

# Multiplicative updates for symmetric-cone factorizations

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## Abstract

Given a matrix  $X \in \mathbb{R}_+^{m \times n}$  with non-negative entries, the cone factorization problem over a cone  $\mathcal{K} \subseteq \mathbb{R}^k$  concerns computing  $\{a_1, \dots, a_m\} \subseteq \mathcal{K}$  and  $\{b_1, \dots, b_n\} \subseteq \mathcal{K}^*$  belonging to its dual so that  $X_{ij} = \langle a_i, b_j \rangle$  for all  $i \in [m], j \in [n]$ . Cone factorizations are fundamental to mathematical optimization as they allow us to express convex bodies as feasible regions of linear conic programs. In this paper, we introduce and analyze the symmetric-cone multiplicative update (SCMU) algorithm for computing cone factorizations when  $\mathcal{K}$  is symmetric; i.e., it is self-dual and homogeneous. Symmetric cones are of central interest in mathematical optimization as they provide a common language for studying linear optimization over the nonnegative orthant (linear programs), over the second-order cone (second order cone programs), and over the cone of positive semidefinite matrices (semidefinite programs). The SCMU algorithm is multiplicative in the sense that the iterates are updated by applying a meticulously chosen automorphism of the cone computed using a generalization of the geometric mean to symmetric cones. Using an extension of Lieb’s concavity theorem and von Neumann’s trace inequality to symmetric cones, we show that the squared loss objective is non-decreasing along the trajectories of the SCMU algorithm. Specialized to the nonnegative orthant, the SCMU algorithm corresponds to the seminal algorithm by Lee and Seung for computing Nonnegative Matrix Factorizations.

## 1 Introduction

A fundamental problem in mathematical optimization is to maximize a linear function over a convex constraint set. An important consideration for the development of tractable numerical algorithms for such problems is that the constraint set admits a compact description. A powerful paradigm for obtaining compact descriptions of convex sets is based on *lifts* over structured convex cones. More precisely, given a convex set  $P$ , an *extended formulation* or lift of  $P$  over a (full-dimensional closed) convex cone  $\mathcal{K}$  is a description of  $P$  as the projection of an affine slice of the cone  $\mathcal{K}$ ; i.e.,  $P = \pi(\mathcal{K} \cap \mathcal{A})$ , where  $\pi$  is a linear projection and  $\mathcal{A}$  an affine subspace.

Given a  $\mathcal{K}$ -lift, the problem of maximizing linear functions over  $P$  reduces to one of solving a *linear conic program* over the cone  $\mathcal{K}$ . Linear conic programs (LCPs) capture many important classes of optimization problems – for instance, LCPs where the cone  $\mathcal{K} = \mathbb{R}_+^k$  is the  $k$ -dimensional nonnegative orthant correspond to Linear Programs (LPs), LCPs where  $\mathcal{K} = \mathbb{S}_+^k$  is the cone of  $k \times k$  positive semidefinite matrices correspond to Semidefinite Programs (SDPs), and linear conic programs where  $\mathcal{K} = \mathbb{L}_k = \{(x, t) \in \mathbb{R}^k \times \mathbb{R} : \|x\|_2 \leq t\}$  is the  $k$ -dimensional second order cone

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correspond to Second Order Cone programs (SOCPs). We often refer to extended formulations over  $\mathbb{R}_+^k, \mathbb{S}_+^k, \mathbb{L}_k$  as LP, SDP, and SOCP-lifts respectively.

Lifted descriptions are powerful tool for optimization because there are many examples of convex sets whose descriptions (at least on the surface) appear to be complex (such as by being specified by a number of inequalities that is exponential in the ambient dimensions) but do in fact admit compact lifted descriptions. Examples of such constraint sets include the unit ball of the  $\ell_1$  norm, the spanning tree polytopes [20, 27], and the permutahedra [10].

The choice of cone  $\mathcal{K}$  also matters – the smallest known LP-lift for the stable set polytope of a perfect graph with  $n$  vertices is  $n^{O(\log n)}$  [28] whereas there exists an SDP-lift of size  $n + 1$  using the theta body [21], and in fact it is the latter representation that gives rise to the *only known* polynomial-time algorithm for finding the largest stable set in a perfect graph. Moreover, the dimension of the cone  $\mathcal{K}$  is fundamentally linked to the computational complexity required to solve the associated linear conic program. As such, for an extended formulation to be practically useful for optimization, it is important to seek extended formulations involving cones of low dimensionality.

Despite the usefulness of extended formulations, it is not immediately clear how one systematically searches for such descriptions. To this end, Yannakakis shows that LP-lifts of polytopes are fundamentally linked to the existence of a structured matrix factorization of a slack matrix [28], while Gouveia, Parrilo, and Thomas [11] extend this connection to lifts using more general cones.

We explain this connection more precisely: Given an entrywise non-negative matrix  $X \in \mathbb{R}_+^{n \times m}$  and a (full-dimensional convex) cone  $\mathcal{K}$  lying in inner product space  $(V, \langle \cdot, \cdot \rangle)$ , the *cone factorization problem* concerns finding two collections of vectors  $a_1, \dots, a_n$  in the cone  $\mathcal{K}$  and  $b_1, \dots, b_m$  in the dual cone  $\mathcal{K}^*$  where

$$X_{ij} = \langle a_i, b_j \rangle, \text{ for all } i \in [n], j \in [m]. \quad (1)$$

In the case where  $\mathcal{K} = \mathbb{R}_+^k$ , the cone factorization problem reduces to the *Non-negative Matrix Factorization* (NMF) problem in which we seek a collection of non-negative vectors  $a_1, \dots, a_m \in \mathbb{R}_+^k$  and  $b_1, \dots, b_n \in \mathbb{R}_+^k$  such that  $X_{ij} = \langle a_i, b_j \rangle$  for all  $i \in [m]$  and  $j \in [n]$ . The smallest  $k \in \mathbb{N}$  for which  $X$  has an  $k$ -dimensional NMF is the *nonnegative rank* of the matrix  $X$ . NMFs were initially studied within the field of linear algebra (see, e.g., [4]), and later gained prominence as a dimensionality reduction tool providing interpretable parts-based representations of non-negative data [18].

In the context of LP-lifts, Yannakakis [28] showed that the existence of an  $\mathbb{R}_+^k$ -lift for a polytope  $P = \{x \in \mathbb{R}^d : \langle c_i, x \rangle \leq d_i, i \in [\ell]\}$  is equivalent to the existence of a  $k$ -dimensional NMF of its *slack matrix*  $S_P$ , a nonnegative rectangular matrix where the rows are indexed by the bounding hyperplanes  $\{x : \langle c_i, x \rangle \leq d_i\}$  of  $P$ , the columns are indexed by the extreme points  $v_j$ , and the  $ij$ -entry is given by  $S_{ij} = d_i - \langle c_i, v_j \rangle$ . Subsequently, Gouveia, Parrilo, and Thomas [11] showed that a polytope  $P$  admits an extended formulation over a (full-dimensional) closed convex cone  $\mathcal{K}$  if its slack matrix  $P$  admits an exact cone factorization over  $\mathcal{K}$ . In fact, this relationship holds for convex sets beyond polyhedral ones, although it is then necessary to extend the notion of a slack matrix to that of a *slack operator* indexed over the infinite set of extreme points and/or facets.

Cone factorizations also have important applications beyond optimization. One such prominent example is in quantum information science, where factorizations over the PSD cone  $\mathbb{S}_+^k$  are relevant to the quantum correlation generation problem [13] and one-way quantum communication complexity [9]. Specifically, given a non-negative matrix  $X$ , the *positive semidefinite matrix factorization* (PSDMF) problem concerns computing two families of  $k \times k$  PSD matrices  $A_1, \dots, A_m$  and  $B_1, \dots, B_n$  such that  $X_{ij} = \text{tr}(A_i B_j)$  for all  $i \in [m], j \in [n]$ . The smallest  $k$  for which  $X$  admits an  $k$ -dimensional PSDMF is known as the *PSD-rank* of  $X$  [7].

**Computing cone factorizations.** The task of computing an exact cone factorization is in general intractable. Concretely, in the specific setting where  $\mathcal{K}$  is the non-negative orthant, Vavasis showed that even computing the non-negative rank is NP-hard [26]. On the positive side, Arora et al. propose an algorithm for computing  $k$ -dimensional NMFs whose complexity is polynomial time in the dimensions of the matrix, provided the rank parameter  $k$  is held constant [1].

Despite the hardness of NMFs, a wide range of numerical algorithms for computing (approximate) NMFs have been developed and implemented in a wide range of data analytical applications. Most, if not all, algorithmic approaches are based on the principle of alternating minimization. One of the most prominent approaches for computing NMFs is the Multiplicative Update (MU) algorithm proposed by Lee and Seung [19]. The update scheme is based on the majorization-minimization framework [17], and it operates by performing pointwise scaling by carefully selected non-negative weights. In fact, it is the simplicity of the MU scheme that drives its popularity. Other alternative methods include variants of projected gradient methods and coordinate descent.

There are also a wide range of methods for computing PSDMFs, which operate in a similar vein by alternating minimization. In particular, Vandaele et al. propose algorithmic approaches based on the projected gradient method and coordinate descent [24], while the authors [14, 15, 16] develop algorithms by drawing connections between PSDMFs to the affine rank minimization and the phase retrieval problems from the signal processing literature. Recently, Soh and Varvitsiotis introduced the Matrix Multiplicative Update (MMU) method, which is the analogue of the MU scheme for computing PSDMFs in which updates are performed by congruence scaling with appropriately chosen PSD matrices [22]. In particular, the Matrix Multiplicative Update scheme retains the simplicity that the MU update scheme for NMF offers.

**Contributions.** In this work, we introduce the *symmetric-cone multiplicative update* (SCMU) algorithm for computing cone factorizations (1) in the case where the cone  $\mathcal{K}$  is *symmetric*; that is, it is self dual and homogenous (i.e., the automorphism group of  $\mathcal{K}$  acts transitively on  $\text{int}(\mathcal{K})$ ). The SCMU algorithm corresponds to Lee-Seung’s Multiplicative update algorithm [19] for NMF when  $\mathcal{K}$  is the non-negative orthant, and to the Matrix Multiplicative update algorithm in [22] for PSDMF when  $\mathcal{K}$  is the cone of PSD matrices.

The SCMU scheme is based on the principles of the majorization-minimization framework, and is *multiplicative* in the sense that iterates are updated by applying an appropriately chosen automorphism of (the interior) of the cone  $\mathcal{K}$ . As a result, the SCMU algorithm ensures that iterates remain in the interior of the cone  $\mathcal{K}$ , provided it is initialized in the interior. In terms of performance guarantees for the SCMU algorithm, we prove is that the Euclidean squared loss is non-increasing along its trajectories and moreover, we also show that fixed points of our scheme correspond to first-order stationary points. Additionally, if the starting factorization lies in a direct sum of simple symmetric cones, the direct sum structure is preserved throughout the execution of the SCMU algorithm. As such, the SCMU algorithm specifies a method for computing hybrid cone factorizations.

In terms of applications, when applied to computing cone factorizations of slack matrices, the SCMU algorithm provides a practical way to compute conic lifts of convex sets – for instance, it can be used to compute approximate SOCP lifts when  $\mathcal{K}$  is chosen to be (products of) the second order cone  $\mathbb{L}_k = \{(x, t) \in \mathbb{R}^k \times \mathbb{R} : \|x\|_2 \leq t\}$ . In particular, we give explicit examples where we apply the SCMU algorithm for computing SOCP lifts of regular polygons.

**Paper Organization.** In Section 2, we provide necessary background material concerning Euclidean Jordan Algebras and Symmetric Cones. In Section 3, we describe our approach and derive

the multiplicative update algorithm for symmetric-cone factorizations. In Section 4, we show that the Euclidean square loss is nonincreasing along the algorithms' trajectories and that fixed points satisfy the first-order optimality conditions. In Section 5, we conclude with numerical experiments focusing on lifts over the second-order cone.

## 2 Euclidean Jordan Algebras and Symmetric Cones

In this section, we provide brief background on symmetric cones to describe our algorithm and our analysis. Our discussion requires the formal language of *Euclidean Jordan algebras* (EJAs) from which symmetric cones arise. For further details and omitted proofs, we refer the reader to [5, 25].

**Jordan Algebras.** Let  $\mathcal{J}$  be a finite-dimensional vector space endowed with a bilinear product  $\circ : \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ . We say that the pair  $(\mathcal{J}, \circ)$  form a *Jordan algebra* if the following properties hold:

$$\begin{aligned} x \circ y &= y \circ x, \\ x^2 \circ (x \circ y) &= x \circ (x^2 \circ y). \end{aligned}$$

Here, we use the shorthand notation  $x^2 = x \circ x$ . In the remainder of this paper, we assume that the Jordan algebra  $(\mathcal{J}, \circ)$  has an identity element; that is, there exists  $e \in \mathcal{J}$  such that  $e \circ x = x \circ e = x$ .

**Euclidean Jordan Algebras.** A Jordan algebra  $(\mathcal{J}, \circ)$  over  $\mathbb{R}$  which is equipped with an associative inner product  $(\cdot, \cdot)$ , (i.e.,  $(x \circ y, z) = (y, x \circ z)$ , for all  $x, y, z$ ) is called *Euclidean*.

**Spectral decomposition and powers.** Our algorithm requires the computation of square-roots as well as inverses of Jordan Algebra elements. To explain how these are defined, we require a *spectral decomposition theorem* for EJAs. In what follows, we describe a version of this theorem based on *Jordan frames* (see Type-II spectral decomposition [5, Theorem III.1.2]).

More concretely, an *idempotent* is an element  $x \in \mathcal{J}$  satisfying  $x^2 = x$ . We say that an idempotent is *primitive* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a collection of primitive idempotents  $\{c_1, \dots, c_m\} \subseteq \mathcal{J}$  form a *Jordan frame* if they satisfy (i)  $c_i \circ c_j = 0$  if and only if  $i \neq j$ , and (ii)  $\sum_i c_i = e$ . It follows that distinct elements of a Jordan frame  $c_i$  are orthogonal with respect to the inner product  $(\cdot, \cdot)$ .

For every  $x \in \mathcal{J}$  there exists a Jordan frame  $c_1, \dots, c_r$ , and real numbers  $\lambda_1, \dots, \lambda_r$  with  $x = \sum_{j=1}^r \lambda_j c_j$ . Here, the value  $r$  is called the *rank* of  $x$ . The scalars  $\{\lambda_1, \dots, \lambda_r\}$  are called the *eigenvalues* of  $x$  and – up to re-ordering and accounting for multiplicities – are uniquely specified.

Given  $x \in \mathcal{J}$  with spectral decomposition  $x = \sum_i \lambda_i c_i$ , we define its  $a$ -th power as  $x^a = \sum_i \lambda_i^a c_i$  whenever  $\lambda_i^a$  exists for all  $i$ . We say that an element  $x \in \mathcal{J}$  is *invertible* if all its eigenvalues are nonzero.

**The canonical trace inner product.** In our description of an EJA so far, we have only assumed the existence of an inner product  $(\cdot, \cdot)$ . There is in fact a canonical choice: Given the spectral decomposition  $x = \sum_{j=1}^r \lambda_j c_j$ , define the *trace* to be the sum of the eigenvalues  $\text{tr}(x) = \sum_{j=1}^r \lambda_j$ . Then, the bilinear mapping  $\text{tr}(x \circ y)$  is positive definite, symmetric, and also satisfies the associativity property (see, for instance, [5, Proposition II.4.3]). Furthermore, in the case where the EJA is *simple* (i.e., it cannot be expressed as the direct sum of smaller EJAs), the inner product  $(\cdot, \cdot)$  is a positive scalar multiple of the canonical one  $\text{tr}(x \circ y)$  ([5, Proposition III.4.1]). In the remainder of this paper we use the notation  $\langle x, y \rangle = \text{tr}(x \circ y)$  to denote the canonical EJA inner product.

**The Lyapunov transformation and the quadratic representation.** Consider an EJA  $(\mathcal{J}, \circ)$ . As the algebra product  $\circ$  is bilinear, given any  $x$ , there is a matrix  $L(x)$  satisfying  $x \circ y = L(x)y$  for

all  $y \in \mathcal{J}$ . We note that it is fairly common in the literature to express the defining properties of an EJA in terms of the Lyapunov operator  $L(x)$ . For instance, the requirement  $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$  is equivalent to the requirement that the operator  $L(x)$  commutes with  $L(x^2)$ , while the requirement  $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$  is equivalent to the requirement  $\langle L(x)y, z \rangle = \langle y, L(x)z \rangle$ , i.e., the operator  $L(x)$  is symmetric with respect to the trace inner product.

The *quadratic representation* of  $x \in \mathcal{J}$  is defined by  $P(x) = 2L^2(x) - L(x^2)$ . The term quadratic alludes to the fact that  $P(x)e = x^2$ . The quadratic representation satisfies the following properties:

$$\langle P(x)y, z \rangle = \langle y, P(x)z \rangle \tag{2}$$

$$P(\lambda x) = \lambda^2 P(x) \tag{3}$$

$$P(P(x)y) = P(x)P(y)P(x) \tag{4}$$

$$(P(x))^a = P(x^a), \text{ if } x^a \text{ is defined} \tag{5}$$

$$P(x)(x^{-1}) = x, \text{ if } x \text{ is invertible} \tag{6}$$

$$P(x)(S_{\mathcal{J}}) = S_{\mathcal{J}} \text{ and } P(x)(\text{int}(S_{\mathcal{J}})) = \text{int}(S_{\mathcal{J}}), \text{ for all invertible } x \in \mathcal{J} \tag{7}$$

$$P(x) \succeq 0, \text{ for any } x \in S_{\mathcal{J}} \tag{8}$$

$$P(x) \succ 0, \text{ for any } x \in \text{int}(S_{\mathcal{J}}) \tag{9}$$

**Cone of squares and symmetric cones.** Given an inner product space  $(V, \phi(\cdot, \cdot))$ , we say that a cone  $\mathcal{K} \subset V$  is *symmetric* if:

(i)  $\mathcal{K}$  is *self-dual*; i.e.,  $\mathcal{K}^* = \{y \in V : \phi(y, x) \geq 0 \forall x \in \mathcal{K}\} = \mathcal{K}$ , and

(ii)  $\mathcal{K}$  is *homogeneous*; i.e., given  $u, v \in \text{int}(\mathcal{K})$ , there is an invertible linear map  $T$  such that  $T(u) = v$ , and  $T(\mathcal{K}) = (\mathcal{K})$  (or  $\text{Aut}(\mathcal{K})$  acts transitively on  $\text{int}(\mathcal{K})$ ).

Given an EJA  $(\mathcal{J}, \circ, (\cdot, \cdot))$ , the cone of squares is defined as the set

$$S_{\mathcal{J}} = \{x^2 : x \in \mathcal{J}\}.$$

The set  $S_{\mathcal{J}}$  is self-dual with respect to the inner product  $\langle x, y \rangle = \text{tr}(x \circ y)$ , and hence closed and convex.

The key connection between symmetric cones and EJAs is that *all* symmetric cones arise as cone of squares of some EJA; see e.g., [5, Theorem III.3.1]. In particular, the homogeneity property of the cone of squares follows from the existence of a *scaling point*; i.e., for  $x, y \in \text{int}(S_{\mathcal{J}})$  it holds

$$P(w)x = y, \text{ where } w = P(x^{-1/2})(P(x^{1/2})y)^{1/2}. \tag{10}$$

As both  $x, y \in \text{int}(S_{\mathcal{J}})$ , it follows by (9) that  $P(y^{1/2})$  and  $P(x^{-1/2})$  are positive definite. Consequently, the product  $P(w) = P(y^{1/2})P(x^{-1/2})$  is also positive definite and thus invertible. By (7),  $P(w)$  defines an automorphism of  $S_{\mathcal{J}}$ .

**Classification of EJAs and Symmetric Cones.** There is in fact a complete classification of all EJAs. Let  $\mathbb{C}, \mathbb{H}, \mathbb{O}$  denote the fields of complex numbers, quaternions, and octonions respectively. Then every *finite*-dimensional EJA  $(\mathcal{J}, \circ)$  is isomorphic to a direct sum of these *simple* EJAs:

1. Symmetric matrices over  $\mathbb{R}$  or Hermitian matrices over  $\mathbb{C}$  or  $\mathbb{H}$  with  $X \circ Y = (XY + YX)/2$ .
2. The space  $\mathbb{R} \times \mathbb{R}^k$  with product  $x \circ y = (x_0, x_1) \circ (y_0, y_1) = (x^T y, x_0 y_1 + y_0 x_1)$ .
3. The space of 3-by-3 Hermitian matrices over  $\mathbb{O}$  with product  $X \circ Y = (XY + YX)/2$ .

Subsequently, since all symmetric cones arise as cones of squares of some EJA, it follows that any symmetric cone corresponds to a direct sum of:

1. Symmetric PSD matrices over  $\mathbb{R}$  or Hermitian PSD matrices over  $\mathbb{C}$  or  $\mathbb{H}$
2. Second-order cones, i.e.,  $\mathbb{L}_d = \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : \|x\|_2 \leq t\}$ .
3. 3-by-3 PSD matrices over octonions  $\mathbb{O}$ .

In Table 1, we summarize the basic information for the EJA corresponding to the cone of positive semidefinite matrices and the second order cone.

**The Geometric Mean.** Given a symmetric cone  $\mathcal{K}$ , the (*metric*) *geometric mean* of two elements  $x, y \in \text{int}(\mathcal{K})$  is defined as the scaling point that takes  $x^{-1}$  to  $y$  [23], i.e.,

$$x \# y = P(x^{1/2})(P(x^{-1/2})y)^{1/2}. \quad (11)$$

As established in (10), the geometric mean satisfies:

$$P(x \# y)x^{-1} = y. \quad (12)$$

In fact, it turns out that the geometric mean is the unique element of  $\text{int}(\mathcal{K})$  satisfying (12). Indeed, suppose  $w \in \text{int}(\mathcal{K})$  satisfies  $P(w)x^{-1} = y$ . We then have

$$P(x^{-1/2})y = P(x^{-1/2})P(w)x^{-1} = P(x^{-1/2})P(w)P(x^{-1/2})e = P(P(x^{-1/2})w)e = (P(x^{-1/2})w)^2. \quad (13)$$

As  $P(x^{-1/2})w$  and  $P(x^{-1/2})y$  are both elements of  $\mathcal{K}$  (recall (7)), the equality in (13) implies that  $P(x^{-1/2})w = (P(x^{-1/2})y)^{1/2}$  and thus  $w = P(x^{1/2})(P(x^{-1/2})y)^{1/2} = x \# y$ .

Finally, the geometric mean satisfies the following useful properties (e.g. see [23]):

$$x \# y = y \# x, \quad (x \# y)^{-1} = y^{-1} \# x^{-1}, \quad x \# e = x^{1/2}. \quad (14)$$

### 3 Deriving the Symmetric-cone Multiplicative Update Algorithm

Let  $\mathcal{K} \subseteq \mathbb{R}^d$  be a symmetric cone, and let  $X \in \mathbb{R}_+^{m \times n}$  be an entrywise non-negative matrix. Our goal is to compute vectors  $a_1, \dots, a_m$  and  $b_1, \dots, b_n$  belonging to the cone  $\mathcal{K}$  such that

$$X_{ij} \approx \langle a_i, b_j \rangle, \quad \text{for all } i \in [m], j \in [n].$$

More formally, we frame our problem as a minimization instance over the squared loss objective:

$$\arg \min_{a_i \in \mathcal{K}, b_j \in \mathcal{K}} \sum_{i,j} (X_{ij} - \langle a_i, b_j \rangle)^2. \quad (15)$$

In order to (approximately) solve (15), we alternate between minimizing the  $a_i$ 's and the  $b_j$ 's. Consider the sub-problem corresponding to fixing the variables  $\{a_i\}$  and minimizing over  $\{b_j\}$ . The objective (15) is separable in the  $b_j$ 's. After dropping the suffix  $j$  to simplify our notation, the problem simplifies to

$$\arg \min_b \|Ab - x\|_2^2 \quad \text{s.t.} \quad b \in \mathcal{K}, \quad (16)$$

Cone of squares	PSD matrices	Second-order cone
Ambient dimension	$X \in \mathbb{R}^{n \times n}$	$(t, x) \in \mathbb{R} \times \mathbb{R}^n$
$\circ$	$X \circ Y = \frac{XY+YX}{2}$	$(t, x) \circ (s, u) = (st + \langle x, u \rangle, tu + sx)$
Identity	$I$	$(1, 0, \dots, 0)$
Trace	$\sum X_{ii}$	$2t$
Rank (of interior)	$n$	$2$
Eigendecomposition	Spectral decomposition	$(t + \ x\ ) \begin{pmatrix} \frac{1}{2} \\ \frac{x}{2\ x\ } \end{pmatrix} + (t - \ x\ ) \begin{pmatrix} \frac{1}{2} \\ -\frac{x}{2\ x\ } \end{pmatrix}$
$x^{1/2}$	$X^{1/2}$	$\sqrt{t + \ x\ } \begin{pmatrix} \frac{1}{2} \\ \frac{x}{2\ x\ } \end{pmatrix} + \sqrt{t - \ x\ } \begin{pmatrix} \frac{1}{2} \\ -\frac{x}{2\ x\ } \end{pmatrix}$
$x^{-1}$	$X^{-1}$	$\frac{1}{t + \ x\ } \begin{pmatrix} \frac{1}{2} \\ \frac{x}{2\ x\ } \end{pmatrix} + \frac{1}{t - \ x\ } \begin{pmatrix} \frac{1}{2} \\ -\frac{x}{2\ x\ } \end{pmatrix}$

Table 1: Summary of key properties of the EJA associated to the cone of positive semidefinite matrices and the second order cone.

where  $x$  is a fixed vector (the  $j$ -th column of  $X$ ) and  $\mathcal{A} : \mathbb{R}^d \rightarrow \mathbb{R}^m$  is the linear mapping

$$x \mapsto \mathcal{A}x = (\langle a_1, x \rangle, \dots, \langle a_m, x \rangle)^\top.$$

We solve the convex optimization problem (16) via the Majorization-Minimization (MM) approach (see, e.g., [17] and references therein). In order to solve a problem  $\min\{f(x) : x \in \mathcal{X}\}$  using the MM framework, we need to identify a family of *auxilliary* functions  $u_x : \text{dom}f \rightarrow \mathbb{R}$ , indexed by  $x \in \text{dom}f$ , satisfying these conditions

$$\begin{aligned} f(y) &\leq u_x(y), \text{ for all } y \in \mathcal{X}, \text{ and} \\ f(x) &= u_x(x). \end{aligned} \tag{17}$$

The MM update scheme is given by

$$x^{k+1} = \arg \min\{u_{x^k}(y) : y \in \mathcal{X}\}.$$

Subsequently, the objective  $f$  is non-increasing along the trajectory of  $x^k$ , which follows easily from the inequalities

$$f(x^{k+1}) \leq u_{x^k}(x^{k+1}) \leq u_{x^k}(x^k) = f(x^k).$$

Recall that in our setting, the goal is to minimize the function  $f(b) = \frac{1}{2}\|\mathcal{A}b - x\|_2^2$  over the symmetric cone  $\mathcal{K}$ . As the objective is quadratic, the Taylor expansion of  $f$  at the past iterate  $b_k$  is

$$f(b) = f(b_k) + (b - b_k)^\top \nabla f(b_k) + \frac{1}{2}\|\mathcal{A}(b - b_k)\|_2^2. \tag{18}$$

Suppose we restrict our search to quadratic auxilliary functions of the form

$$u_{b_k}(b) = f(b_k) + (b - b_k)^\top \nabla f(b_k) + \frac{1}{2}(b - b_k)^\top P(w)(b - b_k), \tag{19}$$

for some appropriate  $w \in \text{int}(\mathcal{K})$ . Using such an auxilliary function, to compute the next iterate, we need to minimize the function in (19) over the cone  $\mathcal{K}$

$$b_{k+1} = \arg \min_{b \in \mathcal{K}} u_{b_k}(b). \tag{20}$$

Note that although the objective (19) is strictly convex since choosing  $w \in \text{int}(\mathcal{K})$  ensures that  $P(w)$  is positive definite, it is not immediately clear how one handles the constraint  $b \in \mathcal{K}$ . Instead, we show that it is possible to make a specific choice of  $w \in \text{int}(\mathcal{K})$  such that the minimizer of the unconstrained auxiliary problem (20) lies in the interior of  $\mathcal{K}$ . In such a setting, we would have

$$\arg \min_{b \in \mathcal{K}} u_{b_k}(b) = \arg \min_{b \in \mathbb{R}^d} u_{b_k}(b), \quad (21)$$

and hence we can simply calculate  $b_{k+1}$  as the unconstrained minimum of (19), namely

$$b_{k+1} = b_k - P(w^{-1})\nabla f(b_k) = b_k - P(w^{-1})(\mathcal{A}^\top \mathcal{A}b_k - \mathcal{A}^\top x). \quad (22)$$

Furthermore, suppose we pick  $w \in \text{int}(\mathcal{K})$  so that

$$b_k - P(w^{-1})\mathcal{A}^\top \mathcal{A}b_k = 0. \quad (23)$$

Then, the MM update  $b_{k+1}$  is given by

$$b_{k+1} = P(w^{-1})\mathcal{A}^\top x = \sum_{i=1}^m x_i P(w^{-1})a_i.$$

Suppose that  $a_i \in \text{int}(\mathcal{K})$ , and that the vector  $x$  is non-zero. From (7), it follows that  $b_{k+1}$  is a non-negative linear sum of elements in  $\text{int}(\mathcal{K})$ , and hence  $b_{k+1} \in \text{int}(\mathcal{K})$  so long as  $x \neq 0$ .

The equation (23) specifies  $w$  uniquely; specifically, (23) is equivalent to  $P(w)b_k = \mathcal{A}^\top \mathcal{A}b_k$ , which by the scaling point interpretation of the geometric mean (recall (12)) implies that

$$w = b_k^{-1} \# \mathcal{A}^\top \mathcal{A}b_k. \quad (24)$$

Note that for (24) to exist, we require  $b_k$  and  $\mathcal{A}^\top \mathcal{A}b_k$  to be in  $\text{int}(\mathcal{K})$ .

Finally, to check that the function  $u_{b_k}(b)$  given in (19) corresponding to  $w = b_k^{-1} \# \mathcal{A}^\top \mathcal{A}b_k$  is indeed an auxiliary function, we need to verify that the two conditions (17) are satisfied. The proof of this fact is deferred to the next section and is the main technical result in this paper.

Summarizing the preceding discussion, employing the MM approach to minimize the function  $f(b) = \frac{1}{2} \|\mathcal{A}b - x\|_2^2$  over the symmetric cone  $\mathcal{K}$  using an auxiliary function of the form (19) with  $w = b_k^{-1} \# \mathcal{A}^\top \mathcal{A}b_k$  leads to the following the update rule:

$$b_{k+1} = P(b_k \# (\mathcal{A}^\top \mathcal{A}b_k)^{-1})\mathcal{A}^\top x.$$

In a similar fashion, we get an update rule for the  $a_i$ 's when the  $b_j$ 's are fixed, where  $\mathcal{A}$  is replaced by

$$x \mapsto \mathcal{B}x = (\langle b_1, x \rangle, \dots, \langle b_m, x \rangle).$$

We summarize our procedure for computing factorizations over symmetric cones in Algorithm 1.

### 3.1 Two important special cases

First, we specialize the SCMU algorithm to the setting where  $\mathcal{K} = \mathbb{R}_+^k$ , in which case the SCMU algorithm gives an iterative method for computing NMFs. The non-negative orthant is the cone of squares of the EJA  $(\mathbb{R}^k, \circ)$ , where the Jordan product is componentwise multiplication, i.e.,  $x \circ y = \text{diag}(x)y$ . Moreover, the trace of  $x$  is just its 1-norm, the Lyapunov transformation is  $L(x) = \text{diag}(x)$  and the quadartic mapping  $P(x) = \text{diag}(x)^2$ . Finally, the metric geometric mean (11) of  $x, y \in \mathbb{R}_+^k$  is given by  $x \# y = (\sqrt{x_1 y_1}, \dots, \sqrt{x_k y_k})$ .



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**Algorithm 1** Symmetric-Cone Multiplicative Update (SCMU) algorithm

---

**Input:** A non-negative matrix  $X \in \mathbb{R}_+^{m \times n}$

**Output:** Vectors  $a_1, \dots, a_m, b_1, \dots, b_n \in \mathcal{K}$  with  $X_{ij} \approx \langle a_i, b_j \rangle, \forall i, j$

While stopping criterion not satisfied:

$$\begin{aligned} a_i &\leftarrow P(a_i \# (\mathcal{B}^\top \mathcal{B} a_i)^{-1}) \mathcal{B}^\top X_{i,:}, \text{ for all } 1 \leq i \leq m \\ b_j &\leftarrow P(b_j \# (\mathcal{A}^\top \mathcal{A} b_j)^{-1}) \mathcal{A}^\top X_{:,j}, \text{ for all } 1 \leq j \leq n \end{aligned}$$


---

Putting everything together, the  $\ell$ -th coordinate of the vector  $b_j$  is updated by:

$$b_j(\ell) \leftarrow b_j(\ell) \frac{\sum_{i=1}^m a_i(\ell) X_{ij}}{\sum_{i=1}^m a_i(\ell) \langle a_i, b_j \rangle}. \quad (25)$$

These updates correspond to the multiplicative update rule introduced by Lee and Seung in [19], which is one of the most widely used approaches for calculating NMFs.

Next, we specialize the SCMU algorithm for cone factorizations with  $k \times k$  PSD factors (with real entries). Letting  $\mathbb{S}^k$  denote the space of  $k \times k$  real symmetric matrices, the (real)  $k \times k$  PSD cone is the cone of squares of the EJA  $(\mathbb{S}^k, \circ)$ , where the Jordan product is given by  $X \circ Y = (XY + YX)/2$ . In this setting, the Lyapunov operator  $L(X)$  and the quadratic representation  $P(X)$  are superoperators (i.e., linear operators acting on a vector space of linear operators), and are concretely given by

$$\begin{aligned} \text{vec}(L(X)(Y)) &= \frac{1}{2}((X \otimes I) + (I \otimes X))\text{vec}(Y) \\ \text{vec}(P(X)Y) &= (X \otimes X)\text{vec}(Y), \end{aligned}$$

where  $\text{vec}(\cdot)$  is the vectorization operator. Using that  $\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B)$ , we get

$$P(X)Y = XYX. \quad (26)$$

Moreover, the trace is just the usual trace of a symmetric matrix (i.e., the sum of its eigenvalues). Lastly, using (26), the metric geometric mean (11) specializes to

$$X \# Y = X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2},$$

the usual geometric mean of two positive definite matrices, e.g. see [3]. Putting everything together, we have that

$$\begin{aligned} \mathcal{A} : \mathbb{S}^k &\rightarrow \mathbb{R}^m & Z &\mapsto (\text{tr}(A_1 Z), \dots, \text{tr}(A_m Z))^\top \\ \mathcal{A}^\top : \mathbb{R}^m &\rightarrow \mathbb{S}^k & x &\mapsto \sum_{i=1}^m x_i A_i \end{aligned}$$

and thus, the SCMU algorithm in the case  $\mathcal{K} = \mathbb{S}_+^k$  specializes to:

$$B_j \leftarrow S_j(\mathcal{A}^\top X_{:,j})S_j, \text{ where } S_j = B_j \# ([\mathcal{A}^\top \mathcal{A}]B_j)^{-1}.$$

This is exactly the Matrix Multiplicative update method derived in [22].

## 4 Performance Guarantees of the SCMU Algorithm

Our first result is to show that, given a symmetric cone  $\mathcal{K} \subseteq \mathbb{R}^d$ , the function  $u_{b_k}(b)$  defined in (19) does indeed parameterize a family of auxiliary functions for  $f(b) = \frac{1}{2}\|\mathcal{A}b - x\|_2^2$  over  $\mathcal{K}$ . To do so, we need to verify that the two properties given in (17) do indeed hold. We obviously have that  $f(b_k) = u_{b_k}(b_k)$ , and hence it remains to verify the domination property, namely that

$$f(b) \leq u_{b_k}(b), \quad \text{for all } b \in \mathcal{K}. \quad (27)$$

In fact, we show in the next theorem that this bound holds for all  $b \in \mathbb{R}^d$ .

**Theorem 4.1.** *Let  $b_k \in \text{int}(\mathcal{K})$  and define  $\mathcal{A}$  to be the linear map*

$$x \mapsto \mathcal{A}x = (\langle a_1, x \rangle, \dots, \langle a_m, x \rangle)^\top,$$

where  $a_1, \dots, a_m \in \text{int}(\mathcal{K})$ . Furthermore, let  $f(b) = \frac{1}{2}\|\mathcal{A}b - x\|_2^2$  and

$$u_{b_k}(b) = f(b_k) + (b - b_k)^\top \nabla f(b_k) + \frac{1}{2}(b - b_k)^\top P(b_k^{-1} \# \mathcal{A}^\top \mathcal{A} b_k)(b - b_k).$$

Then, we have that

$$\mathcal{A}^\top \mathcal{A} \preceq P(b_k^{-1} \# \mathcal{A}^\top \mathcal{A} b_k). \quad (28)$$

In particular, this immediately implies that

$$f(b) \leq u_{b_k}(b), \quad \text{for all } b \in \mathbb{R}^d. \quad (29)$$

*Proof.* We first focus on simple EJAs. We need to show the validity of the generalized inequality (28) with respect to the Euclidean inner product. Nevertheless, as for simple EJAs, any inner product is a positive multiple of the canonical one, it suffices to prove (28) for the canonical one.

The proof of (28) is broken down in two steps. First, we show that it suffices to consider the special case where  $b_k = e$ . More precisely, we first prove the inequality

$$\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} \preceq P(e \# \tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} e) \quad (30)$$

for all  $\mathcal{A}$  such that  $\tilde{a}_i \in \text{int}(\mathcal{K})$ . Assuming that (30) holds, we let  $\tilde{\mathcal{A}} = \mathcal{A}P(b^{1/2})$ , and let  $\tilde{w} = (\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} e)^{1/2}$ . Then

$$\tilde{w} = (\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} e)^{1/2} = (P(b^{1/2})\mathcal{A}^\top \mathcal{A} P(b^{1/2})e)^{1/2} = (P(b^{1/2})\mathcal{A}^\top \mathcal{A} b)^{1/2}.$$

Thus, by (11) we have

$$P(b^{-1/2})\tilde{w} = P(b^{-1/2})(P(b^{1/2})\mathcal{A}^\top \mathcal{A} b)^{1/2} = b^{-1} \# \mathcal{A}^\top \mathcal{A} b.$$

By (30) we have  $\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} \preceq P(\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} e)^{1/2} = P(\tilde{w})$ . Consequently, we have

$$P(b^{-1} \# \mathcal{A}^\top \mathcal{A} b) = P(P(b^{-1/2})\tilde{w}) = P(b^{-1/2})P(\tilde{w})P(b^{-1/2}) \succeq P(b^{-1/2})\tilde{\mathcal{A}}^\top \tilde{\mathcal{A}} P(b^{-1/2}) = \mathcal{A}^\top \mathcal{A},$$

where in the second equality we used (4). To conclude the proof, we prove (30) by showing the following two properties in Lemma 4.2 and Lemma 4.3 respectively:

$$\langle b, P^{\frac{1}{2}}(a_1)b \rangle + \langle b, P^{\frac{1}{2}}(a_2)b \rangle \leq \langle b, P^{\frac{1}{2}}((a_1 + a_2))b \rangle, \quad \text{for all } b \in \mathbb{R}^d. \quad (31)$$

$$\langle a, b \rangle^2 \leq \text{tr}(a) \langle b, P^{\frac{1}{2}}(a)b \rangle, \quad \text{for all } b \in \mathbb{R}^d. \quad (32)$$

Assuming the validity of (31) and (32), it is now easy to conclude that (30) holds. By definition of the map  $\mathcal{A}$  and (14) we have

$$e\#(\mathcal{A}^\top \mathcal{A}e) = (\mathcal{A}^\top \mathcal{A}e)^{1/2} = \left( \sum_{i=1}^m \langle a_i, e \rangle a_i \right)^{\frac{1}{2}} = \left( \sum_{i=1}^m \text{tr}(a_i \circ e) a_i \right)^{\frac{1}{2}} = \left( \sum_{i=1}^m \text{tr}(a_i) a_i \right)^{\frac{1}{2}}.$$

Consequently, (30) is equivalent to

$$\sum_{i=1}^m \langle a_i, b \rangle^2 \leq \left\langle b, P^{\frac{1}{2}} \left( \sum_{i=1}^m \text{tr}(a_i) a_i \right) b \right\rangle \quad (33)$$

Lastly, the proof of (33) follows from the following chain of inequalities:

$$\begin{aligned} \sum_{i=1}^m \langle a_i, b \rangle^2 &\leq \sum_{i=1}^m \text{tr}(a_i) \langle b, P(a_i^{1/2}) b \rangle \\ &= \sum_{i=1}^m \langle b, P^{1/2}(\text{tr}(a_i) a_i) b \rangle \\ &\leq \left\langle b, P^{\frac{1}{2}} \left( \sum_{i=1}^m \text{tr}(a_i) a_i \right) b \right\rangle, \end{aligned}$$

where for the first inequality we use (32), for the second equality we use property (3) of the quadratic representation, and for the last inequality we use (31).

Lastly, we consider the case where the EJA is a direct sum of simple ones. In this case, the cone of squares  $\mathcal{K}$  is a direct of simple symmetric cones, i.e.,  $\mathcal{K} = \mathcal{K}_1 \oplus \dots \oplus \mathcal{K}_k$  and the Jordan product is given by  $(x_1, \dots, x_k) \circ (y_1, \dots, y_k) = (x_1 \circ y_1, \dots, x_k \circ y_k)$  and  $\text{tr}(x_1, \dots, x_k) = \sum_{i=1}^k \text{tr}(x_i)$ . Then, it is easy to check that the operators  $\mathcal{A}^\top \mathcal{A}$  and  $P(\mathcal{A}^\top \mathcal{A}e)^{1/2}$  are separable with respect to the blocks corresponding to these individual simple EJAs. Thus, if the operator inequality  $\mathcal{A}^\top \mathcal{A} \preceq P(\mathcal{A}^\top \mathcal{A}e)^{1/2}$  holds for each individual block, it holds for the full-sized operators.  $\square$

Next we proceed with the proof of (31).

**Lemma 4.2.** *For any  $a_1, a_2 \in \text{int}(\mathcal{K})$  we have that*

$$P^{\frac{1}{2}}(a_1) + P^{\frac{1}{2}}(a_2) \preceq P^{\frac{1}{2}}(a_1 + a_2).$$

*Proof.* We will show that

$$\langle b, P^{\frac{1}{2}}(a_1) b \rangle + \langle b, P^{\frac{1}{2}}(a_2) b \rangle \leq \langle b, P^{\frac{1}{2}}(a_1 + a_2) b \rangle, \quad \text{for all } b \in \mathbb{R}^d;$$

For any EJA  $(\mathcal{J}, \circ)$  the function

$$f : S_{\mathcal{J}} \times S_{\mathcal{J}} \rightarrow \mathbb{R}, \quad (a, b) \mapsto \text{tr}(P(k) a^p \circ b^{1-p}),$$

is jointly concave for any fixed  $k \in \mathcal{J}$  and  $0 \leq p \leq 1$  [8, Theorem 3.1]. This the extension of Lieb's Concavity Theorem in the more general setting of EJAs. Also, by [8, Lemma 3.1] we have

$$\text{tr}(P(k) a^p \circ b^{1-p}) = \langle k, P(a^p, b^{1-p}) k \rangle,$$

and thus, it follows that for any fixed  $k \in \mathcal{J}$ , the mapping

$$(a, b) \mapsto \langle k, P(a^{1/2}, b^{1/2})k \rangle,$$

is concave; i.e., for any  $\lambda \in [0, 1]$  and  $(a_1, b_1), (a_2, b_2) \in S_{\mathcal{J}} \times S_{\mathcal{J}}$  we have

$$\begin{aligned} & \langle k, P((\lambda a_1 + (1 - \lambda)a_2)^{1/2}, (\lambda b_1 + (1 - \lambda)b_2)^{1/2})k \rangle \\ & \geq \lambda \langle k, P(a_1^{1/2}, b_1^{1/2})k \rangle + (1 - \lambda) \langle k, P(a_2^{1/2}, b_2^{1/2})k \rangle. \end{aligned}$$

Setting  $a_1 = b_1 = a/\lambda$  and  $a_2 = b_2 = b/(1 - \lambda)$ , and using that  $P(x, x) = P(x)$  we get

$$\langle k, P^{\frac{1}{2}}(a + b)k \rangle \geq \langle k, P^{\frac{1}{2}}(a)k \rangle + \langle k, P^{\frac{1}{2}}(b)k \rangle,$$

which is exactly (31).  $\square$

Next we proceed with the proof of (32).

**Lemma 4.3.** *For any  $a \in \text{int}(\mathcal{K})$  we have that*

$$\langle a, b \rangle^2 \leq \text{tr}(a) \langle b, P(a^{1/2})b \rangle, \quad \text{for all } b \in \mathbb{R}^d.$$

*Proof.* For this, note that:

$$\langle a, b \rangle = \text{tr}(a \circ b) = \text{tr}(P(a^{1/4})a^{1/2} \circ b) = \text{tr}(a^{1/2} \circ P(a^{1/4})b),$$

where for the last equality we used (2). Using that  $\text{tr}(a \circ b)^2 \leq \text{tr}(a^2)\text{tr}(b^2)$  (see [12]), we get

$$\langle a, b \rangle^2 = \text{tr}(a \circ b)^2 \leq \text{tr}(a) \text{tr}(P(a^{1/4})b \circ P(a^{1/4})b). \quad (34)$$

Lastly, we have that

$$\text{tr}(P(a^{1/4})b \circ P(a^{1/4})b) = \langle P(a^{1/4})b, P(a^{1/4})b \rangle = \langle b, P^2(a^{1/4})b \rangle = \langle b, P(a^{1/2})b \rangle, \quad (35)$$

where for the last equality we use (5). Combining (34) with (35) we get  $\langle a, b \rangle^2 \leq \text{tr}(a) \langle b, P(a^{1/2})b \rangle$ .  $\square$

In our last result in this section we show that fixed points of our update scheme correspond to first-order stationary points of the optimization problem (15).

**Theorem 4.4.** *Let  $\{a_i\}_{i \in [m]}, \{b_j\}_{j \in [n]} \in \text{int}(\mathcal{K})$  be fixed points of the multiplicative update rule:*

$$\begin{aligned} a_i & \leftarrow P(a_i \# (\mathcal{B}^\top \mathcal{B} a_i)^{-1}) \mathcal{B}^\top X_{:i}, \quad \text{for all } 1 \leq i \leq m \\ b_j & \leftarrow P(b_j \# (\mathcal{A}^\top \mathcal{A} b_j)^{-1}) \mathcal{A}^\top X_{:j}, \quad \text{for all } 1 \leq j \leq n. \end{aligned}$$

*Then  $\{a_i\}_{i \in [m]}, \{b_j\}_{j \in [n]}$  satisfy*

$$\mathcal{B}^\top(X_{:i}) = [\mathcal{B}^\top \mathcal{B}](a_i), \quad i \in [m] \quad \text{and} \quad \mathcal{A}^\top(X_{:j}) = [\mathcal{A}^\top \mathcal{A}](b_j), \quad j \in [n]. \quad (36)$$

*Proof.* We only focus on the  $b_j$ 's as the argument for the  $a_i$ 's is similar. Assume that

$$b_j = P(w) \mathcal{A}^\top X_{:j}, \quad \text{where } w = b_j \# (\mathcal{A}^\top \mathcal{A} b_j)^{-1} \text{ for all } 1 \leq j \leq n.$$

Since  $a_i, b_j \in \text{int}(\mathcal{K})$ , we have  $\langle a_i, b_j \rangle > 0$  and hence  $\mathcal{A}^\top \mathcal{A} b_j = \sum_i \langle a_i, b_j \rangle a_i \in \text{int}(\mathcal{K})$ . Thus  $w$  is invertible and

$$P(w^{-1})b_j = \mathcal{A}^\top X_{:j}. \quad (37)$$

By noting  $w = b_j \# (\mathcal{A}^\top \mathcal{A} b_j)^{-1}$  and (14), we have

$$w^{-1} = (b_j)^{-1} \# (\mathcal{A}^\top \mathcal{A} b_j).$$

It follows from (12) that

$$P(w^{-1})b_j = \mathcal{A}^\top \mathcal{A} b_j. \quad (38)$$

By combining (37) and (38), we have  $\mathcal{A}^\top \mathcal{A} b_j = \mathcal{A}^\top X_{:j}$  for all  $j \in [n]$ .  $\square$

## 5 Numerical Experiments

In this section we proceed from theory to practise and use the SCMU algorithm for computing SOCP-lifts of regular  $n$ -gons. To the best of our knowledge there are no algorithms developed specifically for computing SOCP-lifts. In terms of negative results, Fawzi established in [6] that the  $3 \times 3$  positive semidefinite cone does not admit any second-order cone representation.

In the case of the second-order cone, there are two different types of lifts that can be considered. The first possibility is to consider lifts over  $\mathbb{L}_n = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \|x\|_2 \leq t\}$ , whereas the second possibility is to consider some fixed  $n$ , say  $n = 2$ , and consider lifts over Cartesian products of  $\mathbb{L}_2$ . Nevertheless, there is a close relationship between these two types of SOC-lifts. Specifically, in view of the “tower of variables” construction given in [2], a lift over  $\mathbb{L}_k$  (where  $k = 2^\theta$ ) can be transformed into a lift over the symmetric cone

$$\underbrace{\mathbb{L}_2 \times \dots \times \mathbb{L}_2}_{k-1 \text{ times}},$$

by adding  $k - 2$  additional variables. As an example of this,  $(x_1, x_2, x_3, x_4, t) \in \mathbb{L}_4$  iff

$$\exists y_1, y_2 \text{ where } (x_1, x_2, y_1) \in \mathbb{L}_2, (x_3, x_4, y_2) \in \mathbb{L}_2, (y_1, y_2, t) \in \mathbb{L}_2.$$

### 5.1 Implementation details

**Damped updates.** Recall that the update rule of the SCMU algorithm is:

$$b \leftarrow P(b\#(\mathcal{A}^\top \mathcal{A}b)^{-1})\mathcal{A}^\top x.$$

In performing the update, it is necessary to compute square-roots and inverses of certain elements. The conditioning of these steps depend on how close the eigenvalues of these elements are to zero. As such, it is advisable to apply a small amount of damping when performing these steps. We summarize these steps in Algorithm 5.1 – here,  $e$  is the EJA identity element, while  $J$  is the identity map. In our numerical experiments, we apply  $\epsilon = 10^{-6}$  as our choice of damping.

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**Algorithm 2** Damped multiplicative updates for cone factorizations

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**Input:** EJA element  $a$ , linear map  $\mathcal{B}$ , damping parameter  $\epsilon$

1.  $z \leftarrow (\mathcal{B}^\top \mathcal{B}a + \epsilon e)^{1/2}$
  2.  $T \leftarrow P(z) + \epsilon I$
  3.  $h \leftarrow T^{1/2}(T^{-1/2}a + \epsilon e)^{1/2}$
  4.  $b \leftarrow P(h)\mathcal{A}^\top x$
- 

**Strategic initializations.** We apply the following two stage initialization strategy. In the first stage, we apply our algorithm for 100 iterations over 100 different random initializations. We keep track of the final iterate and the residual error corresponding to all initializations. We then eliminate all but 10 iterates with smallest 10 residual errors. In the second stage, we apply our algorithm for an additional 900 iterations starting from the final iterate of these 10 iterates. We report the smallest residual error obtained in the second stage.

## 5.2 SOCP-lifts for regular $n$ -gons

Denoting by  $\times^l \mathbb{L}_k$  the Cartesian product of  $l$  copies of where  $\mathbb{L}_k = \{(x, t) \in \mathbb{R}^k \times \mathbb{R} : \|x\|_2 \leq t\}$ , in our experiments we compute  $\times^l \mathbb{L}_k$  factorizations of the slack matrices of regular 4-gons, 5-gons, 6-gons, and 8-gons for various values of  $l$  and  $k$ . Subsequently, based on our numerical results, we formulate several conjectures on the (non)existence of certain types of SOCP-lifts.

Our guiding principle is a simple heuristic that was explicitly formulated and used in [24] in the setting of PSD factorizations. Specifically, for a fixed value of  $k$ , we would like to find the least  $l$  for which our matrix has a  $\times^l \mathbb{L}_k$  factorization. Denoting by  $l^*$  the least possible value such a lift exists, we expect to see a noticeable “phase transition” with respect to the error of the SCMU algorithm, namely it should be positive for  $l < l^*$  and zero for  $l \geq l^*$ .

We note that for a fixed value of  $k$ , there always exists a  $\times^l \mathbb{L}_k$ -lift for a large enough value of  $l$ . Indeed, for any non-negative matrix  $X \in \mathbb{R}_+^{m \times n}$  we always have a non-negative factorization with vectors of dimension  $\min\{n, m\}$ . Considering any NMF with an even dimension  $d$ , we can pair the non-negative coordinates in pairs of two to get a factorization  $\underbrace{\mathbb{R}_+^2 \times \dots \times \mathbb{R}_+^2}_{d/2 \text{ times}}$ . Lastly, as  $\mathbb{R}_+^2$  can

be rotated to  $\mathbb{L}_1$ , this construction leads to a  $\times^{d/2} \mathbb{L}_k$  factorization.

In practise, for fixed  $k$  and increasing  $l$ , we notice that the errors steadily decrease up to a point where the error stagnates. Based on this, we conjecture that the first instance where the error stabilizes corresponds to smallest  $l$  for which there is an exact  $\times^l \mathbb{L}_k$ -lift.

In the first instance, we compute factorizations of the slack matrix of the regular 4-gon. In Figure 1, we show the final residual errors obtained using our method. For  $l = 1$  and increasing  $k$ ,

	$\times^1$	$\times^2$	$\times^3$
$\mathbb{L}_1$	0.50	0.0019	0.0025
$\mathbb{L}_2$	0.17	0.0020	0.0025
$\mathbb{L}_3$	0.17	0.0021	0.0027
$\mathbb{L}_4$	0.17	0.0021	0.0027

Figure 1: Best error for regular 4-gon.

we noticed that the errors decrease from  $k = 1$  to  $k = 2$ , but stagnate right after. This suggests that the 4-gon does not admit a  $\mathbb{L}_k$ -lift for any  $k$ . In the case of the 4-gon, we have an explicit factorization of the slack matrix using  $\mathbb{L}_1 \times \mathbb{L}_1$ :

$$\sqrt{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \frac{\sqrt{2}}{12} \begin{pmatrix} 3 & -3 & & \\ 3 & 3 & & \\ & & 2 & 2 \\ & & 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ -2 & & 2 \\ 6 & 3 & 3 \\ -3 & & 3 \end{pmatrix}.$$

**5-gon.** In our second example, we compute a factorization of the slack matrix of the regular 5-gon. In Figure 2, we show the final residual errors obtained using the SCMU method and based on these, conjecture that the regular 5-gon admits a  $\times^l \mathbb{L}_k$ -lift if and only if  $l \geq 3$ .

**6-gon.** In the third instance, we compute factorization of the slack matrix of the regular 6-gon. In Figure 3, we show the final residual errors obtained using the SCMU method and conjecture that the regular 5-gon admits a  $\times^l \mathbb{L}_k$ -lift if and only if  $l \geq 3$ .

	$\times^1$	$\times^2$	$\times^3$	$\times^4$
$\mathbb{L}_1$	0.47	0.12	0.0024	0.0026
$\mathbb{L}_2$	0.10	0.018	0.0026	0.0027
$\mathbb{L}_3$	0.10	0.018	0.0034	0.0033
$\mathbb{L}_4$	0.10	0.018	0.0040	0.0035

Figure 2: Best error for regular 5-gon

	$\times^1$	$\times^2$	$\times^3$	$\times^4$
$\mathbb{L}_1$	0.45	0.095	0.0023	0.0027
$\mathbb{L}_2$	0.069	0.021	0.0034	0.0033
$\mathbb{L}_3$	0.070	0.023	0.0036	0.0036
$\mathbb{L}_4$	0.071	0.022	0.0044	0.0033

Figure 3: Best error for regular 6-gon

**8-gon.** In our fourth example, we factorize the slack matrix of the regular 8-gon. In Figure 4, we show the final residual errors obtained using our method. We observe similar trends as we did with the previous instances and conjecture that the regular 8-gon admits a  $\times^l \mathbb{L}_k$ -lift if and only if  $l \geq 4$ .

	1 copy	2 copies	3 copies	4 copies
$\mathbb{L}_1$	0.44	0.073	0.029	0.0040
$\mathbb{L}_2$	0.038	0.028	0.010	0.0059
$\mathbb{L}_3$	0.040	0.027	0.0096	0.0068
$\mathbb{L}_4$	0.043	0.025	0.0093	0.0060

Figure 4: Best error for regular 8-gon

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