

On an Ergodic Two-Sided Singular Control Problem

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March 1, 2022

Abstract

Motivated by applications in natural resource management, risk management, and finance, this paper is focused on an ergodic two-sided singular control problem for a general one-dimensional diffusion process. The control is given by a bounded variation process. Under some mild conditions, the optimal reward value as well as an optimal control policy are derived by the vanishing discount method. Moreover, the Abelian and Cesàro limits are established. Then a direct solution approach is provided at the end of the paper.

Key Words. Ergodic singular control, vanishing discount, Abelian limit, Cesàro limit.

2000 MR Subject Classification. 93E20, 60H30, 60J70.

1 Introduction

This work is motivated by applications in reversible investment problem (De Angelis and Ferrari (2014), Guo and Pham (2005)), optimal harvesting and renewing problems (Hening and Tran (2020), Hening et al. (2019)), and dividend payment and capital injection problems (Jin et al. (2013), Lindensjö and Lindskog (2020)). In the aforementioned applications, the systems of interests are controlled by bounded variation processes in order to achieve certain economic benefits. It is well known that any bounded variation process can be written as a difference of two nondecreasing and càdlàg processes. Often the two nondecreasing processes will introduce rewards and costs to the optimization problems, respectively. For example, in natural renewable resource management problems such as forestry, while harvesting brings profit, it is costly to renew the natural renewable resource. One needs to balance the harvesting and renewing decisions so as to achieve an optimal reward. Similar considerations prevail

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in reversible investment and dividend payment and capital injection problems. In terms of the reward (or cost) functional, the two singular control processes have different signs. We call such problems two-sided mixed singular control problems. In contrast, in the traditional singular control formulation, the reward or cost corresponding to the singular control is expressed in terms of the expectation of its total variation.

We note that most of the existing works on two-sided mixed singular control problems are focused on discounted criterion. In many applications, however, optimizations using discounted criteria are not appropriate. For example, in optimal harvesting problems, discounted criteria are largely in favor of current interests and disregard the future effect. As a result, they may lead to myopic harvesting policy and extinction of the species. We refer to [Alvarez and Shepp \(1998\)](#), [Lungu and Øksendal \(1997\)](#), [Song et al. \(2011\)](#) for examples in which the optimal policy under the discounted criterion is to harvest all at time 0, resulting in an immediate extinction.

In view of these considerations, this paper aims to investigate a two-sided mixed singular control problem for a general one-dimensional diffusion on $[0, \infty)$ using a long-term average criterion. Ergodic singular control problems have been extensively studied in the literature; see, for example, [Hynd \(2013\)](#), [Jack and Zervos \(2006\)](#), [Karatzas \(1983\)](#), [Menaldi and Robin \(2013\)](#), [Weerasinghe \(2002, 2007\)](#), [Wu and Chen \(2017\)](#) and many others. In the aforementioned references, the cost (or reward) associated with the singular control is expressed in terms of expectation of the total variation. Our formulation in (2.3) is different. In particular, in (2.3), while $\eta(T)$ contributes positively toward the reward, the contribution from $\xi(T)$ is negative. On the one hand, this is motivated by the specific applications in areas such as reversible investment, harvesting and renewing, and dividend payment and capital injection problems. On the other hand, the mixed signs also add an interesting twist to singular control theory as the gradient constraints in the corresponding HJB equation are different from those in the traditional setup as those in [Karatzas \(1983\)](#), [Menaldi and Robin \(2013\)](#), [Weerasinghe \(2002, 2007\)](#).

To find the optimal value λ_0 of (2.3) and an optimal control policy, one may attempt to use the guess-and-check approach. That is, one first finds a smooth solution to the HJB equation (3.1) and then uses a verification argument to derive the optimal long-term average value as well as an optimal control. Indeed this is the approach used in [Karatzas \(1983\)](#), in which the Abelian and Cesàro limits are established for a singular control problem when the underlying process is a one-dimensional Brownian motion. The result was further extended to an ergodic singular control problem for a one-dimensional diffusion process in [Weerasinghe \(2002\)](#) in which the drift and diffusion coefficients satisfy certain symmetry properties.

While this approach seems plausible, it is not easy to guess the right solution. In particular, since the underlying process X_0 is a general one-dimensional diffusion on $[0, \infty)$ (to be defined in (2.1)) and the rewards associated with the singular controls ξ and η have mixed signs in (2.3), our problem does not have the same kind of symmetry as in [Karatzas \(1983\)](#) and [Weerasinghe \(2002\)](#). In addition, the gradient constraint $c_1 \leq u'(x) \leq c_2$ brings much difficulty and subtlety in finding the free boundaries that separates the action and non-action regions. Besides, this approach does not reveal how the ergodic, discounted and finite-time horizon problems (to be defined in (2.3), (2.6), and (2.7), respectively) are related to each other.

In view of the above considerations, we adapt the vanishing discount approach developed in [Menaldi and Robin \(2013\)](#) and [Weerasinghe \(2007\)](#). First, we observe that under Assumption 3.2, careful analysis using reflected diffusion process on an appropriate interval $[a, b]$ reveals that $\lambda_0 > 0$. We next use Assumptions 4.2 and 4.3 to show that $\lim_{r \downarrow 0} rV_r(x) = \lambda_0$ for any $x \geq 0$ and that there exist two positive constants $a_* < b_*$ for which the reflected diffusion process on $[a_*, b_*]$ is an optimal state process, where $V_r(x)$ is the value function of the discounted problem (2.6). These results are summarized in Theorem 4.4. Furthermore, we show in Theorem 5.1 that the Cesàro limit holds as well. As an illustration, we study an ergodic two-sided singular control problem for a geometric Brownian motion model using this framework in Section 6. This approach fails when some of our assumptions are violated. Indeed, the case study in subsection 6.1 indicates that the long-term average reward can be arbitrarily large under suitable settings.

Upon the completion of this paper, we learned the recent paper [Alvarez \(2018\)](#), which deals with an ergodic two-sided singular control problem for a general one-dimensional diffusion on $(-\infty, \infty)$. The formulation in [Alvarez \(2018\)](#) is different from ours because the rewards associated with the singular controls are both positive. Under certain conditions, the paper first constructs a solution to the associated HJB equation and then verifies that the local time reflection policy is optimal. The approach is different from the vanishing discount method used in this paper, which also helps to establish the Abelian and Cesàro limits.

Thanks to the referee who also brought our attention to the paper [Alvarez and Hening \(2019\)](#), which studies the optimal sustainable harvesting of a population that lives in a random environment. It proves that there exists a unique optimal local time reflection harvesting strategy and establishes an Abelian limit under certain conditions. In contrast to the problem considered in this paper, [Alvarez and Hening \(2019\)](#) is focused on a one-sided singular control problem without running rewards and hence it is simpler than our formulation in (2.3).

Motivated by the two papers mentioned above, we add Section 7 to provide a direct solution approach to (2.3). Following the idea in [Alvarez \(2018\)](#) and [Alvarez and Hening \(2019\)](#), we first impose conditions so that the long-term average reward for an (a, b) -reflection policy $\lambda(a, b)$ achieves its maximum value $\lambda_* = \lambda(a_*, b_*)$ at a pair $0 < a_* < b_* < \infty$. The maximizing pair (a_*, b_*) further allows us to derive a C^2 solution to the HJB equation (3.1). This, together with the verification theorem (Theorem 3.1), reveals that $\lambda_0 = \lambda_*$ and the (a_*, b_*) -reflection policy is optimal.

The rest of the paper is organized as follows. Section 2 begins with the formulation of the problem; it also presents the main results of the paper. Section 3 collects some preliminary results. The Abelian and Cesàro limits are established in Sections 4 and 5, respectively. Section 6 is devoted to an ergodic two-sided singular control problem when the underlying process is a geometric Brownian motion with subsection 6.1 providing a case study for $\mu = 0$ and $h(x) = x^p, p \in (0, 1)$. Finally, we provide a direct solution approach in Section 7. An example on ergodic two-sided singular control for Verhulst-Pearl diffusion is studied in Section 7 for illustration.

2 Formulation and Main Results

To begin, suppose the uncontrolled process is given by the following one-dimensional diffusion process with state space $[0, \infty)$:

$$dX_0(s) = \mu(X_0(s))ds + \sigma(X_0(s))dW(s), \quad X_0(0) = x_0 \in [0, \infty), \quad (2.1)$$

where W is a one-dimensional standard Brownian motion, and μ and σ are suitable functions so that a weak solution X_0 exists and is unique in the sense of probability law. We refer to Section 5.5 of [Karatzas and Shreve \(1991\)](#) for such conditions. Assume throughout the paper that 0 is an unattainable boundary point (entrance or natural) and that ∞ is a natural boundary point; see Chapter 15 of [Karlin and Taylor \(1981\)](#) or Section 5.5 of [Karatzas and Shreve \(1991\)](#) for more details on classifications of boundary points for one-dimensional diffusions. Note that if 0 is an entrance point, then it is part of the state space for X_0 ; otherwise, if 0 is a natural point, then it is not in the state space. While an entrance point cannot be reached from the interior, it is possible that the process X_0 will start from an entrance boundary point and quickly move to the interior and never return to it. The choice of $[0, \infty)$ as the state space for X_0 is motivated by the following consideration. The states of interest in applications such as reversible investment, optimal harvesting and renewing, and dividend payment and capital injection problems are all bounded from below. For notational convenience, we then choose $[0, \infty)$ as the state space for X_0 .

Throughout the paper, we further suppose that the process X_0 possesses scale and speed densities given by

$$s(x) := \exp \left\{ - \int_1^x \frac{2\mu(y)}{\sigma^2(y)} dy \right\}, \quad m(x) := \frac{1}{\sigma^2(x)s(x)}, \quad x > 0,$$

respectively. The scale and speed measures of X_0 are

$$S[a, b] := \int_a^b s(x)dx, \quad \text{and} \quad M[a, b] := \int_a^b m(x)dx, \quad \forall [a, b] \in [0, \infty).$$

The infinitesimal generator of the process X_0 is

$$\mathcal{L}f(x) := \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) = \frac{1}{2} \frac{d}{dM} \left(\frac{df(x)}{dS} \right), \quad \forall f \in C^2([0, \infty)).$$

We now introduce a bounded variation process $\varphi = \xi - \eta$ to (2.1), resulting in the following controlled dynamics:

$$\begin{cases} dX(s) = \mu(X(s))ds + \sigma(X(s))dW(s) + d\xi(s) - d\eta(s), & s \geq 0, \\ X(0-) = x \geq 0. \end{cases} \quad (2.2)$$

Throughout the paper, we assume that the control process $\varphi(\cdot) = \xi(\cdot) - \eta(\cdot)$ is an adapted, càdlàg process that admits the minimal Jordan decomposition $\varphi(t) = \xi(t) - \eta(t)$, $t \geq 0$. In particular, ξ, η are nonnegative and nondecreasing processes satisfying $\xi(0-) = \eta(0-) = 0$ such that the associated Borel measures $d\xi$ and $d\eta$ on $[0, \infty)$ are mutually singular. In

addition, it is required that under the control process $\varphi(\cdot)$, (2.2) admits a unique nonnegative weak solution $X(\cdot)$. Such a control process $\varphi(\cdot)$ is said to be *admissible*.

The goal is to maximize the expected long-term average reward:

$$\lambda_0 := \sup_{\varphi(\cdot) \in \mathcal{A}_x} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 \eta(T) - c_2 \xi(T) \right], \quad (2.3)$$

where $c_1 < c_2$ are two positive constants, h is a nonnegative function, and \mathcal{A}_x is the set of *admissible controls*, i.e.,

$$\mathcal{A}_x := \{ \varphi = (\xi, \eta) \text{ is admissible and satisfies } \mathbb{E}_x[\xi(T)] \leq K_1(x)T^n + K_2(x) \text{ for all } T > 0 \}, \quad (2.4)$$

where $K_1(x)$ and $K_2(x)$ are positive real-valued functions, and $n \in \mathbb{N}$ is a positive integer. The requirement that $\mathbb{E}_x[\xi(T)] \leq K_1(x)T^n + K_2(x)$ ensures that the expectation in the right-hand side of (2.3) as well as the discounted and finite-time horizon problems (2.6) and (2.7) are well-defined. The set is clearly nonempty because the “zero control” $\varphi(t) \equiv 0$ is in \mathcal{A}_x . Lemma 3.4 below indicates that \mathcal{A}_x includes local time controls as well. It is also apparent that the value of λ_0 does not depend on the initial condition x since an initial jump does not alter the value of the limit in (2.3). In addition, it is obvious that $\lambda_0 \geq 0$.

In this paper, we aim to find the value λ_0 and an optimal control policy φ^* that achieves the value λ_0 . Motivated by Weerasinghe (2007), we will approach this problem via the vanishing discount method and show that λ_0 is equal to the *Abelian limit* as well as the *Cesàro limit*. In other words, we demonstrate that

$$\lambda_0 = \lim_{r \downarrow 0} r V_r(x) = \lim_{T \rightarrow \infty} \frac{V_T(x)}{T}, \quad (2.5)$$

where $V_r(x)$ and $V_T(x)$ denote respectively the value functions for the related discounted and finite horizon problems

$$V_r(x) := \sup_{\varphi(\cdot) \in \mathcal{A}_x} \mathbb{E}_x \left[\int_0^\infty e^{-rs} [h(X(s)) ds + c_1 d\eta(s) - c_2 d\xi(s)] \right], \quad (2.6)$$

$$V_T(x) := \sup_{\varphi(\cdot) \in \mathcal{A}_x} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 \eta(T) - c_2 \xi(T) \right]. \quad (2.7)$$

The main result of this paper is given next.

Theorem 2.1. *Under Assumptions 3.2, 4.2, and 4.3, the following assertions hold:*

- (i) *There exist $0 < a_* < b_* < \infty$ so that the reflected diffusion process on the interval $[a_*, b_*]$ (if the initial point is outside this interval, then there will be an initial jump to the nearest point of the interval) is an optimal state process for the ergodic control problem (2.3). Hence the optimal control policy is given by $\varphi_* = L_{a_*} - L_{b_*}$, in which L_{a_*} and L_{b_*} denote the local time processes at a_* and b_* , respectively.*

- (ii) *The Abelian and Cesàro limits in (2.5) hold.*

The proof of Theorem 2.1 follows from Theorems 4.4 and 5.1, which will be presented in the subsequent sections.

Remark 2.2. Since the optimal state process is a reflected diffusion on the interval $[a_*, b_*] \subset (0, \infty)$, it follows that it possesses a unique invariant measure π . According to Chapter 15 of Karlin and Taylor (1981), the invariant measure is $\pi(dx) = \frac{1}{M[a_*, b_*]} M(dx)$. Moreover, using the ergodicity for linear diffusions in Chapter II, section 6 of Borodin and Salminen (2002), we have

$$\lambda_0 = \int_{a_*}^{b_*} h(x)\pi(dx) + \frac{c_1}{2M[a_*, b_*]s(b_*)} - \frac{c_2}{2M[a_*, b_*]s(a_*)}.$$

3 Preliminary Results

In this section, we provide some preliminary results.

Theorem 3.1. *Suppose there exists a nonnegative function $u \in C^2([0, \infty))$ and a nonnegative number λ such that*

$$\max \{ \mathcal{L}u(x) + h(x) - \lambda, u'(x) - c_2, -u'(x) + c_1 \} = 0. \quad (3.1)$$

Then $\lambda_0 \leq \lambda$.

Proof. Let $x \in [0, \infty)$ and $\varphi(\cdot) = (\xi(\cdot), \eta(\cdot)) \in \mathcal{A}_x$ be an arbitrary control policy and denote by X the controlled state process with $X(0-) = x$. For $n \in \mathbb{N}$, let $\beta_n := \inf\{t \geq 0 : X(t) \geq n\}$. By Ito's formula we have

$$\begin{aligned} u(X(T \wedge \beta_n)) &= u(x) + \int_0^{T \wedge \beta_n} \mathcal{L}u(X(s))ds - \int_0^{T \wedge \beta_n} u'(X(s-))(d\eta_s^c - d\xi_s^c) \\ &\quad + \int_0^{T \wedge \beta_n} u'(X(s))\sigma(X(s))dW(s) + \sum_{0 \leq s \leq T \wedge \beta_n} [u(X(s)) - u(X(s-))]. \end{aligned} \quad (3.2)$$

The HJB equation (3.1) implies that $c_1 \leq u'(x) \leq c_2$. Note also that $\Delta X(s) = \Delta\xi(s) - \Delta\eta(s)$. Consequently, we can use the mean value theorem to obtain

$$u(X(s)) - u(X(s-)) \leq c_2\Delta\xi(s) - c_1\Delta\eta(s). \quad (3.3)$$

Note that (3.1) also implies that $\mathcal{L}u(x) \leq -h(x) + \lambda$. Plugging this and (3.3) into (3.2) gives us

$$\begin{aligned} u(X(T \wedge \beta_n)) &\leq u(x) - \int_0^{T \wedge \beta_n} h(X(s))ds + \lambda(T \wedge \beta_n) \\ &\quad + \int_0^{T \wedge \beta_n} u'(X(s))\sigma(X(s))dW(s) - c_1\eta(T \wedge \beta_n) + c_2\xi(T \wedge \beta_n). \end{aligned}$$

Rearranging terms and then taking expectations, we arrive at

$$\mathbb{E}_x \left[\int_0^{T \wedge \beta_n} h(X(s))ds + c_1\eta(T \wedge \beta_n) - c_2\xi(T \wedge \beta_n) \right] + \mathbb{E}_x[u(X(T \wedge \beta_n))] \leq u(x) + \lambda\mathbb{E}_x[T \wedge \beta_n].$$

Passing to the limit as $n \rightarrow \infty$ and then dividing both sides by T , we obtain from the nonnegativity of u and the monotone convergence theorem that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 \eta(T) - c_2 \xi(T) \right] \leq \lambda.$$

Finally, taking supremum over $\varphi(\cdot)$ yields the assertion that $\lambda_0 \leq \lambda$. \square

To proceed, we make the following assumption.

Assumption 3.2. (i) $\lim_{x \downarrow 0} [h(x) + c_2 \mu(x)] \leq 0$ and $\lim_{x \rightarrow \infty} [h(x) + c_1 \mu(x)] < 0$.

(ii) There exist $0 < a < b < \infty$ satisfying

$$\int_a^b h(y) m(y) dy + \frac{c_1}{2s(b)} - \frac{c_2}{2s(a)} > 0. \quad (3.4)$$

Lemma 3.3. *Suppose Assumption 3.2 holds. Then there exists a positive number $\lambda = \lambda(a, b)$ so that the following boundary value problem has a solution:*

$$\begin{cases} \frac{1}{2} \sigma^2(x) u''(x) + \mu(x) u'(x) + h(x) = \lambda, & x \in (a, b), \\ u'(a) = c_2, & u'(b) = c_1. \end{cases} \quad (3.5)$$

Proof. For any $0 < a < b < \infty$ and $\lambda \in \mathbb{R}$ given, a solution to the differential equation $\frac{1}{2} \sigma^2(x) u''(x) + \mu(x) u'(x) + h(x) = \lambda$ is given by

$$u(x) = c_1 x + \int_a^x 2s(u) \int_u^b [c_1 \mu(y) + h(y) - \lambda] m(y) dy du, \quad x \in [a, b]. \quad (3.6)$$

Note that $u'(b) = c_1$. The other boundary condition $u'(a) = c_2$ gives

$$c_1 + 2s(a) \int_a^b [c_1 \mu(y) + h(y) - \lambda] m(y) dy = c_2.$$

Note that

$$\int_a^b \mu(y) m(y) dy = \frac{1}{2s(b)} - \frac{1}{2s(a)}, \quad \forall [a, b] \in (0, \infty). \quad (3.7)$$

Hence it follows that

$$\lambda = \lambda(a, b) = \frac{1}{2M[a, b]} \left(\frac{c_1}{s(b)} - \frac{c_2}{s(a)} + 2 \int_a^b h(y) m(y) dy \right), \quad (3.8)$$

where $M[a, b] = \int_a^b m(y) dy > 0$ is the speed measure of the interval $[a, b]$. In particular, it follows from (3.4) that λ is positive. \square

Lemma 3.4. For any $0 < a < b < \infty$, let X be the reflected diffusion process on $[a, b]$:

$$X(t) = x + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) + L_a(t) - L_b(t), \quad (3.9)$$

where without loss of generality we can assume that the initial condition $x \in [a, b]$, and L_a and L_b denote the local time processes at a and b , respectively. Then there exist some positive constants K_1 and K_2 so that

$$\mathbb{E}_x[L_a(t) + L_b(t)] \leq K_1 t + K_2, \quad \text{for any } t \geq 0.$$

In particular, the policy $L_a - L_b \in \mathcal{A}_x$.

Proof. For the given $0 < a < b < \infty$, as in the proof of Lemma 3.3, we can verify that the function given by

$$v(x) := -x + a + 2 \int_a^x s(y) \int_y^b [-\mu(z) + K_1]m(z)dzdy, \quad x \in [a, b]$$

is a solution to the boundary value problem

$$\begin{cases} \frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) + K_1 = 0, & x \in (a, b), \\ v'(a) = 1, \quad v'(b) = -1, \end{cases}$$

where $K_1 = K_1(a, b) = \frac{1}{2M[a, b]}(\frac{1}{s(a)} + \frac{1}{s(b)}) > 0$.

It is well-known that equation (3.9) has a unique solution X ; see, for example, (Harrison, 1985, Section 2.4) or Burdzy et al. (2009). We now apply Itô's formula to the process $v(X(t))$,

$$\begin{aligned} \mathbb{E}_x[v(X(t))] &= v(x) + \mathbb{E}_x \left[\int_0^t \left(\mu(X(s))v'(X(s)) + \frac{1}{2}\sigma^2(X(s))v''(X(s)) \right) ds \right] \\ &\quad + \mathbb{E}_x[v'(a)L_a(t) - v'(b)L_b(t)] \\ &= v(x) - K_1 t + \mathbb{E}_x[L_a(t) + L_b(t)]. \end{aligned}$$

Since v is continuous and $X(t) \in [a, b]$ for all $t \geq 0$, it follows that there exists a positive constant $K_2 = K_2(a, b)$ such that $\mathbb{E}_x[L_a(t) + L_b(t)] \leq K_1 t + K_2$. The lemma is proved. \square

Corollary 3.5. Suppose Assumption 3.2 holds, then $\lambda_0 > 0$.

Proof. To show that $\lambda_0 > 0$, we consider the function u of (3.6) in which we choose $0 < a < b < \infty$ so that $\lambda = \lambda(a, b)$ of (3.8) is positive. Now let X be the reflected diffusion process on $[a, b]$ given by (3.9). Thanks to Lemma 3.4, the policy $L_a - L_b \in \mathcal{A}_x$. Apply Itô's formula to $u(X(t))$ and then take expectations to obtain

$$\begin{aligned} \mathbb{E}_x[u(X(t))] &= u(x) + \mathbb{E}_x \left[\int_0^t \left(\mu(X(s))u'(X(s)) + \frac{1}{2}\sigma^2(X(s))u''(X(s)) \right) ds \right] \\ &\quad + \mathbb{E}_x[u'(a)L_a(t) - u'(b)L_b(t)] \end{aligned}$$

$$= u(x) + \mathbb{E}_x \left[\int_0^t [\lambda - h(X(s))] ds + c_2 L_a(t) - c_1 L_b(t) \right].$$

Rearranging terms, dividing by t , and then passing to the limit as $t \rightarrow \infty$, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_x \left[\int_0^t h(X(s)) ds - c_2 L_a(t) + c_1 L_b(t) \right] = \lambda = \lambda(a, b),$$

where $\lambda(a, b)$ is defined in (3.8). In other words, the long-term average reward of the policy $L_a - L_b$ is $\lambda(a, b) > 0$. Now by the definition of λ_0 , we have $\lambda_0 \geq \lambda > 0$. \square

4 The Abelian Limit

For a given $r > 0$, recall the discounted value function $V_r(x)$ defined in (2.6). Also recall the definition of λ_0 given in (2.3). The following proposition presents a relationship between the discounted and long-term average problems.

Proposition 4.1. *We have $\liminf_{r \downarrow 0} rV_r(x) \geq \lambda_0$ for any $x \in [0, \infty)$.*

Proof. Since $V_r(x) \geq 0$, the relation $\liminf_{r \downarrow 0} rV_r(x) \geq \lambda_0$ holds trivially when $\lambda_0 = 0$. Now assume that $\lambda_0 > 0$. Let $x \in [0, \infty)$. For any $0 < K < \lambda_0$, there exists a policy $\varphi = \xi - \eta \in \mathcal{A}_x$ so that

$$\lambda_0 \geq \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T) \right] \geq K; \quad (4.1)$$

here X is the controlled process corresponding to the policy φ with initial condition $X(0-) = x$. Let $F(T) := \mathbb{E}_x[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T)]$ and $G(T) := \frac{F(T)}{T+1}$ for $T > 0$. Thanks to (4.1), there exists a $T_1 > 0$ such that $F(T) > 0$ for all $T \geq T_1$. Consequently we can use integration by parts and Fubini's theorem to obtain

$$\begin{aligned} \mathbb{E}_x \left[\int_0^T e^{-rs} (h(X(s)) ds - c_2 d\xi(s) + c_1 d\eta(s)) \right] &= e^{-rT} F(T) + r \int_0^T e^{-rs} F(s) ds \\ &\geq r \int_0^T e^{-rs} F(s) ds, \end{aligned}$$

for all $T \geq T_1$. Then it follows from the dynamic programming principle (see Guo and Pham (2005)) that for all $T \geq T_1$,

$$\begin{aligned} rV_r(x) &\geq \sup_{\varphi \in \mathcal{A}_x} r \mathbb{E}_x \left[\int_0^T e^{-rs} (h(X(s)) ds - c_2 d\xi(s) + c_1 d\eta(s)) + e^{-rT} V_r(X(T)) \right] \\ &\geq r \mathbb{E}_x \left[\int_0^T e^{-rs} (h(X(s)) ds - c_2 d\xi(s) + c_1 d\eta(s)) \right] \\ &\geq r^2 \int_0^\infty e^{-rs} F(s) ds = r^2 \int_0^\infty e^{-rs} (s+1) G(s) ds = \int_0^\infty e^{-t} (t+r) G(t/r) dt. \end{aligned}$$

In view of (4.1), for any $\varepsilon > 0$, we can find a $T_2 > 0$ so that $G(T) = \frac{F(T)}{T+1} \geq K - \varepsilon$ for all $T \geq T_2$. Denote $T_0 := T_1 \vee T_2$. Then we can estimate

$$\begin{aligned} rV_r(x) &\geq \int_0^{rT_0} e^{-t}(t+r)G(t/r)dt + \int_{rT_0}^{\infty} e^{-t}(t+r)G(t/r)dt \\ &\geq \int_0^{rT_0} e^{-t}F(t/r)dt + \int_{rT_0}^{\infty} e^{-t}(t+r)(K-\varepsilon)dt \\ &\geq -r^2T_0 \max_{t \in [0, T_0]} F(t) + (K-\varepsilon)(1+r-r^2(T_0+1)). \end{aligned}$$

Letting $r \downarrow 0$, we obtain $\liminf_{r \downarrow 0} rV_r(x) \geq K - \varepsilon$. Since $\varepsilon > 0$ and $K < \lambda_0$ are arbitrary, we obtain $\liminf_{r \downarrow 0} rV_r(x) \geq \lambda_0$, which completes the proof. \square

Usually one can show that the value function V_r of (2.6) is a viscosity solution to the HJB equation

$$\min\{rv(x) - \mathcal{L}v(x) - h(x), -v'(x) + c_2, v'(x) - c_1\} = 0, \quad x \in (0, \infty). \quad (4.2)$$

Under additional assumptions such as concavity of the function h , for specific models (such as geometric Brownian motion in Guo and Pham (2005)), one can further show that V_r is a smooth solution to (4.2) and that there exist $0 < a_r < b_r < \infty$ so that the reflected diffusion process on the interval $[a_r, b_r]$ is an optimal state process. In other words, the policy $L_{a_r} - L_{b_r}$ is an optimal control policy, where L_{a_r} and L_{b_r} denote the local times of the controlled process X at a_r and b_r , respectively. If the initial position $X(0-)$ is outside the interval $[a_r, b_r]$, then an initial jump to the nearest boundary point is exerted at time 0. We also refer to Matomäki (2012) and Weerasinghe (2005) for sufficient conditions for the optimality of such policies for general one-dimensional diffusion processes under different settings.

Motivated by these recent developments, we make the following assumption:

Assumption 4.2. For each $r > 0$, there exist two numbers with $0 < a_r < b_r < \infty$ so that the discounted value function $V_r(x)$ of (2.6) is $C^2([0, \infty))$ and satisfies the following system of equations:

$$\begin{cases} rV_r(x) - \mathcal{L}V_r(x) - h(x) = 0, & c_1 \leq V_r'(x) \leq c_2, & x \in (a_r, b_r), \\ rV_r(x) - \mathcal{L}V_r(x) - h(x) \geq 0, & V_r'(x) = c_1, & x \geq b_r, \\ rV_r(x) - \mathcal{L}V_r(x) - h(x) \geq 0, & V_r'(x) = c_2, & x \leq a_r. \end{cases} \quad (4.3)$$

In addition, the following assumption is needed for the proof of Theorem 4.4.

Assumption 4.3. The functions h , μ , and σ are continuously differentiable and satisfies $\inf_{x \in [a, b]} \sigma^2(x) > 0$ for any $[a, b] \subset (0, \infty)$.

We now state the main result of this section.

Theorem 4.4. *Let Assumptions 3.2, 4.2, and 4.3 hold. Then there exist positive constants $a_* < b_*$ so that the following statements hold true:*

(i) $\lim_{r \downarrow 0} rV_r(x) = \lambda_0$ for all $x \in \mathbb{R}$.

(ii) The reflected diffusion process on the state space $[a_*, b_*]$ (if the initial point is outside this interval, then there will be an initial jump to the nearest point of the interval) is an optimal state process for the ergodic control problem (2.3). Hence the optimal control policy here is given by $\varphi_* = L_{a_*} - L_{b_*}$, in which L_{a_*} and L_{b_*} denote the local time processes at a_* and b_* , respectively.

To prove Theorem 4.4, we first establish a series of technical lemmas.

Lemma 4.5. *Suppose Assumptions 3.2 (i) and 4.2 hold. Then $\lambda_0 > 0$ if and only if $\liminf_{r \downarrow 0} a_r > 0$.*

Proof. Recall that the function $V_r \in C^2(0, \infty)$ satisfies $rV_r(x) - \mu(x)V_r'(x) - \frac{1}{2}\sigma^2(x)V_r''(x) - h(x) = 0$ for $x \in (a_r, b_r)$ and $V_r(x) = c_2x + V_r(0+)$ for $x \leq a_r$, where $V_r(0+) = \lim_{x \downarrow 0} V_r(x) = V_r(a_r) - c_2a_r$. Therefore the smooth pasting principle for V_r at a_r implies that $V_r'(a_r) = c_2$ and $V_r''(a_r) = 0$. Thus it follows that

$$r(c_2a_r + V_r(0+)) - c_2\mu(a_r) - h(a_r) = 0 \quad \text{or} \quad rV_r(0+) = h(a_r) + c_2\mu(a_r) - rc_2a_r. \quad (4.4)$$

(i) Suppose first that $\lambda_0 > 0$. Then Proposition 4.1 implies that any limit point of $\{rV_r(0+)\}$ must be greater than 0. If there exists a sequence $\{r_n\} \subset (0, 1]$ for which $\lim_{n \rightarrow \infty} a_{r_n} = 0$, then passing to the limit as $n \rightarrow \infty$ in (4.4) will give us $\lim_{n \rightarrow \infty} r_n V_{r_n}(0+) = \lim_{n \rightarrow \infty} [h(a_{r_n}) + c_2\mu(a_{r_n})] \leq 0$ thanks to Assumption 3.2 (i). This is a contradiction.

(ii) Now if $\liminf_{r \downarrow 0} a_r = 0$, then using (4.4) again, we obtain $\lim_{n \rightarrow \infty} r_n V_{r_n}(0+) = 0$ for some sequence $\{r_n\}$ such that $\lim_{n \rightarrow \infty} r_n = 0$ and $\lim_{n \rightarrow \infty} a_{r_n} = 0$. This, together with Proposition 4.1, indicates that $\lambda_0 = 0$. \square

Lemma 4.6. *Suppose Assumptions 3.2 (i) and 4.2 hold. Then $\limsup_{r \downarrow 0} b_r < \infty$.*

Proof. As in the proof of Lemma 4.5, we can use the smooth pasting for the function V_r at b_r to obtain

$$0 < rV_r(b_r) = c_1\mu(b_r) + h(b_r). \quad (4.5)$$

Suppose that there exists some sequence $\{r_n\} \subset (0, 1]$ so that $\lim_{n \rightarrow \infty} r_n = 0$ and $\lim_{n \rightarrow \infty} b_{r_n} = \infty$. Now passing to the limit as $n \rightarrow \infty$ in (4.5), we obtain

$$0 \leq \lim_{n \rightarrow \infty} r_n V_{r_n}(b_{r_n}) = \lim_{n \rightarrow \infty} (c_1\mu(b_{r_n}) + h(b_{r_n})) < 0$$

thanks to Assumption 3.2 (i). This is a contradiction. Hence $\limsup_{r \downarrow 0} b_r < \infty$ and the assertion of the lemma follows. \square

The following lemma can be obtained directly from Corollary 3.5 and Lemmas 4.5 and 4.6.

Lemma 4.7. *Suppose Assumptions 3.2 (i) and 4.2 hold. Then there exist constants $0 < r_0 < 1$ and $0 < K_1 < K_2 < \infty$ so that $K_1 \leq a_r < b_r \leq K_2$ for all $0 < r \leq r_0$.*

Lemma 4.8. *Let Assumptions 3.2, 4.2, and 4.3 hold. Then there exists a function $w_r \in C^1((0, \infty)) \cap C^2((0, \infty) \setminus \{a_r, b_r\})$ satisfying*

$$\begin{cases} \frac{1}{2}\sigma^2(x)w_r''(x) + (\sigma(x)\sigma'(x) + \mu(x))w_r'(x) - (r - \mu'(x))w_r(x) + h'(x) = 0, & x \in (a_r, b_r); \\ c_1 \leq w_r(x) \leq c_2, & x \in (a_r, b_r), \\ w_r(x) = c_2, \quad w_r'(x) = 0, & x \leq a_r, \\ w_r(x) = c_1, \quad w_r'(x) = 0, & x \geq b_r. \end{cases} \quad (4.6)$$

Proof. Recall Assumption 4.2 indicates that V_r of (2.6) satisfies

$$\begin{aligned} \frac{1}{2}\sigma^2(x)V_r''(x) + \mu(x)V_r'(x) - rV_r(x) + h(x) &= 0, \quad x \in (a_r, b_r), \\ c_1 < V_r'(x) < c_2, \quad x &\in (a_r, b_r), \end{aligned} \quad (4.7)$$

and

$$V_r'(x) = c_2, \text{ for } x \leq a_r, \quad V_r'(x) = c_1, \text{ for } x \geq b_r, \text{ and } V_r''(x) = 0 \text{ for } x \leq a_r \text{ or } x \geq b_r.$$

Now differentiating (4.7) and denoting $w_r := V_r'$, then w_r satisfies (4.6). \square

Lemma 4.9. *Let Assumptions 3.2, 4.2, and 4.3 hold. Then there exist two positive constants $a_* < b_*$, a constant $l_0 > 0$, and a function $w_0 \in C^1(0, \infty)$ satisfying*

$$\begin{cases} \frac{1}{2}\sigma^2(x)w_0'(x) + \mu(x)w_0(x) + h(x) = l_0, & x \in (a_*, b_*), \\ c_1 \leq w_0(x) \leq c_2, & x \in (a_*, b_*), \\ w_0(x) = c_2, \quad w_0'(x) = 0, & x \leq a_*, \\ w_0(x) = c_1, \quad w_0'(x) = 0, & x \geq b_*. \end{cases} \quad (4.8)$$

Proof. Thanks to (4.6), w_r is uniformly bounded. Next we integrate the first equation of (4.6) from a_r to x ($x \in (a_r, b_r)$) to obtain

$$\frac{1}{2}\sigma^2(x)w_r'(x) = c_2\mu(a_r) + h(a_r) - \mu(x)w_r(x) + r \int_{a_r}^x w_r(y)dy - h(x). \quad (4.9)$$

Thanks to Lemma 4.7 and the continuity of h and μ , the right-hand side of (4.9) is uniformly bounded on $[K_1, K_2]$ for all $r \in (0, r_0]$, where r_0, K_1 , and K_2 are the positive constants found in Lemma 4.7. This, together with Assumption 4.3, implies that $\{w_r'(x), r \in (0, r_0]\}$ is uniformly bounded on $[K_1, K_2]$. Consequently, $\{w_r(x), r \in (0, r_0]\}$ is equicontinuous.

Now we rewrite the first equation of (4.6) as

$$\frac{1}{2}\sigma^2(x)w_r''(x) = -(\sigma(x)\sigma'(x) + \mu(x))w_r'(x) + (r - \mu'(x))w_r(x) - h'(x), \quad x \in (a_r, b_r).$$

Since $\{w_r'\}$ and $\{w_r\}$ are uniformly bounded and σ, σ', μ' , and h' are continuous, it follows from Assumption 4.3 that there exists a positive constant K independent of r such that

$$|w_r''(x)| \leq K, \text{ for all } x \in (a_r, b_r) \subset [K_1, K_2], \text{ for any } r \in (0, r_0].$$

This, together with the fact that $w'_r(x) = 0$ for all $x \in [K_1, K_2] \setminus (a_r, b_r)$ (c.f. (4.6)), implies that $\{w'_r, r \in (0, r_0]\}$ is equicontinuous on $[K_1, K_2]$.

Meanwhile, Lemma 4.7 implies that there exists a sequence $\{r_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} r_n = 0$ for which

$$\lim_{n \rightarrow \infty} a_{r_n} = a_*, \quad \lim_{n \rightarrow \infty} b_{r_n} = b_* \quad (4.10)$$

for some $0 < K_1 \leq a_* \leq b_* \leq K_2 < \infty$. Since μ and h are continuous, we have

$$\lim_{n \rightarrow \infty} (c_2 \mu(a_{r_n}) + h(a_{r_n})) = c_2 \mu(a_*) + h(a_*) =: l_0.$$

Recall that (4.4) indicates $rV_r(0+) = h(a_r) + c_2 \mu(a_r) - rc_2 a_r$. Thus l_0 is a limiting point of $\{rV_r(0+)\}$. This, together with Proposition 4.1, implies that $l_0 \geq \lambda_0$.

Since $\{w_{r_n}\}$ and $\{w'_{r_n}\}$ are equicontinuous and uniformly bounded sequences on $[K_1, K_2]$, by the Arzelà-Ascoli theorem, there exists a continuously differentiable function w_0 on $[K_1, K_2]$ such that for some subsequence of $\{r_n\}$ (still denoted by $\{r_n\}$), we have

$$\lim_{n \rightarrow \infty} w_{r_n}(x) = w_0(x) \quad \lim_{n \rightarrow \infty} w'_{r_n}(x) = w'_0(x). \quad (4.11)$$

Passing to the limit along the subsequence $\{r_n\}$ in (4.9) and noting that w_{r_n} is uniformly bounded, we obtain from (4.10) and (4.11) that

$$\frac{1}{2} \sigma^2(x) w'_0(x) + \mu(x) w_0(x) + h(x) = l_0, \quad x \in (a_*, b_*).$$

Since $\{w_{r_n}\}$ and $\{w'_{r_n}\}$ are equicontinuous, we can extend w_0 to $(0, \infty)$ so that w_0 is continuously differentiable and that the third and fourth lines of (4.8) are satisfied. That $c_1 \leq w_0(x) \leq c_2$ for all $x \in (a_*, b_*)$ is obvious. This completes the proof of the lemma. \square

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. Let the positive constants l_0 and $a_* < b_*$ and the function w_0 be as in the statement of Lemma 4.9. Define the function Q by

$$Q(x) = \int_0^x w_0(u) du, \quad x \in (0, \infty).$$

Since w_0 is positive and satisfies (4.8), it is easy to see that Q is nonnegative and satisfies

$$\begin{cases} \frac{1}{2} \sigma^2(x) Q''(x) + \mu(x) Q'(x) + h(x) = l_0, & x \in (a_*, b_*), \\ Q'(x) = c_2, & x \leq a_*, \\ Q'(x) = c_1, & x \geq b_*. \end{cases} \quad (4.12)$$

Without loss of generality, we assume that x is in the interval $[a_*, b_*]$. Let X_* be the reflected diffusion process on the interval $[a_*, b_*]$ with $X_*(0) = x$; that is,

$$X_*(t) = x + \int_0^t \mu(X_*(s)) ds + \int_0^t \sigma(X_*(s)) dW(s) + L_{a_*}(t) - L_{b_*}(t),$$

where L_{a_*} and L_{b_*} denote the local time processes at a_* and b_* , respectively. Thanks to Lemma 3.4, $L_{a_*} - L_{b_*} \in \mathcal{A}_x$ and X_* is an admissible process. We now apply Itô's formula to $Q(X_*(T))$ and use (4.12) to obtain

$$\mathbb{E}_x [Q(X_*(T))] = Q(x) + l_0 T + \mathbb{E}_x \left[- \int_0^T h(X_*(s)) ds + c_2 L_{a_*}(t) - c_1 L_{b_*}(t) \right].$$

Rearranging terms, dividing both sides by T , and then letting $T \rightarrow \infty$, we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 L_{b_*}(T) - c_2 L_{a_*}(T) \right] = l_0.$$

This implies that $l_0 \leq \lambda_0$. Recall we have observed in the proof of Lemma 4.9 that $l_0 \geq \lambda_0$. Hence we conclude that $l_0 = \lambda_0$ and $\varphi_* = L_{a_*} - L_{b_*}$ is an optimal policy.

Recall from the proof of Lemma 4.9 that l_0 is a limiting point of $\{rV_r(0+)\}$. Since $\lambda_0 = l_0$, it follows that any limiting point of $\{rV_r(0+)\}$ is equal to λ_0 . Moreover, for any $x \in (0, \infty)$, we have

$$\begin{aligned} \lim_{r \downarrow 0} rV_r(x) &= \lim_{r \downarrow 0} r(V_r(x) - V_r(0+)) + \lim_{r \downarrow 0} rV_r(0+) \\ &= \lim_{r \downarrow 0} rw_r(\theta x) + \lambda_0 \\ &= \lambda_0, \end{aligned}$$

where $\theta \in [0, 1]$. Note that we used the fact that w_r is uniformly bounded to obtain the last equality. The proof is complete. \square

5 The Cesàro Limit

This section establishes the Cesàro limit $\lim_{T \rightarrow \infty} \frac{1}{T} V_T(x) = \lambda_0$, where $V_T(x)$ is the value function for the finite-horizon problem defined in (2.7).

Theorem 5.1. *Let Assumptions 3.2, 4.2, and 4.3 hold. Then we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} V_T(x) = \lambda_0. \quad (5.1)$$

Proof. The proof is divided into two steps.

Step 1. We first prove

$$\liminf_{T \rightarrow \infty} \frac{1}{T} V_T(x) \geq \lambda_0. \quad (5.2)$$

Note that (5.2) is obviously true when $\lambda_0 = 0$. Now we consider the case when $\lambda_0 > 0$ and let K be a constant so that $K < \lambda_0$. By (2.3) there exists an admissible control $\varphi := (\xi, \eta) \in \mathcal{A}_x$ so that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 \eta(T) - c_2 \xi(T) \right] > K.$$

For any $T > 0$, by the definition of $V_T(x)$, we have $\mathbb{E}_x[\int_0^T h(X(s))ds + c_1\eta(T) - c_2\xi(T)] \leq V_T(x)$ and hence

$$K < \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s))ds + c_1\eta(T) - c_2\xi(T) \right] \leq \liminf_{T \rightarrow \infty} \frac{1}{T} V_T(x).$$

Since $K < \lambda_0$ is arbitrary we conclude that $\lambda_0 \leq \liminf_{T \rightarrow \infty} \frac{1}{T} V_T(x)$. This gives (5.2).

Step 2. We now show that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} V_T(x) \leq \lambda_0. \quad (5.3)$$

To this end, consider any $\varphi \in \mathcal{A}_x$ for which $\mathbb{E}_x[\int_0^T h(X(s))ds + c_1\eta(T) - c_2\xi(T)] \geq (V_T(x) - 1) \vee 0$, where X is the corresponding controlled process with $X(0-) = x \in (0, \infty)$.

Let $r > 0$ and consider the functions V_r and w_r defined respectively in (2.6) and (4.6). Recall that $w_r = V_r' \in [c_1, c_2]$. Hence for any $x > 0$, we can write

$$V_r(x) = V_r(0) + \int_0^x w_r(y)dy \geq V_r(0) + c_1x. \quad (5.4)$$

We now apply Itô's formula to the process $e^{-rT}V_r(X(T))$ to obtain

$$\begin{aligned} & \mathbb{E}_x[e^{-r(T \wedge \beta_n)}V_r(X(T \wedge \beta_n))] \\ &= V_r(x) + \mathbb{E}_x \left[\int_0^{T \wedge \beta_n} e^{-rs}(-rV_r + \mathcal{L}V_r)(X(s))ds + e^{-rs}V_r'(X(s))(d\xi^c(s) - d\eta^c(s)) \right] \\ & \quad + \mathbb{E}_x \left[\sum_{0 \leq s \leq T \wedge \beta_n} e^{-rs} [V_r(X(s)) - V_r(X(s-))] \right], \end{aligned}$$

where $\beta_n := \inf\{t \geq 0 : X(t) \geq n\}$ and $n \in \mathbb{N}$. Since V_r satisfies the HJB equation (4.2), we have

$$\mathbb{E}_x[e^{-r(T \wedge \beta_n)}V_r(X(T \wedge \beta_n))] \leq V_r(x) + \mathbb{E}_x \left[\int_0^{T \wedge \beta_n} e^{-rs}[-h(X(s))ds + c_2d\xi(s) - c_1d\eta(s)] \right].$$

Rearranging terms and using (5.4) yield

$$\begin{aligned} & \mathbb{E}_x \left[\int_0^{T \wedge \beta_n} e^{-rs}[h(X(s))ds - c_2d\xi(s) + c_1d\eta(s)] \right] \\ & \leq V_r(x) - \mathbb{E}_x[e^{-r(T \wedge \beta_n)}V_r(X(T \wedge \beta_n))] \\ & \leq V_r(x) - \mathbb{E}_x[e^{-r(T \wedge \beta_n)}V_r(0) + e^{-r(T \wedge \beta_n)}c_1X(T \wedge \beta_n)] \\ & \leq V_r(x) - \mathbb{E}_x[e^{-r(T \wedge \beta_n)}V_r(0)]. \end{aligned}$$

Then it follows that

$$\mathbb{E}_x \left[\int_0^T e^{-rs}(h(X(s))ds - c_2d\xi(s) + c_1d\eta(s)) \right] \leq V_r(x) - e^{-rT}V_r(0). \quad (5.5)$$

Using integration by parts, we have

$$\begin{aligned}
& e^{-rT} \mathbb{E}_x \left[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T) \right] \\
&= \mathbb{E}_x \left[\int_0^T e^{-rs} (h(X(s)) ds - c_2 d\xi(s) + c_1 d\eta(s)) \right] \\
&\quad - r \mathbb{E}_x \left[\int_0^T e^{-rt} \left[\int_0^t h(X(s)) ds - c_2 \xi(t) + c_1 \eta(t) \right] dt \right].
\end{aligned}$$

Plugging this observation into (5.5), we obtain

$$\begin{aligned}
& e^{-rT} \mathbb{E}_x \left[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T) \right] \\
&\leq V_r(x) - e^{-rT} V_r(0) - r \mathbb{E}_x \left[\int_0^T e^{-rt} \left[\int_0^t h(X(s)) ds - c_2 \xi(t) + c_1 \eta(t) \right] dt \right] \\
&\leq V_r(x) - e^{-rT} V_r(0) + r c_2 \int_0^T e^{-rt} \mathbb{E}[\xi(t)] dt \\
&\leq V_r(x) - e^{-rT} V_r(0) + r c_2 \int_0^T e^{-rt} (K_1(x) t^n + K_2(x)) dt \\
&= V_r(x) - e^{-rT} V_r(0) + \frac{c_2 K_1(x) n!}{r^n} \left(1 - e^{-rT} \sum_{j=0}^n \frac{(rT)^j}{j!} \right) + K_2(x) (1 - e^{-rT}) \\
&= (1 - e^{-rT}) V_r(0) + [V_r(x) - V_r(0)] + \frac{c_2 K_1(x) n!}{r^n} \left(1 - e^{-rT} \sum_{j=0}^n \frac{(rT)^j}{j!} \right) + K_2(x) (1 - e^{-rT}),
\end{aligned}$$

where the third last step follows from (2.4). Now we pick $r = \frac{\delta}{T}$ for some $\delta \in (0, 1)$ and divide both sides by T to obtain

$$\begin{aligned}
& e^{-\delta} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T) \right] \\
&\leq \frac{1 - e^{-\delta}}{\delta} r V_r(0) + \frac{V_r(x) - e^{-\delta} V_r(0)}{T} + \frac{c_2 K_1(x) n! T^{n-1}}{\delta^n} \left(1 - e^{-\delta} \sum_{j=0}^n \frac{\delta^j}{j!} \right) + \frac{K_2(x) (1 - e^{-\delta})}{T}.
\end{aligned}$$

Since $\lim_{r \downarrow 0} r V_r(0) = \lambda_0$ thanks to Theorem 4.4, we first let $\delta \downarrow 0$ and then let $T \rightarrow \infty$ to obtain

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds - c_2 \xi(T) + c_1 \eta(T) \right] \leq \lambda_0.$$

This is true for any $\varphi \in \mathcal{A}_x$ and hence it follows that $\limsup_{T \rightarrow \infty} \frac{V_T(x)}{T} \leq \lambda_0$; establishing (5.3). The proof is complete. \square

6 Geometric Brownian Motion Models

In this section, we apply the vanishing discount method to study a two-sided singular control problem when the underlying process is a geometric Brownian motion. That is, we consider

the following long-term average singular control problem

$$\left\{ \begin{array}{l} \lambda_0 := \sup_{\varphi(\cdot) \in \mathcal{A}_x} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(s)) ds + c_1 \eta(T) - c_2 \xi(T) \right], \\ \text{subject to } \begin{cases} dX(s) = \mu X(s) ds + \sigma X(s) dW(s) + d\xi(s) - d\eta(s), & s \geq 0, \\ X(0-) = x \geq 0, \end{cases} \end{array} \right. \quad (6.1)$$

where $\mu < 0$, $\sigma > 0$, $0 < c_1 < c_2$ are constants, h is a nonnegative function satisfying Assumption 6.2 below, and the singular control $\varphi = \xi - \eta$ is admissible as in Section 2.

In problem (6.1), the underlying uncontrolled process is a geometric Brownian motion with state space $(0, \infty)$. Using the criteria in Chapter 15 of Karlin and Taylor (1981), we see that both 0 and ∞ are natural boundary points. Moreover, the scale and speed densities are respectively given by $s(x) = x^{-\frac{2\mu}{\sigma^2}}$ and $m(x) = \frac{1}{\sigma^2} x^{\frac{2\mu}{\sigma^2} - 2}$, $x > 0$.

Remark 6.1. Let us explain why we need to assume $\mu < 0$ in (6.1). Suppose that $\mu > 0$, then one can show that $\lambda_0 = \infty$. Indeed, for any $x > 0$ and any sequence $t_k \rightarrow \infty$ as $k \rightarrow \infty$, we can construct an admissible policy $\varphi(\cdot) = (\xi(\cdot), \eta(\cdot))$ as follows. Let $\xi(t) \equiv 0$ for all $t \geq 0$, $\eta(t) = 0$ for $t < t_k$ and $\eta(t) \equiv \Delta\eta(t_k) = X_0(t_k) := x \exp\{(\mu - \frac{1}{2}\sigma^2)t_k + \sigma W(t_k)\}$ for all $t \geq t_k$. In other words, under the policy $\varphi(\cdot)$, the manager does nothing before time t_k and then harvests all at time t_k . Since $X_0(t_k) = x \exp\{(\mu - \frac{1}{2}\sigma^2)t_k + \sigma W(t_k)\}$, we have

$$\lambda_0 \geq \lim_{k \rightarrow \infty} \frac{1}{t_k} \mathbb{E}_x [c_1 \eta(t_k)] = \lim_{k \rightarrow \infty} \frac{1}{t_k} \mathbb{E}_x [c_1 X_0(t_k)] = \lim_{k \rightarrow \infty} \frac{1}{t_k} c_1 x e^{\mu t_k} = \infty.$$

When $\mu = 0$, in the presence of Assumption 6.2 below, we no longer have $\lim_{x \rightarrow \infty} (h(x) + c_1 \mu x) < 0$, thus violating Assumption 3.2 (i). Consequently the vanishing discount method is not applicable. We will present in Section 6.1 a case study which indicates for the Cobb Douglas function $h(x) = x^p$, $p \in (0, 1)$, the optimal long-term average reward can be arbitrarily large.

We need the following assumption throughout this section:

Assumption 6.2. The function h is nonnegative and satisfies the following conditions:

- (i) h is strictly concave, continuously differentiable and nondecreasing on $[0, \infty)$, and satisfies $h(0) = 0$;
- (ii) h has a finite Legendre transform on $(0, \infty)$; that is, for all $z > 0$

$$\Phi_h(z) := \sup_{x \geq 0} \{h(x) - zx\} < \infty. \quad (6.2)$$

- (iii) h satisfies the Inada condition at 0, i.e.,

$$\lim_{x \downarrow 0} \frac{h(x)}{x} = \infty. \quad (6.3)$$

Remark 6.3. Assumption 6.2 is common in the finance literature; see, for example, Guo and Pham (2005) and De Angelis and Ferrari (2014). The Inada condition at 0 implies that the right derivative $h'(0+)$ of the function h is infinity. If this condition is violated, say, there exists some $c \in [c_1, c_2]$ such that

$$h'(0+) + \mu c \leq 0, \quad (6.4)$$

then one can show that $\lambda_0 = 0$.

Indeed, in the presence of (6.4), we can immediately verify that the function $u(x) = cx$ and $\lambda = 0$ satisfy (3.1). Consequently it follows from Theorem 3.1 that $\lambda_0 = 0$. Moreover, we can construct an optimal policy in the following way. Suppose $X(0-) = x \geq 0$. Let $\xi(t) \equiv 0$ and $\eta(t) \equiv \Delta\eta(0) = x$; that is, the policy harvests all and brings the state to 0 at time 0. The long-term average reward for such a policy is 0.

The main result of this section is presented next.

Theorem 6.4. *Under Assumption 6.2, there exist $0 < a_* < b_* < \infty$ such that the reflected geometric Brownian motion on the interval $[a_*, b_*]$ is an optimal state process (if the initial point is outside this interval, then there will be an initial jump to the nearest point of the interval). In other words, the optimal value λ_0 of (6.1) is achieved by $\varphi_*(\cdot) := L_{a_*}(\cdot) - L_{b_*}(\cdot)$, in which $L_{a_*}(\cdot)$ and $L_{b_*}(\cdot)$ denote respectively the local times of X at a_* and b_* . Moreover, the Abelian and the Cesàro limits (2.5) hold.*

Proof. In view of Theorem 2.1, we only need to verify Assumptions 3.2, 4.2, and 4.3 hold. Assumption 4.3 obviously holds. Under Assumption 6.2, it is established in Guo and Pham (2005) that there exist two positive numbers $a_r < b_r$ so that the discounted problem V_r is a classical solution to (4.3), which verifies Assumption 4.2.

It remains to show Assumption 3.2 holds. Since $h(0) = 0$, we have $\lim_{x \downarrow 0} [h(x) + c_2 \mu x] = 0$. In view of (6.2) and the fact that $\mu < 0$, we have $h(x) - \left(\frac{-c_1 \mu}{2}\right)x \leq \Phi_h\left(\frac{-c_1 \mu}{2}\right) < \infty$. Consequently it follows that

$$\lim_{x \rightarrow \infty} [h(x) + c_1 \mu x] \leq \lim_{x \rightarrow \infty} \left[\Phi_h\left(\frac{-c_1 \mu}{2}\right) - \frac{c_1 \mu}{2}x + c_1 \mu x \right] = -\infty.$$

Assumption 3.2 (i) is therefore verified. To verify Assumption 3.2 (ii), we recall that the scale and speed densities of the geometric Brownian motion are respectively given by $s(x) = x^{-\frac{2\mu}{\sigma^2}}$ and $m(x) = \frac{1}{\sigma^2} x^{\frac{2\mu}{\sigma^2}-2}$, $x > 0$. Next we observe that condition (6.3) implies that there exists a positive constant δ so that $\frac{h(y)}{y} > -2\mu c_2$ for all $y \in (0, \delta)$. Now let $b > \delta > a$. Then we have

$$\begin{aligned} \frac{c_1}{2s(b)} - \frac{c_2}{2s(a)} + \int_a^b h(y)m(y)dy &\geq -\frac{c_2}{2s(a)} + \int_a^\delta \frac{h(y)}{y} y \frac{1}{\sigma^2} y^{\frac{2\mu}{\sigma^2}-2} dy \\ &> -\frac{c_2}{2s(a)} + \frac{1}{\sigma^2} \int_a^\delta (-2\mu c_2) y^{\frac{2\mu}{\sigma^2}-1} dy \\ &= -\frac{c_2}{2s(a)} + c_2 \left(\frac{1}{s(a)} - \frac{1}{s(\delta)} \right) \end{aligned}$$

$$= \frac{c_2}{2s(a)} - \frac{c_2}{s(\delta)} > 0,$$

by choosing $a > 0$ sufficiently small so that $\frac{1}{s(a)} > \frac{2}{s(\delta)}$. This gives (3.4) and hence verifies Assumption 3.2 (ii). \square

6.1 Case Study: $\mu = 0$ and $h(x) = x^p, p \in (0, 1)$

In this subsection, we consider the problem (6.1) in which $\mu = 0$ and $h(x) = x^p, p \in (0, 1)$. We will demonstrate via probabilistic arguments that $\lambda_0 = \infty$. To this end, we first present the following lemma.

Lemma 6.5. *Suppose $\mu = 0$ and $h(x) = x^p$ for some $p \in (0, 1)$, then for all $\lambda > 0$ sufficiently large satisfying*

$$\frac{2p}{\sigma^2(1-p)}\lambda^{1-\frac{1}{p}} < \frac{c_2 - c_1}{2} \quad \text{and} \quad c_2\lambda^{\frac{1}{p}-1} > \frac{2}{\sigma^2 p} \ln \lambda, \quad (6.5)$$

the function

$$u(x) := \begin{cases} \frac{2}{\sigma^2 p(1-p)}x^p - \frac{2\lambda}{\sigma^2} \log x + Cx, & \text{if } x > a_*, \\ c_2(x - a_*) + u(a_*), & \text{if } 0 \leq x \leq a_* \end{cases} \quad (6.6)$$

is twice continuously differentiable, nonnegative, and satisfies the following system

$$\begin{cases} \mathcal{L}u(x) + h(x) - \lambda = 0, & \text{if } x > a_*, \\ c_1 \leq u'(x) \leq c_2, & \text{if } x > a_*, \\ \mathcal{L}u(x) + h(x) - \lambda \leq 0, & \text{if } x \leq a_*, \\ u'(x) = c_2, & \text{if } x \leq a_*. \end{cases} \quad (6.7)$$

Here $a_* = \lambda^{1/p}$ and $C = c_2 - \frac{2p}{\sigma^2(1-p)}\lambda^{1-\frac{1}{p}}$.

Proof. Note that condition (6.5) implies that $C > c_2 - \frac{c_2 - c_1}{2} = \frac{c_2 + c_1}{2} > c_1$. The fact that $u \in C^1((0, \infty)) \cap C^2((0, \infty) \setminus \{a_*\})$ is a direct consequence of the definition of u in (6.6). Detailed calculations reveal that $\lim_{x \downarrow a_*} u''(x) = 0$ and hence $u \in C^2(0, \infty)$. In addition, the equalities in (6.7) can be verified by direct computations. We next show the inequalities in (6.7) hold as well.

For $x < a_*$, we have

$$\mathcal{L}u(x) + h(x) - \lambda = h(x) - \lambda \leq h(a_*) - \lambda = 0.$$

For $x > a_*$, we have

$$\begin{aligned} u'(x) &= \frac{2}{\sigma^2(1-p)}x^{p-1} - \frac{2\lambda}{\sigma^2}x^{-1} + C, \\ u''(x) &= -\frac{2}{\sigma^2}x^{p-2} + \frac{2\lambda}{\sigma^2}x^{-2} = \frac{2\lambda}{\sigma^2}x^{-2}(1 - x^p) \leq 0. \end{aligned}$$

These imply that $u'(x) \leq u'(a_*) = c_2$ for all $x > a_*$. Also we notice that $\lim_{x \rightarrow \infty} u'(x) = C > c_1$. Hence it follows that $c_1 < u'(x) \leq c_2$ for all $x > a_*$.

Also observe that $u(a_*) = \frac{2}{\sigma^2 p} \left(\frac{\lambda}{1-p} - \lambda \ln \lambda \right) + C \lambda^{\frac{1}{p}} = \frac{2}{\sigma^2(1-p)} \left(\frac{1}{p} - p \right) \lambda + \left(c_2 \lambda^{\frac{1}{p}-1} - \frac{2}{\sigma^2 p} \ln \lambda \right) \lambda > 0$ thanks to the second inequality of (6.5). Hence $u(x) > 0$ for all $x > 0$. The proof is complete. \square

Proposition 6.6. *Suppose $\mu = 0$ and $h(x) = x^p$ for some $p \in (0, 1)$, then $\lambda_0 = \infty$.*

Proof. For any $\lambda > 0$ satisfying (6.5), we construct a policy whose long-term average reward equals λ . Since λ is arbitrary, it follows that $\lambda_0 = \infty$. To this end, recall that $a_* = \lambda^{1/p}$. Let $b_* := \log a_*$ and consider the drifted Brownian motion reflected at b_* :

$$\psi(t) := b_* - \frac{1}{2} \sigma^2 t + \sigma W(t) + \phi(t) \geq b_*, \quad t \geq 0$$

where $\phi(t) = -\inf_{0 \leq s \leq t} \left\{ -\frac{1}{2} \sigma^2 s + \sigma W(s) \right\}$. One can verify immediately that ϕ is a nondecreasing process that increases only when $\psi(t) = b_*$; i.e.,

$$\phi(t) = \int_0^t \mathbf{1}_{\{b_*\}}(\psi(s)) d\phi(s). \quad (6.8)$$

Now let $X(t) := e^{\psi(t)}$. Note that $X(0) = a_*$. By Itô's formula, we have

$$dX(t) = \left(-\frac{\sigma^2}{2} + \frac{\sigma^2}{2} \right) X(t) dt + \sigma X(t) dW(t) + X(t) d\phi(t) = \sigma X(t) dW(t) + dL_{a_*}(t), \quad (6.9)$$

where $L_{a_*}(t) := \int_0^t X(s) d\phi(s)$. We make the following observations.

- (i) $X(t) \geq a_*$ for all $t \geq 0$. This is obvious since $\psi(t) \geq b_*$ for all $t \geq 0$.
- (ii) L_{a_*} is a nondecreasing process that increases only when $X(t) = a_*$. Indeed, using (6.8) and note that $X(t) = a_*$ if and only if $\psi(t) = b_*$, it follows that

$$\begin{aligned} L_{a_*}(t) &= \int_0^t X(s) d\phi(s) = \int_0^t X(s) \mathbf{1}_{\{b_*\}}(\psi(s)) d\phi(s) \\ &= \int_0^t \mathbf{1}_{\{a_*\}}(X(s)) X(s) d\phi(s) = \int_0^t \mathbf{1}_{\{a_*\}}(X(s)) dL_{a_*}(s). \end{aligned} \quad (6.10)$$

We now apply Itô's formula to the process $u(X(t))$, where X is given by (6.9) with $X(0) = a_*$. Since $X(t) \geq a_*$ for all $t \geq 0$ and u satisfies (6.7), we have

$$\begin{aligned} \mathbb{E}[u(X(t \wedge \beta_n))] &= u(a_*) + \mathbb{E} \left[\int_0^{t \wedge \beta_n} \mathcal{L}u(X(s)) ds + \int_0^{t \wedge \beta_n} u'(X(s)) dL_{a_*}(s) \right] \\ &= u(a_*) + \mathbb{E} \left[- \int_0^{t \wedge \beta_n} h(X(s)) ds + c_2 L_{a_*}(t) \right] + \lambda \mathbb{E}[t \wedge \beta_n], \end{aligned}$$

where $\beta_n := \inf\{t \geq 0 : X(t) \geq n\}$ and $n \in \mathbb{N} \cap (a_*, \infty)$. Since u is nonnegative, rearranging the terms yields

$$\mathbb{E} \left[\int_0^{t \wedge \beta_n} h(X(s)) ds - c_2 L_{a_*}(t) \right] \leq u(a_*) + \lambda \mathbb{E}[t \wedge \beta_n].$$

Now dividing both sides by t , and passing to the limit first as $n \rightarrow \infty$ and then as $t \rightarrow \infty$, we obtain

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[\int_0^t h(X(s)) ds - c_2 L_{a_*}(t) \right] \leq \lambda. \quad (6.11)$$

In view of Section 15.5 of [Karlin and Taylor \(1981\)](#), the process X of (6.9) has a unique stationary distribution $\pi(dx) = a_* x^{-2} dx$, $x \in (a_*, \infty)$. The strong law of large numbers (([Khasminskii, 2012](#), Theorem 4.2)) then implies that as $t \rightarrow \infty$

$$\frac{1}{t} \int_0^t h_n(X(s)) ds \rightarrow \int_{a_*}^{\infty} h_n(x) \pi(dx), \quad \text{a.s.}$$

where $h_n(x) := h(x) \wedge n$ and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the random variables $\{\frac{1}{t} \int_0^t h_n(X(s)) ds, t > 0\}$ are non-negative and bounded above by n . Thus we can apply the bounded convergence theorem to obtain

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h_n(X(s)) ds \right] = \int_{a_*}^{\infty} h_n(x) \pi(dx).$$

On the other hand, since $h \geq h_n$ and $h_n \uparrow h$, we have

$$\begin{aligned} \liminf_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h(X(s)) ds \right] &= \lim_{n \rightarrow \infty} \liminf_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h(X(s)) ds \right] \\ &\geq \lim_{n \rightarrow \infty} \liminf_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h_n(X(s)) ds \right] \\ &= \lim_{n \rightarrow \infty} \int_{a_*}^{\infty} h_n(x) \pi(dx) \\ &= \int_{a_*}^{\infty} h(x) \pi(dx) \\ &= \frac{(a_*)^p}{1-p} = \frac{\lambda}{1-p} \geq \lambda, \end{aligned} \quad (6.12)$$

where the second to the last line follows from the monotone convergence theorem and the equality in the last line follows from the fact that $a_* = \lambda^{1/p}$.

We next estimate $\mathbb{E}[L_{a_*}(t)]$. To this end, we observe from the third equality of (6.10) that

$$L_{a_*}(t) = \int_0^t \mathbf{1}_{\{a_*\}}(X(s)) X(s) d\phi(s) = a_* \phi(t),$$

and hence $\mathbb{E}[L_{a_*}(t)] = a_* \mathbb{E}[\phi(t)]$. On the other hand, since $\phi(t) = -\inf_{0 \leq s \leq t} \{-\frac{1}{2}\sigma^2 s + \sigma W(s)\} = \sup_{0 \leq s \leq t} \{\frac{1}{2}\sigma^2 s + \sigma(-W(s))\}$, it follows from (1.8.11) of [Harrison \(1985\)](#) that

$$\mathbb{P}\{\phi(t) \leq y\} = \Phi\left(\frac{y - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right) - e^y \Phi\left(\frac{-y - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}\right),$$

where Φ is the standard normal distribution function $\Phi(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$. Then we can compute

$$\begin{aligned}\mathbb{E}[\phi(t)] &= \int_0^\infty y \left(\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-t\sigma^2/2)^2}{2\sigma^2 t}} - e^y \Phi\left(\frac{-y-t\sigma^2/2}{\sigma\sqrt{t}}\right) + e^y \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y+t\sigma^2/2)^2}{2\sigma^2 t}} \right) dy \\ &= \sigma^2 t \Phi(\sigma\sqrt{t}/2) + \frac{\sqrt{2t}\sigma e^{-\frac{\sigma^2 t}{8}}}{\sqrt{\pi}} - \int_0^\infty y e^y \Phi\left(\frac{-y-t\sigma^2/2}{\sigma\sqrt{t}}\right) dy.\end{aligned}\quad (6.13)$$

To estimate the last integral, we use the tail estimate for the standard normal distribution function (see, for example, (Feller, 1968, Section 7.1)):

$$1 - \Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \geq [x^{-1} - x^{-3}] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x > 0,$$

to obtain

$$\Phi\left(\frac{-y-t\sigma^2/2}{\sigma\sqrt{t}}\right) = 1 - \Phi\left(\frac{y+t\sigma^2/2}{\sigma\sqrt{t}}\right) \geq \frac{\sigma\sqrt{t}}{y+t\sigma^2/2} \left[1 - \frac{\sigma^2 t}{(y+t\sigma^2/2)^2}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+t\sigma^2/2)^2}{2\sigma^2 t}}.$$

Moreover, we have

$$\lim_{y \rightarrow \infty} y \frac{\sigma\sqrt{t}}{y+t\sigma^2/2} \left[1 - \frac{\sigma^2 t}{(y+t\sigma^2/2)^2}\right] = \sigma\sqrt{t}.$$

Therefore, for any $\varepsilon > 0$, there exists an $M > 0$ so that

$$y \frac{\sigma\sqrt{t}}{y+t\sigma^2/2} \left[1 - \frac{\sigma^2 t}{(y+t\sigma^2/2)^2}\right] \geq \sigma\sqrt{t}(1 - \varepsilon) \quad \text{for all } y \geq M.$$

Then we can compute

$$\begin{aligned}\int_0^\infty y e^y \Phi\left(\frac{-y-t\sigma^2/2}{\sigma\sqrt{t}}\right) dy &\geq \int_M^\infty y e^y \Phi\left(\frac{-y-t\sigma^2/2}{\sigma\sqrt{t}}\right) dy \\ &\geq \int_M^\infty y e^y \frac{\sigma\sqrt{t}}{y+t\sigma^2/2} \left[1 - \frac{\sigma^2 t}{(y+t\sigma^2/2)^2}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+t\sigma^2/2)^2}{2\sigma^2 t}} dy \\ &\geq \int_0^\infty \sigma\sqrt{t}(1 - \varepsilon) \frac{1}{\sqrt{2\pi}} e^y e^{-\frac{(y+t\sigma^2/2)^2}{2\sigma^2 t}} dy \\ &\quad - \int_0^M \sigma\sqrt{t}(1 - \varepsilon) \frac{1}{\sqrt{2\pi}} e^y e^{-\frac{(y+t\sigma^2/2)^2}{2\sigma^2 t}} dy \\ &\geq \sigma\sqrt{t}(1 - \varepsilon) \sigma\sqrt{t} \Phi(\sigma\sqrt{t}/2) - \sigma\sqrt{t}(1 - \varepsilon) \frac{M e^M}{\sqrt{2\pi}}.\end{aligned}$$

Plugging this estimation into (6.13) gives

$$\begin{aligned}\mathbb{E}[\phi(t)] &\leq \sigma^2 t \Phi(\sigma\sqrt{t}/2) + \frac{\sqrt{2t}\sigma e^{-\frac{\sigma^2 t}{8}}}{\sqrt{\pi}} - \sigma\sqrt{t}(1 - \varepsilon) \sigma\sqrt{t} \Phi(\sigma\sqrt{t}/2) + \sigma\sqrt{t}(1 - \varepsilon) \frac{M e^M}{\sqrt{2\pi}} \\ &= \sigma^2 t \Phi(\sigma\sqrt{t}/2) \varepsilon + \frac{\sqrt{2t}\sigma e^{-\frac{\sigma^2 t}{8}}}{\sqrt{\pi}} + \sigma\sqrt{t}(1 - \varepsilon) \frac{M e^M}{\sqrt{2\pi}}.\end{aligned}$$

Hence, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[L_{a_*}(t)] = \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[a_* \phi(t)] \leq a_* \sigma^2 \varepsilon. \quad (6.14)$$

Now combining (6.12) and (6.14) yields

$$\liminf_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h(X(s)) ds - c_2 L_{a_*}(t) \right] \geq \lambda - c_2 a_* \sigma^2 \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{t} \int_0^t h(X(s)) ds - c_2 L_{a_*}(t) \right] \geq \lambda.$$

This, together with (6.11), implies that the policy L_{a_*} of (6.10) has a long-term average reward of λ . The proof is concluded. \square

7 A Direct Solution Approach

In the previous sections, we obtained the solution to the ergodic two-sided singular control problem (2.3) via the vanishing discount method. A critical condition for this approach is that we need to first solve the discounted problem (2.6). In this section, we provide a direct solution approach to the problem (2.3).

Let us briefly explain the strategy on how to derive λ_0 of (2.3). First we focus on a class of policies that keeps the process in the interval $[a, b]$. The proof of Corollary 3.5 shows that the long-term average reward for such a policy is equal to $\lambda(a, b)$ of (3.8). Next we impose conditions so that $\lambda(a, b)$ achieves its maximum value $\lambda_* = \lambda(a_*, b_*)$ at a pair $0 < a_* < b_* < \infty$. The maximizing pair (a_*, b_*) further allows us to derive a C^2 solution to the HJB equation (3.1). This, together with the verification theorem (Theorem 3.1), reveals that $\lambda_0 = \lambda_*$ and the (a_*, b_*) -reflection policy is an optimal policy.

This approach is motivated by the recent paper Alvarez (2018), which solves an ergodic two-sided singular control problem for general one-dimensional diffusion processes. Note that the setups in Alvarez (2018) is different from ours. In particular, the cost rates associated with the singular controls are all positive in Alvarez (2018). By contrast, our formulation in (2.3) has mixed signs for the singular controls η and ξ . We also note that Theorem 2.5 in Alvarez (2018) only proves that the (a_*, b_*) -reflection policy is optimal *in the class of two-point reflection policies*. Here we observe that the (a_*, b_*) -reflection policy is optimal among all admissible controls by the verification theorem.

Recall from the proof of Corollary 3.5 that the long-term average reward for the (a, b) -reflection policy is equal to $\lambda(a, b)$ of (3.8). Now we wish to maximize the long-term average reward $\lambda(a, b)$. First, detailed computations reveal that

$$\begin{aligned} \frac{\partial}{\partial a} \lambda(a, b) &= \frac{m(a)}{M[a, b]} [\lambda(a, b) - \pi_2(a)], \\ \frac{\partial}{\partial b} \lambda(a, b) &= \frac{m(b)}{M[a, b]} [\pi_1(b) - \lambda(a, b)]. \end{aligned}$$

where

$$\pi_1(x) = h(x) + c_1\mu(x), \quad \text{and} \quad \pi_2(x) = h(x) + c_2\mu(x). \quad (7.1)$$

Therefore the first order optimality condition says that a maximizing pair $a_* < b_*$ must satisfy

$$\lambda(a_*, b_*) = \pi_1(b_*) = \pi_2(a_*). \quad (7.2)$$

Furthermore, (7.2) is equivalent to

$$\int_{a_*}^{b_*} (\pi_2(x) - \pi_2(a_*))m(x)dx + \frac{c_1 - c_2}{2s(b_*)} = 0, \quad (7.3)$$

$$\int_{a_*}^{b_*} (\pi_1(x) - \pi_1(b_*))m(x)dx + \frac{c_1 - c_2}{2s(a_*)} = 0. \quad (7.4)$$

To see this, we note that using the definition of $\lambda(a, b)$ in (3.8), we can write

$$\begin{aligned} \frac{\partial}{\partial b}\lambda(a, b) &= \frac{m(b)}{M[a, b]}[\pi_1(b) - \lambda(a, b)] \\ &= \frac{m(b)}{M^2[a, b]} \left(\int_a^b \pi_1(b)m(x)dx - \int_a^b h(x)m(x)dx - \frac{c_1}{2s(b)} + \frac{c_2}{2s(a)} \right) \\ &= \frac{m(b)}{M^2[a, b]} \left(\int_a^b (\pi_1(b) - \pi_1(x))m(x)dx + \int_a^b c_1\mu(x)m(x)dx - \frac{c_1}{2s(b)} + \frac{c_2}{2s(a)} \right) \\ &= \frac{m(b)}{M^2[a, b]} \left(\int_a^b (\pi_1(b) - \pi_1(x))m(x)dx + \frac{c_2 - c_1}{2s(a)} \right). \end{aligned}$$

This gives the equivalence between $\lambda(a_*, b_*) = \pi_1(b_*)$ and (7.4). The equivalence between $\lambda(a_*, b_*) = \pi_2(a_*)$ and (7.3) can be established in a similar fashion.

To show that the first order condition (7.2) or equivalently the system of equations (7.3)–(7.4) has a solution, we impose the following conditions:

Assumption 7.1. (i) The functions h and μ are continuously differentiable with $h'(x) > 0$ for all $x > 0$. In addition, $h(0) = \mu(0) = 0$.

(ii) For $i = 1, 2$, there exists an $\hat{x}_i > 0$ such that $\pi_i(x)$ is strictly increasing on $(0, \hat{x}_i)$ and strictly decreasing on $[\hat{x}_i, \infty)$. In addition $\lim_{x \rightarrow \infty} \pi_1(x) < 0$. Consequently, there exists a $b_0 > \hat{x}_1$ such that $\pi_1(b_0) = 0$.

(iii) It holds true that

$$\liminf_{a \downarrow 0} \left[\int_0^{b_0} [\pi_1(y) - \pi_1(b_0)]m(y)dy + \frac{c_1 - c_2}{2s(a)} \right] > 0. \quad (7.5)$$

Remark 7.2. We remark that Assumption 7.1 (i) is standard in the literature of singular controls; see, for example, Alvarez (2018), Alvarez and Hening (2019) and Weerasinghe (2007). Condition (ii) is motivated by similar conditions in Alvarez (2018) and Weerasinghe (2007). In addition, these conditions are satisfied under the setup in Guo and Pham (2005) as well. Condition (iii) is a technical one and it guarantees the existence of an optimizing pair $0 < a_* < b_* < \infty$ satisfying the system of equations (7.3)–(7.4); see the proof of Proposition 7.4.

Remark 7.3. We also note that the extreme points \hat{x}_2, \hat{x}_1 in Assumption 7.1 must satisfy $\hat{x}_2 < \hat{x}_1$. Suppose on the contrary that $\hat{x}_1 \leq \hat{x}_2$. Then since π_1 achieves its maximum at \hat{x}_1 and π_2 is strictly increasing on $(0, \hat{x}_2)$, we have

$$\pi_1'(\hat{x}_1) = h'(\hat{x}_1) + c_1\mu'(\hat{x}_1) = 0, \quad \text{and} \quad \pi_2'(\hat{x}_1) = h'(\hat{x}_1) + c_2\mu'(\hat{x}_1) \geq 0. \quad (7.6)$$

On the other hand, Assumption 6.2 (i) says that $h'(\hat{x}_1) > 0$. This, together with fact that $0 < c_1 < c_2 < \infty$, implies that

$$\pi_2'(\hat{x}_1) = h'(\hat{x}_1) + c_2\mu'(\hat{x}_1) = h'(\hat{x}_1) - c_2 \frac{h'(\hat{x}_1)}{c_1} = -\frac{c_2 - c_1}{c_1} h'(\hat{x}_1) < 0;$$

resulting in a contradiction to (7.6).

Proposition 7.4. *Let Assumption 7.1 hold. Then there exists a unique pair $0 < a_* < b_* < \infty$ satisfying the system of equations (7.3)–(7.4).*

Proof. Since $\hat{x}_2 < \hat{x}_1$ thanks to Remark 7.3, we only need to consider two cases: $\pi_2(\hat{x}_2) \geq \pi_1(\hat{x}_1)$ and $\pi_2(\hat{x}_2) < \pi_1(\hat{x}_1)$.

Case (i): $\pi_2(\hat{x}_2) \geq \pi_1(\hat{x}_1)$. In this case, thanks to Assumption 7.1, there exists a $y_1 \in (0, \hat{x}_2]$ such that $\pi_2(y_1) = \pi_1(\hat{x}_1)$. In view of Remark 7.3, we have $y_1 \leq \hat{x}_2 < \hat{x}_1$. In addition, Assumption 7.1 also implies that for any $a \in [0, y_1]$, there exists a unique $b_a \in [\hat{x}_1, b_0)$ such that $\pi_1(b_a) = \pi_2(a)$. Consequently we can consider the function

$$\ell(a) := \int_a^{b_a} h(x)m(x)dx - \pi_1(b_a)M[a, b_a] + \frac{c_1}{2s(b_a)} - \frac{c_2}{2s(a)}, \quad a \in [0, y_1]. \quad (7.7)$$

Using (3.7) and the fact that $\pi_1(b_a) = \pi_2(a)$, we can rewrite $\ell(a)$ as

$$\ell(a) = \int_a^{b_a} [\pi_2(x) - \pi_2(a)]m(x)dx + \frac{c_1 - c_2}{2s(b_a)} = \int_a^{b_a} [\pi_1(x) - \pi_1(b_a)]m(x)dx + \frac{c_1 - c_2}{2s(a)}.$$

Since $b_{y_1} = \hat{x}_1$, we have

$$\ell(y_1) = \int_{y_1}^{\hat{x}_1} [\pi_1(x) - \pi_1(\hat{x}_1)]m(x)dx + \frac{c_1 - c_2}{2s(y_1)}.$$

Recall from Assumption 7.1 that π_1 is strictly increasing on $(0, \hat{x}_1)$ and $c_1 < c_2$. Hence it follows that $\ell(y_1) < 0$.

Next we show that ℓ is decreasing on $[0, y_1]$ and hence $\ell(0+) := \lim_{a \downarrow 0} \ell(a)$ exists. To this end, we consider $0 \leq a_1 < a_2 \leq y_1$ and denote $b_i = b_{a_i}$ for $i = 1, 2$. Since $\pi_1(b_1) = \pi_2(a_1) < \pi_2(a_2) = \pi_1(b_2)$ and π_1 is decreasing on $[\hat{x}_1, \infty)$, it follows that $\hat{x}_1 \leq b_2 < b_1$. As a result, we can compute

$$\begin{aligned} \ell(a_1) - \ell(a_2) &= \int_{a_1}^{a_2} h(x)m(x)dx + \int_{b_2}^{b_1} h(x)m(x)dx - \pi_1(b_1)M[a_1, b_1] + \pi_1(b_2)M[a_2, b_2] \\ &\quad + \frac{c_1}{2s(b_1)} - \frac{c_2}{2s(a_1)} - \frac{c_1}{2s(b_2)} + \frac{c_2}{2s(a_2)} \end{aligned}$$

$$\begin{aligned}
&= \int_{a_1}^{a_2} [\pi_2(x) - \pi_2(a_1)]m(x)dx + \int_{b_2}^{b_1} [\pi_1(x) - \pi_1(b_1)]m(x)dx \\
&\quad + [\pi_1(b_2) - \pi_1(b_1)]M[a_2, b_2].
\end{aligned}$$

Thanks to Assumption 7.1, the terms $\pi_2(x) - \pi_2(a_1)$, $\pi_1(x) - \pi_1(b_1)$, and $\pi_1(b_2) - \pi_1(b_1)$ are all positive. Therefore $\ell(a_1) - \ell(a_2) > 0$ as desired.

Using Assumption 7.1 (iii), we have $\ell(0+) > 0$. Since the function ℓ is also continuous, there exists a unique $a_* \in (0, y_1]$ such that $\ell(a_*) = 0$. Denote $b_* = b_{a_*} \in [\hat{x}_1, b_0]$. Then

$$\begin{aligned}
0 = \ell(a_*) &= \int_{a_*}^{b_*} h(x)m(x)dx - \pi_1(b_*)M[a_*, b_*] + \frac{c_1}{2s(b_*)} - \frac{c_2}{2s(a_*)} \\
&= \int_{a_*}^{b_*} [\pi_2(x) - \pi_2(a_*)]m(x)dx - \frac{c_2}{2s(b_*)} + \frac{c_2}{2s(a_*)} + \frac{c_1}{2s(b_*)} - \frac{c_2}{2s(a_*)} \\
&= \int_{a_*}^{b_*} [\pi_2(x) - \pi_2(a_*)]m(x)dx + \frac{c_1 - c_2}{2s(b_*)}.
\end{aligned}$$

This gives (7.3). Similar calculations reveal that

$$0 = \ell(a_*) = \int_{a_*}^{b_*} [\pi_1(x) - \pi_1(b_*)]m(x)dx + \frac{c_1 - c_2}{2s(a_*)},$$

establishing (7.4).

Case (ii): $\pi_2(\hat{x}_2) < \pi_1(\hat{x}_1)$. In this case, for any $a \in [0, \hat{x}_2]$, there exists a $b_a \in [\hat{x}_1, \infty)$ such that $\pi_2(a) = \pi_1(b_a)$. Consequently we can define the function $\ell(a)$ as in (7.7) for all $a \in [0, \hat{x}_2]$. Let $y_2 \in [\hat{x}_1, \infty)$ be such that $\pi_2(\hat{x}_2) = \pi_1(y_2)$. Then we have $b_{\hat{x}_2} = y_2$ and

$$\ell(\hat{x}_2) = \int_{\hat{x}_2}^{y_2} [\pi_2(x) - \pi_2(\hat{x}_2)]m(x)dx + \frac{c_1 - c_2}{2s(y_2)} < 0.$$

The rest of the proof is very similar to that of Case (i). We shall omit the details for brevity. \square

Proposition 7.5. *Let Assumption 7.1 hold. Then there exist a function $u \in C^2([0, \infty))$ and a positive number λ_* satisfying the HJB equation (3.1).*

Proof. Let $0 < a_* < b_* < \infty$ be as in Proposition 7.4 and define

$$\lambda_* := \frac{1}{M[a_*, b_*]} \left[\int_{a_*}^{b_*} h(x)m(x)dx + \frac{c_1}{2s(b_*)} - \frac{c_2}{2s(a_*)} \right]. \quad (7.8)$$

In addition, we consider the function

$$u(x) = \begin{cases} c_1x + \int_{a_*}^x 2s(u) \int_u^{b_*} [\pi_1(y) - \lambda_*]m(y)dydu, & x \in [a_*, b_*], \\ c_1(x - b_*) + u(b_*), & x > b_*, \\ c_2(x - a_*) + u(a_*), & x < a_*. \end{cases} \quad (7.9)$$

We next verify that the pair (u, λ_*) satisfies (3.1). First it is obvious that u is continuously differentiable and satisfies $u'(x) = c_1$ for $x \geq b_*$ and $u'(x) = c_2$ for $x \leq a_*$.

Next we show that u is C^2 and satisfies $\mathcal{L}u(x) + h(x) - \lambda_* = 0$ for $x \in (a_*, b_*)$. To this end, we compute for $x \in (a_*, b_*)$:

$$\begin{aligned} u'(x) &= c_1 + 2s(x) \int_x^{b_*} [\pi_1(y) - \lambda_*] m(y) dy, \\ u''(x) &= -4s(x) \frac{\mu(x)}{\sigma^2(x)} \int_x^{b_*} [\pi_1(y) - \lambda_*] m(y) dy + 2s(x) [-\pi_1(x) + \lambda_*] m(x). \end{aligned}$$

Consequently it follows that u satisfies $\mathcal{L}u(x) + h(x) - \lambda_* = 0$ for $x \in (a_*, b_*)$.

To show that u is C^2 , it suffices to show that u is C^2 at the points a_* and b_* . Recall that $a_* < b_*$ and λ_* satisfy (7.2) thanks to Proposition 7.4. Therefore it follows immediately that $u'(b_*-) = c_1$ and $u''(b_*-) = 0$. In addition, we can use (7.4) to compute

$$u'(a_*+) = c_1 + 2s(a_*) \int_{a_*}^{b_*} [\pi_1(y) - \lambda_*] m(y) dy = c_1 + 2s(a_*) \frac{c_2 - c_1}{2s(a_*)} = c_2,$$

and

$$\begin{aligned} u''(a_*+) &= -4s(a_*) \frac{\mu(a_*)}{\sigma^2(a_*)} \int_{a_*}^{b_*} [\pi_1(y) - \lambda_*] m(y) dy + 2s(a_*) [-\pi_1(a_*) + \lambda_*] m(a_*) \\ &= -4s(a_*) \frac{\mu(a_*)}{\sigma^2(a_*)} \cdot \frac{c_2 - c_1}{2s(a_*)} + \frac{2}{\sigma^2(a_*)} [-\pi_1(a_*) + \lambda_*] \\ &= -\frac{2}{\sigma^2(a_*)} [(c_2 - c_1)\mu(a_*) + \pi_1(a_*) - \lambda_*] \\ &= 0, \end{aligned}$$

where the last equality follows from (7.2). Therefore we have shown that $u \in C^2((0, \infty))$.

Recall that the function π_1 is decreasing on $[\hat{x}_1, \infty)$ and $b_* \in [\hat{x}_1, \infty)$ by the proof of Proposition 7.4. Therefore for any $x \geq b_*$,

$$\mathcal{L}u(x) + h(x) - \lambda_* = c_1\mu(x) + h(x) - \lambda_* = \pi_1(x) - \lambda_* \leq \pi_1(b_*) - \lambda_* = 0.$$

Similar argument reveals that $\mathcal{L}u(x) + h(x) - \lambda_* \leq 0$ for all $x \in a_*$.

It remains to show that $u'(x) \in [c_1, c_2]$ for $x \in (a_*, b_*)$. To this end, we first consider the function

$$k(x) := \int_x^{b_*} [\pi_1(y) - \lambda_*] m(y) dy, \quad x \in [a_*, b_*].$$

We claim that k is nonnegative, which, in turn, implies that $v'(x) = c_1 + 2s(x)k(x) \geq c_1$ on $[a_*, b_*]$. To see the claim, we consider two cases. (i) If $\pi_1(y) \geq \lambda_*$ for all $y \in [a_*, b_*]$, then $k(x) \geq 0$ for all $x \in [a_*, b_*]$. (ii) Otherwise, since $a_* \leq \hat{x}_2 < \hat{x}_1$, $b_* > \hat{x}_1$, the monotonicity of π_1 implies that there exists a $y_1 \in (a_*, b_*)$ so that $\pi_1(y) - \lambda_*$ is negative on $[a_*, y_1)$ and positive on (y_1, b_*) . Consequently the function k is increasing on $[a_*, y_1)$ and then decreasing

on (y_1, b_*) . On the other hand, we have $k(b_*) = 0$, and thanks to (7.4), $k(a_*) = \frac{c_2 - c_1}{2s(a_*)} > 0$. Therefore we again have the claim that $k(x) \geq 0$ for all $x \in [a_*, b_*]$.

Finally, to show that $v'(x) \leq c_2$ on $[a_*, b_*]$, we consider the function

$$\rho(x) := k(x) + \frac{c_1 - c_2}{2s(x)}, \quad x \in [a_*, b_*].$$

Note that $\rho(b_*) = \frac{c_1 - c_2}{2s(b_*)} < 0$. In addition, $\rho(a_*) = \ell(a_*) = 0$ thanks to the proof of Proposition 7.4. Next we compute

$$\rho'(x) = -(\pi_1(x) - \lambda_*)m(x) + \frac{c_1 - c_2}{2} \cdot \frac{2\mu(x)}{\sigma^2(x)s(x)} = (\lambda_* - \pi_2(x))m(x).$$

As a result, if $\lambda_* - \pi_2(x) < 0$ for all $x \in [a_*, b_*]$, then $\rho(x) \leq \rho(a_*) = 0$ for all $x \in [a_*, b_*]$. Otherwise, using the monotonicity assumption of π_2 in Assumption 7.1, there exists some $y_2 \in (a_*, b_*)$ so that $\lambda_* - \pi_2(x) < 0$ for $x \in [a_*, y_2)$ and > 0 for $x \in (y_2, b_*]$. This, in turn, implies that $\rho(x)$ is decreasing on $[a_*, y_2)$ and increasing on $(y_2, b_*]$. In such a case, we still have $\rho(x) \leq 0$ for all $x \in [a_*, b_*]$. Then it follows that $2s(x)k(x) \leq c_2 - c_1$ and hence $v'(x) = c_1 + 2s(x)k(x) \leq c_2$.

To summarize, we have shown that the function u of (7.9) is C^2 and satisfies

$$\begin{cases} \mathcal{L}u(x) + h(x) - \lambda_* = 0, & c_1 \leq u'(x) \leq c_2, & x \in (a_*, b_*), \\ \mathcal{L}u(x) + h(x) - \lambda_* \leq 0, & u'(x) = c_2, & x \leq a_*, \\ \mathcal{L}u(x) + h(x) - \lambda_* \leq 0, & u'(x) = c_1, & x \geq b_*. \end{cases}$$

In particular, (u, λ_*) satisfies the HJB equation 3.1. This concludes the proof. \square

Now we present the main result of this section.

Theorem 7.6. *Let Assumption 7.1 hold. Then there exist $0 < a_* < b_* < \infty$ so that the (a_*, b_*) -reflection policy is optimal for problem 2.3. In addition, $\lambda_0 = \lambda_*$, where λ_* is defined in (7.8).*

Proof. Since (u, λ_*) is a solution to the HJB equation, where the function u of (7.9) is bounded from below, it follows from Theorem 3.1 that $\lambda_0 \leq \lambda_*$. Furthermore, the proof of Corollary 3.5 says that the long-term average reward of the (a_*, b_*) -policy is equal to λ_* and therefore optimal. The proof is complete. \square

Example 7.7. In this example, we consider the ergodic two-sided singular control problem for the Verhulst-Pearl diffusion:

$$dX(t) = \mu X(t)(1 - \gamma X(t))dt + \sigma X(t)dW(t) + d\xi(t) - d\eta(t),$$

where $\mu > 0$ is the per-capita growth rate, $\frac{1}{\gamma} > 0$ is the carrying capacity, and $\sigma^2 > 0$ is the variance parameter measuring the fluctuations in the per-capita growth rate. The singular control $\varphi := \xi - \eta$ is as in Section 2. Here, we can regard $\xi(t)$ and $\eta(t)$ as the cumulative

renewing and harvesting amount up to time t , respectively. The scale and speed densities are

$$s(x) = x^{-\alpha} e^{\gamma\alpha(x-1)}, \quad m(x) = \frac{1}{\sigma^2} x^{\alpha-2} e^{-\gamma\alpha(x-1)}, \quad x > 0.$$

where $\alpha = \frac{2\mu}{\sigma^2}$. Detailed computations using the criteria given in Chapter 15 of Karlin and Taylor (1981) reveal that both 0 and ∞ are natural boundary points.

For positive constants $c_1 < c_2$ and a nonnegative function h satisfying Assumption 6.2 (i) and (ii), we consider the problem

$$\lambda_0 := \sup_{\varphi(\cdot) \in \mathcal{A}_x} \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T h(X(t)) dt + c_1 \eta(T) - c_2 \xi(T) \right], \quad (7.10)$$

where \mathcal{A}_x is the set of admissible controls as defined in (2.4).

We now claim that Assumption 7.1 is satisfied and hence in view of Theorem 7.6, the optimal value λ_0 of (7.10) is achieved by the (a_*, b_*) -reflection policy, where $0 < a_* < b_* < \infty$. Assumption 7.1 (i) is obviously satisfied. We now verify Assumption 7.1 (ii). For $i = 1, 2$, we have $\pi_i(x) = h(x) + c_i \mu x(1 - \gamma x)$ and hence

$$\pi'_i(x) = h'(x) + c_i \mu - 2c_i \mu \gamma x, \quad x \geq 0.$$

Noting that $h'(x) > 0$ and $c_i, \mu > 0$, we see from the above equation that $\pi'_i(x) > 0$ for $x > 0$ in a neighborhood of 0. Since h satisfies Assumption 6.2 (i) and (ii), it follows that there exists some positive number M so that $h'(x) < 1$ for all $x \geq M$. Thus $\lim_{x \rightarrow \infty} \pi'_i(x) = -\infty$. Then the continuity of $\pi'_i(x)$ implies that there exists a $\hat{x}_i > 0$ so that $\pi'_i(\hat{x}_i) = 0$. Furthermore, since h is strictly concave, $h'(x)$ is decreasing. Therefore $\pi'_i(x) < 0$ for all $x > \hat{x}_i$. Hence π_i is first increasing on $[0, \hat{x}_i)$ and then decreasing on $[\hat{x}_i, \infty)$. On the other hand, using Assumption 6.2 (ii), we have $\lim_{x \rightarrow \infty} \pi_i(x) = -\infty$ and hence there exists a $b_0 > \hat{x}_1$ so that $\pi_1(b_0) = 0$. Assumption 7.1 (ii) is verified.

Finally, note that the arguments in the previous paragraph also reveal that π_1 is strictly positive on $(0, b_0)$. This, together with facts that $\pi_1(b_0) = 0$ and $\lim_{a \downarrow 0} \frac{1}{s(a)} = 0$, leads to (7.5) and hence Assumption 7.1 (iii) is in force. The proof is complete.

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