & s_d C &= -i \mathcal{B}, & s_d \mathcal{B} &= 0, & s_d \phi &= 0, \\ s_d B &= 0, & s_d \tilde{\phi} &= \mp m \bar{C}, & s_d (\partial \cdot A) &= 0, & s_d E &= \square \bar{C}.\end{aligned}\quad (30)$$

Let us dwell a bit on the derivation of (30) from (28) *and vice-versa* by exploiting the sheer beauty of the discrete *duality* symmetry transformations (29). Let us first focus on the gauge field A_μ which transforms under the (anti-)BRST symmetry transformations as: $s_{ab} A_\mu = \partial_\mu \bar{C}$, $s_b A_\mu = \partial_\mu C$. From these nilpotent transformations, we can derive

^{||}We would like to point out that our choice of the 2D Lagrangian density in Eq. (27) is quite different from the thread-bare discussions on various forms of the Lagrangian densities in our earlier work [7].

the off-shell nilpotent (i.e. fermionic) (anti-)co-BRST symmetry transformations $s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C$, $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$ by using the transformations (29) and the replacements $s_b \rightarrow s_d$, $s_{ab} \rightarrow s_{ad}$. It can be explicitly checked that $s_b A_\mu = \partial_\mu C$ goes to $s_d (\mp i \varepsilon_{\mu\nu} A^\nu) = \partial_\mu (\pm i \bar{C})$ under (29) which, ultimately, implies that $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$. In exactly similar fashion, we note that $s_{ab} A_\mu = \partial_\mu C$ goes to $s_{ad} (\mp i \varepsilon_{\mu\nu} A^\nu) = \partial_\mu (\pm i C)$ under (29) and the replacement $s_{ab} \rightarrow s_{ad}$, which finally, leads to $s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C$. We can repeat this exercise with $s_b \bar{C} = i B$, $s_{ab} C = -i B$ which, due to (29) and replacements $s_b \rightarrow s_d$, $s_{ab} \rightarrow s_{ad}$, lead to the derivation of $s_d C = -i \mathcal{B}$, $s_{ad} \bar{C} = +i \mathcal{B}$. We lay emphasis on the fact that the reciprocal relationships *also* exist**. To be precise, we can obtain $s_b A_\mu = \partial_\mu C$ from the co-BRST symmetry transformation $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$ by the replacement $s_d \rightarrow s_b$ and the discrete *duality* transformations (29). In other words, we have: $s_b (\mp i \varepsilon_{\mu\nu} A^\nu) = -\varepsilon_{\mu\nu} \partial^\nu (\pm i C)$ from $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$ which implies $s_b A_\mu = \partial_\mu C$. Thus, we can derive (28) from (30), too. In exactly similar manner, we observe that the concise forms of the (anti-)co-BRST charges $Q_{(a)d}$, for our present 2D theory, can be derived from the concise forms of the (anti-)BRST charges (10), for $D = 2$, as

$$Q_d = \int dx [\mathcal{B} \dot{\bar{C}} - \dot{\mathcal{B}} \bar{C}], \quad Q_{ad} = \int dx [\mathcal{B} \dot{C} - \dot{\mathcal{B}} C], \quad (31)$$

due to the discrete symmetry transformations (29). Hence, the nilpotent symmetry transformations (28) and (30) as well as the conserved charges (10) and (31) are interconnected due to the *duality* symmetry transformations (29). To be more precise, in the mathematical language, we have a set of very interesting mappings: $s_{(a)b} \leftrightarrow s_{(a)d}$ and $Q_{(a)b} \leftrightarrow Q_{(a)d}$ due to the existence of a couple of very beautiful *duality* symmetry transformations (29) in our theory. It is an elementary exercise to note that we can *also* derive the concise forms of Q_b and Q_{ab} for our 2D theory from Q_d and Q_{ad} [cf. Eq. (31)] by using the discrete duality symmetry transformations (29).

We end this section with a couple of final remarks. First, the (anti-)co-BRST symmetry transformations (30), derived from the (anti-)BRST symmetry transformations (28), are the *symmetry* transformations for the Lagrangian density (27) as evident from the following:

$$\begin{aligned} s_{ad} \mathcal{L}_{(B)} &= \partial_\mu \left[(\mathcal{B} \pm m \tilde{\phi}) \partial^\mu C \pm m \varepsilon^{\mu\nu} (\phi \partial_\nu C \mp m C A_\nu) \right], \\ s_d \mathcal{L}_{(B)} &= \partial_\mu \left[(\mathcal{B} \pm m \tilde{\phi}) \partial^\mu \bar{C} \pm m \varepsilon^{\mu\nu} (\phi \partial_\nu \bar{C} \pm m \bar{C} A_\nu) \right]. \end{aligned} \quad (32)$$

As a consequence of the the above observations, it is clear that the action integral $S = \int d^2x \mathcal{L}_{(B)}$ remains invariant under (30) for the physical fields which vanish-off as $x \rightarrow \pm\infty$ due to Gauss's divergence theorem. Second, we note that all the *exact* forms of the (anti-)BRST charges [that have been expressed in Eq. (11)] can be replicated in the context of conserved and nilpotent (anti-)co-BRST charges. The *latter* can be expressed in terms of the nilpotent (anti-)co-BRST symmetry transformations due to the replacements $Q_b \rightarrow Q_d$, $Q_{ab} \rightarrow Q_{ad}$, $s_b \rightarrow s_d$, $s_{ab} \rightarrow s_{ad}$ along with the *discrete* duality symmetry^{††} transformations that have been listed in Eq. (29).

**It is worthwhile to mention that, in an earlier work (see, for e.g. [7]), the derivation of $s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$ from $s_b A_\mu = \partial_\mu C$ has been carried out for the *on-shell* nilpotent s_d and s_b . However, the reciprocal relationship has *not* been established in [7].

^{††}Though the duality symmetry transformations (29) are able to provide the relationships between the

5 Conclusions

One of the main highlights and key results of our present investigation is the *modification* in the Stückelberg-technique [cf. Eq. (15)] for our 2D *massive* Abelian 1-form gauge theory which has led to the derivation of the Lagrangian density (21) that is the fulcrum of our whole discussion in our present investigation. For our 2D *modified* version of Proca theory (i.e. a *massive* Abelian 1-form theory) the most *fundamental* symmetries are the *continuous* (anti-)BRST and (anti-)co-BRST transformations along with the *discrete* duality transformations (29). From these *fermionic* continuous symmetries, one can define a *unique* bosonic continuous symmetry transformation. There is a *continuous* ghost-scale symmetry, too, in our theory. An elegant blend of *continuous* and *discrete* symmetry transformations provide the physical realizations (see, e.g. [6, 7] for details) of the de Rham cohomological operators of the differential geometry [8-10]. The *discrete* duality symmetry transformations [cf. Eq. (29)], in particular, provide the physical realization of the Hodge duality $*$ operation of the differential geometry [6, 7] where it has been established that $s_{(a)d} = \pm * s_{(a)b} *$. This relationship provides the analogue of the mathematical relationship $\delta = \pm * d *$ of differential geometry that exists between the co-exterior derivative (δ) and the exterior derivative (d). In our present investigation, we have shown the direct utility of the *duality* transformations (29) in establishing the mapping $s_{(a)d} \leftrightarrow s_{(a)b}$ which has *not* been accomplished in [7] for the off-shell nilpotent (anti-)BRST ($s_{(a)b}$) and (anti-)co-BRST symmetry transformations ($s_{(a)d}$). Furthermore, the Lagrangian density (27) is different from the Lagrangian density considered in [7]. The physical state of our theory is the *harmonic* state (in the Hodge decomposed state) that is annihilated by the (anti-)BRST and (anti-)co-BRST charges which lead to the annihilation of the physical state (out of *total* states of the *quantum* Hilbert space) by the operator forms of the *first-class* constraints and their *dual* versions. For instance, it is evident that $Q_{(a)b}|phys\rangle = 0$ implies that $B|phys\rangle = 0$ and $\dot{B}|phys\rangle = 0$. In other words, the first-class constraints $B(= \Pi^0)|phys\rangle = 0$ and $\dot{B}(= \vec{\nabla} \cdot \vec{E} \mp m \Pi_\phi)|phys\rangle = 0$ annihilate the physical states which are consistent with the Dirac quantization condition^{††}. In exactly similar fashion, the condition: $Q_{(a)d}|phys\rangle = 0$ leads to the $\mathcal{B}|phys\rangle = 0$ and $\dot{\mathcal{B}}|phys\rangle = 0$ which imply that $(E \mp m \tilde{\phi})|phys\rangle = 0$ and $\frac{d}{dt}(E \mp m \tilde{\phi})|phys\rangle = 0$ which are nothing but the *dual* versions of the first-class constraints [as can be seen by applying (29) on the *first-class* constraints]. These conditions are very sacrosanct and these are valid at any arbitrary loop-level diagrams in perturbation theory. In our very recent work [18], we have been able to show the importance of the annihilation of the *physical state* by the *dual* versions of the first-class constraints which, ultimately, imply that the 2D ABJ anomaly term (i.e. the

continuous symmetry transformations $s_{(a)b}$ and $s_{(a)d}$ and corresponding conserved charges, these continuous symmetries and conserved charges have their own *independent* identity as do the exterior and co-exterior derivatives of differential geometry [8-10].

^{††}It can be *also* seen that the conditions: $B|phys\rangle = 0$ and $\dot{B}|phys\rangle = 0$ (which emerge out from the physicality criteria: $Q_{(a)b}|phys\rangle = 0$) imply that: $(\partial \cdot A \pm m \phi)|phys\rangle = 0$ and $\frac{d}{dt}(\partial \cdot A \pm m \phi)|phys\rangle = 0$ where $(\partial \cdot A \pm m \phi)$ is the *dual* of the quantity $(E \mp m \tilde{\phi})$. The *latter* restriction [i.e. $(E \mp m \tilde{\phi})|phys\rangle = 0$] on the physical state emerges out from $Q_{(a)d}|phys\rangle = 0$. The discussion on the various aspects of constraints, gauge symmetry transformations, symmetry generators, etc., has been clarified very beautifully in a couple of earlier works (see, e.g. [16, 17]).

pseudo-scalar electric field E) is *trivial* in an *interacting* 2D gauge theory where there is a coupling between the gauge field and matter (Dirac) fields. In fact, we have proved that our 2D *interacting* Proca theory (with the Dirac fields) is *also* a tractable field-theoretic example of Hodge theory (see, e.g. [18] for details). Our present discussion can be generalized for the *massive* 4D Abelian 2-form and 6D Abelian 3-form gauge theories where there will be *modifications* of the celebrated Stückelberg formalism because the *above* higher p -form ($p = 2, 3, \dots$) gauge theories have been proven to be the tractable examples of Hodge theory [19, 20]. It will be a nice future endeavor to discuss the Stückelberg-modified SUSY QED where the question of infrared problem and the existence of the ultralight particles as the dark matter candidates have been discussed in a very nice piece of recent work [21].

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