A UNIFIED FRAMEWORK FOR DISTRIBUTED OPTIMIZATION ALGORITHMS OVER TIME-VARYING DIRECTED GRAPHS

WOOCHEOL CHOI, DOHEON KIM, AND SEOK-BAE YUN

Abstract. In this paper, we propose a framework under which the decentralized optimization algorithms suggested in [\[8,](#page-23-0) [11,](#page-24-0) [14,](#page-24-1) [15\]](#page-24-2) can be treated in a unified manner. More precisely, we show that the distributed subgradient descent algorithms [\[8,](#page-23-0) [14\]](#page-24-1), the subgradient-push algorithm [\[15\]](#page-24-2), and the distributed algorithm with row-stochastic matrix [\[11\]](#page-24-0) can be derived by making suitable choices of consensus matrices, step-size and subgradient from the decentralized subgradient descent proposed in [\[14\]](#page-24-1). As a result of such unified understanding, we provide a convergence proof that covers the algorithms in [\[8,](#page-23-0) [11,](#page-24-0) [14,](#page-24-1) [15\]](#page-24-2) under a novel algebraic condition that is strictly weaker than the conventional graphtheoretic condition in [\[14\]](#page-24-1). This unification also enables us to derive a new distributed optimization scheme.

1. Introduction

In this paper, we consider N agents cooperating with each other to solve the following optimization problem:

Find a minimizer
$$
x^* \in \mathbb{R}^d
$$
 of the function $f(x) := \sum_{i=1}^N f_i(x)$. (1.1)

Here, for each $1 \leq i \leq N$, the function $f_i : \mathbb{R}^d \to \mathbb{R}$ is local cost known only to the *i*-th agent. The N agents are connected by a network and each agent can receive information from its neighboring agents. Distributed optimization has received a lot of attention due to its application for various problems containing wireless sensor network, multi-agent control, machine learning. In distributed optimization, many agents have their own local cost and try to find a minimizer of the sum of those local cost functions in a collaborative way. The algorithms consist of optimization step for local function and consensus step by communication.

In the seminal work [\[14\]](#page-24-1), Nedic et al introduced the concept of the decentralized (sub)gradient descent (DGD) to solve (1.1) :

$$
x_i(t+1) = \sum_{j=1}^{N} p_{ij}(t)x_j(t) - \delta_i(t)g_i(t), \quad i = 1, ..., N, \quad t = 0, 1, 2, ..., \tag{1.2}
$$

with the $N \times N$ row-stochastic matrices $P(t) := (p_{ij}(t))$, the step-size $\delta_i(t)$, and the subgradients $g_i(t) \in \partial f_i(x_i(t))$ $(1 \leq i \leq N, t = 0, 1, 2, ...)$. The matrix P contains the information on the connectivity of the underlying network: $p_{ij}(t)$ vanishes if the *i*-th agent does not receive any information from the j-th agent at time t. Otherwise p_{ij} is always positive.

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Many varaints and extensions of [\(1.2\)](#page-0-1) were proposed in the literature, and their convergence properties were extensively studied. In [\[8\]](#page-23-0), the authors designed a distributed subgradient algorithm to solve [\(1.1\)](#page-0-0) given by

$$
x_i(t+1) = \sum_{j=1}^{N} p_{ij}(t) \Big(x_j(t) - \delta_j(t) g_j(t) \Big), \tag{1.3}
$$

The convergence of the above algorithms were investigated for undirected graph under the assumption that $\{p_{ij}(t)\}\$ is a symmetric and doubly-stochastic matrix.

On the other hand, it is more practical to consider directed communications between agents under certain environments. The network of agents is then represented by a directed graph. To handle the distributed optimization on directed graph, Nedic-Olshevsky [\[15\]](#page-24-2) employed the push-sum protocol [\[9\]](#page-23-1) to design the subgradient-push method:

$$
\begin{cases}\nw_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)x_j(t) \in \mathbb{R}^d, \quad i = 1, ..., N, \quad t = 0, 1, 2, ..., \\
y_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)y_j(t) \in \mathbb{R}_+, \\
z_i(t+1) = \frac{w_i(t+1)}{y_i(t+1)} \in \mathbb{R}^d, \\
x_i(t+1) = w_i(t+1) - \delta_i(t+1)g_i(t+1), \quad g_i(t+1) \in \partial f_i(z_i(t+1)),\n\end{cases}
$$
\n(1.4)

where ${a_{ij}(t)}$ is a column-stochastic matrix used for the direct communication. When stated in terms of $w_i(t)$ and $y_i(t)$, [\(1.4\)](#page-1-0) can be reformulated in a more succinct form:

$$
\begin{cases}\nw_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)(w_j(t) - \delta_j(t)g_j(t)), & g_i(t) \in \partial f_i\left(\frac{w_i(t)}{y_i(t)}\right) \\
y_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)y_j(t), & 1 \le i \le N, \quad t \ge 0.\n\end{cases}
$$
\n(1.5)

This algorithm has been extended to various problems containing the stochastic distributed optimization [\[16,](#page-24-3) [20\]](#page-24-4), online distributed optimization [\[1\]](#page-23-2).

Recently the authors in [\[11\]](#page-24-0) designed a distributed subgradient algorithm for directed graph by using row-stochastic matrix for communication

$$
\begin{cases}\n x_i(t+1) = \sum_{j=1}^{N} p_{ij} x_j(t) - \frac{\delta(t)}{\mathbf{e}_i^{\top} z_i(t)} g_i(t), \quad g_i(t) \in \partial f_i(x_i(t)), \\
 z_i(t+1) = \sum_{j=1}^{N} p_{ij} z_j(t), \quad z_i(0) = \mathbf{e}_i, \quad 1 \le i \le N, \quad t \ge 0.\n\end{cases}
$$
\n(1.6)

We also refer to [\[12\]](#page-24-5) for further details and the convergence analysis of the above algorithm.

The main goal of this paper is two-fold. First, we show that the algorithms (1.3) - (1.6) can be exploited in a unified way in the framework [\(1.2\)](#page-0-1). More pecisely, we show that the above mentioned algorithms $(1.3)-(1.6)$ $(1.3)-(1.6)$ can all be derived from (1.2) by making specific choices of the row-stochastic matrix $P(t)$ and the scalars $\delta_i(t)$ in [\(1.2\)](#page-0-1). This unified perspective also enables us to design a new algorithm which resembles the subgradient-push method [\[15\]](#page-24-2), but differs in the order of the consensus step and the gradient descent step:

$$
\begin{cases}\nw_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)w_j(t) - \theta_i(t)g_i(t), & g_i(t) \in \partial f_i\left(\frac{w_i(t)}{y_i(t)}\right), \\
y_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)y_j(t), & 1 \le i \le N, \quad t \ge 0,\n\end{cases}
$$
\n(1.7)

where $A(t) = \{a_{ij}(t)\}\$ are column stochastic. We refer to [\[2,](#page-23-3) [7,](#page-23-4) [18,](#page-24-6) [22,](#page-24-7) [23,](#page-24-8) [24\]](#page-24-9) for unifications of gradient tracking type algorithms.

Secondly, as a result of such unified understanding of the above schemes, we provide a convergence proof that covers the optimization schemes (1.3) - (1.7) in a unified manner. The novelty of this convergence proof, aside from the unification itself, lies in providing the convergence of the distributed algorithm (1.2) for which the matrices $P(t)$ are row-stochastic, but not necessarily column-stochastic. While the convergence analysis of (1.2) is well-understood when $P(t)$ are doubly stochastic (both rowstochastic and column-stochastic) and the stepsize $\delta_i(t)$ satisfies a sutiable decaying property and is independent of i, i.e., $\delta_i(t) = \delta(t)$, the convergence analysis of [\(1.2\)](#page-0-1) for general row-stochastic $P(t)$ is, to the best knowledge of the authors, not established yet. Moreover, when $P(t)$ is only row-stochastic and $\delta_i(t) = \delta(t)$, the distributed algorithm [\(1.2\)](#page-0-1) may fail to converge to an optimizer of the problem even though a suitable decaying property is given.

We establish the convergence of (1.2) to a minimizer for row-stochastic $P(t)$ under a general condition on the stepsize $\delta_i(t)$ which is associated to the sequence of absolute probability vectors [\[10\]](#page-23-5). We mention that the sequence of absolute probability vectors has been used importantly in the literature containing consensus algorithms [\[13\]](#page-24-10) and distributed optimization algorithms [\[19\]](#page-24-11).

Another contribution of our convergence result is that we successfully replace the conventional graph theoretic condition [\[15\]](#page-24-2) on the underlying graph with an algebraic condition on the consensus matrix in the convergence proof, and show that the latter is strictly more general than the former. More precisely, the previous works assumed that each vertex of graph $G(t)$ has a self-loop, and finite unions of time-varying graphs,

$$
\bigcup_{t \le s < t + T} G(s) := \left(V, \bigcup_{t \le s < t + T} E(s) \right) \quad (\forall \ t \ge 0) \quad \text{ for some fixed } T \in \mathbb{N},
$$

are strongly connected. We replace these conditions with a strictly weaker non-vaninishing condition on the time-varying consensus matrices up to a finite product:

$$
P(t+T,t):=P(t+T-1)P(t+T-2)\cdots P(t)>0\quad(\forall\;t\geq 0)\quad \ \, \text{for some fixed}\;T\in\mathbb{N}.
$$

We also provide an explicit example which satisfies the latter but not the former (see Section 4). Here $P(t + T, t) > 0$ means that every element of $P(t + T, t)$ is positive. We note that our condition does not require the existence of self-loop at each vertex of graph $G(t)$, which was necessary in most of the previous works.

Notation. Before we finish this introduction we set up notational convention that is kept throughout the paper.

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• For
$$
1 \le p \le \infty
$$
, $v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^{d \times 1}$ and $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, define

$$
||v||_p := \left(\sum_{l=1}^d |v_l|^p\right)^{\frac{1}{p}} \quad (1 \le p < \infty), \quad ||v||_{\infty} = \max_{1 \le l \le d} |v_l|, \quad ||A||_p := \sup_{x \ne 0} \frac{||Ax||_p}{||x||_p} \quad (1 \le p \le \infty).
$$

Note that

$$
||v||_2 \le ||v||_1
$$
, $||A||_2 = ||A^\top||_2$, $||A||_{\infty} = \max_{1 \le i \le n} \sum_{j=1}^n |a_{ij}|$, $||A||_1 = \max_{1 \le j \le n} \sum_{i=1}^n |a_{ij}|$.

- Unless a specification is needed, we use $\|\cdot\|$ generically to denote any sub-multiplicative matrix norm throughout the paper: If we write a statement with this norm, then it means that it holds for any sub-multiplicative matrix norm since they are all equivalent.
- Lastly, we denote by I_N , O_N and 1_N the $N \times N$ identity matrix, the $N \times N$ zero matrix and the $N \times 1$ matrix whose entries are all 1, respectively.

The paper is organized as follows. In Section 2, we derive the algorithms $(1.3)-(1.7)$ $(1.3)-(1.7)$ from the distributed algorithm [\(1.2\)](#page-0-1) choosing suitable parameters. In Section 3, we review some fundamental properties of row-stochastic matrices. In Section 4, we state the convergence result for [\(1.2\)](#page-0-1) and discuss the assumptions used in the convergence result. In Section 5, we apply the result of Section 4 to derive the convergence estimates of the algorithms $(1.3)-(1.7)$ $(1.3)-(1.7)$. Section 6 is devoted to give the proof of the convergence result stated in Section 4.

2. Derivation of distributed algorithms from [\(1.2\)](#page-0-1)

In this section, we show that the distributed algorithms $(1.3)-(1.6)$ $(1.3)-(1.6)$ can be recovered from (1.2) by making specific choices of $P(t)$ and $\delta_i(t)$. For this purpose, we write [\(1.2\)](#page-0-1) as

$$
X(t+1) = P(t)X(t) - \Delta(t)G(t), \quad t \ge 0,
$$
\n(2.1)

with the following notations:

$$
X(t) = [x_1(t) \dots x_N(t)]^\top \in \mathbb{R}^{N \times d}, \qquad P(t) = (p_{ij}(t)) \in \mathbb{R}^{N \times N},
$$

\n
$$
\Delta(t) = \text{diag}(\delta_1(t), \dots, \delta_N(t)) \in \mathbb{R}^{N \times N}, \qquad G(t) = [g_1(t) \dots g_N(t)]^\top \in \mathbb{R}^{N \times d}.
$$
\n(2.2)

Example 2.1. Derivation of (1.3) : We divide (2.1) in odd and even cases:

$$
X(2t + 1) = P(2t)X(2t) - \Delta(2t)G(2t),
$$

\n
$$
X(2t + 2) = P(2t + 1)X(2t + 1) - \Delta(2t + 1)G(2t + 1).
$$
\n(2.3)

We choose $P(2t) \equiv I_N$ in $(2.3)_1$ to get

$$
X(2t+1) = X(2t) - \Delta(2t)G(2t).
$$
\n(2.4)

On the other hand, we set $\Delta(2t+1) \equiv O_N$ to reduce $(2.3)_2$ into

$$
X(2t+2) = P(2t+1)X(2t+1).
$$
\n(2.5)

Inserting (2.4) into (2.5) , we obtain

$$
X(2t + 2) = P(2t + 1)(X(2t) - \Delta(2t)G(2t)).
$$

Finally, we make the following choices of $P(t)$, $\Delta(t)$, and $G(t)$:

$$
\tilde{P}(t) = (\tilde{p}_{ij}(t)) := P(2t+1), \quad \tilde{\Delta}(t) = \text{diag}(\tilde{\delta}_1(t), \dots, \tilde{\delta}_N(t)) := \Delta(2t),
$$

$$
\tilde{G}(t) := [\tilde{g}_1(t) \dots \tilde{g}_N(t)]^\top := G(2t).
$$

and introduce the new optimizing variable:

$$
\tilde{X}(t) = [\tilde{x}_1(t) \dots \tilde{x}_N(t)]^\top := X(2t),
$$

to obtain the distributed algorithm [\(1.3\)](#page-1-1):

$$
\tilde{X}(t+1) = \tilde{P}(t) \big(\tilde{X}(t) - \tilde{\Delta}(t) \tilde{G}(t) \big), \quad t \ge 0,
$$

or equivalently,

$$
\tilde{x}_i(t+1) = \sum_{j=1}^N \tilde{p}_{ij}(t)(\tilde{x}_j(t) - \delta_j(t)\tilde{g}_i(t)), \quad \tilde{g}_i(t) \in \partial f_i(\tilde{x}_i(t)), \quad 1 \le i \le N, \quad t \ge 0.
$$

Example 2.2. Derivation of [\(1.6\)](#page-1-2): We let the matrix $P(t)$ of [\(2.1\)](#page-3-0) independent of time, i.e.,

$$
P(0) = P(1) = P(2) = \cdots =: P,
$$

and introduce a new variable $Z(t)$ defined as the t-th power of P with the convention $P^0 := I_N$:

$$
Z(t) = [z_1(t) \dots z_N(t)]^{\top} := P^t, \text{ with } z_i(t) = [z_{i1}(t) \dots z_{iN}(t)]^{\top}, \quad 1 \le i \le N, \quad t \ge 0.
$$

Then, we have

$$
Z(t+1) = P^{t+1} = PP^t = PZ(t).
$$
\n(2.6)

We then choose our new step-size by

$$
\delta_i(t) \equiv \frac{\delta(t)}{i\text{-th diagonal entry of } P^t}, \quad t \ge 0,
$$

for some scalars $\delta(t)$ $(t \geq 0)$, to get from (2.1) that

$$
X(t+1) = PX(t) - \frac{\delta(t)}{P_{ii}^t} G(t).
$$
\n(2.7)

From (2.6) and (2.7) , we derive the distributed algorithm (1.6) :

$$
\begin{cases}\n x_i(t+1) = \sum_{j=1}^N p_{ij} x_j(t) - \frac{\delta(t)}{z_{ii}(t)} g_i(t), & g_i(t) \in \partial f_i(x_i(t)), \\
 z_i(t+1) = \sum_{j=1}^N p_{ij} z_j(t), & z_i(0) = \mathbf{e}_i, & 1 \le i \le N, \quad t \ge 0,\n\end{cases}
$$

where $\mathbf{e}_i \in \mathbb{R}^N$ denotes the *i*-th standard unit vector.

Example 2.3. Derivation of a new algorithm: In this example, we propose a new algorithm that generalizes the subgradient-push method in [\[15\]](#page-24-2) (see Example [2.4\)](#page-6-0). Let the sequence $\{A(t)\}_{t\geq0}$ of $N \times N$ column-stochastic matrices and the sequence $\{Y(t)\}_{t \geq 0}$ of $N \times 1$ matrices satisfying

$$
Y(t+1) = A(t)Y(t), \quad Y(t) > 0, \quad t \ge 0
$$

be given. Here we remark that one may replace the condition " $Y(t) > 0$ ($\forall t \ge 0$)" by a stronger condition "each $A(t)$ $(t \ge 0)$ has no zero row and $Y(0) > 0$ ". Indeed, we have

$$
\min_{1 \le i \le N} y_i(t+1) = \min_{1 \le i \le N} \left(\sum_{j=1}^N a_{ij}(t) y_j(t) \right) \ge \min_{1 \le i \le N} \left(\sum_{j=1}^N a_{ij}(t) \right) \cdot \left(\min_{1 \le k \le N} y_k(t) \right),
$$

and by the assumption that $A(t)$ has no zero row, we have

$$
\min_{1 \le i \le N} \left(\sum_{j=1}^N a_{ij}(t) \right) > 0.
$$

So we can prove $Y(t) > 0$ by inducting on t.

We then make the following choice for $\{P(t)\}_{t\geq 0}$:

$$
P(t) \equiv \text{diag}(Y(t+1))^{-1}A(t)\,\text{diag}(Y(t)), \quad t \ge 0,
$$

which is row-stochastic by construction:

$$
P(t)1_N = \text{diag}(Y(t+1))^{-1} A(t) \text{ diag}(Y(t))1_N
$$

=
$$
\text{diag}(Y(t+1))^{-1} A(t) Y(t)
$$

=
$$
\text{diag}(Y(t+1))^{-1} Y(t+1)
$$

=
$$
1_N.
$$
 (2.8)

We choose this $P(t)$ in [\(2.1\)](#page-3-0) and, finally we multiply diag($Y(t + 1)$) to the left of both sides of (2.1)

$$
diag(Y(t+1))X(t+1) = A(t) diag(Y(t))X(t) - diag(Y(t+1))\Delta(t)G(t), \quad t \ge 0,
$$

and introduce

$$
W(t) := \text{diag}(Y(t))X(t), \quad \Theta(t) := \text{diag}(Y(t+1))\Delta(t), \quad t \ge 0
$$

to get

$$
\begin{cases} Y(t+1) = A(t)Y(t), \\ W(t+1) = A(t)W(t) - \Theta(t)G(t), \quad t \ge 0. \end{cases}
$$
\n(2.9)

If we denote

$$
W(t) = [w_1(t) \dots w_N(t)]^\top, \qquad Y(t) = [y_1(t) \dots y_N(t)]^\top, \n\Theta(t) = \text{diag}(\theta_1(t) \dots \theta_N(t)), \qquad A(t) = (a_{ij}(t)),
$$
\n(2.10)

then we can rewrite (2.9) as

$$
\begin{cases}\ny_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)y_j(t), & 1 \le i \le N, \quad t \ge 0, \\
w_i(t+1) = \sum_{j=1}^{N} a_{ij}(t)w_j(t) - \theta_i(t)g_i(t), & g_i(t) \in \partial f_i\left(\frac{w_i(t)}{y_i(t)}\right).\n\end{cases} (2.11)
$$

$$
Y(2t + 1) = A(2t)Y(2t)
$$

\n
$$
W(2t + 1) = A(2t)W(2t) - \Theta(2t)G(2t)
$$

\n
$$
Y(2t + 2) = A(2t + 1)Y(2t + 1)
$$

\n
$$
W(2t + 2) = A(2t + 1)W(2t + 1) - \Theta(2t + 1)G(2t + 1)
$$
\n(2.12)

and set

$$
A(2t) = I_N
$$
, $\Delta(2t + 1) = O_N$, $t \ge 0$.

It leads to

$$
Y(2t + 1) = Y(2t)
$$

\n
$$
W(2t + 1) = W(2t) - \Theta(2t)G(2t)
$$

\n
$$
Y(2t + 2) = A(2t + 1)Y(2t + 1)
$$

\n
$$
W(2t + 2) = A(2t + 1)W(2t + 1),
$$
\n(2.13)

which is reduced to

$$
Y(2t + 2) = A(2t + 1)Y(2t)
$$

$$
W(2t + 2) = A(2t + 1)(W(2t) - \Theta(2t)G(2t)).
$$
 (2.14)

This, together with the following choices:

$$
\hat{X}(t) = [\hat{x}_1(t) \dots \hat{x}_N(t)]^\top := W(2t+1), \qquad \hat{G}(t) := [\hat{g}_1(t) \dots \hat{g}_N(t)]^\top := G(2t), \n\hat{Y}(t) = [\hat{y}_1(t) \dots \hat{y}_N(t)]^\top := Y(2t), \qquad \hat{A}(t) = (\hat{a}_{ij}(t)) := A(2t+1), \n\hat{\Theta}(t) = \text{diag}(\hat{\theta}_1(t), \dots, \hat{\theta}_N(t)) := \Theta(2t), \qquad \hat{W}(t) = [\hat{w}_1(t) \dots \hat{w}_N(t)]^\top := W(2t),
$$
\n(2.15)

lead to the subgradient-push method in [\[15\]](#page-24-2)

$$
\hat{Y}(t+1) = \hat{A}(t)\hat{Y}(t) \n\hat{W}(t+1) = \hat{A}(t)(\hat{W}(t) - \hat{\Theta}(t)\hat{G}(t)),
$$
\n(2.16)

which can be written as

$$
\begin{cases}\n\hat{y}_i(t+1) = \sum_{j=1}^N \hat{a}_{ij}(t)\hat{y}_j(t) \in \mathbb{R}_+, \\
\hat{x}_i(t) = \hat{w}_i(t) - \hat{\theta}_i(t)\hat{g}_i(t), \quad \hat{g}_i(t) \in \partial f_i\left(\frac{\hat{w}_i(t)}{\hat{y}_i(t)}\right), \\
\hat{w}_i(t+1) = \sum_{j=1}^N \hat{a}_{ij}(t)\hat{x}_j(t) \in \mathbb{R}^d.\n\end{cases}
$$

In Table 1, we provide a systematic summary of this section.

3. Properties of row-stochastic matrices

In this section, we study some properties of ergodic sequences of row-stochastic matrices, which is crucially used in the sequel.

Definition 3.1.

- (1) A matrix A is non-negative (positive) if all of its entries are non-negative (positive), and we write $A \geq 0$ (> 0).
- (2) A non-negative square matrix is row-stochastic if all of its row has sum equal to 1.
- (3) A non-negative vector is a probability vector if the sum of its entries is equal to 1.

(4) A sequence $\{P(t)\}_{t\geq0}$ of row-stochastic matrices is *ergodic* if there exists a sequence $\{v(t)\}_{t\geq0}$ of probability vectors satisfying

$$
\lim_{t \to \infty} P(t)P(t-1)\dots P(t_0) = 1_N v(t_0)^{\top}, \quad \forall \ t_0 \ge 0.
$$

(5) A sequence $\{\pi(t)\}_{t\geq 0}$ of probability vectors is a set of absolute probability vectors for the sequence $\{P(t)\}_{t\geq 0}$ of row stochastic matrices if

$$
\pi(t+1)^{\top} P(t) = \pi(t)^{\top}, \quad t \ge 0.
$$
\n(3.1)

The following proposition establishes a relation between ergodic sequences and sets of absolute probability vectors.

Proposition 3.2 ([\[10\]](#page-23-5)). Let $\{P(t)\}_{t\geq0}$ be a sequence of $N \times N$ row stochastic matrices, and let ${\{\pi(t)\}}_{t\geq0}$ be a sequence of $N\times1$ probability vectors. Then the following are equivalent.

(1) $\{P(t)\}_{t\geq 0}$ is ergodic and satisfies

$$
\lim_{t \to \infty} P(t)P(t-1)\dots P(t_0) = 1_N \pi(t_0)^{\top}, \quad \forall \ t_0 \ge 0.
$$

(2) $\{\pi(t)\}_{t\geq 0}$ is a unique set of absolute probability vectors for $\{P(t)\}_{t\geq 0}$.

Next, we introduce a useful tool to study ergodicity of sequences of row-stochastic matrices.

Definition 3.3 ([\[5,](#page-23-6) [6\]](#page-23-7)). For each $N \times N$ row-stochastic matrix $P = (p_{ij})$, we define its ergodicity coefficient by

$$
\tau(P) := \frac{1}{2} \max_{i_1, i_2} \sum_{j=1}^{N} |p_{i_1 j} - p_{i_2 j}| = 1 - \min_{i_1, i_2} \sum_{j=1}^{N} \min\{p_{i_1 j}, p_{i_2 j}\}.
$$
 (3.2)

The identity [\(3.2\)](#page-8-0) can be derived in the following way:

$$
\frac{1}{2} \max_{i_1, i_2} \sum_{j=1}^N |p_{i_1 j} - p_{i_2 j}| = \frac{1}{2} \max_{i_1, i_2} \sum_{j=1}^N (p_{i_1 j} + p_{i_2 j} - 2 \min\{p_{i_1 j}, p_{i_2 j}\})
$$

$$
= \frac{1}{2} \max_{i_1, i_2} \left(2 - 2 \sum_{j=1}^N \min\{p_{i_1 j}, p_{i_2 j}\}\right) = 1 - \min_{i_1, i_2} \sum_{j=1}^N \min\{p_{i_1 j}, p_{i_2 j}\}.
$$

For any $N \times N$ row-stochastic matrix P, it is clear from the definition that $0 \leq \tau(P) \leq 1$, and that

 $\tau(P) = 0$ if and only if $P = 1_N v^{\top}$ for some $v \in \mathbb{R}^{N \times 1}$,

which reveals a close relationship between τ and ergodicity. We finish this section by introducing some useful properties of τ .

Lemma 3.4 ([\[5,](#page-23-6) [6\]](#page-23-7)).

(1) For any row-stochastic matrix P , we have

$$
\tau(P) = \sup_{\substack{u \neq 0, \\ u^{\top}1_N = 0}} \frac{\|P^{\top}u\|_1}{\|u\|_1}
$$

(2) For any row-stochastic matrices P_1, P_2 of the same size, we have

$$
\tau(P_1P_2) \leq \tau(P_1)\tau(P_2).
$$

Lemma 3.5 ([\[4\]](#page-23-8)). A sequence $\{P(t)\}_{t\geq 0}$ of $N \times N$ row stochastic matrices is ergodic if and only if

$$
\lim_{t \to \infty} \tau(P(t)P(t-1)\dots P(t_0)) = 0, \quad \forall \ t_0 \ge 0.
$$

4. Conditions for convergence to an optimum

In this section, we present the main theorem of this paper, which states a general sufficient condition for which the algorithm (2.1) finds a minimizer x^* of the distributed optimization problem (1.1) .

For our convergence result, we make the following assumptions for f_i :

- (1) For each $i = 1, \ldots, N$, f_i is convex, and we have $L_i := \sup$ $z \in \mathbb{R}^d$ sup $\sup_{g \in \partial f_i(z)} \|g\|_1 < \infty.$
- (2) There is at least one solution to the minimization problem $x^* \in \operatorname{argmin}_{x \in \mathbb{R}^d} f(x)$.

From now on, to the sequence $\{P(t)\}_{t>0}$, we associate the backward products

$$
P(t, t_0) := P(t - 1)P(t - 2) \dots P(t_0), \quad t \ge t_0 \ge 0
$$

with the convention $P(t_0, t_0) := I_N$ ($t_0 \geq 0$). Below, we introduce the main assumptions which constitute a sufficient condition to guarantee the convergence of the algorithm [\(2.1\)](#page-3-0) to a minimizer of f.

• (A1): The sequence $\{P(t)\}_{t\geq 0}$ of $N \times N$ row-stochastic matrices satisfies

$$
p^{+} := \inf_{t \ge 0} \left[\min_{i,j: \ p_{ij}(t) > 0} p_{ij}(t) \right] > 0 \tag{4.1}
$$

and there exists $T \in \mathbb{N}$ such that

$$
P(t+T,t) > 0 \quad \forall \ t \ge 0.
$$

- (A2): Each $\delta_i(\cdot)$ is nonnegative, and $\sum_{t=0}^{\infty} (||\Delta(t)|| \sup_{\ell \geq t} ||\Delta(\ell)||) < \infty$.
- (A3): The set of absolute probability vectors for $\{P(t)\}_{t\geq0}$ denoted by

$$
\pi(t) = (\pi_1(t), \dots, \pi_N(t))^\top \quad (t \ge 0)
$$

satisfies
$$
\sum_{t=0}^{\infty} ||\Delta(t)\pi(t+1)|| = \infty \text{ and } \sum_{t=0}^{\infty} \sqrt{t} \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) < \infty.
$$

We remark that the existence and uniqueness of the vector $\pi(t)$ in (A3) is guaranteed by (A1) as proved in Lemma 4.2 below.

Now we state the main theorem of this paper.

Theorem 4.1. Let $X(t)$ be a solution to [\(2.1\)](#page-3-0). Suppose that $(A1)-(A3)$ hold. Then there exists some minimizer x^* of f such that $\lim_{t \to \infty} x_i(t) = x^*$ for all $1 \le i \le N$.

In the following lemma, we prove that if there exists a sequence $\{P(t)\}_{t\geq 1}$ satisfying $(A1)$ is given, then there exist sequences $\{\Delta(t)\}_{t>0}$ and $\{\pi(t)\}_{t>0}$ satisfying $(A2)$ - $(A3)$.

Lemma 4.2. Suppose that $(A1)$ holds. Then the following assertions hold.

(1) For all $t \ge t_0 \ge 0$, the function τ defined in [\(3.2\)](#page-8-0) satisfies

$$
\tau(P(t,t_0)) \le C\lambda^{t-t_0}
$$

for some constants $C > 0$ and $0 < \lambda < 1$, both independent of t and t_0 .

(2) The set $\{\pi(t)\}_{t\geq0}$ of absolute probability vectors defined in (A3) uniquely exists, and satisfies

$$
\pi_i(t) \ge (p^+)^T, \quad \forall \ 1 \le i \le N, \quad \forall \ t \ge 0.
$$

(3) If we set

$$
\delta_i(t) := \frac{c\overline{\delta}(t)}{\pi_i(t+1) + \epsilon_i(t)}, \quad \forall \ 1 \le i \le N, \ \forall \ t \ge 0
$$
\n(4.2)

for some $c > 0$, $\bar{\delta}(t) \ge 0$ satisfying $\sum_{t \ge 0} \bar{\delta}(t) \sup_{\ell \ge t} \bar{\delta}(t) < \infty$ and $\sum_{t \ge 0} \bar{\delta}(t) = \infty$, and $\epsilon_i(t) \in (-\pi_i(t+1),\infty)$ approaching zero geometrically fast as $t \to \infty$, then $\{\Delta(t)\}_{t\geq 0}$ satisfies $(A2)$ - $(A3)$.

Proof. (1) For any row-stochastic $Q = (q_{ij})$, we deduce from [\(3.1\)](#page-8-1) the following inequality

$$
\tau(Q) = 1 - \min_{i_1, i_2} \sum_{j=1}^{N} \min\{q_{i_1j}, q_{i_2j}\} \le 1 - N \min_{i,j} q_{ij}.
$$
\n(4.3)

For any $s \geq 0$, every entry of $P(s+T, s)$ is greater than or equal to $(p^+)^T$ defined in [\(4.1\)](#page-9-0). Combining this with [\(4.3\)](#page-10-0) we deduce

$$
\tau(P(s+T,s)) \le 1 - N(p^+)^T =: \gamma < 1.
$$

Suppose $t - t_0 = qT + m$ with $q \in \mathbb{N} \cup \{0\}$ and $0 \leq m < T$. By Proposition 2.4, we have

$$
\tau(P(t, t_0)) \le \tau(P(t, t_0 + qT)) \prod_{k=1}^q \tau(P(t_0 + kT, t_0 + (k-1)T))
$$

$$
\le \gamma^q = \gamma^{\frac{t-t_0 - m}{T}} = \gamma^{-\frac{m}{T}} \cdot \left(\gamma^{\frac{1}{T}}\right)^{t-t_0} < \gamma^{-1} \cdot \left(\gamma^{\frac{1}{T}}\right)^{t-t_0} = C\lambda^{t-t_0}.
$$

(2) By (1), Lemma [3.5](#page-9-1) and Proposition [3.2,](#page-8-2) the existence and uniqueness of $\{\pi(t)\}_{t\geq0}$ are guaranteed. By $(A1)$, we have

$$
\pi_i(t) = \pi(t)^{\top} \mathbf{e}_i = \pi(t+T)^{\top} P(t+T,t) \mathbf{e}_i \ge \sum_{j=1}^{N} \pi_j(t+r_0) (p^+)^T = (p^+)^T.
$$

(3) By (2) and [\(4.2\)](#page-10-1), there exist positive constants c_1, c_2 indepedent of t such that

$$
c_1\overline{\delta}(t) \le \delta_i(t) \le c_2\overline{\delta}(t).
$$

Combining this with the given assumptions, we have

$$
\sum_{t=0}^{\infty} \left(\|\Delta(t)\| \sup_{\ell \ge t} \|\Delta(\ell)\| \right) < \infty \quad \text{and} \quad \sum_{t=0}^{\infty} \|\Delta(t)\pi(t+1)\| = \infty.
$$

Also note that $\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)|$ approaches zero geometrically fast, since

$$
|\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| = \frac{c\overline{\delta}(t)}{(\pi_i(t+1) + \epsilon_i(t))(\pi_j(t+1) + \epsilon_j(t))} \cdot |\pi_i(t+1)\epsilon_j(t) - \pi_j(t+1)\epsilon_i(t)|
$$

$$
\leq \tilde{C} \cdot (|\epsilon_j(t)| + |\epsilon_i(t)|),
$$

for some constant $\tilde{C} > 0$. Hence

$$
\sum_{t=0}^{\infty} \sqrt{t} \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) < \infty.
$$
 (4.4)

The proof is finished. \Box

Remark 4.3.

- (1) We may set $\delta_i(t) = \frac{ct^{\alpha}}{\pi_i(t+1)}$ with $\alpha \in [-1, -1/2)$ and $c > 0$ in Lemma [4.2](#page-9-2) (3).
- (2) When all $P(t)$ are doubly-stochastic, then $\pi(0) = \pi(1) = \cdots = \frac{1}{N}1_N$. So we may simply set $\delta_i(t) = ct^{\alpha}$ with $\alpha \in [-1, -1/2)$ and $c > 0$ in Lemma [4.2](#page-9-2) (3).

In the following proposition, we provide a sufficient condition $(\mathcal{A}1)'$ for $\{P(t)\}_{t\geq0}$ to satisfy $(\mathcal{A}1)$, described in terms of directed graphs. The condition $(A1)'$ is adapted from [\[15\]](#page-24-2). We mention that however, $(A1)$ is not equivalent to $(A1)'$. (See the remark that follows Proposition [4.5](#page-11-0) below.) We begin with the following definition.

Definition 4.4. For graph $G = (V, E)$ with $V = \{1, \dots, N\}$ and $E \subset V \times V$, we call G strongly connected if for any $(i, j) \in V \times V$, there exists a path from i to j, i.e., there exists a finite sequence of vertices $i = i_0, i_1, \ldots, i_n = j$ in V such that $(i_{l-1}, i_l) \in E$ for $l = 1, \ldots, n$.

Proposition 4.5. Given the sequence of $N \times N$ matrices $\{P(t)\}_{t\geq0}$ with nonnegative entries, define the directed graphs $G(t) = (V, E(t))$ $(t \ge 0)$ with $V = \{1, ..., N\}$ and $E(t) \subseteq V \times V$ in the following way:

$$
(i,j) \in E(t) \quad \text{if and only if} \quad p_{ij}(t) > 0.
$$

Then the following assumption implies $(A1)$:

 $(A1)'$ The sequence $\{P(t)\}_{t>0}$ of $N \times N$ row-stochastic matrices with positive diagonal entries satisfy

$$
p^{+} := \inf_{t \ge 0} \left[\min_{i,j: \ p_{ij}(t) > 0} p_{ij}(t) \right] > 0
$$
\n(4.5)

and there exists $t_0 \in \mathbb{N}$ such that for all $t \geq 0$, the union graph

$$
\bigcup_{t \le s < t + t_0} G(s) := \left(V, \bigcup_{t \le s < t + t_0} E(s) \right)
$$

is strongly connected.

Proof. Define $A(t) = (a_{ij}(t))$ $(t \ge 0)$ by

$$
a_{ij}(t) = \begin{cases} 1 & \text{if } (i,j) \in E(t) \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases}
$$

Then we have $P(t) \ge p^+(I_N + A(t))$, for any $t \ge 0$ by definition of p^+ given in [\(4.5\)](#page-11-1), and so

$$
P(t + t_0, t) \ge (p^+)^{t_0} (I_N + A(t + t_0 - 1)) \dots (I_N + A(t))
$$

$$
\ge (p^+)^{t_0} \left(I_N + \sum_{k=0}^{t_0 - 1} A(t + k) \right) =: B(t, t_0).
$$
 (4.6)

$$
(i,j) \in \bigcup_{0 \le k \le t_0 - 1} E(t + k) \quad \text{or} \quad i = j.
$$

By the strong connectivity of $(A1)'$, the matrix $B(t,t_0)$ is an irreducible matrix, i.e., for any pair of indices (i, j) there exists a finite sequence $i = i_0, i_1, \ldots, i_n = j$ such that (i_{m-1}, i_m) -entry of the matrix $B(t, t_0)$ is positive for all $1 \leq m \leq n$, and has positive diagonal entries. So, the same is true for $P(t + t_0, t)$ by [\(4.1\)](#page-9-0). Noting that $P(t + (N - 1)t_0, t)$ is equal to

$$
P(t + (N-1)t_0, t) = \prod_{k=0}^{N-1} P(t + (k+1)t_0, t + kt_0)
$$
\n(4.7)

which is a product of $N-1$ irreducible matrices with positive diagonal entries, we can see that proving the following claim would ensure the positivity of $P(t + (N-1)t_0, t)$, thereby finishing the proof. \Box

Lemma 4.6. Each row of a product of k irreducible matrices with positive diagonal entries has at least $k+1$ positive entries. $(1 \leq k \leq N-1)$

Proof. Let $A^{(\ell)} = (a_{ij}^{(\ell)})$ $(\ell = 1, ..., k)$ be such matrices, and consider the product $A^{(1)} \dots A^{(k)}$. We proceed by induction on k.

(i) $k = 1$: For each i, we have $a_{ii}^{(1)} > 0$, and the existence of $j \neq i$ satisfying $a_{ij}^{(1)} > 0$ can be shown in the following way: Choose any $q \neq i$. Since $A^{(1)}$ is irreducible, there exists a finite sequence $i = i_0, i_1, \ldots, i_n = q$ of indices such that $a_{i_m}^{(1)}$ $\sum_{i_{m-1}i_m}^{(1)}$ > 0 for all $1 \leq m \leq n$. Set $j = i_s$, where s is the minimal index satisfying $i_s \neq i$.

(ii) Suppose that the claim holds for $k-1 \leq N-2$, Set $A^{(1)} \dots A^{(\ell)} = (a_{ij}^{(1,\ell)})$. Note that if $a_{ij}^{(1,k-1)}$ is positive then so is $a_{ij}^{(1,k)}$, as can be seen in the following relation:

$$
a_{ij}^{(1,k)} = \sum_{\ell} a_{i\ell}^{(1,k-1)} a_{\ell j}^{(k)} \ge a_{ij}^{(1,k-1)} a_{jj}^{(k)}.
$$
\n(4.8)

Fix i. If the i-th row of $A^{(1)} \dots A^{(k-1)}$ had at least $k+1$ positive entries, then by [\(4.8\)](#page-12-0), the same is true for $A^{(1)} \dots A^{(k)}$. Now suppose that the *i*-th row of $A^{(1)} \dots A^{(k-1)}$ has exactly k positive entries $(k \leq$ N −1). To complete the induction step, it suffices to show that there exists j with $a_{ij}^{(1,k)} > 0 = a_{ij}^{(1,k-1)}$. Pick any ℓ with $a_{i\ell}^{(1,k-1)} = 0$. Since $A^{(k)}$ is irreducible, there exists a finite sequence $i = i_0, i_1, \ldots, i_n = q$ of indices such that $a_{i_m}^{(k)}$ $\sum_{i_{m-1}i_m}^{(k)}$ > 0 for all $1 \leq m \leq n$. Set $j = i_s$, where s is the minimal index satisfying $a_{ii_s}^{(1,k-1)} = 0$, whose existence is guaranteed by the fact that $a_{i\ell}^{(1,k-1)} = 0$. Then

$$
a_{ii_s}^{(1,k)} = \sum_p a_{ip}^{(1,k-1)} a_{pi}^{(k)} \ge a_{ii_{s-1}}^{(1,k-1)} a_{i_{s-1}i_s}^{(k)} > 0.
$$

Hence the claim holds for $k + 1$.

We close this section with showing that the condition $(A1)$ is strictly more general than the condition (A1)'. For this, we consider time independent $P(t) \equiv P$, where P is *irreducible* and *aperiodic*. In other words, the corresponding directed graph $G(t) \equiv G = (V, E)$ (as described in Proposition [4.5\)](#page-11-0) is

(i) strongly connected: for any pair of vertices (i, j) , G contains a path from i to j, i.e., a finite sequence $i = i_0, i_1, \ldots, i_n = j$ such that $(i_{m-1}, i_m) \in E$ for all $1 \leq m \leq n$, and

(ii) of period 1: for each vertex $i \in V$, the greatest common divisor of the lengths of all paths from i to i is equal to 1,

which is necessary and sufficient for the matrix P to be *primitive*, i.e., $P^T > 0$ for some $T \in \mathbb{N}$ (see [\[17\]](#page-24-12), for example). It implies that our example $P(t) \equiv P$ satisfies the condition (A1). In addition, if P has at least one zero diagonal entry, then our example does not satisfy the positive diagonal condition of $(\mathcal{A}1')$. Here is an example of such P:

$$
P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

5. Application of the main theorem to several algorithms

In this section, we apply the result of Theorem [4.1](#page-9-3) to derive the convergence results for the four examples discussed in Section 2.

5.1. Convergence theorem for Example [2.1.](#page-3-4)

Corollary 5.1. Let $\tilde{X}(t)$ be a solution to the following algorithm, originated from [\[8\]](#page-23-0):

$$
\tilde{x}_i(t+1) = \sum_{j=1}^N \tilde{p}_{ij}(t)(\tilde{x}_j(t) - \tilde{\delta}_j(t)\tilde{g}_i(t)), \quad \tilde{g}_i(t) \in \partial f_i(\tilde{x}_i(t)), \quad 1 \le i \le N, \quad t \ge 0.
$$

Suppose that the following conditions $(\tilde{A}1),(\tilde{A}2),(\tilde{A}3)$ hold.

• (\tilde{A} 1): The sequence $\{\tilde{P}(t)\}_{t\geq 0}$ of $N \times N$ row-stochastic matrices and its backward products satisfy

$$
\tilde{p}^+ := \inf_{t \ge 0} \left[\min_{i,j \colon \tilde{p}_{ij}(t) > 0} \tilde{p}_{ij}(t) \right] > 0
$$

and

$$
\tilde{P}(t+T,t) > 0 \quad (\forall \ t \ge 0)
$$

for some $T \in \mathbb{N}$.

- ($\tilde{A}2$): Each $\tilde{\delta}_i(\cdot)$ is nonnegative, and $\sum_{t=0}^{\infty} \left(\|\tilde{\Delta}(t)\| \sup_{\ell \geq t} \|\tilde{\Delta}(\ell)\| \right) < \infty$.
- ($\tilde{A}3$): The set of absolute probability vectors for $\{\tilde{P}(t)\}_{t\geq0}$, denoted by

$$
\tilde{\pi}(t) = (\tilde{\pi}_1(t), \dots, \tilde{\pi}_N(t))^\top \quad (t \ge 0),
$$

 $satisfy\ \sum_{t=0}^{\infty} \|\tilde{\Delta}(t)\tilde{\pi}(t)\| = \infty\ and\ \sum_{t=0}^{\infty} \sqrt{t} \left(\max_{i,j} |\tilde{\pi}_i(t)\tilde{\delta}_i(t) - \tilde{\pi}_j(t)\tilde{\delta}_j(t)| \right) < \infty.$ Then there exists some minimizer x^* of f such that $\lim_{t\to\infty} \tilde{x}_i(t) = x^*$ for all $1 \le i \le N$.

Proof. For $t \geq 0$ we set $x_i(2t) = \tilde{x}_i(t)$ and $x_i(2t+1) = \tilde{x}_i(t) - \tilde{\delta}_i(t)\tilde{g}_i(t)$. We also define

$$
P(2t) \equiv I_N, \quad P(2t+1) = \tilde{P}(t), \quad \Delta(2t+1) \equiv O_N, \quad \Delta(2t) = \tilde{\Delta}(t) \quad (t \ge 0)
$$
 (5.1)

and

$$
\delta_i(2t) = \delta_i(t) \quad \text{and} \quad \delta_i(2t+1) = 0. \tag{5.2}
$$

Then $X(t) = [x_1(t), \cdots, x_N(t)]^{\top}$ satisfies

$$
X(t+1) = P(t)X(t) - \Delta(t)G(t).
$$
\n(5.3)

Using [\(5.1\)](#page-13-0) it is direct to check that $(A1)$ and $(A2)$ are equivalent to $(A1)$ and $(A2)$ respectively. Since $P(2t) = I_N$ and $P(2t + 1) = \tilde{P}(t)$ we find that $\pi(2t + 1) = \pi(2t) = \tilde{\pi}(t)$. Using this and [\(5.2\)](#page-13-1) we see that (A3) and ($\tilde{A}3$) are equivalent. Combining this with Theorem [4.1,](#page-9-3) we get the desired result. \square

5.2. Convergence theorem for Example [2.2.](#page-4-3)

Corollary 5.2. Let $X(t)$ be a solution to the following algorithm, originated from [\[11\]](#page-24-0):

$$
\begin{cases}\n x_i(t+1) = \sum_{j=1}^N p_{ij} x_j(t) - \frac{\delta(t)}{z_{ii}(t)} g_i(t), & g_i(t) \in \partial f_i(x_i(t)), \\
 z_i(t+1) = \sum_{j=1}^N p_{ij} z_j(t), & z_i(0) = \mathbf{e}_i, & 1 \le i \le N, \quad t \ge 0.\n\end{cases}
$$

Suppose that $P = (p_{ij})$ is row-stochastic and primitive. Choose $\delta(t) = ct^{\alpha}$ with $\alpha \in [-1, -1/2)$ and c > 0. Then there exists some minimizer x^* of f such that $\lim_{t\to\infty} x_i(t) = x^*$ for all $1 \le i \le N$.

Proof. In Example [2.2,](#page-4-3) we see that $x_i(t)$ satisfies [\(1.2\)](#page-0-1) with $P(t) \equiv P$ and $\delta_i(t) = \frac{\delta(t)}{z_{ii}(t)}$. The condition (A1) is satisfied for any primitive P, and we have $\pi(0) = \pi(1) = \cdots =: \pi$. A standard theory of rowstochastic matrices yields that as $t \to \infty$, $Z(t) = P^t \to 1_N \pi$, i.e., $z_{ii}(t) \to \pi_i$ ($\forall i$) geometrically fast (see [\[17\]](#page-24-12)). Hence if we choose $\delta(t) = ct^{\alpha}$ with $\alpha \in [-1, -1/2)$ and $c > 0$, then $(A2, 3)$ are satisfied. \square

5.3. Convergence theorem for Example [2.3.](#page-4-2)

Corollary 5.3. Let $Y(t)$, $W(t)$ be a solution to the following algorithm:

$$
\begin{cases}\ny_i(t+1) = \sum_{j=1}^N a_{ij}(t)y_j(t), & 1 \le i \le N, \quad t \ge 0, \\
w_i(t+1) = \sum_{j=1}^N a_{ij}(t)w_j(t) - ct^\alpha g_i(t), & g_i(t) \in \partial f_i\left(\frac{w_i(t)}{y_i(t)}\right),\n\end{cases}
$$

with $\alpha \in [-1, -1/2)$ and $c > 0$. Suppose that $Y(0) > 0$ and $(\mathcal{A}1*)$ hold.

• (A1∗): The sequence ${A(t)}_{t\geq0}$ of $N\times N$ column-stochastic matrices with no zero rows satisfy

$$
a^+ := \inf_{t \ge 0} \left[\min_{i,j: \ a_{ij}(t) > 0} a_{ij}(t) \right] > 0
$$

and

$$
A(t + T, t) := A(t + T - 1) \dots A(t + 1)A(t) > 0 \quad (\forall t \ge 0)
$$

for some $T \in \mathbb{N}$.

Then there exists some minimizer x^* of f such that $\lim_{t\to\infty}$ $\frac{w_i(t)}{y_i(t)} = x^*$ for all $1 \leq i \leq N$. Proof. We may write the above scheme as

$$
\frac{w_i(t+1)}{y_i(t+1)} = \sum_{j=1}^{N} \frac{a_{ij}(t)y_j(t)}{y_i(t+1)} \frac{w_j(t)}{y_j(t)} - \frac{ct^{\alpha}}{y_i(t+1)} g_i(t).
$$
\n(5.4)

We set $P(t) = \text{diag}(Y(t+1))^{-1}A(t)\text{diag}(Y(t))$ for $t \ge 0$. Then $P(t)$ is row-stochastic as checked in [\(2.8\)](#page-5-1). We take $\delta_i(t) = \frac{ct^{\alpha}}{y_i(t+1)}$ with $\alpha \in [-1, -1/2)$ and $c > 0$. Then $x_i(t) = \frac{w_i(t)}{y_i(t)}$ satisfies

$$
x_i(t+1) = \sum_{j=1}^{N} p_{ij}(t)x_j(t) - \delta_i(t)g_i(t), \quad g_i(t) \in \partial f_i(x_i(t)).
$$
\n(5.5)

We aim to show that $P(t)$ and $\delta_i(t)$ satify the assumption $(A1)-(A3)$. To find the absolute probability vectors of $P(t)$, we note that

$$
Y(t+1)^{\top} P(t) = Y(t+1)^{\top} \operatorname{diag}(Y(t+1))^{-1} A(t) \operatorname{diag}(Y(t))
$$

= $(1_N)^{\top} A(t) \operatorname{diag}(Y(t))$
= $(1_N)^{\top} \operatorname{diag}(Y(t))$
= $Y(t)^{\top}$.

By multiplying 1_N to both sides, we also get

$$
Y(t+1)^{\top} 1_N = Y(t)^{\top} 1_N = \dots = Y(0)^{\top} 1_N.
$$

Therefore, to show that $\pi(t) = \frac{1}{Y(0)^T 1_N} Y(t)$, it only remains to prove that the condition $(\mathcal{A}1*)$ combined with $Y(0) > 0$ is a sufficient condition for $(A1)$, which implies the uniqueness of the set of absolute probability vectors. Indeed, we have

$$
A(t+T, t) > 0 \quad \Rightarrow \quad P(t+T, t) = \text{diag}(Y(t+T))^{-1}A(t+T, t)\,\text{diag}(Y(t)) > 0.
$$

The condition $p^+ > 0$ can be proved in the following way: for $t \geq T$, we have

$$
Y(t) = A(t, t - T)Y(t - T) \ge (a^+)^T 1_N(1_N)^T Y(t - T) = ((a^+)^T (1_N)^T Y(0)) 1_N
$$

and

$$
Y(t+1) \le 1_N(1_N)^{\top} Y(t+1) = ((1_N)^{\top} Y(0)) 1_N
$$

so

$$
P(t) = \text{diag}(Y(t+1))^{-1}A(t)\,\text{diag}(Y(t)) \ge (a^+)^T A(t), \quad t \ge T.
$$

Hence

$$
p^{+} \ge \min \left\{ \min_{0 \le t < T} \left[\min_{i,j: a_{ij}(t) > 0} a_{ij}(t) \right], (a^{+})^{T+1} \right\} > 0.
$$

Finally, note that $\delta_i(t) = \frac{ct^{\alpha}}{y_i(t+1)}$ with $\alpha \in [-1, -1/2)$ and $c > 0$, i.e., $\theta_i(t) = ct^{\alpha}$ satisfy the condition in Lemma [4.2](#page-9-2) (3), thereby satisfying $(A2)$, $(A3)$. Summing up, we obtain the desired result.

Recall that in Example [2.3](#page-4-2) we have used the substitution $x_i(t) = \frac{w_i(t)}{y_i(t)}$. We remark that we can get an analogous statement of Proposition [4.5](#page-11-0) for $(\mathcal{A}1*)$ instead of $(\mathcal{A}1)$.

5.4. Convergence theorem for Example [2.4.](#page-6-0)

Corollary 5.4. Let $\hat{X}(t), \hat{Y}(t), \hat{W}(t)$ be a solution to the following algorithm, originated from [\[15\]](#page-24-2):

$$
\begin{cases}\n\hat{y}_i(t+1) = \sum_{j=1}^N \hat{a}_{ij}(t)\hat{y}_j(t) \in \mathbb{R}_+, \\
\hat{x}_i(t) = \hat{w}_i(t) - ct^\alpha \hat{g}_i(t), \quad \hat{g}_i(t) \in \partial f_i\left(\frac{\hat{w}_i(t)}{\hat{y}_i(t)}\right), \\
\hat{w}_i(t+1) = \sum_{j=1}^N \hat{a}_{ij}(t)\hat{x}_j(t) \in \mathbb{R}^d.\n\end{cases}
$$

with $\alpha \in [-1, -1/2)$ and $c > 0$. Suppose that $\hat{Y}(0) > 0$ and $(\hat{A}1*)$ hold.

• ($\hat{A}1*$): The sequence $\{\hat{A}(t)\}_{t>0}$ of $N \times N$ column-stochastic matrices with no zero rows satisfy

$$
\hat{a}^+ := \inf_{t \ge 0} \left[\min_{i,j: \ \hat{a}_{ij}(t) > 0} \hat{a}_{ij}(t) \right] > 0
$$

and

$$
\hat{A}(t+T,t) := \hat{A}(t+T-1)\dots\hat{A}(t+1)\hat{A}(t) > 0 \quad (\forall \ t \ge 0)
$$

for some $T \in \mathbb{N}$.

Then there exists some minimizer x^* of f such that $\lim_{t\to\infty}$ $\frac{\hat{w}_i(t)}{\hat{y}_i(t)} = \lim_{t \to \infty}$ $\frac{\hat{x}_i(t)}{\hat{y}_i(t)} = x^*$ for all $1 \leq i \leq N$.

Proof 1. We let

$$
w_i(2t) = \hat{w}_i(t), \quad w_i(2t+1) = \hat{x}_i(t) \quad \text{and} \quad y_i(2t+1) = y_i(2t) = \hat{y}_i(t).
$$
 (5.6)

We write down the first and the second lines of the algorithm as

$$
\begin{cases} y_i(2t+1) = y_i(2t), & 1 \le i \le N, \quad t \ge 0, \\ w_i(2t+1) = w_i(2t) - ct^{\alpha}g_i(2t), & g_i(2t) \in \partial f_i\left(\frac{w_i(2t)}{y_i(2t)}\right). \end{cases}
$$

Next we write the first and the third lines of the algorithm as

$$
\begin{cases}\ny_i(2t+2) = \sum_{j=1}^N \hat{a}_{ij}(t)y_j(2t+1), & 1 \le i \le N, \quad t \ge 0, \\
w_i(2t+2) = \sum_{j=1}^N \hat{a}_{ij}(t)w_j(2t+1).\n\end{cases}
$$

Now we set $\theta_i(2t) = c(2t)^{\alpha}, \theta_i(2t+1) = 0, A(2t) = I_N$, and $A(2t+1) = \hat{A}(t)$.

Then we see that $W(t)$, $Y(t)$ satisfies the algorithm [\(2.11\)](#page-5-2). By following the same argument as in the proof of Corollary [5.3,](#page-14-0) we get the desired result. \Box

Proof 2. We let

$$
\tilde{x}_i(t+1) = \frac{\hat{w}_i(t+1)}{\hat{y}_i(t+1)}.
$$
\n(5.7)

Then the scheme is written as

$$
\tilde{x}_i(t+1) = \sum_{j=1}^N \frac{\hat{a}_{ij}(t)}{\hat{y}_i(t+1)} \Big(\hat{y}_j(t)\tilde{x}_j(t) - ct^\alpha \tilde{g}_j(t) \Big) \n= \sum_{j=1}^N \frac{\hat{a}_{ij}(t)\hat{y}_j(t)}{\hat{y}_i(t+1)} \Big(\tilde{x}_j(t) - \frac{ct^\alpha}{\hat{y}_j(t)} \tilde{g}_j(t) \Big), \quad \tilde{g}_i(t) \in \partial f_i(\tilde{x}_i(t))
$$
\n(5.8)

Thus we have

$$
\tilde{x}_i(t+1) =: \sum_{j=1}^N \tilde{p}_{ij}(t) \Big(\tilde{x}_j(t) - \tilde{\delta}_j(t) \tilde{g}_j(t) \Big), \quad \tilde{g}_i(t) \in \partial f_i \left(\tilde{x}_i(t) \right) \tag{5.9}
$$

where $\tilde{p}_{ij}(t) = \frac{\hat{a}_{ij}(t)\hat{y}_j(t)}{\hat{y}_i(t+1)}$ and $\tilde{\delta}_j(t) = \frac{ct^{\alpha}}{\hat{y}_j(t)}$. It corresponds to the algorithm in Corollary [5.1.](#page-13-2) Moreover, from the proof of Corollary [5.3](#page-14-0) we see that $(\tilde{A}1) - (\tilde{A}3)$ are satisfied for \tilde{P} and $\tilde{\Delta}$. Therefore we may apply Corollary 5.1 to obtain the convergence result. apply Corollary [5.1](#page-13-2) to obtain the convergence result.

Proof 3. We let

$$
x_i(2t) = \frac{\hat{w}_i(t)}{\hat{y}_i(t)} \quad \text{and} \quad x_i(2t+1) = \frac{\hat{x}_i(t)}{\hat{y}_i(t)}.
$$
 (5.10)

We write the second line of the algorithm as

$$
\frac{\hat{x}_i(t)}{\hat{y}_i(t)} = \frac{\hat{w}_i(t)}{\hat{y}_i(t)} - \frac{ct^{\alpha}}{\hat{y}_i(t)}\hat{g}_i(t), \quad \hat{g}_i(t) \in \partial f_i\left(\frac{\hat{w}_i(t)}{\hat{y}_i(t)}\right). \tag{5.11}
$$

Now we set $\delta_i(2t) = \frac{ct^{\alpha}}{\hat{y}_i(t)}$, $g_i(2t) = \hat{g}_i(t)$, and $p_{ij}(2t) = \delta_{ij}$. Then the above equality is written as

$$
x_i(2t+1) = x_j(2t) - \delta_i(2t)g_i(2t), \quad g_i(2t) \in \partial f_i(x_i(2t)).
$$
\n(5.12)

Next we write down the third line of the algorithm as

$$
\frac{\hat{w}_i(t+1)}{\hat{y}_i(t+1)} = \sum_{j=1}^N \frac{\hat{a}_{ij}(t)\hat{y}_j(t)}{\hat{y}_i(t+1)} \frac{\hat{x}_j(t)}{\hat{y}_j(t)}.
$$
\n(5.13)

Let $p_{ij}(2t+1) = \frac{\hat{a}_{ij}(t)\hat{y}_j(t)}{\hat{y}_i(t+1)}$ and $\delta_i(2t+1) = 0$. Then this is written in terms of x_i as

$$
x_i(2t+2) = \sum_{j=1}^{N} p_{ij}(2t+1)x_i(2t+1).
$$
\n(5.14)

From [\(5.12\)](#page-17-0) and [\(5.14\)](#page-17-1) we see that $x_i(t)$ satisfies [\(1.2\)](#page-0-1). Now it remains to check that $P(t) = (p_{ij}(t))$ and $\Delta(t) = \text{diag}(\delta_1(t), \ldots, \delta_N(t))$ satisfy the conditions $(\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$. Note that

$$
P(t) = \text{diag}(Y(t+1))^{-1} A(t) \text{diag}(Y(t)),
$$

where

$$
A(2t) = I_N
$$
, $A(2t + 1) = \hat{A}(t)$, $Y(2t + 1) = Y(2t) = \hat{Y}(t)$.

By following the same argument as in the proof of Corollary [5.3,](#page-14-0) we can see that $\pi(t) = \frac{1}{Y(0)^{\top}1_N}Y(t)$, and thus $(A1) - (A3)$ hold. Thus we have the desired result.

6. Proof of Theorem [4.1](#page-9-3)

In this section we give the proof of Theorem [4.1.](#page-9-3) For this aim, we state and prove preliminary lemmas which are inspired by [\[15\]](#page-24-2).

Lemma 6.1. Suppose (A1) hold. Assume that a sequence $\{b(t)\}_{t\geq 0}$ of vectors in $\mathbb{R}^{N\times 1}$ satisfy

$$
\sup_{t\geq 0} \|b(t)\|_1 < \infty \quad and \quad b(t)^{\top} 1_N = 0 \quad (\forall \ t \geq 0).
$$

Then the sequence $\{X(t)\}_{t\geq 0}$ generated by [\(2.1\)](#page-3-0) satisfy the following:

(1) We have

$$
||X(t)^{\top}b(t)||_1 \leq C\lambda^t + C\sum_{k=0}^{t-1} \lambda^{t-k-1} ||\Delta(k)||_{\infty},
$$

for some constant $C > 0$ independent of $t \geq 1$.

(2) If $\lim_{t\to\infty} ||\Delta(t)||_{\infty} = 0$ holds, then

$$
\lim_{t \to \infty} \left\| X(t)^\top b(t) \right\|_1 = 0.
$$

 (3) If $(A2)$ holds, then

$$
\sum_{t=1}^{\infty} \|\Delta(t)\|_{\infty} \left\| X(t)^{\top} b(t) \right\|_{1} < \infty.
$$

Proof. (1) Using (1.2) iteratively, we have

$$
X(t) = P(t,0)X(0) - \sum_{k=0}^{t-1} P(t,k+1)\Delta(k)G(k), \quad t \ge 1.
$$

Multiplying $b(t)$ to both sides and using the triangle inequality, we deduce

$$
\|X(t)^{\top}b(t)\|_{1} = \left\|X(0)^{\top}P(t,0)^{\top}b(t) - \sum_{k=0}^{t-1}G(k)^{\top}\Delta(k)^{\top}P(t,k+1)^{\top}b(t)\right\|_{1}
$$

\n
$$
\leq \|X(0)^{\top}\|_{1}\|P(t,0)^{\top}b(t)\|_{1} + \sum_{k=0}^{t-1}\|G(k)^{\top}\|_{1}\|\Delta(k)^{\top}\|_{1}\|P(t,k+1)^{\top}b(t)\|_{1}
$$

\n
$$
\leq \|X(0)\|_{\infty}\tau(P(t,0))\|b(t)\|_{1} + \sum_{k=0}^{t-1}\|G(k)\|_{\infty}\|\Delta(k)\|_{\infty}\tau(P(t,k+1))\|b(t)\|_{1},
$$

\n(6.1)

where we used Lemma [3.4](#page-8-3) in the second inequality. By Lemma [4.2](#page-9-2) (1) we have $\tau(P(t,0)) \leq \tilde{C} \lambda^t$ for some $\tilde{C} > 0$ and $0 < \lambda < 1$. We also have

$$
\sup_{k\geq 0} ||G(k)||_{\infty} = \sup_{k\geq 0} \max_{1\leq i\leq N} ||g_i(k)||_1 \leq \max_{1\leq i\leq N} L_i < \infty.
$$

Using this we bound the right hand side of [\(6.1\)](#page-18-0) as

$$
\left\| X(t)^\top b(t) \right\|_1 \le C\lambda^t + C\sum_{k=0}^{t-1} \lambda^{t-k-1} \|\Delta(k)\|_\infty. \tag{6.2}
$$

(2) By (1), it suffices to show that

$$
\lim_{t \to \infty} \sum_{k=0}^{t-1} \lambda^{t-k-1} \|\Delta(k)\|_{\infty} = 0.
$$
\n(6.3)

For any $1\leq m\leq t-1$ we have

$$
\sum_{k=0}^{t-1} \lambda^{t-k-1} \|\Delta(k)\|_{\infty} \le \sum_{k=0}^{t-1} \lambda^{t-k-1} \sup_{\ell \ge k} \|\Delta(\ell)\|_{\infty}
$$

$$
\le \left(\sup_{\ell \ge 0} \|\Delta(\ell)\|_{\infty}\right) \sum_{k=0}^{m-1} \lambda^{t-k-1} + \left(\sup_{\ell \ge m} \|\Delta(\ell)\|_{\infty}\right) \sum_{k=m}^{t-1} \lambda^{t-k-1}
$$

$$
\le \left(\sup_{\ell \ge 0} \|\Delta(\ell)\|_{\infty}\right) \frac{\lambda^{t-m}}{1-\lambda} + \left(\sup_{\ell \ge m} \|\Delta(\ell)\|_{\infty}\right) \frac{1}{1-\lambda}
$$

By plugging in $m = \lfloor \frac{t}{2} \rfloor$ $\frac{t}{2}$ and sending $t \to \infty$, we can see that [\(6.3\)](#page-19-0) holds.

(3) We apply [\(6.2\)](#page-18-1) to obtain

$$
\sum_{t=1}^{\infty} ||\Delta(t)||_{\infty} ||X(t)^{\top}b(t)||_{1}
$$
\n
$$
\leq \sum_{t=1}^{\infty} ||\Delta(t)||_{\infty} \left(C\lambda^{t} + C \sum_{k=0}^{t-1} ||\Delta(k)||_{\infty} \lambda^{t-k-1} \right)
$$
\n
$$
= C \sum_{t=1}^{\infty} ||\Delta(t)||_{\infty} \lambda^{t} + C \sum_{t=1}^{\infty} \sum_{k=0}^{t-1} ||\Delta(t)||_{\infty} ||\Delta(k)||_{\infty} \lambda^{t-k-1}.
$$

We bound this as follows

$$
\sum_{t=1}^{\infty} ||\Delta(t)||_{\infty} ||X(t)^{\top}b(t)||_{1}
$$
\n
$$
\leq C \sum_{t=1}^{\infty} \left(\sup_{\ell \geq 1} ||\Delta(\ell)||_{\infty} \right) \lambda^{t} + C \sum_{t=1}^{\infty} \sum_{k=0}^{t-1} \left(\sup_{\ell \geq k} ||\Delta(\ell)||_{\infty} \right) ||\Delta(k)||_{\infty} \lambda^{t-k-1}
$$
\n
$$
= C \frac{\lambda}{1-\lambda} \left(\sup_{\ell \geq 1} ||\Delta(\ell)||_{\infty} \right) + C \sum_{k=0}^{\infty} \sum_{t=k+1}^{\infty} \left(\sup_{\ell \geq k} ||\Delta(\ell)||_{\infty} \right) ||\Delta(k)||_{\infty} \lambda^{t-k-1}
$$
\n
$$
= C \frac{\lambda}{1-\lambda} \left(\sup_{\ell \geq 1} ||\Delta(\ell)||_{\infty} \right) + C \frac{1}{1-\lambda} \sum_{k=0}^{\infty} \left(\sup_{\ell \geq k} ||\Delta(\ell)||_{\infty} \right) ||\Delta(k)||_{\infty} < \infty,
$$

where we used $(A2)$ in the last inequality. The proof is done.

Lemma 6.2. Suppose $(A1)$ - $(A2)$ holds. Then we have

$$
||X(t)|| = O(\sqrt{t}), \quad t \to \infty.
$$

Proof. Using [\(1.2\)](#page-0-1) and the fact that $L_i = \sup_{z \in \mathbb{R}^d} \sup_{g \in \partial f_i(z)} ||g||_1 < \infty$, we find

$$
||X(t+1)||_{\infty} = ||P(t)X(t) - \Delta(t)G(t)||_{\infty}
$$

\n
$$
\leq ||P(t)||_{\infty} ||X(t)||_{\infty} + ||\Delta(t)||_{\infty} ||G(t)||_{\infty}
$$

\n
$$
= ||X(t)||_{\infty} + ||\Delta(t)||_{\infty} ||G(t)||_{\infty}.
$$

Iterating gives us the following estimate:

$$
||X(t)||_{\infty} \le ||X(0)||_{\infty} + \sum_{k=0}^{t-1} ||\Delta(k)||_{\infty} ||G(k)||_{\infty}
$$

$$
\le ||X(0)||_{\infty} + \left(\max_{1 \le i \le N} L_i\right) \sum_{k=0}^{t-1} ||\Delta(k)||_{\infty}.
$$

Now we use the Cauchy-Schwarz inequality to deduce

$$
||X(t)||_{\infty} \le ||X(0)||_{\infty} + \left(\max_{1 \le i \le N} L_i\right) \left(\sum_{k=0}^{\infty} ||\Delta(k)||_{\infty}^2\right)^{\frac{1}{2}} \sqrt{t} = O(\sqrt{t}),
$$

where $(A2)$ is used. The proof is done.

Lemma 6.3. Suppose $(A1)$ - $(A2)$ holds. Then for any $u \in \mathbb{R}^{d \times 1}$, we have

$$
\left\| X(t+1)^\top \pi(t+1) - u \right\|_2^2 \le \left\| X(t)^\top \pi(t) - u \right\|_2^2 + \|\Delta(t)\|_\infty^2 \left(\max_{1 \le i \le N} L_i \right)^2 + \frac{2}{N} \|\Delta(t)\pi(t+1)\|_1 \sum_{i=1}^N \left(f_i(u) - f_i(X(t)^\top \pi(t)) \right) + 2 \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) \sum_{i=1}^N L_i \left\| X(t)^\top \pi(t) - u \right\|_2 + 4 \|\Delta(t)\|_\infty \sum_{i=1}^N L_i \left\| X(t)^\top (\pi(t) - \mathbf{e}_i) \right\|_2.
$$

Proof. We recall from [\(3.1\)](#page-8-1) that $\pi(t)$ satisfies $\pi(t+1)^\top P(t) = \pi(t)^\top$. Combining this with [\(1.2\)](#page-0-1), we find the following equality

$$
X(t+1)^{\top} \pi(t+1) - u = X(t)^{\top} P(t)^{\top} \pi(t+1) - G(t)^{\top} \Delta(t)^{\top} \pi(t+1) - u
$$

= $X(t)^{\top} \pi(t) - G(t)^{\top} \Delta(t)^{\top} \pi(t+1) - u.$

Using this identitiy, we compute

$$
\left\| X(t+1)^\top \pi(t+1) - u \right\|_2^2 = \left\| X(t)^\top \pi(t) - G(t)^\top \Delta(t)^\top \pi(t+1) - u \right\|_2^2
$$

$$
= \left\| X(t)^\top \pi(t) - u \right\|_2^2 + \left\| \sum_{i=1}^N \pi_i(t+1) \delta_i(t) g_i(t) \right\|_2^2
$$

$$
- 2 \sum_{i=1}^N \pi_i(t+1) \delta_i(t) \left\langle X(t)^\top \pi(t) - u, g_i(t) \right\rangle
$$

$$
=: \left\| X(t)^\top \pi(t) - u \right\|_2^2 + I_1 + I_2.
$$

We estimate \mathcal{I}_1 as

$$
I_1 \leq \left(\sum_{i=1}^N \pi_i(t+1)\delta_i(t) \|g_i(t)\|_2\right)^2
$$

$$
\leq \|\Delta(t)\|_{\infty}^2 \left(\sum_{i=1}^N \pi_i(t+1)L_i\right)^2
$$

$$
\leq \|\Delta(t)\|_{\infty}^2 \left(\max_{1 \leq i \leq N} L_i\right)^2,
$$

where $\sum_{i=1}^{N} \pi_i(t+1) = 1$ is used in the third inequality. Next we decompose I_2 as follows:

$$
I_2 = -2 \sum_{i=1}^{N} \pi_i(t+1) \delta_i(t) \left\langle X(t)^{\top} \mathbf{e}_i - u, g_i(t) \right\rangle
$$

- 2 $\sum_{i=1}^{N} \pi_i(t+1) \delta_i(t) \left\langle X(t)^{\top} (\pi(t) - \mathbf{e}_i), g_i(t) \right\rangle$
=: $I_{21} + I_{22}$.

By the convexity of f_i , we obtain

$$
I_{21} = -2 \sum_{i=1}^{N} \pi_i(t+1) \delta_i(t) \langle x_i(t) - u, g_i(t) \rangle
$$

\n
$$
\leq 2 \sum_{i=1}^{N} \pi_i(t+1) \delta_i(t) (f_i(u) - f_i(x_i(t)))
$$

\n
$$
= 2 \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \pi_j(t+1) \delta_j(t) \right) (f_i(u) - f_i(X(t)^{\top} \pi(t)))
$$

\n
$$
+ 2 \sum_{i=1}^{N} \left(\pi_i(t+1) \delta_i(t) - \frac{1}{N} \sum_{j=1}^{N} \pi_j(t+1) \delta_j(t) \right) (f_i(u) - f_i(X(t)^{\top} \pi(t)))
$$

\n
$$
+ 2 \sum_{i=1}^{N} \pi_i(t+1) \delta_i(t) (f_i(X(t)^{\top} \pi(t)) - f_i(X(t)^{\top} \mathbf{e}_i)).
$$

Here we recall that $\sup_{g \in \partial f_i(z)} \|g\|_1 \leq L_i$ for all $z \in \mathbb{R}^d$ and achieve the following estimates:

$$
I_{21} \leq \frac{2}{N} \|\Delta(t)\pi(t+1)\|_1 \sum_{i=1}^N \left(f_i(u) - f_i(X(t)^{\top}\pi(t)) \right)
$$

+
$$
2 \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) \sum_{i=1}^N L_i \|X(t)^{\top}\pi(t) - u\|_2
$$

+
$$
2 \|\Delta(t)\|_{\infty} \sum_{i=1}^N L_i \|X(t)^{\top}(\pi(t) - \mathbf{e}_i)\|_2
$$

and

$$
I_{22} \leq 2||\Delta(t)||_{\infty} \sum_{i=1}^{N} L_i ||X(t)^{\top} (\pi(t) - \mathbf{e}_i)||_2.
$$

Combining the above estimates finishes the proof. \Box

We recall from [\[15,](#page-24-2) Lemma 7] the following result.

Lemma 6.4. Consider a minimization problem $min_{x \in \mathbb{R}^d} f(x)$, where $f : \mathbb{R}^d \to \mathbb{R}$ is continuous. Suppose that the solution set X^* of the problem is nonempty, and let $\{x(t)\}_{t\geq0}$ be a sequence such that for all $x^* \in X^*$ and $t \geq 0$,

$$
||x(t+1) - x^*||^2 \le (1 + b(t))||x(t) - x^*||^2 - a(t)(f(x(t)) - f(x^*)) + c(t),
$$

where

$$
a(t), b(t), c(t) \ge 0 \quad (\forall \ t \ge 0), \quad \sum_{t=0}^{\infty} a(t) = \infty, \quad \sum_{t=0}^{\infty} b(t) < \infty, \quad \sum_{t=0}^{\infty} c(t) < \infty.
$$

Then the sequence $\{x(t)\}_{t\geq 0}$ converges to some solution $x^* \in X^*$.

We conclude this section by presenting the proof of the main theorem.

Proof of Theorem [4.1.](#page-9-3) We let $x(t) := X(t)^\top \pi(t)$. Then the estimate of Lemma [6.3](#page-20-0) with $u = x^*$ gives

$$
||x(t+1) - x^*||^2 \le ||x(t) - x^*||^2 - a(t)(f(x(t)) - f(x^*)) + c(t),
$$
\n(6.4)

where $a(t) := \frac{2}{N} ||\Delta(t)\pi(t+1)||_1$ and

$$
c(t) := \|\Delta(t)\|_{\infty}^{2} \left(\max_{1 \leq i \leq N} L_{i}\right)^{2}
$$

+
$$
2 \left(\max_{i,j} |\pi_{i}(t+1)\delta_{i}(t) - \pi_{j}(t+1)\delta_{j}(t)|\right) \sum_{i=1}^{N} L_{i} \left\|X(t)^{\top}\pi(t) - x^{*}\right\|_{2}
$$

+
$$
4\|\Delta(t)\|_{\infty} \sum_{i=1}^{N} L_{i} \left\|X(t)^{\top}(\pi(t) - \mathbf{e}_{i})\right\|_{2}.
$$

By $(A3)$ we see that

$$
\sum_{t=0}^{\infty} a(t) = \frac{2}{N} \sum_{t=0}^{\infty} \|\Delta(t)\pi(t+1)\|_1 = \infty.
$$
\n(6.5)

Next we estimate $\sum_{t=0}^{\infty} c(t)$. It follows from $(\mathcal{A}2)$ that

$$
\sum_{t=0}^{\infty} \|\Delta(t)\|_{\infty}^{2} \le \sum_{t=0}^{\infty} \left(\|\Delta(t)\| \sup_{l \ge t} \|\Delta(t)\| \right) < \infty.
$$
\n(6.6)

Combining Lemma [6.2](#page-19-1) and $(A3)$, we derive

$$
\sum_{t=0}^{\infty} 2 \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) \sum_{i=1}^{N} L_i \left\| X(t)^\top \pi(t) - x^* \right\|_2
$$
\n
$$
\leq 2C \sum_{t=0}^{\infty} \sqrt{t} \left(\max_{i,j} |\pi_i(t+1)\delta_i(t) - \pi_j(t+1)\delta_j(t)| \right) < \infty.
$$
\n(6.7)

Next we apply Lemma [6.1](#page-18-2) (3) to find

$$
\sum_{t=0}^{\infty} 4\|\Delta(t)\|_{\infty} \sum_{i=1}^{N} L_i \|X(t)^{\top} (\pi(t) - \mathbf{e}_i)\|_2 < \infty,\tag{6.8}
$$

where we used that $(\pi(t) - \mathbf{e}_i)^\top \mathbf{1}_N = 1 - 1 = 0$ for all $t \geq 0$ and $1 \leq i \leq N$. Combining the above estimates we find that

$$
\sum_{t=0}^{\infty} c(t) < \infty. \tag{6.9}
$$

Given the estimates [\(6.5\)](#page-22-0) and [\(6.9\)](#page-23-9), we may apply Lemma [6.4](#page-22-1) to conclude that $x(t) = X(t)^{\top} \pi(t)$ converges to some minimizer $x^* \in \mathbb{R}^d$. In addition, we have

$$
\lim_{t \to \infty} ||X(t)^{\top} \pi(t) - x_i(t)||_2 = \lim_{t \to \infty} ||X(t)^{\top} (\pi(t) - \mathbf{e}_i)||_2 = 0,
$$
\n(6.10)

which yields that $\lim_{t\to\infty} x_i(t) = x^*$. The proof is complete.

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