

Decentralized Observation of Discrete-Event Systems: At Least One Can Tell*

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1 Introduction

The move towards multi-agent, autonomous systems in daily living, including decentralized micro-grids for power, autonomous robots, self-driving cars, and smart small appliances and electronics, requires truly decentralized decision-making, obviating the need for a centralized decision point or fusion rule or coordinator. In this work we explore agents who make decentralized observations and we examine under what conditions the agents' decisions suffice to determine if some behavior is legal or not.

In particular, we introduce a new decentralized observation condition which we call *at least one can tell* and which attempts to capture the idea that for any possible behavior that a system can generate, at least one decentralized observation agent can tell whether that behavior was “good” or “bad”, for given formal specifications of “good” and “bad”. We provide several equivalent formulations of the *at least one can tell* condition, and we relate it to (and show that it is different from) previously introduced *joint observability* [7, 9]. In fact, contrary to joint observability which is undecidable [7, 9], we show that the *at least one can tell* condition is decidable. We also show that when the condition holds, finite-state decentralized observers exist.

2 Background

The setting of this work is that of systems whose behavior can be thought of as sequences of actions or events, also referred to as *discrete-event systems* (DESS) and modelled as automata or languages. For more details on automata theory and languages, see the classic book by Hopcroft and Ullman [3] and for details on discrete-event systems the book by Wonham and Cai [11].

2.1 Preliminaries

A finite set of *letters* Σ is called an *alphabet*. The set of all finite sequences over Σ , also called *words*, is denoted by Σ^* . The empty word (i.e., the sequence of length 0) is denoted ϵ . Given a subalphabet $\Sigma_1 \subseteq \Sigma$, the *projection function from Σ onto Σ_1* is the function $P_1 : \Sigma^* \rightarrow \Sigma_1^*$ that removes from words in Σ all letters except those in Σ_1 . For example, if $\Sigma = \{a, b\}$ and $\Sigma_1 = \{a\}$, then $P_1(abbab) = aa$ and $P_1(bb) = \epsilon$. A language L over Σ is a subset of all possible words and will be used to denote the behavior of some system

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or process, where elements of Σ are events. In that context, the projection function will be used to capture observations of the behavior of that system or process.

A *finite automaton* over Σ is a tuple $A = (\Sigma, Q, Q_0, F, \Delta)$, where Q is a finite set of *states*; $Q_0 \subseteq Q$ is a set of *initial* states; $F \subseteq Q$ is a set of *final* or *accepting* states; and $\Delta \subseteq Q \times \Sigma \times Q$ is the *transition relation*. When the relation Δ can be represented as a function $\Delta : Q \times \Sigma \rightarrow Q$, we say that A is a *deterministic finite automaton* (DFA) and otherwise it is a *nondeterministic finite automaton* (NFA). If we think of elements of Σ as events then a DFA is one where at a given state, an event leads to only one other state, whereas an NFA allows for the same event from a state to lead to multiple other states. A *run* of A is a finite sequence of states and transitions $q_0 \xrightarrow{a_1} q_1 \cdots \xrightarrow{a_n} q_n$, for $n \geq 0$, such that $q_0 \in Q_0$ and $(q_i, a_{i+1}, q_{i+1}) \in \Delta$ for all $i = 0, \dots, n-1$. The run is called *accepting* if $q_n \in F$. The run is said to *generate* the word $a_1 \cdots a_n$. A word $\rho \in \Sigma^*$ is said to be *accepted* or *recognized* by A if there is an accepting run of A generating ρ . The language accepted or recognized by A is the set of all words in Σ^* accepted by A .

Since projection can be used to capture the observations of a system, it is natural to ask whether the projection of a language recognized by a finite automaton A can also be recognized by a finite automaton. In fact, the answer is “yes”. Informally the process is as follows. Let A be an automaton over Σ and let $\Sigma_1 \subseteq \Sigma$. We wish to find an automaton A' over Σ_1 that recognizes $P_1(L(A))$, where $P_1 : \Sigma^* \rightarrow \Sigma_1^*$, and for a language $L \subseteq \Sigma^*$, $P_1(L) = \{P_1(\rho) \mid \rho \in L\}$. We start by replacing all alphabet labels in A by ε . This results in an NFA A' with ε -transitions. A' already recognizes $P_1(L(A))$ and is a solution to our problem, provided having a non-deterministic automaton is not a problem. If A' is required to be deterministic, we can use the standard subset construction of [3] to convert an NFA with ε -transitions into a DFA.

The *inverse projection* $P_1^{-1}(L) : \Sigma_1^* \rightarrow \Sigma^*$ of a given language returns all words whose projection is in that language, i.e., $P_1^{-1}(L) = \{\rho \mid P_1(\rho) \in L\}$. Inverse projection allows us to speak about the words an agent thinks could have been produced based on the agent’s observations. Inverse projection of a regular language can also be recognized by a finite automaton. For $\Sigma_1 \subseteq \Sigma$, let A be an automaton over Σ_1 . To find an automaton that recognizes $P_1^{-1}(L(A))$, add self-loops of all events in $\Sigma \setminus \Sigma_1$ at all states of A .

2.2 Previous work: joint observability (JO)

Previous work [7, 9] introduced the following definition of *joint observability*:

Definition 1 (Joint Observability). *Given alphabet Σ and subalphabets $\Sigma_i \subseteq \Sigma$, for $i = 1, \dots, n$, and given regular languages $K \subseteq L \subseteq \Sigma^*$, K is called jointly observable if there exists a total function*

$$f : \Sigma_1^* \times \cdots \times \Sigma_n^* \rightarrow \{0, 1\}$$

such that

$$\forall \rho \in L : \rho \in K \iff f(P_1(\rho), \dots, P_n(\rho)) = 1$$

where $P_i : \Sigma^* \rightarrow \Sigma_i^*$ is the projection function onto Σ_i , for $i = 1, \dots, n$.

In the definition above L models the plant, and K models the good behaviors of the plant. We want to know whether the behavior of the plant was good or bad. But we can’t observe the behavior of the plant directly, so we have to rely on the decentralized projections. Joint observability says that whether a behavior is good can be completely determined based only on the decentralized observations of agents.

The following is a necessary and sufficient condition for joint observability:

Theorem 1 ([7, 9]). *K is jointly observable iff*

$$\exists \rho, \rho' \in L : \rho \in K \wedge \rho' \in L - K \wedge \forall i \in \{1, \dots, n\} : P_i(\rho) = P_i(\rho') \quad (1)$$

Informally, Condition (1) says that whether strings are good or not cannot possibly be determined if there are two strings—one good and one bad—that look the same to all agents.

We call Condition (1) the JO condition, or simply JO. It turns out that JO is undecidable [7, 9].

3 Decentralized Observability: *At Least One Can Tell*

Even though JO has been used in [7, 9] as a stepping stone to showing the undecidability of decentralized supervisory control problems, JO itself is not really decentralized, because it requires a centralized decision point f . In this paper, we investigate “truly decentralized” observation conditions. We begin by a definition that tries to capture the *at least one can tell* property: namely, that there is no centralized decision point, but for every behavior of the plant, at least one of the decentralized observers can tell whether this behavior is good or bad, i.e., whether it belongs in K or not. We call this condition *OCT*. We compare OCT to JO, and we examine equivalent formulations of OCT.

3.1 The OCT condition

Definition 2 (At Least One Can Tell). *Given alphabet Σ and subalphabets $\Sigma_i \subseteq \Sigma$, for $i = 1, \dots, n$, and given regular languages $K \subseteq L \subseteq \Sigma^*$, the at least one can tell condition (or simply *OCT*) is defined as follows:*

$$\forall \rho \in L : \exists i \in \{1, \dots, n\} : \text{CanTell}(i, \rho) \quad (2)$$

where *CanTell* is defined as follows:

$$\text{CanTell}(i, \rho) = \rho \in K \Rightarrow \exists \rho' \in L - K : P_i(\rho) = P_i(\rho') \quad (3)$$

$$\wedge \rho \in L - K \Rightarrow \exists \rho' \in K : P_i(\rho) = P_i(\rho') \quad (4)$$

3.2 The negation of OCT

In view of what follows, it is useful to state explicitly the negation of the OCT condition:

Lemma 1. *The negation of OCT is equivalent to:*

$$\left(\exists \rho \in K : \forall i \in \{1, \dots, n\} : \exists \rho_i \in L - K : P_i(\rho) = P_i(\rho_i) \right) \quad (5)$$

$$\vee \left(\exists \rho \in L - K : \forall i \in \{1, \dots, n\} : \exists \rho_i \in K : P_i(\rho) = P_i(\rho_i) \right) \quad (6)$$

Proof. The negation of OCT is:

$$\exists \rho \in L : \forall i \in \{1, \dots, n\} : \neg \text{CanTell}(i, \rho)$$

Since $\rho \in L$ is equivalent to $\rho \in K \vee \rho \in L - K$, the above is equivalent to:

$$\left(\exists \rho \in K : \forall i \in \{1, \dots, n\} : \neg \text{CanTell}(i, \rho) \right) \vee \left(\exists \rho \in L - K : \forall i \in \{1, \dots, n\} : \neg \text{CanTell}(i, \rho) \right)$$

which is by definition of *CanTell* equivalent to the disjunction of (5) and (6). \square

3.3 Comparison of OCT with JO

It is easy to show that OCT is a stronger condition than JO:

Theorem 2. *OCT implies JO.*

Proof. By contrapositive. Suppose JO doesn't hold. Then, by Theorem 1, there exist ρ, ρ' such that $\rho \in K$, $\rho' \in L - K$, and for all $i = 1, \dots, n$, $P_i(\rho) = P_i(\rho')$. We will show that (5) holds. Indeed, this is done by setting ρ_i to ρ' for each $i = 1, \dots, n$. By Lemma 1, (5) implies the negation of OCT. \square

The two conditions are *not* equivalent, however:

Theorem 3. *JO does not generally imply OCT.*

Proof. Consider the following example: $\Sigma = \{a, b\}$, $\Sigma_1 = \{a\}$, $\Sigma_2 = \{b\}$, $K = (ab)^*$ and $L = (ab)^*b^*$. We have two observers, and in this case JO holds: if the numbers of a 's and b 's are equal, we know that the word was in K , otherwise there are more b 's than a 's, and the word must have been in $L - K$. But the decentralized observers alone cannot tell:¹ one observer sees a bunch of a 's, the other a bunch of b 's. There is no way of comparing the number of a 's and b 's (since there is no centralized decision point). So OCT shouldn't hold here. Indeed, take $\rho = abb$, $\rho_1 = ab$, and $\rho_2 = abab$. Note that $\rho \in L - K$, $\rho_1 \in K$ and $\rho_2 \in K$. We will show that (6) holds. Indeed, $P_1(\rho) = P_1(\rho_1) = a$, so observer 1 cannot tell. Similarly, $P_2(\rho) = P_2(\rho_2) = bb$, so observer 2 cannot tell. By Lemma 1, (6) implies that OCT does not hold. \square

The distinction between JO and OCT can also be seen by noticing that a counterexample to JO is a pair ρ, ρ' , such that $P_i(\rho) = P_i(\rho')$ for all $i = 1, \dots, n$ (Theorem 1), whereas as Lemma 1 indicates and as we saw in the proof of Theorem 3, a counterexample to OCT is a set of $n + 1$ words, $\rho, \rho_1, \dots, \rho_n$, such that for each $i = 1, \dots, n$, we have $P_i(\rho) = P_i(\rho_i)$. Crucially, ρ' is the same word which must “match” ρ in every projection, whereas the words ρ_1, \dots, ρ_n need not be the same.

3.4 Functional characterization of OCT

Definition 2 captures our intuition about the *at least one can tell* property, namely, that at least one of the decentralized agents can be sure whether the behavior of the plant was good or bad. However, Definition 2 does not make explicit the existence of such decentralized observation agents. We rectify this by giving the definition that follows, which we then prove equivalent to OCT.

Definition 3 (Alternative OCT). *We say that the alternative OCT (ALTOCT) condition holds iff there exist total functions $f_i : \Sigma_i^* \rightarrow \{Y, N, U\}$, such that*

$$\begin{aligned}
(\forall \rho \in L : \exists i \in \{1, \dots, n\} : & \quad \rho \in K \Rightarrow f_i(P_i(\rho)) = Y \\
& \quad \wedge \\
& \quad \rho \in L - K \Rightarrow f_i(P_i(\rho)) = N) \\
& \quad \wedge \\
(\forall \rho \in L : \forall i \in \{1, \dots, n\} : & \quad f_i(P_i(\rho)) = Y \Rightarrow \rho \in K \\
& \quad \wedge \\
& \quad f_i(P_i(\rho)) = N \Rightarrow \rho \in L - K)
\end{aligned} \tag{7}$$

Y means that f_i knows that ρ was in K , N means that f_i knows that ρ was not in K , and U means f_i doesn't know. The bottom, $\forall \rho \dots \forall i \dots$ part says that no observer can “lie”, namely, if it says Y then it's really the case that $\rho \in K$, and if it says N then it's really the case that $\rho \in L - K$. The top, $\forall \rho \dots \exists i \dots$ part says that at least one observer can tell.

Theorem 4. *The ALTOCT condition is equivalent to the OCT condition.*

Proof. Suppose the condition of Theorem 4 holds. We will show that OCT holds. Pick some $\rho \in L$. By the first conjunct of Condition 7, there is some $i \in \{1, \dots, n\}$ such that function f_i satisfies the following:

$$(\rho \in K \Rightarrow f_i(P_i(\rho)) = Y) \quad \wedge \quad (\rho \in L - K \Rightarrow f_i(P_i(\rho)) = N) \tag{8}$$

We reason by cases:

¹We found this out thanks to a student in the DES school, Martijn Goorden, who cleverly observed that in an example presented in Tripakis' lecture, there doesn't seem to be a way for *at least one observer to tell*, yet joint observability holds. We thank Martijn for this observation.

- $\rho \in K$: Then $f_i(P_i(\rho)) = Y$. We claim that $CanTell(i, \rho)$ holds. Suppose not. Then there exists $\rho' \in L - K$ such that $P_i(\rho) = P_i(\rho')$. So $f_i(P_i(\rho')) = Y$. But then, by the second conjunct of Condition 7, we must have

$$(f_i(P_i(\rho')) = Y \Rightarrow \rho' \in K) \quad \wedge \quad (f_i(P_i(\rho')) = N \Rightarrow \rho' \in L - K) \quad (9)$$

so $\rho' \in K$, which is a contradiction. Therefore $CanTell(i, \rho)$ holds.

- $\rho \in L - K$: Then $f_i(P_i(\rho)) = N$. We claim that $CanTell(i, \rho)$ holds. Suppose not. Then there exists $\rho' \in K$ such that $P_i(\rho) = P_i(\rho')$. So $f_i(P_i(\rho')) = N$. But then, by the second conjunct of Condition 7, we must again have (9), so $\rho' \in L - K$, which is a contradiction. Therefore $CanTell(i, \rho)$ holds.

In both cases we have established $CanTell(i, \rho)$, which proves OCT. Thus, the condition of Theorem 4 implies OCT.

We now prove the converse, namely, that OCT implies the condition of Theorem 4. Suppose OCT holds. In order to establish the condition of Theorem 4 we need to define total functions $f_i : \Sigma_i^* \rightarrow \{Y, N, U\}$ such that Condition 7 is satisfied. We define each f_i as follows. Let $\sigma \in \Sigma_i^*$. Then:

$$f_i(\sigma) = \begin{cases} Y, & \text{if } \exists \rho \in K : P_i(\rho) = \sigma \wedge \forall \rho' \in L - K : P_i(\rho') \neq \sigma \\ N, & \text{if } \exists \rho \in L - K : P_i(\rho) = \sigma \wedge \forall \rho' \in K : P_i(\rho') \neq \sigma \\ U, & \text{otherwise.} \end{cases} \quad (10)$$

It remains to show that the functions defined in (10) satisfy Condition 7. We prove each of the two conjuncts of Condition 7 separately.

For the first conjunct, pick some $\rho \in L$. We must find $i \in \{1, \dots, n\}$ such that (8) holds. Pick an i such that $CanTell(i, \rho)$ holds. We know that such an i must exist, by OCT. We now claim that (8) holds:

- $\rho \in K$: Then, by $CanTell(i, \rho)$, it must be the case that $\nexists \rho' \in L - K : P_i(\rho) = P_i(\rho')$. So $\forall \rho' \in L - K : P_i(\rho) \neq P_i(\rho')$. Then, by (10), $f_i(P_i(\rho)) = Y$.
- $\rho \in L - K$: Then, by $CanTell(i, \rho)$, it must be the case that $\nexists \rho' \in K : P_i(\rho) = P_i(\rho')$. So $\forall \rho' \in K : P_i(\rho) \neq P_i(\rho')$. Then, by (10), $f_i(P_i(\rho)) = N$.

This proves (8) and the first conjunct of Condition 7.

For the second conjunct of Condition 7, pick some $\rho \in L$ and some $i \in \{1, \dots, n\}$. We must show that (9) holds. We reason by cases:

- $CanTell(i, \rho)$ holds: Then, by the same analysis as above we can show that either $\rho \in K$ and $f_i(P_i(\rho)) = Y$, or $\rho \in L - K$ and $f_i(P_i(\rho)) = N$, i.e., that (9) holds.
- $CanTell(i, \rho)$ does not hold: Then we claim that $f_i(P_i(\rho)) = U$. Notice that $\neg CanTell(i, \rho)$ is equivalent to

$$(\rho \in K \wedge \exists \rho' \in L - K : P_i(\rho) = P_i(\rho')) \vee (\rho \in L - K \wedge \exists \rho' \in K : P_i(\rho) = P_i(\rho')).$$

Therefore, the first two cases of (10) do not hold, so it must be that $f_i(P_i(\rho)) = U$. Then, (9) holds since both implications hold trivially.

In both cases, (9) holds. Thus we have shown that OCT implies the condition of Theorem 4. This completes the proof of the theorem. \square

Theorem 4 justifies the definition of OCT as a truly decentralized observation condition. Indeed, OCT holds if and only if decentralized functions satisfying condition (7) exist. In fact, we could equivalently have defined OCT to be the existence of functions satisfying condition (7). Then, Definition 2 could be seen as a necessary and sufficient condition for *at least one can tell* observability.

3.5 DES reformulations of OCT

This section explores the relationship between OCT and work in the supervisory control of discrete-event systems. In the DES literature, alphabet symbols represent events and words over an alphabet represent sequences of events. In this section we adopt the convention used in the DES literature of denoting words by letters in the Roman alphabet s, t, \dots and only denoting events by Greek letters, as opposed to the convention in theoretical computing of using Greek letters such as ρ to also denote words. We have preserved this difference in notation to highlight the point that there is resemblance between our work on OCT which can be situated entirely within theoretical computing (i.e., automata theory and formal languages) without reference to control-theoretic properties and existing work in the supervisory control of DES community.

There are decentralized control conditions within the discrete-event systems literature that bear some resemblance to OCT but are not the same. Consider the following definition, which on first blush looks similar to a combination of *co-observability* [6] and *DEA co-observability* [12]. As we will see in Theorem 5, it is actually decomposability [5] (which, informally, says that whether a string is legal can be determined from the observations of legal strings) plus a counterpart to decomposability that roughly says that whether a string is illegal can be determined from the observations of illegal strings.

Definition 4 (Discrete-Event Systems OCT). *Given alphabet Σ and subalphabets $\Sigma_1, \Sigma_2, \dots, \Sigma_n \subseteq \Sigma$, and given regular languages $K \subseteq L \subseteq \Sigma^*$, we say that the discrete-event systems OCT (DESOCT) condition holds if*

$$\begin{aligned} \forall s, s_1, s_2, \dots, s_n \in \Sigma^* : \left(\bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \right) \Rightarrow & \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \wedge s \in L \right) \Rightarrow s \in K \right) \\ & \wedge \\ & \left(\left(\bigwedge_{i=1, \dots, n} s_i \in L - K \wedge s \in L \right) \Rightarrow s \notin K \right) \end{aligned} \quad (11)$$

where $P_i : \Sigma^* \rightarrow \Sigma_i^*$ is the projection function onto Σ_i , for $i = 1, \dots, n$.

It can be shown, via logical transformations, that Definition 4 is equivalent to OCT.

Lemma 2. *The DESOCT condition is equivalent to the OCT condition.*

Proof. First, (11) is equivalent to:

$$\begin{aligned} \forall s, s_1, s_2, \dots, s_n \in \Sigma^* : \left(\bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \wedge s \in L \right) \Rightarrow & \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \Rightarrow s \in K \right) \right. \\ & \wedge \\ & \left. \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \Rightarrow s \notin K \right) \right) \end{aligned} \quad (12)$$

Next, (12) is equivalent to:

$$\begin{aligned} \forall s \in L, \forall s_1, s_2, \dots, s_n \in \Sigma^* : \bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \Rightarrow & \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \Rightarrow s \in K \right) \right. \\ & \wedge \\ & \left. \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \Rightarrow s \notin K \right) \right) \end{aligned} \quad (13)$$

Next, (13) is equivalent to:

$$\forall s \in L, \forall s_1, s_2, \dots, s_n \in \Sigma^* : \bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \Rightarrow \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \Rightarrow s \in K \right) \right)$$

$$\begin{aligned} & \wedge \\ & \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \Rightarrow s \in L - K \right) \end{aligned} \quad (14)$$

Now let's take the negation of (14):

$$\begin{aligned} \exists s \in L, \exists s_1, s_2, \dots, s_n \in \Sigma^* : & \bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \quad \wedge \quad \left(\neg \left(\bigwedge_{i=1, \dots, n} s_i \in K \Rightarrow s \in K \right) \right. \\ & \vee \\ & \left. \neg \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \Rightarrow s \in L - K \right) \right) \end{aligned} \quad (15)$$

Then (15) is equivalent to:

$$\begin{aligned} \exists s \in L, \exists s_1, s_2, \dots, s_n \in \Sigma^* : & \bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \quad \wedge \quad \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \wedge s \notin K \right) \right. \\ & \vee \\ & \left. \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \wedge s \notin L - K \right) \right) \end{aligned} \quad (16)$$

Finally, (16) can be rewritten by observing that $s \in L$ and $s \notin K$ is equivalent to $s \in L - K$ (and similarly for $s \in L$ and $s \notin L - K$):

$$\begin{aligned} \exists s \in L, \exists s_1, s_2, \dots, s_n \in \Sigma^* : & \bigwedge_{i=1, \dots, n} P_i(s) = P_i(s_i) \quad \wedge \quad \left(\left(\bigwedge_{i=1, \dots, n} s_i \in K \wedge s \in L - K \right) \right. \\ & \vee \\ & \left. \left(\bigwedge_{i=1, \dots, n} s_i \in L - K \wedge s \in K \right) \right) \end{aligned} \quad (17)$$

We can see from Lemma 1 that (17) is the negation of OCT. Thus, Definition 4 is equivalent to OCT. \square

Definition 4 can be rewritten as a combination of language inclusions, which will make it immediately apparent that DESOCT is decidable for regular language inputs.

Definition 5 (Language OCT). *Given alphabet Σ and subalphabets $\Sigma_1, \Sigma_2, \dots, \Sigma_n \subseteq \Sigma$, and given regular languages $K \subseteq L \subseteq \Sigma^*$, we say that the language OCT (LANGOCT) condition holds if*

$$\bigcap_{i=1, \dots, n} P_i^{-1}(P_i(K)) \cap L \subseteq K \quad (18)$$

$$\bigcap_{i=1, \dots, n} P_i^{-1}(P_i(L - K)) \cap L \subseteq L - K \quad (19)$$

Theorem 5. *The DESOCT condition is equivalent to the LANGOCT condition.*

Proof. We will show that (11) implies the conjunction (18) \wedge (19) and vice versa.

(11) implies (18) \wedge (19)

Consider $s \in \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(K)) \cap L$. Then $s \in L$ and $s \in P_i^{-1}(P_i(K))$, i.e., $P_i(s) \in P_i(K)$, for all $i = 1, \dots, n$. This means that there exist $s_1, s_2, \dots, s_n \in K$ such that $P_i(s_i) = P_i(s)$ for all $i = 1, \dots, n$. By the first conjunct in the consequent of (11), this means that $s \in K$. Therefore, (18) holds.

Similarly, consider $s \in \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(L-K)) \cap L$. Then there exist s_1, s_2, \dots, s_n such that $P_i(s_i) = P_i(s)$ and $s_1, s_2, \dots, s_n \in L-K$ and $s \in L$. By the second conjunct in the consequent of (11), this means that $s \in L-K$. Therefore, (19) holds.

(18) \wedge (19) implies (11)

Now consider $s, s_1, \dots, s_n \in \Sigma^*$ that satisfy the antecedent of (11), i.e., $P_i(s) = P_i(s_i)$ for $i = 1, \dots, n$. First, suppose that for all $i = 1, \dots, n$, $s_i \in K$ and $s \in L$. Since, for all $i = 1, \dots, n$, $P_i(s) = P_i(s_i)$ and $s_i \in K$, this means that $s \in P_i^{-1}(P_i(K))$. Since $s \in \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(K))$ and $s \in L$, by (18), we have $s \in K$.

Second, suppose that for all $i = 1, \dots, n$, $s_i \in L-K$ and $s \in L$. Since, for all $i = 1, \dots, n$, $P_i(s) = P_i(s_i)$ and $s_i \in L-K$, this means that $s \in P_i^{-1}(P_i(L-K))$. Since $s \in \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(L-K))$ and $s \in L$, by (19), we have $s \in L-K$. \square

4 Decidability, computational complexity, finite implementation

We now show that OCT is decidable for regular languages and we provide an asymptotic computational complexity analysis. We also show that, when the OCT condition is met, there exists a finite implementation.

4.1 OCT decidability and computational complexity

Using the formulation of OCT given by LANGOCT ((18) and (19)), we can demonstrate decidability of OCT. In addition, we will show that OCT can be decided in time $O(p^{n+2} \cdot m^{n+1})$ where n is the number of agents, and p and m are the number of states of the automata recognizing the languages L and K , respectively.

Theorem 6. *If K and L are regular languages, OCT is decidable. Moreover, if K and L are recognized by DFAs whose state sets have cardinality m and p , respectively, then for n agents, OCT is decidable in time $O(p^{n+2} \cdot m^{n+1})$.*

Proof. Decidability follows from Lemma 2, Theorem 5, and the fact that the operations of projection, inverse projection, intersection, and checking set containment are decidable for regular languages.

Let us now examine the computational complexity. Let K be recognized by a DFA A_K with m states and let L be recognized by a DFA A_L with p states. We can compute a DFA A_{L-K} recognizing $L-K$ by noting that $L-K = L \cap \overline{K}$, where \overline{K} denotes the complement of set K .² \overline{K} is recognized by a DFA $A_{\overline{K}}$ with exactly the same states as A_K , as it suffices to turn the accepting states of A_K into rejecting states and vice-versa. Then, A_{L-K} can be built as the Cartesian product of A_L and $A_{\overline{K}}$ [3]. The number of states of A_{L-K} is $p \cdot m$.

Consider first condition (18). The language $P_i(K)$ is recognized by the same automaton that recognizes K but with all events not in Σ_i replaced by ε . This process can be completed in $O(m)$ time and the number of states of the resulting automaton is still m . The inverse projection $P_i^{-1}(P_i(K))$ is achieved by adding in self-loops of events not in Σ_i to the resulting automaton. This can be done in $O(m)$ time and the number of states of the resulting automaton is still m . The intersection of the n terms $P_i^{-1}(P_i(K))$, for $i = 1, \dots, n$, together with L can be done in $O(m^n \cdot p)$ time, since intersection of automata can be represented with an automaton whose state space is the Cartesian product of the constituent automata. The result is an NFA A_1 with $m^n \cdot p$ states, such that $L(A_1) = \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(K)) \cap L$. For condition (18) we need

to check whether $L(A_1) \subseteq K$. This is equivalent to checking $L(A_1) \cap \overline{K} = \emptyset$, which in turn amounts to

²In the DES literature the overbar notation is used to denote prefix-closure. Here, we are adopting the standard math convention of using overbar to denote set complement.

computing the product of A_1 with $A_{\overline{K}}$.³ Therefore, the overall time complexity of checking condition (18) is $O(m^n \cdot p \cdot m) = O(m^{n+1} \cdot p)$.

Next, consider condition (19). Reasoning similarly as above, we first construct an NFA A_2 such that $L(A_2) = \bigcap_{i=1, \dots, n} P_i^{-1}(P_i(L - K)) \cap L$. This can be done in $O((p \cdot m)^n \cdot p)$ time and the resulting automaton A_2 has $(p \cdot m)^n \cdot p$ states. It remains to check whether $L(A_2) \subseteq L - K$, which is equivalent to checking $L(A_2) \cap \overline{L - K} = \emptyset$. Recall that $L - K$ is recognized by DFA A_{L-K} with $p \cdot m$ states. Because A_{L-K} is deterministic, the complement $\overline{L - K}$ is also recognized by a DFA $A_{\overline{L-K}}$ with exactly the same states. Therefore, checking $L(A_2) \subseteq L - K$ can be done by building the product of A_2 with $A_{\overline{L-K}}$, where this product has $(p \cdot m)^n \cdot p \cdot (p \cdot m)$ states. Thus, the overall complexity for checking condition (19) is $O((p \cdot m)^n \cdot p \cdot (p \cdot m)) = O(p^{n+2} \cdot m^{n+1})$.

We can see that the complexity of condition (19) is higher than that of condition (18), therefore, the overall complexity of OCT is $O(p^{n+2} \cdot m^{n+1})$. \square

4.2 Finite-state OCT observers

The functions f_i in Theorem 4 can be seen as decentralized observers, each outputting Y, N, U depending on whether they can tell whether the original behavior ρ was in K , not in K , or unknown. Each of these functions takes as input the entire projection $P_i(\rho)$ which is generally unbounded in length. So a brute-force implementation of these functions requires unbounded memory. In this section, we show that we can also implement these functions using finite memory.

A *finite-state observer* is a DFA O_i over subalphabet Σ_i , such that the states of O_i are labeled by one of Y, N, U , corresponding to the three observation outcomes of function f_i . The requirement is that for every $\sigma \in \Sigma_i^*$, the label of the state that O_i ends up in after reading σ is exactly $f_i(\sigma)$. Note that O_i is deterministic, so for a given σ there is a unique state that O_i ends up in after reading σ .

Theorem 7. *If OCT holds, then finite-state observers as above exist.*

Proof. We build O_i as follows:

- Let $P_i(K)$ be the projection of regular language K onto Σ_i . $P_i(K)$ is a regular language. Let A_1 be a DFA recognizing $P_i(K)$. Without loss of generality we can assume that A_1 is complete, meaning there is a transition from every state of A_1 for every input letter.
- Similarly, let A_2 be a DFA recognizing $P_i(L - K)$.
- Both A_1, A_2 are DFA over the same alphabet Σ_i . Let A be the synchronous product of A_1 and A_2 . Synchronous product means that each transition of A corresponds to a pair of transitions for each of A_1, A_2 , labeled with the same letter.
- A state q of A is a pair of states (q_1, q_2) , where q_1 is a state of A_1 and q_2 is a state of A_2 . We label q as follows:
 - q is labeled with Y if q_1 is accepting and q_2 is rejecting.
 - q is labeled with N if q_1 is rejecting and q_2 is accepting.
 - q is labeled with U otherwise.

We claim that O_i as constructed above satisfies our requirements for a correct local observer. To see this, suppose that OCT holds and consider some $\sigma \in \Sigma_i^*$. We distinguish cases:

- There is some $\rho \in L$ such that $\sigma = P_i(\rho)$. We distinguish subcases:

³It is well-known that checking subset inclusion for languages L_1 and L_2 can be done in polynomial time when L_1 is represented by an NFA and L_2 by a DFA. A table in [2] provides a nice summary of the computational complexity of checking subset inclusion for various different automata representations of L_1 and L_2 .

- $\rho \in K$: Suppose that O_i , after reading the input word σ , ends up in a state $q = (q_1, q_2)$. Because $\rho \in K$, we have $\sigma \in P_i(K)$, so q_1 is accepting. There are two subcases:
 - * q_2 is accepting: This means that σ is also in $P_i(L - K)$. So there must exist some $\rho' \in L - K$ such that $P_i(\rho') = \sigma$, so $P_i(\rho') = P_i(\rho)$. This means that $CanTell(i, \rho)$ does not hold and that $f_i(\sigma)$ must be equal to U . Indeed, q is also labeled U .
 - * q_2 is rejecting: This means that $\sigma \notin P_i(L - K)$, which implies that $\forall \rho' \in L - K : P_i(\rho') \neq \sigma$, $\forall \rho' \in L - K : P_i(\rho') \neq P_i(\rho)$. This means that $CanTell(i, \rho)$ holds and that $f_i(\sigma)$ must be equal to Y . Indeed, q is also labeled Y .
- $\rho \in L - K$: the analysis is similar to the case $\rho \in K$ and is omitted.
- There is no $\rho \in L$ such that $\sigma = P_i(\rho)$. Then, observe that Condition (7) does not impose any restrictions on what $f_i(\sigma)$ can be, meaning that function f_i can return any of the three symbols Y, N, U . As a result, any behavior of O_i on σ is acceptable.

□

We see in the construction in the proof of Theorem 7 that we produce DFA recognizing $P_i(K)$ and $P_i(L - K)$. While we have not proven that a polynomial-time algorithm for finite-state observers does not exist, we speculate that it does not since the first conjunct in the consequent of (11) is the same expression that appears in co-observability and producing finite-state supervisors for co-observable systems cannot be done in polynomial time [4]. The result from Theorem 6 and the proof from Theorem 7 are in keeping with results in the field of DES where, if the number of agents is fixed, verification of the necessary and sufficient conditions for supervisory control solutions to exist can be decided in polynomial time in the size of the state sets but where, even when the conditions are satisfied, the synthesis of corresponding supervisors cannot be done in polynomial time (cf., [10] for centralized control and [4] for decentralized control).

5 Related Work

Decentralized observation problems similar to the ones we examine here are also studied in [8]. In particular, the so-called *local* observation problems defined in [8] are similar in spirit to the ALTOCT condition, but with a crucial difference. In ALTOCT, the local decision functions f_i can each return three possible values, Y, N, U , whereas in the local observation problems defined in [8], the local decision functions f_i are only allowed two possible return values, 0 or 1. Another difference is that the local observation problems defined in [8] include a global combination function B which can be any Boolean function, whereas in our setting of OCT and ALTOCT, the combination is implicitly disjunction: if at least one local observer says Y then this is enough to ensure that the behavior ρ of the plant was good, and if at least one local observer says N then we can be sure that ρ was bad. (By definition, it is impossible to have the case where some observer says Y and another says N .)

Decentralized control problems under partial observation are investigated in [1, 6, 12]. For decentralized discrete-event systems problems that require control, problem solutions require that the agents' observations together with their control capabilities are enough to effect the necessary control. Co-observability and other variations (such as D & A co-observability [12]) differ from joint observability and the one can tell condition in the following way. Co-observability roughly says “based on what sequence of events has occurred so far, can decentralized supervisors know enough about an upcoming event to know whether to prevent it from occurring”. Together with controllability, co-observability ensures that decentralized supervisory control problems can be solved because for any event that could lead somewhere illegal, at least one agent that *can* stop that event from occurring knows enough from the agent's observations to do so. As such, co-observability and conditions like it typically are expressed in the form “for all s that is in $K \dots$ if $P(s) = \dots$ and some conditions on string s and on event σ , then $s\sigma \in K$ ”. The conditions on s and σ capture what is necessary and sufficient to determine if $s\sigma$ is in K (or in $L - K$ for D & A co-observability). These conditions are used so that supervisors make decisions about each upcoming event σ and enact control over σ . In contrast, joint

observability, one can tell and decentralized observability, are divorced from a particular control problem at hand. Rather, they speak to whether a string that has already occurred is or isn't in K and whether or not decentralized agents are able to determine that.

The construction in [4] used to show that co-observability can be verified in polynomial time also uses a product construction that bears some resemblance to the one used to show that OCT is decidable. In both constructions, the paths through the states in the Cartesian product produce the strings that serve as a violation of the relevant condition (OCT here and co-observability in [4]). In [4], to check co-observability you must check strings $s\sigma$, $s'\sigma$ and $s''\sigma$ and determine if s is in L before you can even check if $s\sigma \in K$. So the product automaton in [4] requires an extra $(n+2)$ th element in its Cartesian product that captures the states of L , which is not needed here. In [4] it is not assumed that the FSA representing languages are complete so instead at each move it is necessary to check if there is a transition on σ . An additional dump state d is added to the product there; the purpose of d is to keep track of whether or not co-observability is violated by the strings thus far generated; in contrast, in our construction here we take the n automata $A_{L-K}^1 \dots A_{L-K}^n$ to be recognizers of $L - K$ and not K and so a violation of OCT is determined by the presence of strings that land at a terminal state of the product.

6 Conclusions

We have shown that checking decentralized observability amounts to ensuring that at least one agent can tell if a word is legal or not. This condition is decidable for regular languages and if it is satisfied then finite-state observers can be constructed (albeit not likely in polynomial time).

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