A differential relation between the energy and electric charge of a dyon

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Abstract

The differential relation between the energy and electric charge of a dyon is derived. The relation expresses the derivative of the energy with respect to the electric charge in terms of the boundary value for the temporal component of the dyon's electromagnetic potential. The use of the Hamiltonian formalism and transition to the unitary gauge make it possible to show that this derivative is proportional to the phase frequency of the electrically charged massive gauge fields forming the dyon's core. It follows from the differential relation that the energy and electric charge of the non-BPS dyon cannot be arbitrarily large. Finally, the dyon's properties are investigated numerically at different values of the model parameters.

Keywords: magnetic monopole, dyon, electric charge, magnetic charge, Noether charge

1. Introduction

Electrically charged solitons exist in both $(2+1)$-dimensional [1–12] and $(3+1)$-dimensional [13–26] gauge field models. The $(2+1)$-dimensional field models permitting the existence of electrically charged solitons include the Chern–Simons gauge term [27–29], and therefore their gauge fields are topologically massive, resulting in the short-range electric field. In contrast, the three-dimensional electrically charged solitons possess a long-range electric field because the corresponding $(3+1)$-dimensional field models include only the Maxwell gauge term, leading to the massless gauge fields.

The three-dimensional electrically charged solitons can be both topological [13–17] and nontopological [18–26] type. The properties of these two types of solitons are substantially different. The existence of the electrically charged nontopological solitons is due to the presence of the conserved Noether (electric) charge and the special form of the self-interaction potential of scalar fields. The basic property of the nontopological soliton is that its field configuration is a stationary (saddle or minimum) point of the total energy functional at a given fixed value of the Noether charge [30–32]. This property results in the differential relation between the energy and the Noether charge of the nontopological soliton, which, in turn, determines a number of the soliton's properties.

On the other hand, the existence of the topological solitons, including the electrically charged ones, is due to the topological nontriviality of their field configurations, which prevents the transitions of topological solitons into the states with the lower energy. In particular, the presence of a potential term is not a necessary condition for the existence of topological solitons [33]. The best known example of the three-dimensional topological solitons possessing an electric charge is the dyon solution [13] of the Georgi–Glashow model [34]. In this Letter we derive the differential relation between the energy and electric charge of this dyon solution. We also ascertain that the differential relation determines a number of properties of the dyon. In particular, we show that it does not allow the existence of the non-BPS dyons possessing the arbitrary large electric charge and energy.

2. Lagrangian and field equations of the model

The Lagrangian density of the Georgi–Glashow model is

$$\mathcal{L} = -\frac{1}{4}F^{a}_{\mu\nu}F^{a\,\mu\nu} + \frac{1}{2}\left(D_{\mu}\phi^{a}\right)\left(D^{\mu}\phi^{a}\right) - V\left(\phi\right), \quad (1)$$

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where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e\epsilon^{abc} A^b_\mu A^c_\nu \tag{2}$$

is the non-Abelian field strength,

$$D_\mu \phi^a = \partial_\mu \phi^a - e\epsilon^{abc} A^b_\mu \phi^c \tag{3}$$

is the covariant derivative of the Higgs field $\boldsymbol{\phi}$, and

$$V(\phi) = \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2 \tag{4}$$

is the self-interaction potential of the Higgs field $\boldsymbol{\phi}$. In Eqs. (2) – (4), e is the gauge coupling constant, λ is the self-interaction coupling constant of the Higgs field, and v is the Higgs field vacuum expectation value. The field equations of model (1) have the form

$$D_\nu F^{a\,\mu\nu} + e\epsilon^{abc}\phi^b D^\mu \phi^c = 0, \tag{5}$$
$$D_\mu D^\mu \phi^a + \lambda\left(\phi^b\phi^b - v^2\right)\phi^a = 0, \tag{6}$$

and the symmetric energy-momentum tensor is

$$T_{\mu\nu} = -F^a_{\mu\rho}F^{a\,\rho}_{\nu} + \left(D_\mu\phi^a\right)\left(D_\nu\phi^a\right) + \eta_{\mu\nu}\left[\frac{1}{4}F^a_{\rho\tau}F^{a\,\rho\tau} - \frac{1}{2}\left(D_\rho\phi^a\right)\left(D^\rho\phi^a\right) + V(\phi)\right], \tag{7}$$

where $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ is the metric tensor.

In model (1), finite energy field configurations must satisfy the asymptotic condition $\lim\limits_{r\to\infty}|\boldsymbol{\phi}| = v$, which is equivalent to the mapping of the infinitely distant space sphere S^2_∞ to the vacuum sphere $S^2_{\text{vac}} : |\boldsymbol{\phi}| = v$. It is well known that the mappings $S^2 \to S^2$ are split into different topological classes, which are characterised by the integer winding number n according to the sphere's second homotopy group $\pi_2\left(S^2\right) = \mathbb{Z}$. In the topological sector with the winding number $n = 1$, model (1) has the two well known topological soliton solutions: the 't Hooft–Polyakov monopole [35, 36] and Julia–Zee dyon [13]. Both the 't Hooft–Polyakov monopole and Julia–Zee dyon possess the minimum possible magnetic charge $g = 4\pi/e$. At the same time, the electric charge of the 't Hooft–Polyakov monopole is equal to zero, whereas that of the dyon is nonzero. When the electric charge tends to zero, the dyon field configuration smoothly goes into the 't Hooft–Polyakov monopole.

The field configuration of the dyon is described by the spherically symmetric ansatz

$$A^{a0} = n^a v j(r), \tag{8a}$$
$$A^{ai} = \epsilon^{aim} n^m \frac{1-u(r)}{er}, \tag{8b}$$
$$\phi^a = n^a v h(r), \tag{8c}$$

where $n^a = x^a/r$. We now introduce the dimensionless radial variable $\rho = m_V r$, where $m_V = ev$ is the mass of the electrically charged gauge bosons. Then, the ansatz functions $u(r)$, $j(r)$, and $h(r)$ will satisfy the system of nonlinear differential equations of the second order:

$$u''(\rho) - \frac{u(\rho)\left(u(\rho)^2 - 1\right)}{\rho^2} - \left(h(\rho)^2 - j(\rho)^2\right)u(\rho) = 0, \tag{9}$$

$$j''(\rho) + \frac{2}{\rho}j'(\rho) - \frac{2}{\rho^2}u(\rho)^2 j(\rho) = 0, \tag{10}$$

$$h''(\rho) + \frac{2}{\rho}h'(\rho) - \frac{2}{\rho^2}u(\rho)^2 h(\rho) + \kappa\left(1 - h(r)^2\right)h(r) = 0, \tag{11}$$

where the dimensionless parameter $\kappa = \lambda e^{-2}$, and the prime indicates the derivative with respect to ρ. Substituting ansatz (8) into the expression for the T_{00} component of symmetric energy-momentum tensor (7) and integrating it over the space, we obtain the expression for the energy of the dyon

$$E = m_M \int\limits_0^\infty \left[\frac{u'(\rho)^2}{\rho^2} + \frac{1}{2}\left(h'(\rho)^2 + j'(\rho)^2\right) + \frac{\left(u(\rho)^2 - 1\right)^2}{2\rho^4} + \frac{\left(h(\rho)^2 + j(\rho)^2\right)u(\rho)^2}{\rho^2} + \frac{\kappa}{4}\left(1 - h(\rho)^2\right)^2\right]\rho^2 d\rho, \tag{12}$$

where $m_M = 4\pi v e^{-1}$ is the mass of the BPS monopole [14].

The regularity of the dyon solution at $r = 0$ and the finiteness of dyon energy (12) lead us to the boundary conditions for the ansatz functions:

$$j(0) = 0, \qquad \lim_{\rho\to\infty} j(r) = c, \tag{13a}$$
$$u(0) = 1, \qquad \lim_{\rho\to\infty} u(r) = 0, \tag{13b}$$
$$h(0) = 0, \qquad \lim_{\rho\to\infty} h(r) = 1, \tag{13c}$$

where c is a finite value. It follows from Eqs. (9) and (13) that at large ρ, the ansatz function

$u(\rho)\propto\exp\left[-\left(1-c^2\right)^{1/2}\rho\right]$. Hence, the limiting value c of the ansatz function $j(\rho)$ must satisfy the condition $|c|<1$.

The Higgs field (8c) is not invariant under the initial $SU(2)$ gauge group of model (1), but it remains invariant under the $U(1)$ gauge subgroup that corresponds to local rotations around the unit vector $n^a = x^a/r$ in the isospace. This leads to the existence of the long-range gauge field that can be described by the field strength tensor $F_{\mu\nu} = v^{-1}\phi^a F^a_{\mu\nu}$. The corresponding intensities of the electric and magnetic fields of the dyon are

$$E_i = v^{-1}F^a_{0i}\phi^a = -ev^2 j' n_i \tag{14}$$

and

$$B_i = (2v)^{-1}\epsilon_{ijk}F^a_{jk}\phi^a = ev^2\frac{1-u^2}{\rho^2}n_i, \tag{15}$$

respectively. We define the electric (magnetic) charge of the dyon Q_E (Q_M) as the flux of the electric (magnetic) field through the infinitely distant sphere S^2_∞ and obtain the expressions

$$\begin{aligned} Q_E &= \oint_{S^2_\infty} d^2S_n E_n = -\frac{4\pi}{e}\lim_{\rho\to\infty}\rho^2 j'(\rho) \\ &= -\frac{8\pi}{e}\int_0^\infty j(\rho)u(\rho)^2 d\rho \end{aligned} \tag{16}$$

and

$$Q_M = \oint_{S^2_\infty} d^2S_n B_n = \frac{4\pi}{e}. \tag{17}$$

To obtain the second line in Eq. (16), we use Gauss's law (10) written in the form $\left(\rho^2 j'\right)' = 2ju^2$.

The dyon's energy (12) can be written as the sum of terms

$$E = E^{(E)} + E^{(B)} + E^{(G)} + E^{(P)}, \tag{18}$$

where

$$\begin{aligned} E^{(E)} &= m_M\int\limits_0^\infty\left[\frac{1}{2}j'(\rho)^2 + \frac{u(\rho)^2 j(\rho)^2}{\rho^2}\right]\rho^2 d\rho \\ &= -\frac{1}{2}vcQ_E \end{aligned} \tag{19}$$

is the electric field's energy,

$$E^{(B)} = m_M\int\limits_0^\infty\left[u'(\rho)^2 + \frac{(u(\rho)^2-1)^2}{2\rho^2}\right]d\rho \tag{20}$$

is the magnetic field's energy,

$$E^{(G)} = m_M\int\limits_0^\infty\left[\frac{1}{2}h'(\rho)^2 + \frac{h(\rho)^2 u(\rho)^2}{\rho^2}\right]\rho^2 d\rho \tag{21}$$

is the gradient part of the soliton's energy, and

$$E^{(P)} = m_M\int\limits_0^\infty\left[\frac{1}{4}\kappa\left(1-h(\rho)^2\right)^2\right]\rho^2 d\rho \tag{22}$$

is the potential part of the soliton's energy. Any solution of Eqs. (9) – (11) satisfying boundary conditions (13) is a stationary point of the action $S = \int\mathcal{L}d^3x dt$. However, the Lagrangian density (1) does not depend on time if the field configurations are those of ansatz (8). Hence, any solution of Eqs. (9) – (11) and (13) is a stationary point of the Lagrangian

$$L = \int\mathcal{L}d^3x = E^{(E)} - E^{(B)} - E^{(G)} - E^{(P)}. \tag{23}$$

After the scale transformation $\rho\to\varkappa\rho$ of the argument of the solution, the Lagrangian L becomes a function of the scale parameter $\varkappa$. Note that this transformation is valid because Gauss's law (10) remains true even after the rescaling $\rho\to\varkappa\rho$. Since the function $L(\varkappa)$ has a stationary point at $\varkappa=1$, its derivative vanishes at this point: $dL/d\varkappa|_{\varkappa=1} = 0$. It can easily be shown that $E^{(E)}\to\varkappa^{-1}E^{(E)}$, $E^{(B)}\to\varkappa E^{(B)}$, $E^{(G)}\to\varkappa^{-1}E^{(G)}$, and $E^{(P)}\to\varkappa^{-3}E^{(P)}$ under the rescaling $\rho\to\varkappa\rho$. Using this fact and Eqs. (19) – (23), we obtain the virial relation for the dyon solution

$$E^{(E)} + E^{(B)} - E^{(G)} - 3E^{(P)} = 0. \tag{24}$$

In the important case of the BPS dyon [14], we have the analytical expressions for the energy components and the total energy:

$$E^{(E)} = \frac{v}{2}\frac{Q_E^2}{\sqrt{Q_M^2+Q_E^2}}, \tag{25a}$$

$$E^{(B)} = \frac{v}{2}\frac{Q_M^2}{\sqrt{Q_M^2+Q_E^2}}, \tag{25b}$$

$$E^{(G)} = \frac{v}{2}\sqrt{Q_M^2+Q_E^2}, \tag{25c}$$

$$E^{(P)} = 0, \tag{25d}$$

$$E = v\sqrt{Q_M^2+Q_E^2}. \tag{25e}$$

It can easily be checked that Eqs. (25) satisfy Eq. (18) and virial relation (24).

3. A differential relation between the energy and electric charge and its consequences

To obtain the differential relation between the energy and electric charge, we begin with the consideration of the BPS limit $\kappa = 0$. In this case, there is the analytical solution [14] of system (9) – (11)

$$u(\rho) = \frac{\tau\rho}{\sinh(\tau\rho)}, \tag{26}$$

$$j(\rho) = -\frac{Q_E}{Q_M}\tau\left[\coth(\tau\rho) - (\tau\rho)^{-1}\right], \tag{27}$$

$$h(\rho) = \coth(\tau\rho) - (\tau\rho)^{-1}, \tag{28}$$

where the ratio

$$\tau = \frac{Q_M}{\sqrt{Q_M^2 + Q_E^2}}. \tag{29}$$

Eq. (27) tells us that in the BPS case

$$c \equiv j(\infty) = -\frac{Q_E}{\sqrt{Q_M^2 + Q_E^2}}. \tag{30}$$

Using Eqs. (25e) and (30), we obtain successively

$$\frac{dE}{dc} = m_M c\left(1 - c^2\right)^{-3/2}, \tag{31a}$$

$$\frac{dQ_E}{dc} = -\frac{m_M}{v}\left(1 - c^2\right)^{-3/2}, \tag{31b}$$

$$\frac{dE}{dQ_E} = \frac{dE/dc}{dQ_E/dc} = -vc, \tag{31c}$$

where $m_M = vQ_M = 4\pi v/e$ is the mass of the BPS monopole. We see that in the BPS case, the derivatives dE/dc and dQ_E/dc satisfy the condition $dE/dc + vcdQ_E/dc = 0$.

Now we shall show that this condition is also valid in the general case $\kappa \neq 0$. To do this, we go from the ansatz function $j(\rho)$ to the new one $J(\rho) = j(\rho) - c$ satisfying the homogeneous boundary condition $J(\infty) = 0$ at infinity. We consider ansatz functions (8) as functions of the radial variable ρ and the parameter c. Next, we calculate the value $dE/dc + vcdQ_E/dc$ using Eqs. (12) and (16) for the energy and the electric charge, respectively. The resulting expression contains, among others, the two terms: $m_M\rho^2\left(\partial h/\partial\rho\right)\left(\partial^2 h/\partial\rho\partial c\right)$ and $2m_M\left(\partial u/\partial\rho\right)\left(\partial^2 u/\partial\rho\partial c\right)$. Using boundary conditions (13) and integration by parts, we transform these two terms to $-m_M\left[2\rho\left(\partial h/\partial\rho\right)\left(\partial h/\partial c\right) + \rho^2\left(\partial^2 h/\partial\rho^2\right)\left(\partial h/\partial c\right)\right]$ and $-2m_M\left(\partial^2 u/\partial\rho^2\right)\left(\partial u/\partial c\right)$, respectively. After that, the expression $dE/dc + vcdQ_E/dc$ can be written in the form

$$\frac{dE}{dc} + vc\frac{dQ_E}{dc} = -m_M\int_0^\infty\left(2\frac{\partial u}{\partial c}e_1 + \rho^2 J\frac{\partial e_2}{\partial c} + \rho^2\frac{\partial h}{\partial c}e_3 - \frac{\partial}{\partial\rho}\left[\rho^2 J\frac{\partial^2 J}{\partial\rho\partial c}\right]\right)d\rho, \tag{32}$$

where e_1, e_2, and e_3 are the left hand sides of Eqs. (9), (10), and (11), respectively. It is obvious that e_i and their derivatives with respect to the parameter c vanish when $u(\rho, c)$, $j(\rho, c)$, and $h(\rho, c)$ is a solution of system (9) – (11). The last term $m_M\int_0^\infty\partial\left[\rho^2 J\left(\partial^2 J/\partial\rho\partial c\right)\right]/\partial\rho d\rho$ also vanishes because $J\left(\partial^2 J/\partial\rho\partial c\right) \sim \rho^{-3}$ as $\rho \to \infty$. We see that the dyon's energy and the electric charge satisfy differential relation (31c) also in the general case $\kappa \neq 0$. Eq. (31c) can be written in the form

$$\frac{dE}{dQ_N} \equiv e\frac{dE}{dQ_E} = \Omega, \tag{33}$$

where $Q_N = e^{-1}Q_E$ is the Noether charge and the parameter $\Omega = -evc = -m_V c$ is some function of Q_N.

The differential relation (33) has the same form as the differential relations for the nontopological solitons in Refs. [23, 30–32]. In the latter case, the differential relation results from the fact that the nontopological soliton is a stationary point of the total energy functional under the condition that the Noether charge of field configurations is fixed. A similar situation takes place for the dyon. To show this, we give an interpretation of the parameter Ω in Eq. (33). Using the second line of Eq. (19), which is a consequence of Gauss's law (10), we can write the Lagrangian $L = E^{(E)} - E^{(B)} - E^{(G)} - E^{(P)}$ in the form

$$L = \Omega Q_N - E. \tag{34}$$

If boundary conditions are fixed then variations of L must vanish on the dyon solution of field equations. The fixation of the boundary conditions means that the parameter $\Omega = -evc = -evj(\infty)$ remains fixed when varying the Lagrangian L, and hence

$$-\delta L = \delta E - \Omega\delta Q_N = 0. \tag{35}$$

It follows from Eq. (35) that the dyon solution is a stationary point of the total energy functional E provided that the Noether (electric) charge Q_N (Q_E) of field configurations is fixed, and the parameter Ω plays the role of the Lagrange multiplier in Eq. (35).

We now turn to the unitary gauge $\boldsymbol{\phi} = (0, 0, \chi)$ and shall use the Hamiltonian formalism. In the unitary gauge, the canonical fields are

$$A_i^a,\ P_i^a = F_{0i}^a = E_i^a,\ \chi,\ p = \partial_t \chi, \qquad (36)$$

and the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2} P_i^a P_i^a + \frac{1}{2} p^2 + \frac{1}{4} F_{ij}^a F_{ij}^a \\ &+ \frac{1}{2} e^2 \chi^2 \left(A_0^1 A_0^1 + A_i^1 A_i^1 + A_0^2 A_0^2 + A_i^2 A_i^2\right) \\ &+ \frac{1}{2} \left(\nabla \chi\right)^2 + \frac{\lambda}{4} \left(\chi^2 - v^2\right)^2 . \end{aligned} \qquad (37)$$

Gauss's law (the zeroth component of Eq. (5) written in terms of canonical fields (36))

$$\begin{aligned} A_0^a - \delta_3^a A_0^3 &= -\left(e^2 \chi^2\right)^{-1} \left(\partial_i P_i^a - e \epsilon^{abc} A_i^b P_i^c\right) \\ &= -\left(e^2 \chi^2\right)^{-1} D_i P_i^a \end{aligned} \qquad (38)$$

is used to express $A_0^{1,2}$ in Eq. (37). Note that the boundary condition $A_0^3(\infty) = 0$, which completely fixes the unitary gauge, is used to obtain Eq. (38) within the framework of the Hamiltonian formalism.

In the unitary gauge, the unbroken electromagnetic $U(1)$ subgroup corresponds to rotations about the third axis in the isospace. The corresponding Noether charge

$$Q_N = e^{-1} Q_E = \int \epsilon^{3bc} A_i^b P_i^c d^3x \qquad (39)$$

is consistent with the definition of Eq. (16) because of the third component of Gauss's law (38) and the definition $E_i \equiv E_i^3 = P_i^3$ following from Eq. (36). Further, it can be shown that

$$H = \int \mathcal{H} d^3x = \int T_{00} d^3x = E \qquad (40)$$

for field configurations satisfying Gauss's law (38). Now we use the trick used in Refs. [31, 32, 37]. Taking into account Eqs. (35), (39), and (40), the Hamilton equations can be written in the form

$$\partial_t A_i^a = \frac{\delta H}{\delta P_i^a} = \frac{\delta E}{\delta P_i^a} = \Omega \frac{\delta Q_N}{\delta P_i^a} = \Omega \epsilon^{a3b} A_i^b, \qquad (41)$$

$$\begin{aligned}\partial_t P_i^a &= -\frac{\delta H}{\delta A_i^a} = -\frac{\delta E}{\delta A_i^a} = -\Omega \frac{\delta Q_N}{\delta A_i^a} \\ &= \Omega \epsilon^{a3b} P_i^b. \end{aligned} \qquad (42)$$

From Eq. (41) it is easy to obtain the time dependence of the fields $W_\mu^{\pm} = 2^{-1/2}\left(A_\mu^1 \mp i A_\mu^2\right)$ corresponding to electrically charged vector bosons:

$$W_\mu^{\pm}\left(t, \mathbf{x}\right) = \exp\left(\mp i \Omega t\right) W_\mu^{\pm}\left(\mathbf{x}\right), \qquad (43)$$

whereas the remaining canonical fields do not depend on time in the unitary gauge. Thus, we conclude that the parameter Ω entering in Eq. (33) is the phase frequency of the charged vector boson fields in the unitary gauge.

Eqs. (25) – (30) tell us that the energy and the Noether (electric) charge of the BPS dyon increase indefinitely as $\Omega \to m_V$. At the same time, it was shown numerically in Refs. [16, 38] that in the non-BPS case, the energy and electric charge of the dyon cannot exceed the maximum allowable values, which depend on the model's parameters. Let us show the impossibility of arbitrary large values for the dyon's energy and electric charge in the non-BPS case $\kappa \neq 0$ using differential relation (33). For this, we differentiate Eq. (18) with respect to Ω. Taking into account Eqs. (19) and (33), we obtain the relation

$$\frac{\Omega}{2} = \frac{Q_N}{2} \frac{d\Omega}{dQ_N} + \frac{dE^{(B)}}{dQ_N} + \frac{dE^{(G)}}{dQ_N} + \frac{dE^{(P)}}{dQ_N}. \qquad (44)$$

Let us suppose that in the non-BPS case, the energy and the Noether charge of the dyon tend to infinity as $\Omega \to m_V$. It can be shown that in this case, the term $Q_N \left(d\Omega/dQ_N\right)$ must tend to zero. To do this, we write the differential equation $Q_N \left(d\Omega/dQ_N\right) = F\left(Q_N\right)$ and integrate it:

$$\Omega\left(Q_N\right) = \Omega\left(\bar{Q}_N\right) + \int_{\bar{Q}_N}^{Q_N} \bar{\bar{Q}}_N^{-1} F\big(\bar{\bar{Q}}_N\big) d\bar{\bar{Q}}_N. \qquad (45)$$

Because $\lim\limits_{Q_N \to \infty} \Omega\left(Q_N\right) = m_V$, the integral on the right side of Eq. (45) must remain finite as $Q_N \to \infty$. This is only possible if $F\left(Q_N\right) = Q_N \left(d\Omega/dQ_N\right)$ tends to zero as $Q_N \to \infty$.

Next, we estimate the derivative $dE^{(B)}/dQ_N$ as $Q_N \to \infty$. By analogy with electrostatics, the energy of the dyon's magnetic field can be written in the form $E^{(B)} = Q_M^2/\left(8\pi R_M\right)$, where $Q_M = 4\pi/e$ is the dyon's magnetic charge and R_M is the dyon's effective magnetic radius. With the increase in the Noether (electric) charge Q_N (Q_E), the effective size of the dyon also increases as in the BPS case (26) – (29). Hence, the effective magnetic radius R_M also increases (or, at least, remains bounded from below) when $Q_N \to \infty$. It follows that the magnetic field's energy $E^{(B)}(Q_N) =$

$Q_M^2/\left(8\pi R_M(Q_N)\right)$ is a bounded function on the interval $Q_N \in [0,\infty)$. But the derivative of a function bounded on the semi-infinite interval $Q_N \in [0,\infty)$ must tend to zero as $Q_N \to \infty$. The only exception is an oscillating function of Q_N, but it is clear that this variant cannot be realised. Thus, we conclude that $dE^{(B)}/dQ_N \to 0$ as $Q_N \to \infty$.

Next, we consider the behaviour of the derivatives $dE^{(G)}/dQ_N$ and $dE^{(P)}/dQ_N$ as $Q_N \to \infty$. For this, we rewrite Eq. (21) integrating by parts the term $h'^2/2$ and use Eq. (11) to obtain

$$E^{(G)} = m_M \frac{\kappa}{2} \int\limits_0^\infty \left[\left(1-h\left(\rho\right)^2\right) h\left(\rho\right)^2\right] \rho^2 d\rho. \quad (46)$$

Note that Eq. (46) is valid only for the non-BPS case because the integral diverges in the BPS case. Using Gauss's law (10), it can easily be shown that $j(\rho)$ is a monotonic function on the interval $\rho \in [0,\infty)$. Furthermore, boundary conditions (13a) tell us that $j(\rho)$ is a bounded function. Then, it follows from Eq. (16) that the integral $\int_0^\infty u\left(\rho\right)^2 d\rho$ diverges as $Q_N \to \infty$. Eq. (9) tells us that this is only possible if, on the arbitrary large interval $\rho \in [0,\varrho]$, the function $u(\rho)$ is in the infinitesimal neighbourhood of 1 and $h(\rho) \approx j(\rho)$. It follows from Eq. (10) that, in this case, the function $j\left(\rho\right) \approx \bar{c}\rho$, where $\bar{c}$ is a constant. For $\rho \gtrsim \varrho$, the function $u(\rho) \approx 0$ and Eq. (10) together with boundary condition (13a) tell us that $j\left(\rho\right) \approx c + e^2 Q_N/(4\pi\rho)$ in this case. We estimate the constant $\bar{c}$ by equating the two expressions for $j\left(\rho\right)$ at $\rho = \varrho$ and obtain that $\bar{c} = c/\left(2\varrho\right) = -\Omega/(2m_V\varrho)$. Using this expression and Eq. (16), we obtain the leading asymptotic behaviour of the Noether charge $Q_N = e^{-1}Q_E$

$$Q_N \approx \frac{2\pi\varrho\Omega}{e^2 m_V}. \quad (47)$$

We now have everything we need to find the leading asymptotic behaviour of $E^{(G)}$ and $E^{(P)}$ as $Q_N \to \infty$. Using Eqs. (22), (46), and (47) and considering that $h(\rho) \approx j(\rho)$ when $\rho \in [0,\varrho]$, we obtain the expressions:

$$E^{(P)} \approx 2.95\, E^{(G)} \approx 0.000244\, m_M \kappa e^6 Q_N^3. \quad (48)$$

We see that both $E^{(P)}$ and $E^{(G)}$ are $\propto Q_N^3$ in the limit of large Q_N. It follows that the derivatives $dE^{(P)}/dQ_N$ and $dE^{(G)}/dQ_N$ are $\propto Q_N^2$, and hence diverge as $Q_N \to \infty$. But this is incompatible with Eq. (44) (and therefore with differential relation (33)), which implies that the derivatives $dE^{(P)}/dQ_N$ and $dE^{(G)}/dQ_N$ must be finite because $\Omega \in (-m_V, m_V)$. It is obvious that virial relation (24) also cannot be satisfied in this case. It follows that the energy and Noether (electric) charge of the dyon cannot be arbitrarily large in the non-BPS case.

In the BPS case, the potential part $E^{(P)}$ of the dyon's energy is absent and Eq. (46) becomes inapplicable along with our conclusion that $E^{(G)} \propto Q_N^3$ in the limit of large Q_N. Here we need to use Eq. (21) in order to find that $E^{(G)} \to m_V Q_N/2$ as $Q_N \to \infty$ in accordance with Eq. (25c). Furthermore, using Eqs. (25) it can be shown that the combination $(Q_N/2)\left(d\Omega/dQ_N\right) + dE^{(B)}/dQ_N$ vanishes in the BPS case and Eq. (44) takes the form

$$\frac{dE^{(G)}}{dQ_N} = \frac{\Omega}{2}. \quad (49)$$

The correctness of Eq. (49) can be easily verified using Eqs. (25). It follows that Eqs. (33) and (44) do not impose any restrictions in the BPS case, and therefore the energy and Noether (electric) charge of the BPS dyon can be arbitrarily large as $\Omega \to m_V$.

4. Numerical results

Now we present some numerical results concerning the dyon. For numerical calculations, we use the natural units $c = 1$ and $\hbar = 1$. It follows from Eqs. (9) – (11) and (13) that the ansatz functions $u(\rho)$, $j(\rho)$, and $h(\rho)$ depend only on the two dimensionless parameters $\kappa = e^{-2}\lambda$ and $c = -m_V^{-1}\Omega \equiv -\tilde{\Omega}$. Further, Eqs. (12) and (16) tell us that the dimensionless combinations $\tilde{Q}_E = eQ_E$ and $\tilde{E} = e^2 E m_V^{-1}$ also depend only on $\kappa = e^{-2}\lambda$ and $\tilde{\Omega} = m_V^{-1}\Omega$.

Figure 1 presents the dependence of $\tilde{Q}_E$ on the dimensionless phase frequency $\tilde{\Omega}$ for different values of the parameter $\kappa = e^{-2}\lambda$. We see that for any κ, the dyon's electric charge increases monotonically with an increase in Ω. The electric charge of the BPS dyon ($\kappa = 0$) tends to infinity as $Q_E \sim 2\sqrt{2}\pi e^{-1} m_V^{1/2} \left(m_V - \Omega\right)^{-1/2}$ when $\Omega \to m_V$, in accordance with Eq. (30). At the same time, the electric charge of the non-BPS dyon remains finite as $\Omega \to m_V$. For any fixed Ω, the dyon's electric charge decreases monotonically with an increase in κ and tends to some limiting value as $\kappa \to \infty$.

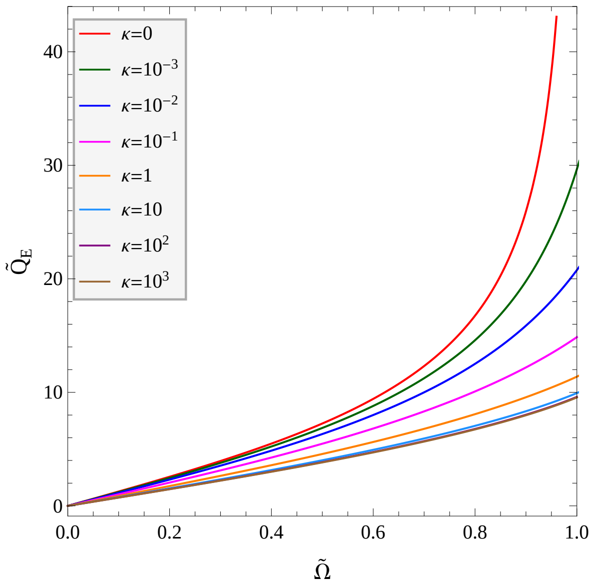

Fig. 1. Dependence of the dimensionless combination $\tilde{Q}_E = eQ_E$ on $\tilde{\Omega} = m_V^{-1}\Omega$ for different values of the parameter $\kappa = e^{-2}\lambda$.

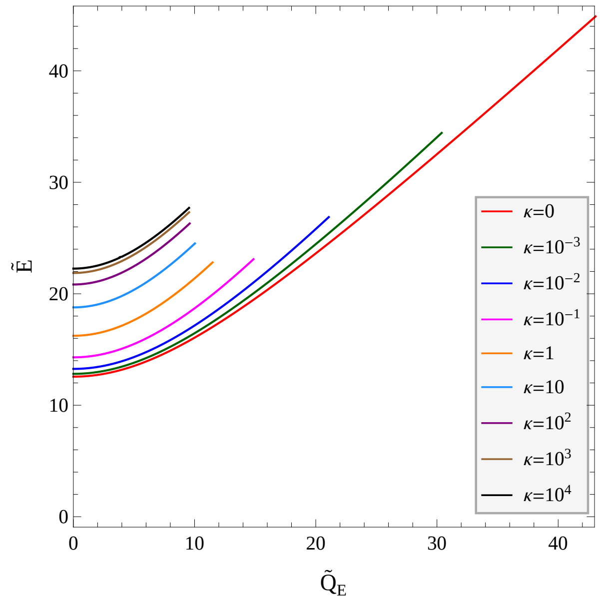

Fig. 2. Dependence of the dimensionless combination $\tilde{E} = e^2 E m_V^{-1}$ on $\tilde{Q}_E = eQ_E$ for different values of the parameter $\kappa = e^{-2}\lambda$.

Figure 2 presents the dependence of the dimensionless scaled energy $\tilde{E}$ on the scaled electric charge $\tilde{Q}_E$ for different values of the parameter $\kappa = e^{-2}\lambda$. It follows from Fig. 2 that for any fixed κ, the dyon's energy increases monotonically with an increase in the electric charge Q_E. Further, for any nonzero κ, there exist the maximum permissible values for the dyon's energy E and electric charge Q_E. They correspond to the rightmost points on the curves in Fig. 2, where the derivative $d\tilde{E}/d\tilde{Q}_E = 1$ according to Eq. (33). For any fixed Q_E, the dyon's energy increases monotonically with an increase in κ, but the maximum allowable energy depends on Q_E. As $\kappa \to \infty$, the curves $\tilde{E}(\tilde{Q}_E)$ tend to the limiting curve. Note that the dyon's energy E is an even function of the electric charge Q_E due to the C-invariance of model (1).

It follows from Fig. 2 that for any fixed κ, the function $E(Q_E)$ is a convex downward, and therefore the second derivative d^2E/dQ_N^2 is positive. Hence, the function $E(Q_E)$ satisfies the inequality

$$E\left(Q_E\right) \leq E\left(Q_E'\right) + m_V e^{-1}\left(Q_E - Q_E'\right). \quad (50)$$

It follows that the dyon having the electric charge Q_E is stable against decay into the the dyon with the smaller electric charge Q_E' and the massive gauge bosons with the total electric charge $Q_E - Q_E'$ and mass $m_V e^{-1}(Q_E - Q_E') = m_V(Q_N - Q_N')$.

In Fig. 3, we can see the dependence of the maximum possible values of the dimensionless scaled energy $\tilde{E}$ and the scaled electric charge $\tilde{Q}_E$ on the parameter $\kappa = e^{-2}\lambda$. For better visualisation of the κ-dependences, we show them on the log-linear plot. It follows from Fig. 3 that the maximum allowable value of $\tilde{Q}_E$ decreases monotonically with an increase in κ in accordance with Ref. [16]. At the same time, the maximum allowable value of $\tilde{E}$ also decreases with an increase in κ until it reaches the minimum point with $(\kappa, \tilde{E}) = (0.418, 22.532)$; after this, it increases with an increase in κ. As $\kappa \to 0$, the maximum allowable values of $\tilde{E}$ and $\tilde{Q}_E$ increase indefinitely according to the power law:

$$\tilde{E} \sim \tilde{Q}_E \sim \alpha\kappa^{-\beta}, \quad (51)$$

where α is a positive constant and $\beta \approx 0.165$. When $\kappa \to \infty$, these maximum allowable values tend asymptotically to the finite limits:

$$\tilde{E} \underset{\kappa\to\infty}{\longrightarrow} \tilde{E}(\infty) \approx 27.704, \quad (52)$$

$$\tilde{Q}_E \underset{\kappa\to\infty}{\longrightarrow} \tilde{Q}_E(\infty) \approx 9.546. \quad (53)$$

Let us consider the limit in which the self-interaction constant λ is fixed and the gauge coupling constant e tends to zero. It follows that the combination $\kappa = e^{-2}\lambda$ increases indefinitely in this

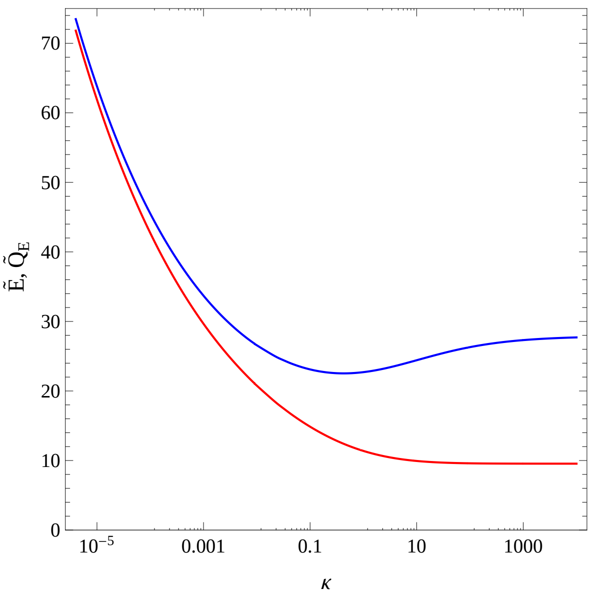

Fig. 3. Dependence of the maximum possible values of $\tilde{E}$ (upper line) and $\tilde{Q}_E$ (lower line) on the parameter $\kappa = e^{-2}\lambda$.

limit. Recalling the definition of $\tilde{E}$ and $\tilde{Q}_E$ and taking into account limiting values (52) and (53), we obtain the leading asymptotic behaviour of the dyon's energy E and electric charge Q_E in the limit $e \to 0$:

$$E \sim \tilde{E}(\infty) v e^{-1}, \quad (54)$$

$$Q_E \sim \tilde{Q}_E(\infty) e^{-1}. \quad (55)$$

It follows that the dyon's energy and electric charge increase indefinitely when λ is fixed and $e \to 0$.

5. Conclusions

In the present paper, we have obtained the differential relation (33) between the energy E and electric charge Q_E of the dyon solution. The differential relation expresses the derivative dE/dQ_E in terms of the boundary value for the temporal component of the dyon's electromagnetic potential. Using the Hamiltonian formalism, it is shown that, in the unitary gauge, the derivative dE/dQ_E is proportional to the phase rotation frequency of the electrically charged boson fields of the dyon. It follows from the differential relation that the dyon is a stationary point of the total energy functional provided that the Noether (electric) charge of field configurations is fixed. The latter property is a characteristic feature of the nontopological solitons, and we therefore conclude that the dyon possesses the properties of both topological and nontopological solitons.

The differential relation (33) results in the boundedness of the derivative dE/dQ_E. It follows that the energy and the electric charge of the non-BPS dyon cannot be arbitrarily large. The numerical study reveals that the dyon is stable against decay into the dyon of smaller electric charge and massive electrically charged gauge bosons. It also shows that the dyon's energy and electric charge increase indefinitely when the gauge coupling constant tends to zero.

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