Worst-Case Services and State-Based Scheduling

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Abstract—In this paper, we shed new light on a classical problem: given a slot-timed, constant-capacity server, to provide long-run service guarantees to competing flows of unit-sized tasks, what short-run scheduling decisions must be made? We model each flow's long-run guarantee as a worst-case service that maps each queued arrival vector recording the flow's cumulative task arrivals, including those initially queued, to a worst-case acceptable departure vector lower-bounding its cumulative task departures. We show these services to be states that can be updated as tasks arrive and depart, and introduce state-based scheduling. We find the schedulability condition that must be preserved to maintain all flows' long-run guarantees, and then use this condition to identify, in each slot, all short-run scheduling decisions that preserve schedulability.

This framework is general but computationally complex. To reduce its complexity, we consider three specializations. On the one hand, we show that when satisfactory short-run scheduling decisions exist, some special ones can be efficiently identified by maximizing the server's capacity slack. On the other hand, we show that a special class of worst-case services, min-plus services, can be efficiently specified and updated using properties of the min-plus algebra, and that this efficiency can be further improved to verge on practical viability by restricting attention to a further specialization, dual-curve services, which turn out to be dynamic extensions of service curves.

Index Terms—Service guarantees, scheduling, cumulative vectors, state-space approach, polymatroid theory, EDF schedules, min-plus algebra, service curves.

I. INTRODUCTION

A constant-capacity server, shared by multiple task flows, is a common resource allocation model. Within this model, given the server's limited capacity, a classical scheduling problem can be formulated, namely that of deciding which tasks to serve and which to defer to meet competing flow service requests. Generally, service is a long-run concept in that service guarantees cover, if not entireties, significant portions of flow lifetimes. In contrast, scheduling is a short-run concept in that scheduling decisions must be made slot-by-slot as tasks arrive and depart. The challenge is to determine how to maintain these long-run service guarantees using short-run scheduling decisions. In this paper, we shed new light on this problem by providing novel answers to two key questions: What to guarantee? and How to guarantee it?

A. A General Framework

What to guarantee? This is foremost a question of service specification because we can only guarantee what we can specify. In [1], [2], [3], [4], cumulative curves were introduced

to characterize long-run flow traffic. They become cumulative vectors in slot-timed systems. In particular, arrival vectors can be used to record a flow's cumulative task arrivals, and departure vectors, its cumulative task departures. We extend the definition of arrival vectors to include initially queued tasks and define a worst-case service to be a map from each such queued arrival vector to a worst-case acceptable departure vector. While this definition encompasses a wide spectrum of guarantees, and leaves open endless intricacies, it is precisely this generality that underlies our framework's generality.

How to guarantee it? This is a question of methodology and our answer is state-based scheduling. Our motivating insight is that, fundamentally, the worst-case service guaranteed to each flow is a state that can be updated as tasks arrive and depart. The key to scheduling is then finding the schedulability condition on the flows' aggregate state necessary and sufficient to ensure that all flows' long-run guarantees can be met. Once this condition is found, it can be used, on the one hand, to admit or deny new service requests, and on the other, to identify all feasible schedules, that is, all short-run scheduling decisions that can be made without endangering any long-run service guarantee, because all such decisions must preserve schedulability. In this way, state-based scheduling allows us to systematically identify all scheduling policies that can simultaneously guarantee all flows their respective services. This is quite a contrast to the traditional approach according to which scheduling policies are first proposed and only thereafter are their capabilities for guaranteeing services examined, verified and refined.

To find the schedulability condition, we introduce the concept of the spectrum of a worst-case service. During any period, the least capacity that must be reserved to guarantee a worst-case service is specified by a spectral value. If all guarantees are to be maintained, the total capacity to be reserved, that is, the sum of the spectral values of all worstcase services, cannot exceed the server's capacity during the given period. This turns out to be the schedulability condition that we seek, and we use it to identify all feasible schedules. The principal constraint imposed on the feasible schedules so identified is determined by a baseline function that specifies the least number of tasks that must be served from any given subset of flows if, during every period, the sum of all spectral values is to remain below the available capacity. The baseline function is supermodular, so the polymatroid theory can be applied. In particular, we show that when the total service is fixed, the feasible schedules form a permutohedron, a special polytope from polymatroid theory.

B. Three Specializations

A downside of our framework's generality is its complexity. This complexity is two-fold. On the one hand, to fully exploit

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the flexibility of selecting any feasible schedule, we must fully determine a feasible permutohedron, which is combinatorially difficult. On the other hand, worst-case services, as full-blown maps between cumulative vectors, are challenging to specify and update. To address these difficulties, but also for their own merits, we consider three specializations: max-slack schedules, min-plus services and dual-curve services.

Max-slack schedules maximize the server's capacity slack, that is, they leave maximum room for it to admit new service requests. According to the schedulability condition, this is achieved by simultaneously minimizing all sums of all spectral values during all periods. This, with a little reflection, suggests that the max-slack schedule is feasible if a feasible schedule ever exists, and that its identification is independent of the feasible permutohedron that contains it. Both conjectures will be confirmed. Aggregating flows into classes so that, intraclass, flows are max-slack scheduled, enables intermediate tradeoffs of flexibility and efficiency, because when the total service is fixed, the feasible class schedules form vet another permutohedron, but with a lower dimension. When all flows' worst-case services allow static deadlines to be assigned to all tasks as they arrive, the max-slack schedule becomes the well-known earliest-deadline-first (EDF) schedule.

Min-plus services are of interest because, using properties of the min-plus algebra, they can be completely identified by their spectra. To specify and update these services, instead of an uncountably infinite full-blown map between cumulative vectors, we need only specify and update a countably infinite spectral matrix. Among all worst-case services that share the same spectrum, there turns out to be one min-plus service that is maximum in that it maps each queued arrival vector to the maximum worst-case departure vector possible. Therefore, every schedulable set of non-min-plus services is dominated by a schedulable set of min-plus services. By upgrading the former to the latter, we can then improve the services, simplify their specification and update, and yet, preserve schedulability.

The efficiency of min-plus services can be further improved to verge on practical viability by restricting attention to a further specialization, dual-curve services. To specify and update these services, instead of a spectral matrix, we need only specify and update a pair of cumulative vectors, one static and one dynamic. The reason that we call them dual-curve, instead of dual-vector, services is to highlight their connection to service curves. First suggested in [3], [4], service curves were fully developed in [5]. Each, according to [5], can be specified by a static cumulative vector. Adding a dynamic vector, yields its dynamic extension, a dual-curve service. This extension allows dual-curve services to be updated as tasks arrive and depart, which is essential for state-based scheduling.

The rest of the paper is organized as follows. In Section II, we introduce our service model. In Section III, we define worst-case services and their spectra. In Sections IV and V, we introduce state-based scheduling, find the schedulability condition, and then use this condition to identify all feasible schedules. In Sections VI, VII, and VIII, we introduce, respectively, max-slack schedules, min-plus services and dual-curve services. In Section IX, we conclude with remarks on possible extensions. The paper also includes an appendix in which two



Fig. 1. The service model, composed of a buffer system, a scheduler and a constant-capacity server.

lemmas from polymatroid theory are proved for completeness.

II. THE SERVICE MODEL

Our service model is discrete in that time is slotted and all tasks are of unit size. As illustrated in Figure 1, the tasks arrive from n distinct flows indexed by $\Omega = \{1, 2, ..., n\}$. Each flow's tasks are buffered separately, either physically or virtually. They, however, all share a single server with a constant capacity of serving c tasks per slot, access to which is controlled by a scheduler.

For each flow, indexed by $\omega \in \Omega$, at the beginning of slot t, a^{ω} tasks arrive and are immediately queued behind the b^{ω} tasks left unserved after slot t - 1. At this point, there are

$$q^{\omega} := a^{\omega} + b^{\omega} \tag{1}$$

tasks queued in flow ω 's buffer. During slot t, the scheduler determines d^{ω} , the number of these tasks to serve. As tasks cannot be served before they arrive,

$$d^{\omega} \le q^{\omega}. \tag{2}$$

Within each flow, the service order is first-come-first-serve so that, at the end of slot t, the first d^{ω} tasks queued in flow ω 's buffer are served and depart, which leaves

$$\dot{b}^{\omega} = q^{\omega} - d^{\omega} \tag{3}$$

tasks unserved after slot t. Then b^{ω} is a state variable that can be updated using (3). As in the above equations, unless noted otherwise, all variables are implicitly indexed by current slot t. To index t + 1, we add a dot, as in \dot{b}^{ω} . To index t + 2, we add two dots, and so on.

The scheduler's choices are constrained. Denoting the ensemble of flow variables, $[x^1, x^2, \ldots, x^n]$, by $x^{[\Omega]}$, and the sum, $\sum_{\omega \in \Omega} x^{\omega}$, by $x^{\langle \Omega \rangle}$, it follows that the selected schedule, $d^{[\Omega]}$, must satisfy the *causality constraint*,

$$d^{[\Omega]} \le q^{[\Omega]},\tag{4}$$

which rewrites (2) in the ensemble form, and the *capacity constraint*,

$$d^{\langle \Omega \rangle} \le c,\tag{5}$$

which says that the total number of tasks served cannot exceed the server's capacity. We call $d^{[\Omega]}$ that satisfies both (4) and (5) a *valid schedule*. The scheduler may only select among valid schedules.

III. WORST-CASE SERVICES

In this section, we use cumulative vectors to define worstcase services and show these services to be states that can be updated as tasks arrive and depart. We also define the spectrum of a worst-case service to specify the capacities to be reserved for it.

A. Definition

A *cumulative vector* is a semi-infinite, non-decreasing vector that starts with 0. Let \mathbb{U} be the set of cumulative vectors and \mathbb{N} be the set of natural numbers. Then $\boldsymbol{x} = [x_j]_{j \in \mathbb{N}} = [x_0, x_1, x_2, \ldots] \in \mathbb{U}$ if $x_0 = 0$, and for all $j \in \mathbb{N}$, $x_j \leq x_{j+1}$. For later uses, we also need two subsets of \mathbb{U} ,

$$\mathbb{U}|x := \{ \boldsymbol{x} \in \mathbb{U} | x_1 = x \},\tag{6}$$

and

$$\mathbb{U}\langle x := \{ \boldsymbol{x} \in \mathbb{U} | \boldsymbol{x} \ge x \boldsymbol{\delta} \}, \tag{7}$$

where $\boldsymbol{\delta} = [\delta_j]_{j \in \mathbb{N}} := [0, 1, 1, \ldots] \in \mathbb{U}$, i.e.,

$$\delta_j := \begin{cases} 0 & \text{if } j = 0\\ 1 & \text{if } j > 0 \end{cases} .$$
(8)

For each flow, we use the *arrival vector*, $a = [a_j]_{j \in \mathbb{N}} \in \mathbb{U}$, and the *departure vector*, $d = [d_j]_{j \in \mathbb{N}} \in \mathbb{U}$, to record its cumulative task arrivals and departures respectively. For all j > 0, a_j and d_j count the tasks that, respectively, arrive and depart during period [t, t+j), that is, from slot t to t+j-1. Here in referencing a generic flow, we suppress its flow index. Notice that a and d completely characterize flow traffic in that the arrival and departure slot of each task can be accurately identified by them. To be precise, for all h > 0, let

$$\tau_h(\boldsymbol{x}) := \max\{j \in \mathbb{N} | x_j < h\}.$$
(9)

Then the *h*th task of *a* arrives in slot $t + \tau_h(a)$, while that of *d* departs in slot $t + \tau_h(d)$.

Although it is up to the scheduler to determine the specific relation between a and d, as tasks cannot be served before they arrive, the number of departures during any given period cannot exceed that of arrivals during the same period plus that of remaining unserved tasks. That is to say, for all $j \in \mathbb{N}$,

$$d_j \le q_j := \begin{cases} 0 & \text{if } j = 0\\ a_j + b & \text{if } j > 0 \end{cases},$$
 (10)

or in the vector form,

$$\boldsymbol{d} \le \boldsymbol{q} := \boldsymbol{a} + b\boldsymbol{\delta},\tag{11}$$

which is the vector extension of (2) and (1). By definition, $q \in \mathbb{U} \rangle b$. As *b* is entirely fixed by the flow's past arrivals and departures, *q* can be viewed as a bijective function of *a* mapping \mathbb{U} to $\mathbb{U} \rangle b$. Comparing to *a*, it is as if the *b* tasks left unserved in the buffer were miscounted by *q* as new arrivals, and we call *q* the *queued arrival vector*.

A scheduler, through its scheduling decisions, maps each q to some d. A natural way of service specification is then in terms of a map from each q to a worst-case acceptable d.



Fig. 2. Performance bounds and the update rule of a worst-case service.

Definition 1: For a flow with b tasks left unserved in its buffer, $\psi : \mathbb{U} \rangle b \rightarrow \mathbb{U}$ is a worst-case service if

$$\psi(q) \le q$$
 for all $q \in \mathbb{U} > b$. (12)

The flow is said to be guaranteed worst-case service ψ if

$$d \ge \psi(q) \text{ for all } q \in \mathbb{U} \rangle b.$$
 (13)

Since ψ is conditioned on b according to this definition, whenever we refer to ψ , we implicitly refer to the pair, (ψ, b) . As illustrated in Figure 2, to guarantee ψ , d must lie between q and $\psi(q)$. This is impossible unless (12) holds because otherwise, (11) and (13) would contradict each other. Subject to this rather natural constraint, (12), a worst-case service is but an arbitrary map between all queued arrival vectors and their corresponding worst-case departure vectors. In principle, to specify a worst-case service, we need only create a table that specifies $\psi(q)$ for each q. This is not practical in general because such a table would contain an uncountably infinite number of entries. Nonetheless, the theoretical possibility of specifying services so broadly itself makes it possible for us to frame the question of "What to guarantee?" in a most general way.

In [1], [2], [3], [4], cumulative curves were shown to be useful tools for performance analysis. The same methods can be applied here with cumulative vectors. Consider first flow backlogs. For all j > 0, the number of tasks left unserved after slot t + j - 1 is

$$b_j := q_j - d_j, \tag{14}$$

according to which, $b_1 = \dot{b}$, $b_2 = \ddot{b}$, and so on. Then, as illustrated in Figure 2, if the flow is guaranteed ψ , this backlog is bounded by

$$b_j \le q_j - \psi_j(\boldsymbol{q}) \le \max_{j>0} (q_j - \psi_j(\boldsymbol{q})).$$
(15)

Consider next task delays. Recall that, for all h > 0, the *h*th task of *d* departs in slot $t + \tau_h(d)$. Notice that this departure corresponds to the *h*th task of *q*, instead of *a*, because the *b* tasks left unserved after slot t - 1 have to be served before any new arrival. One subtlety here is that there is not enough information to determine when the *b* tasks arrived. However,

if we disregard delays experienced prior to slot t and treat the b tasks as if they were new arrivals, the hth task of d can be viewed as arriving in $t + \tau_h(q)$, so its delay is

$$\theta_h := \tau_h(\boldsymbol{d}) - \tau_h(\boldsymbol{q}).$$
 (16)

Then, as illustrated in Figure 2, if the flow is guaranteed ψ , this delay is bounded by

$$\theta_h \le \tau_h(\boldsymbol{\psi}(\boldsymbol{q})) - \tau_h(\boldsymbol{q}) \le \max_{h>0}(\tau_h(\boldsymbol{\psi}(\boldsymbol{q})) - \tau_h(\boldsymbol{q})).$$
(17)

Both bounds in (15) and (17) are tied to a specific q, but they prepare for the case that q is uncertain. In [1], [2], [3], [4], q is confined to a deterministic subset of $\mathbb{U} \rangle b$ so that the worst case of the above worst-case bounds can be determined. Alternatively, if we can endow a probability measure on $\mathbb{U} \rangle b$, a probability distribution of the above bounds can be derived.

We can also design worst-case services to meet certain performance bounds.

Example 2: Given $\overline{b} \in \mathbb{N}$ *, let*

$$\psi^{\mathrm{UB}}(\boldsymbol{q}) := (\boldsymbol{q} - \bar{b}\boldsymbol{\delta})^+ \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b,$$
 (18)

where x^+ denotes $\max\{x, 0\}$. We call ψ^{UB} a uniform-backlog service, because by guaranteeing it, flow backlogs are bounded by \bar{b} uniformly.

Example 3: Given $\overline{\theta} \in \mathbb{N}$ *, let*

$$\psi^{\mathrm{UD}}(\boldsymbol{q}) := \mathcal{R}^{\theta}(\boldsymbol{q} - b\boldsymbol{\delta}) + \boldsymbol{r} \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b, \qquad (19)$$

where $r \in \mathbb{U}$ satisfies that

$$\mathcal{R}^{\theta}b\boldsymbol{\delta} \le \boldsymbol{r} \le b\boldsymbol{\delta},\tag{20}$$

and $\mathcal{R}: \mathbb{U} \to \mathbb{U}|0$ is the right-shift operator defined by

$$[\mathcal{R}\boldsymbol{x}]_{j+1} := x_j. \tag{21}$$

We call ψ^{UD} a **uniform-delay service**, because by guaranteeing it, task delays are bounded by $\overline{\theta}$ uniformly. Notice that, for the b tasks left unserved after slot t - 1, (20) ensures their delays to be bounded between 0 and $\overline{\theta}$.

Using (15) and (17), we can even design worst-case services to fine-tune the backlog bound for each slot and the delay bound for each task, though we will not delve into the details here.

B. The Update Rule

To find $\psi(q)$, we need to know the entire q, but this is not practical for the scheduler. In this paper, as in practice, we assume the scheduler to be *causal* in the sense that it cannot foresee future arrivals. Therefore, scheduling decisions must be made when q is only partially known, which implies that a causal scheduler, always assuming the worst, may, in hindsight, over-guarantee services.

To be specific, we use $x \stackrel{[j]}{=} y$ to denote the relation that $x_i = y_i$ for all $i \leq j$. Then, as long as $q \stackrel{[j]}{=} q'$, the scheduler simply cannot tell the difference between q and q' before slot t + j. So it has to ensure

$$d_j \geq \max\{\psi_j(\boldsymbol{q}), \psi_j(\boldsymbol{q'})\}$$

to ensure that worst-case service ψ can be guaranteed no matter whether q or q' will realize. Applying this logic repeatedly, it is easy to verify that to guarantee ψ , the scheduler has to guarantee a second service, ψ^{C} , defined by

$$\psi_{j}^{\mathcal{C}}(\boldsymbol{q}) := \max_{0 < i \leq j} \max_{\boldsymbol{q'} \stackrel{[i]}{=} \boldsymbol{q}} \psi_{i}(\boldsymbol{q'}) \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b \text{ and } j > 0.$$
(22)

Clearly $\psi^{C} \geq \psi$, i.e., $\psi^{C}(q) \geq \psi(q)$ for all $q \in \mathbb{U} \rangle b$, so ψ may be over-guaranteed.

By definition, ψ^{C} is not any service but one such that

$$\psi^{\mathrm{C}}(\boldsymbol{q}) \stackrel{[j]}{=} \psi^{\mathrm{C}}(\boldsymbol{q'}) \text{ for all } \boldsymbol{q}, \boldsymbol{q'} \in \mathbb{U} \rangle b, \ j \in \mathbb{N} \text{ and } \boldsymbol{q} \stackrel{[j]}{=} \boldsymbol{q'}.$$
(23)

That is to say, $\psi_j^{C}(q)$ depends on q_1, q_2, \ldots, q_j alone, and we call ψ^{C} *causal*. For instance, it is easy to verify that both uniform-backlog and uniform-delay services in Examples 2 and 3 are causal. According to our reasoning, for a causal scheduler, it would not be restrictive at all to restrict all worst-case services to be causal, though we will not do so here because non-causality gives us the flexibility to specify each $\psi(q)$ independently.

When a tasks arrive in slot t, the possible range of a shrinks from \mathbb{U} to $\mathbb{U}|a$, and consequently, due to (11) and (1), that of q shrinks from $\mathbb{U} \rangle b$ to $\mathbb{U}|q$. Now, for all $q \in \mathbb{U}|q$, according to (22),

$$\psi_1^{\mathsf{C}}(\boldsymbol{q}) = \max_{\boldsymbol{q'} \stackrel{[1]}{=} \boldsymbol{q}} \psi_1(\boldsymbol{q'}) = \max_{\boldsymbol{q'} \in \mathbb{U}|q} \psi_1(\boldsymbol{q'}).$$

Then a causal scheduler may only select

$$d \ge p := \max_{\boldsymbol{q} \in \mathbb{U}|q} \psi_1(\boldsymbol{q}), \tag{24}$$

to ensure that ψ can be guaranteed no matter which $q \in \mathbb{U}|q$ will realize.

The immediate portion of ψ to be met by d is denoted by pin (24), but what about the remaining portion? As we will see, it turns out to be yet another worst-case service. Intuitively, as illustrated in Figure 2, when d tasks depart, this allows a reexpression of $\psi(q)$ in a translated coordinate frame in which the origin moves from O to O' at (1, d) in the original frame. Discounting the immediate portion met by d, $\psi(q)$ is truncated in this new frame. A new worst-case departure vector can then, roughly speaking, be constructed by splicing the line segment $\overline{O'A}$ to the truncated $\psi(q)$, that is, replacing \overline{OA} by $\overline{O'A}$.

To formalize the above intuition, observe first that since the counting process of d starts from slot t while that of \dot{d} starts from t+1, d_{j+1} and \dot{d}_j count the numbers of tasks that depart during periods [t, t+j+1) and [t+1, t+j+1) respectively. Therefore, when d tasks depart in slot t, for all $j \in \mathbb{N}$,

$$d_{j+1} = d_j + d. (25)$$

Using (21), this can be rewritten in the vector form as

$$d = \mathcal{R}\dot{d} + d\delta. \tag{26}$$

Similar to (25), when a tasks arrive in slot t, $a_{j+1} = \dot{a}_j + a$, so using (10), (1) and (3), we have

$$q_{j+1} = a_{j+1} + b = \dot{a}_j + q = \begin{cases} q & \text{if } j = 0\\ \dot{q}_j + d & \text{if } j > 0 \end{cases}$$
(27)

Using (21) and (11), this can be rewritten in the vector form as

$$\boldsymbol{q} = \mathcal{R}\dot{\boldsymbol{a}} + q\boldsymbol{\delta} = \mathcal{R}(\dot{\boldsymbol{q}} - b\boldsymbol{\delta}) + q\boldsymbol{\delta}.$$
 (28)

Observe next that to guarantee ψ , for all $q \in \mathbb{U}|q$, there must be $d \geq \psi(q)$. Then, as $\mathcal{R}\dot{d} \in \mathbb{U}|0$, it is immediate from (26) that

$$\dot{d} \ge \mathcal{R}^{-1}(\psi(q) - d\delta)^+, \tag{29}$$

where $\mathcal{R}^{-1}: \mathbb{U}|0 \to \mathbb{U}$ is the inverse of \mathcal{R} so that

$$[\mathcal{R}^{-1}\boldsymbol{x}]_j = x_{j+1}. \tag{30}$$

Notice that (28) establishes q as a bijective function of \dot{q} , mapping $\mathbb{U}\rangle\dot{b}$ to $\mathbb{U}|q$. This fact, together with (29), leads to the following update rule.

Theorem 4: When a tasks arrive and d tasks depart in slot t, if both (2) and (24) hold, that is, if $q \ge d \ge p$, the flow is guaranteed worst-case service ψ if and only if for all $\dot{\mathbf{q}} \in \mathbb{U} \backslash \dot{b}$,

$$\dot{\boldsymbol{d}} \ge \dot{\boldsymbol{\psi}}(\dot{\boldsymbol{q}}) := \mathcal{R}^{-1}(\boldsymbol{\psi}(\boldsymbol{q}) - d\boldsymbol{\delta})^+ \tag{31}$$

$$= \mathcal{R}^{-1}(\psi(\mathcal{R}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{b}}\boldsymbol{\delta}) + q\boldsymbol{\delta}) - d\boldsymbol{\delta})^+, \quad (32)$$

where $\dot{\psi}$ is a worst-case service for the flow in slot t+1.

According to this theorem, ψ is a state that can be updated to $\dot{\psi}$, which is the remaining portion of ψ to be guaranteed after slot t. Recall that ψ is conditioned on b, so $\dot{\psi}$ is conditioned on b. Accordingly, whenever we update ψ to $\dot{\psi}$, we implicitly update (ψ, b) to $(\dot{\psi}, \dot{b})$ through (31) and (3). A class of worst-case services is **update invariant** if it is preserved by the update rule, that is, if ψ belongs to the class, so does $\dot{\psi}$. For instance, using (18), (19) and (23), it can be verified that uniform-backlog, uniform-delay and causal services are all update invariant, which is not surprising because if a property holds from slot t on, it must also hold from t + 1 on.

Proof of Theorem 4: The necessity of (31) follows directly from the derivation of (29). We need only show its sufficiency. As $d \ge p$, (24) implies that $(\psi(q) - d\delta)^+ \in \mathbb{U}|0$ for all $q \in \mathbb{U}|q$, justifying the use of \mathcal{R}^{-1} in (31). Now, if (31) holds, for all $q \in \mathbb{U}|q$, according to (26),

$$\boldsymbol{d} = \mathcal{R}\boldsymbol{\dot{d}} + d\boldsymbol{\delta} \ge (\boldsymbol{\psi}(\boldsymbol{q}) - d\boldsymbol{\delta})^+ + d\boldsymbol{\delta} \ge \boldsymbol{\psi}(\boldsymbol{q}).$$

Then ψ must be guaranteed.

It remains for us to show that $\dot{\psi}$ is a worst-case service. For all $\dot{q} \in \mathbb{U} \rangle \dot{b}$ and j > 0, using (31), (30), (12) and (27), we have

$$\dot{\psi}_j(\dot{q}) = (\psi_{j+1}(q) - d)^+ \le (q_{j+1} - d)^+ = \dot{q}_j^+ \stackrel{(\dagger)}{=} \dot{q}_j,$$

where (†) holds because, according to (3), $d \leq q$ guarantees that $\dot{b} \geq 0$ and thus $\dot{q}_j \geq 0$. It follows that $\dot{\psi}(\dot{q}) \leq \dot{q}$, so according to Definition 1, $\dot{\psi}$ is indeed a worst-case service.

C. The Spectrum

A full-blown map between cumulative vectors is hard to comprehend. The concept of its spectrum helps us distill essential information from it. The motivating question is: for a causal scheduler, to guarantee ψ , how much capacity must be reserved during any given period? According to (13), $d_j \ge \psi_j(\boldsymbol{q})$, while according to (10), $d_i \le q_i$. So there has to be

$$(d_j - d_i)^+ \ge (\psi_j(q) - q_i)^+,$$

where $(d_j - d_i)^+$ counts the number of tasks served during period [t + i, t + j), being 0 by default if $i \ge j$. Since this applies to all $q \in \mathbb{U} > b$, it leads to the following definition.

Definition 5: Given worst-case service ψ , for all $i, j \in \mathbb{N}$, the **spectral value** of ψ indexed by i and j is

$$\Lambda_{ij}(\boldsymbol{\psi}) := \max_{\boldsymbol{q} \in \mathbb{U} \backslash b} (\psi_j(\boldsymbol{q}) - q_i)^+,$$
(33)

and we call the collection of all such values the **spectrum** of ψ .

According to our reasoning, $\lambda_{ij}(\psi)$ is the least capacity to be reserved during period [t + i, t + j) to ensure that ψ can be guaranteed no matter which $q \in \mathbb{U} \rangle b$ will realize. If $\psi \geq \psi'$, i.e., $\psi(q) \geq \psi'(q)$ for all $q \in \mathbb{U} \rangle b$, it is immediate that $\lambda_{ij}(\psi) \geq \lambda_{ij}(\psi')$. Therefore, the better service to be guaranteed, the more capacity to be reserved. Our choice of the term here, spectrum, is fundamentally arbitrary. Still let us contextualize it by providing an analogy to the spectrum of a normal matrix. If N is a normal matrix, i.e., $N^{\dagger}N = NN^{\dagger}$, and λ_{\max} is its eigenvalue with the maximum magnitude, it is well known that

$$|\lambda_{\max}| = \max_{\boldsymbol{x} \neq \boldsymbol{0}} (||N\boldsymbol{x}|| \div ||\boldsymbol{x}||).$$

Comparing this to (33), it is discernable that λ_{\max} , N and \boldsymbol{x} loosely corresponds to, respectively, $\lambda_{ij}(\boldsymbol{\psi})$, $\boldsymbol{\psi}$ and \boldsymbol{q} in (33). There is even a correspondence between operators \div and -. In the so-called min-plus algebra to be introduced later, we use + to replace \times in the standard algebra, so it is only natural to replace \div by -.

We will denote $\lambda_{ij}(\psi)$ by λ_{ij} when no confusion can be introduced. For later uses, some basic properties of λ_{ij} are listed below.

Theorem 6: For all $i, j \in \mathbb{N}$,

$$\lambda_{ij} = 0 \quad \text{if} \ i \ge j, \tag{34}$$

$$\lambda_{ij} \le \lambda_{i,j+1},\tag{35}$$

$$\lambda_{ij} \ge \lambda_{i+1,j},\tag{36}$$

$$\lambda_{ij} \le (\lambda_{0j} - b\delta_i)^+. \tag{37}$$

Proof: Firstly, if $i \geq j$, since $\psi_j(\boldsymbol{q}) \leq \psi_i(\boldsymbol{q}) \leq q_i$, according to (33), $\lambda_{ij} = 0$. Secondly, since $\psi_j(\boldsymbol{q}) \leq \psi_{j+1}(\boldsymbol{q})$ and $q_i \leq q_{i+1}$, according to (33), $\lambda_{ij} \leq \lambda_{i,j+1}$ and $\lambda_{ij} \geq \lambda_{i+1,j}$. Finally, since $\boldsymbol{q} \geq b\boldsymbol{\delta}$ for all $\boldsymbol{q} \in \mathbb{U} \rangle b$, using (33), we have

$$\lambda_{ij} \leq \max_{\boldsymbol{q} \in \mathbb{U} \setminus b} (\psi_j(\boldsymbol{q}) - b\delta_i)^+ = \left(\max_{\boldsymbol{q} \in \mathbb{U} \setminus b} \psi_j(\boldsymbol{q}) - b\delta_i\right)^+.$$

But we also have

$$\lambda_{0j} = \max_{\boldsymbol{q} \in \mathbb{U} \rangle b} \psi_j(\boldsymbol{q}),$$

so (37) must be true.

According to Theorem 4, we can update ψ to $\dot{\psi}$. We will denote $\lambda_{ij}(\dot{\psi})$ by $\dot{\lambda}_{ij}$. Then, using (33), (31) and (30), we have

$$\begin{split} \dot{\lambda}_{ij} &= \max_{\dot{\boldsymbol{q}} \in \mathbb{U} \backslash \dot{b}} (\dot{\psi}_j (\dot{\boldsymbol{q}}) - \dot{q}_i)^+ \\ &\stackrel{(\dagger)}{=} \max_{\boldsymbol{q} \in \mathbb{U} \mid q} ([\mathcal{R}^{-1} (\boldsymbol{\psi}(\boldsymbol{q}) - d\boldsymbol{\delta})^+]_j - \dot{q}_i)^+ \\ &= \max_{\boldsymbol{q} \in \mathbb{U} \mid q} (\psi_{j+1}(\boldsymbol{q}) - d - \dot{q}_i)^+, \end{split}$$

where (†) holds because, according to (28), $\dot{q} \in \mathbb{U} \rangle \dot{b}$ is equivalent to $q \in \mathbb{U} | q$. This implies that in the case that i = 0,

$$\dot{\lambda}_{0j} = \max_{\boldsymbol{q} \in \mathbb{U}|q} (\psi_{j+1}(\boldsymbol{q}) - d)^+ = \left(\max_{\boldsymbol{q} \in \mathbb{U}|q} \psi_{j+1}(\boldsymbol{q}) - d\right)^+,$$
(38)

and in the case that i > 0, according to (27),

$$\dot{\lambda}_{ij} = \max_{\boldsymbol{q} \in \mathbb{U}|q} (\psi_{j+1}(\boldsymbol{q}) - q_{i+1})^+.$$
(39)

Let us introduce

$$\lambda_{ij}(\boldsymbol{\psi}|q) := \max_{\boldsymbol{q} \in \mathbb{U}|q} (\psi_j(\boldsymbol{q}) - q_i)^+.$$
(40)

which we will denote by λ_{ij} . It allows us to rewrite (38) and (39) as the next theorem.

Theorem 7: For all $i, j \in \mathbb{N}$,

$$\dot{\lambda}_{ij} = \begin{cases} (\hat{\lambda}_{0,j+1} - d)^+ & \text{if } i = 0\\ \hat{\lambda}_{i+1,j+1} & \text{if } i > 0 \end{cases}$$
(41)

According to this theorem, we can identify λ_{ij} through λ_{ij} , which is why the latter will play a key role subsequently. Comparing (40) to (33), the only difference is that the range of \boldsymbol{q} shrinks from $\mathbb{U} \rangle \boldsymbol{b}$ to $\mathbb{U} | \boldsymbol{q}$. For this reason, we call $\hat{\lambda}_{ij}$ the **conditional spectral value** of $\boldsymbol{\psi}$ and the collection of all such values the **conditional spectrum** of $\boldsymbol{\psi}$. For later uses, some basic properties of $\hat{\lambda}_{ij}$ are listed below.

Theorem 8: For all $i, j \in \mathbb{N}$ *,*

$$\hat{\lambda}_{ij} = 0 \quad \text{if} \ i \ge j, \tag{42}$$

$$\hat{\lambda}_{ij} \le \hat{\lambda}_{i,j+1},\tag{43}$$

$$\hat{\lambda}_{ij} \ge \hat{\lambda}_{i+1,j},\tag{44}$$

$$\hat{\lambda}_{ij} \le (\hat{\lambda}_{0j} - q\delta_i)^+, \tag{45}$$

$$\hat{\lambda}_{1j} = (\hat{\lambda}_{0j} - q)^+, \tag{46}$$

$$\hat{\lambda}_{01} = p, \tag{47}$$

and

$$\hat{\lambda}_{ij} \le \lambda_{ij}.\tag{48}$$

Proof: Firstly, (42), (43) and (44) are direct counterparts of (34), (35) and (36) respectively, and they can be proved in exactly the same way. Secondly, since $q_i \ge q\delta_i$ and $q_1 = q$ for all $q \in \mathbb{U}|q$, using (40), we have

$$\hat{\lambda}_{ij} \leq \max_{\boldsymbol{q} \in \mathbb{U}|q} (\psi_j(\boldsymbol{q}) - q\delta_i)^+ = \left(\max_{\boldsymbol{q} \in \mathbb{U}|q} \psi_j(\boldsymbol{q}) - q\delta_i\right)^+,$$

and

$$\hat{\lambda}_{1j} = \max_{\boldsymbol{q} \in \mathbb{U}|q} (\psi_j(\boldsymbol{q}) - q_1)^+ = \left(\max_{\boldsymbol{q} \in \mathbb{U}|q} \psi_j(\boldsymbol{q}) - q\right)^+.$$

But we also have

$$\hat{\lambda}_{0j} = \max_{\boldsymbol{q} \in \mathbb{U}|q} \psi_j(\boldsymbol{q}),$$

so (45) and (46) must be true. Thirdly, comparing the last equation above to (24), it is immediate that $\hat{\lambda}_{01} = p$. Finally, since $\mathbb{U}|q \subset \mathbb{U}\rangle b$, comparing (40) to (33), it is immediate that $\hat{\lambda}_{ij} \leq \lambda_{ij}$.

Now let us revisit (41) to give it an intuitive interpretation. By definition, both $\dot{\lambda}_{ij}$ and $\hat{\lambda}_{i+1,j+1}$ are the least capacity to be reserved during period [t + i + 1, t + j + 1), so they should be identical. In the case that i = 0, however, $\hat{\lambda}_{1,j+1}$ has to compete with $(\hat{\lambda}_{0,j+1} - d)^+$ because $\hat{\lambda}_{0,j+1}$ is the least capacity to be reserved during [t, t + j + 1), it is reduced by dafter slot t, and like $\hat{\lambda}_{1,j+1}$, this reduced capacity is also left to be reserved during [t + 1, t + j + 1). It is $\hat{\lambda}_{1,j+1}$ that will lose this competition because, according to (46) and (2),

$$\hat{\lambda}_{1,j+1} = (\hat{\lambda}_{0,j+1} - q)^+ \le (\hat{\lambda}_{0,j+1} - d)^+$$

which explains the curious absence of $\hat{\lambda}_{1,j+1}$ in (41). A corollary is that

$$\dot{\lambda}_{0j} \ge \hat{\lambda}_{1,j+1},\tag{49}$$

the lower bound of which is achieved when d = q.

IV. STATE-BASED SCHEDULING

In our service model, if each flow is guaranteed a worstcase service, it is called a *worst-case system*. For all $\omega \in \Omega$, let ψ^{ω} be the worst-case service guaranteed to flow ω , and we denote the system by $\psi^{[\Omega]}$, which can be viewed as the aggregate state for all flows in the system. Recall that ψ^{ω} is conditioned on b^{ω} , so $\psi^{[\Omega]}$, on $b^{[\Omega]}$. Accordingly, whenever we refer to $\psi^{[\Omega]}$, we implicitly refer to $(\psi^{[\Omega]}, b^{[\Omega]})$.

Given $\psi^{[\Omega]}$, can the server guarantee all flows their respective worst-case services simultaneously? If yes, how? There is no problem if the server's capacity is infinite, because we can simply select $d^{[\Omega]} \ge p^{[\Omega]}, \ d^{[\Omega]} \ge \dot{p}^{[\Omega]}, \ \ddot{d}^{[\Omega]} \ge \ddot{p}^{[\Omega]},$ and so on. But since the server's capacity is inevitably finite, we need to be more strategic. Given any $a^{[\Omega]}$, for $d^{[\Omega]} \ge p^{[\Omega]}$ to be possible, due to capacity constraint (5), there must be $p^{\langle \Omega \rangle} \leq c$. But $d^{[\Omega]} \geq p^{[\Omega]}$ is not enough by itself, because to further ensure the possibility for $\dot{d}^{[\Omega]} \ge \dot{p}^{[\Omega]}$, $d^{[\Omega]}$ must also induce $\dot{\psi}^{[\Omega]}$ such that given any $\dot{a}^{[\Omega]}, \dot{p}^{\langle \Omega \rangle} \leq c$. Even this is not enough, because by the same logic, $d^{[\overline{\Omega}]}$ must induce $\dot{\psi}^{[\Omega]}$ such that given any $\dot{a}^{[\Omega]}$, there exists $\dot{d}^{[\Omega]}\geq\dot{p}^{[\Omega]}$ to induce $\ddot{\psi}^{[\Omega]}$ such that given any $\ddot{a}^{[\Omega]}$, $\ddot{p}^{\langle\Omega\rangle} \leq c$. This can go on indefinitely and the problem will soon manifest itself to be unmanageable. Fortunately there is a neat solution to this problem and the cornerstone of that solution lies in the idea of state-based scheduling.

In the rest of this section, we first outline the general idea of state-based scheduling. We then use the spectrum to formulate the schedulability condition for a worst-case system, which is the key to state-based scheduling.



Fig. 3. A basic operating cycle of state-based scheduling.

A. The General Idea

Since the server's capacity is finite, no matter how smart the scheduler is, it will not be able to guarantee an arbitrary worst-case service. For instance, according to (18) and (19), it is simply impossible to guarantee ψ^{UB} if $\boldsymbol{q} = (\bar{b} + c + 1)\boldsymbol{\delta}$ or ψ^{UD} if $\boldsymbol{q} = (\bar{\theta}c + 1)\boldsymbol{\delta}$. We need some condition to tell what is guaranteeable from what is not. Such a condition should be able to perpetuate itself because a guaranteeable $\psi^{[\Omega]}$ implies a guaranteeable $\dot{\psi}^{[\Omega]}$, a guaranteeable $\ddot{\psi}^{[\Omega]}$, and so on. This motivates the following definition.

Definition 9: Given a condition on $\psi^{[\Omega]}$, P, a valid schedule, $d^{[\Omega]}$, is a **perpetuator** of P if it meets (24) for all $\omega \in \Omega$, i.e., $d^{[\Omega]} \ge p^{[\Omega]}$, and through (31), induces $\dot{\psi}^{[\Omega]}$ that will meet P. We call P **perpetuatable** if $\psi^{[\Omega]}$ meets P implies that given any $a^{[\Omega]}$, there must exist at least one perpetuator of P.

According to this definition, once perpetuatable conditions are met, they can always be met. Perpetuatable conditions can be rather trivial. For instance, it is perpetuatable to require that $\psi^{\omega}(q^{\omega}) = \mathbf{0}$ for all $\omega \in \Omega$ and $q^{\omega} \in \mathbb{U}|b^{\omega}$, where $\mathbf{0}$ is the all-zero vector, but this is nothing but a disguised way to say that there should be no guarantee to any flow at all. Perpetuatable conditions are also not unique. For instance, any condition that is perpetuatable for a server with capacity c' < cmust also be so for that with capacity c. However, there does exist a unique, non-trivial perpetuatable condition. To see this, let P1 and P2 both be perpetuatable. By definition, if $\psi^{[\Omega]}$ meets "P1 or P2", given any $a^{[\Omega]}$, there must exist at least one $d^{[\Omega]}$ that is a perpetuator of "P1 or P2". That is to say, "P1 or P2" should also be perpetuatable, so the union of two perpetuatable conditions is still perpetuatable, which leads to the next definition.

Definition 10: The schedulability condition is the union of all perpetuatable conditions. We call $\psi^{[\Omega]}$ schedulable if it meets the schedulability condition and call $d^{[\Omega]}$ a feasible schedule if it is a perpetuator of the schedulability condition.

The key to state-based scheduling is finding the schedulability condition. Once it is found, state-based scheduling works iteratively. Figure 3 illustrates a basic operating cycle of it:

- S1 existing services together with, if any, newly admitted services form a schedulable $\psi^{[\Omega]}$;
- S2 given $\psi^{[\Omega]}$ and $a^{[\Omega]}$, a feasible schedule, $d^{[\Omega]}$, is selected; and finally,
- S3 given $a^{[\Omega]}$ and $d^{[\Omega]}$, $\psi^{[\Omega]}$ is updated to a schedulable $\dot{\psi}^{[\Omega]}$ which, after a one-slot delay, is fed back to be used in the next cycle.



Fig. 4. A state-space path for a two-flow system.

Here the role of the schedulability condition is two-fold. On the one hand, it is used to admit or deny new service requests to ensure $\psi^{[\Omega]}$, the current state, to be schedulable. On the other hand, $d^{[\Omega]}$ must be selected only among feasible schedules to ensure $\psi^{[\Omega]}$, the next state, to remain schedulable.

It is enlightening to trace state-based scheduling in the state space. In Figure 4, a path from state A to D is illustrated for a two-flow system. Notice that all states visited never leave the schedulable region encircled by the **Pareto frontier**, on which we have to reduce the service guarantee to one flow to improve that to the other. From this perspective, a statebased scheduler works very much like a state-based regulator that control engineers use to regulate an uncertain system. In particular, $a^{[\Omega]}$ corresponds to the uncertain noise that perturbs the system; $d^{[\Omega]}$, the control signal that tries to combat the noise and regulate the state; and the schedulability condition, the constraint on the state that the regulator tries to enforce constantly.

One advantage of state-based scheduling is that it is fully dynamic by its nature. Subject to the schedulability condition, $\psi^{[\Omega]}$ can be configured and reconfigured in each slot, that is, worst-case services can be negotiated and renegotiated on the fly. For instance, in Figure 4, the jump from state B to B' is due to the admission of a new service request from flow 2. Another advantage is that through systematically identifying all feasible schedules, state-based scheduling allows us to identify, not one ad hoc policy, but all scheduling policies that can simultaneously guarantee all flows their respective worstcase services. Notice that although we do have the flexibility to select any feasible schedule in each slot, different selections will induce different future states. For instance, in Figure 4, had a different feasible schedule been selected at C, the path would have diverged to D' and thus been entirely different from that point on.

B. The Schedulability Condition

Whether we can fulfill the promise of state-based scheduling hinges on whether we can find the schedulability condition. Unfortunately, despite its central importance, the definition for the schedulability condition is not constructive. An intelligent guess must be made in the first place, and this is where the spectrum comes to the rescue.

Theorem 11: The schedulability condition for a worst-case system, $\psi^{[\Omega]}$, is that

$$\lambda_{ij}^{\langle \Omega \rangle} \le c_{ij} := (j-i)^+ c \text{ for all } i, j \in \mathbb{N},$$
 (50)

where $\lambda_{ij}^{\langle\Omega\rangle}$ is the short-hand for $\sum_{\omega\in\Omega}\lambda_{ij}(\psi^{\omega})$, and c_{ij} is the server's capacity during period [t+i,t+j), being 0 by default if $i \ge j$. According to Definitions 10 and 9, this implies that:

- I1 $\psi^{[\Omega]}$ is schedulable if $\lambda_{ij}^{\langle\Omega\rangle} \leq c_{ij}$ for all $i, j \in \mathbb{N}$; I2 a valid schedule, $d_{ij}^{[\Omega]}$, is a feasible schedule if $d_{ij}^{[\Omega]} \geq p_{ij}^{[\Omega]}$ and it induces $\dot{\psi}^{[\Omega]}$ such that $\dot{\lambda}_{ij}^{\langle\Omega\rangle} \leq \dot{c}_{ij} = c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$, where $\dot{c}_{ij} = c_{i+1,j+1}$ because both are the server's capacity during [t+i+1, t+j+1); and finally, I3 if $\psi^{[\Omega]}$ is schedulable, given any $a^{[\Omega]}$, there must exist at
- least one $d^{[\Omega]}$ that is a feasible schedule.

The first thing to notice here is that (50) is necessary for $\psi^{[\Omega]}$ to be schedulable. Recall that to guarantee ψ^{ω} , λ_{ij}^{ω} is the least capacity to be reserved during period [t + i, t + j). Since c_{ij} is the server's capacity during the same period, (50) basically says that during any period, the total capacity to be reserved cannot exceed the server's capacity, which is necessary of course. Therefore, to prove the above theorem, we need only show (50) to be perpetuatable, because according to Definition 10, the schedulability condition is the weakest among all perpetuatable conditions, while a necessary, perpetuatable condition has to be the weakest. It then all boils down to establishing I3 in the above theorem, which we will leave for the next section.

Even standing alone, (50) can be used as a normative formula to guide the design of real multiplexing systems very much like how various capacity formulas in the information theory have been used to guide the design of real communication systems. In particular, it is easy to verify that (50) can be rewritten as

$$c \ge \rho(\Omega) := \max_{i < j} \frac{\lambda_{ij}^{\langle \Omega \rangle}}{j - i},\tag{51}$$

which implies that, if $c < \rho(\Omega)$, no matter how smart the scheduler is, it is simply impossible to guarantee all flows their respective worst-case services simultaneously.

We can extend the definition of $\rho(\Omega)$ in (51) to all $\Gamma \subseteq \Omega$ so that (**D**)

$$\rho(\Gamma) := \max_{i < j} \frac{\lambda_{ij}^{\langle i \rangle}}{j - i},\tag{52}$$

where we use $x^{\langle \Gamma \rangle}$ to denote $\sum_{\omega \in \Gamma} x^{\omega},$ with $x^{\langle \phi \rangle} := 0.$ If all flows in Γ are served by a single server, $\rho(\Gamma)$ is the least capacity to be possessed by this server to guarantee ψ^{ω} for all $\omega \in \Gamma$.¹ If each flow is served by a separate server, then,

 $\sum_{\omega \in \Omega} \rho(\{\omega\})$ is the least total capacity to be possessed by all these servers. Let us introduce

$$\eta := \frac{\sum_{\omega \in \Omega} \rho(\{\omega\})}{\rho(\Omega)}.$$
(53)

Clearly, the larger η is, the greater capacity utilization can be achieved by multiplexing. For this reason, we call η the multiplexing gain.

Let us use $\Gamma\Gamma'$ and $\Gamma + \Gamma'$ to denote $\Gamma \cap \Gamma'$ and $\Gamma \cup \Gamma'$ respectively. For all $\Gamma, \Gamma' \subseteq \Omega$ and $\Gamma\Gamma' = \phi$, we have

$$\rho(\Gamma + \Gamma') \le \rho(\Gamma) + \rho(\Gamma'), \tag{54}$$

because

$$\max_{i < j} \frac{\lambda_{ij}^{\langle \Gamma + \Gamma' \rangle}}{j - i} = \frac{\lambda_{i_* j_*}^{\langle \Gamma + \Gamma' \rangle}}{j_* - i_*} = \frac{\lambda_{i_* j_*}^{\langle \Gamma \rangle}}{j_* - i_*} + \frac{\lambda_{i_* j_*}^{\langle \Gamma' \rangle}}{j_* - i_*}$$
$$\leq \max_{i < j} \frac{\lambda_{ij}^{\langle \Gamma \rangle}}{j - i} + \max_{i < j} \frac{\lambda_{ij}^{\langle \Gamma' \rangle}}{j - i},$$

where i_* and j_* maximize $\frac{\lambda_{ij}^{\langle \Gamma+\Gamma'\rangle}}{j-i}$. It follows from (53) and (54) that $\eta \ge 1$. In particular, it is easy to see that $\eta = 1$ if for all $\omega, \omega' \in \Omega$, $\lambda_{ij}^{\omega} \propto \lambda_{ij}^{\omega'}$, while η increases as λ_{ij}^{ω} becomes less proportional to $\lambda_{ij}^{\omega'}$. Roughly speaking, the more diverse $\psi^{[\Omega]}$ becomes, the larger η is.

Example 12: Consider the case that for all $\omega \in \Omega$, $b^{\omega} = 0$, and for all $q^{\omega} \in \mathbb{U}$,

$$\psi^{\omega}(\boldsymbol{q}^{\omega}) = \begin{cases} \mathcal{R}^{i+\bar{\theta}^{\omega}}\boldsymbol{\delta} & \text{if } \boldsymbol{q}^{\omega} = \mathcal{R}^{i}\boldsymbol{\delta} \text{ for some } i \in \mathbb{N} \\ \boldsymbol{0} & \text{otherwise} \end{cases},$$
(55)

which says that, if and only if the arrival traffic of flow ω consists of a single task, the task will be served with a delay no more than $\bar{\theta}^{\omega}$. Using (33), it is immediate that for all $i, j \in \mathbb{N}$,

$$\lambda_{ij}^{\omega} = \begin{cases} 1 & \text{if } j - i > \theta^{\omega} \\ 0 & \text{if } j - i \le \bar{\theta}^{\omega} \end{cases} .$$
 (56)

Without loss of generality, assume that $\bar{\theta}^1 \leq \bar{\theta}^2 \leq \cdots \leq \bar{\theta}^n$. Then, using (52), it can be verified that

$$\rho(\{\omega\}) = \frac{1}{\bar{\theta}^{\omega} + 1} \text{ and } \rho(\Omega) = \max_{\omega \in \Omega} \frac{\omega}{\bar{\theta}^{\omega} + 1}.$$
(57)

According to (53), η is now a function of $\bar{\theta}^{\omega}$'s. The more diverse $\bar{\theta}^{\omega}$'s become, the larger η is. For instance, if $\bar{\theta}^1 =$ $\bar{\theta}^2 = \dots = \bar{\theta}^n, \ \eta = 1, \ \text{while if } \bar{\theta}^1 + 1 = \frac{\bar{\theta}^2 + 1}{2} = \dots = \frac{\bar{\theta}^n + 1}{n},$ $\eta = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$

The complexity of many scheduling policies grows so fast as the number of flows grows that, even if η is large enough to justify multiplexing, a divide-and-conquer strategy remains attractive. To be precise, let $\mathcal{P} \subseteq 2^{\Omega}$ be a *partition* of Ω , i.e., $\bigcup_{\Gamma \in \mathcal{P}} \Gamma = \Omega$ and $\Gamma \Gamma' = \phi$ for all distinct $\Gamma, \Gamma' \in \mathcal{P}$. If, for each $\Gamma \in \mathcal{P}$, all flows in Γ are served by a separate server, then, $\sum_{\Gamma \in \mathcal{P}} \rho(\Gamma)$ is the least total capacity to be possessed by all these servers. Similar to η , let us introduce

$$\eta^{\mathcal{P}} := \frac{\sum_{\omega \in \Omega} \rho(\{\omega\})}{\sum_{\Gamma \in \mathcal{P}} \rho(\Gamma)}.$$
(58)

Using (54) and (53), it is easy to verify that $1 \le \eta^{\mathcal{P}} \le \eta$. Now, if we can identify some \mathcal{P} such that $\max_{\Gamma \in \mathcal{P}} |\Gamma|$ is sufficiently small and $\eta^{\mathcal{P}}$ is sufficiently close to η , we can greatly reduce

¹As ours is a discrete traffic model, $\rho(\Gamma)$ should be rounded up to $\lceil \rho(\Gamma) \rceil$. But we will not do so here. Besides being a good approximation of $\left[\rho(\Gamma)\right]$, $\rho(\Gamma)$ is also more tractable analytically. Furthermore, it can be shown that $\rho(\Gamma)$ is exactly the least capacity to be possessed, if we slightly tweak our model so that a task can be fractionally served, though it still needs to arrive and depart as a whole.

the scheduling complexity without sacrificing too much of the multiplexing gain.

Example 13: In Example 12, consider the case that n = 3and $\bar{\theta}^1 + 1 = \frac{\bar{\theta}^2 + 1}{2} = \frac{\bar{\theta}^3 + 1}{2}$. Using (57), we have

$$\rho(\{1\}) = 2\rho(\{2\}) = 2\rho(\{3\}) = \frac{2}{3}\rho(\{1,2,3\}) = \frac{1}{\bar{\theta}^1 + 1}$$

so $\eta = \frac{4}{3}$. Using (52), we also have

$$\rho(\{1,2\}) = \rho(\{1,3\}) = \frac{1}{\overline{\theta}^1 + 1},$$

so $\eta^{\mathcal{P}} = \frac{4}{3}$ if $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ or $\{\{1, 3\}, \{2\}\}$. In this way, there is no loss of the multiplexing gain.

V. FEASIBLE SCHEDULES

In this section, we not only show that feasible schedules exist but also identify all of them. To get a flavor of what lies ahead, let us first look at the single-flow case. In this case, if ψ is schedulable, (50) requires that

$$\lambda_{ij} \le c_{ij} = (j-i)^+ c \text{ for all } i, j \in \mathbb{N}.$$
(59)

According to I2 in Theorem 11, we need to identify d that induces $\dot{\psi}$ such that $\dot{\lambda}_{ij} \leq c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$. If ψ is schedulable, this is satisfied automatically in the case that i > 0, because using (41), (48) and (59), we have

$$\lambda_{ij} = \lambda_{i+1,j+1} \le \lambda_{i+1,j+1} \le c_{i+1,j+1}.$$

So we need only focus on the case that i = 0. In this case, according to (41), $\dot{\lambda}_{0j} \leq c_{1,j+1}$ implies that

$$c_{1,j+1} \ge \dot{\lambda}_{0j} = (\hat{\lambda}_{0,j+1} - d)^+ \ge \hat{\lambda}_{0,j+1} - d,$$

which in turn implies that

$$d \ge \alpha := \max_{j \in \mathbb{N}} (\hat{\lambda}_{0,j+1} - c_{1,j+1}).$$
(60)

If $d \ge \alpha$, by reversing the above reasoning, it is easy to verify that $\lambda_{ij} \le c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$. It is also immediate from (60) and (47) that $\alpha \ge \lambda_{01} = p$. Then, according to I2 in Theorem 11, d is a feasible schedule if $\alpha \le d \le \min\{q, c\}$. This is impossible unless $\alpha \le q$ and $\alpha \le c$. But fortunately, both are guaranteed if ψ is schedulable because, using (60), (46), (48) and (59), we have

$$\begin{aligned} \alpha &= q + \max_{j \in \mathbb{N}} (\hat{\lambda}_{0,j+1} - q - c_{1,j+1}) \\ &\leq q + \max_{j \in \mathbb{N}} (\hat{\lambda}_{1,j+1} - c_{1,j+1}) \\ &\leq q + \max_{j \in \mathbb{N}} (\lambda_{1,j+1} - c_{1,j+1}) \leq q, \end{aligned}$$

and using (60), (48) and (59), we also have

$$\alpha \le \max_{j \in \mathbb{N}} (\lambda_{0,j+1} - c_{1,j+1}) \le \max_{j \in \mathbb{N}} (c_{0,j+1} - c_{1,j+1}) \le c.$$

Although the above single-flow case is highly simplified, it nonetheless sketches the main skeleton for the general multiple-flow case. In a certain sense, what we are going to do next is but adding flesh to this skeleton. Of course, the general case does pose new technical challenges. In particular, α as defined in (60) will explode from a scalar to a full-blown set function over Ω , and it is in dealing with such functions that the polymatroid theory comes to our attention.

In the rest of this section, we first give a primer for permutohedra, a special type of polytopes from polymatroid theory. We then introduce the baseline function and use it to show the set of feasible schedules to be a polytope that can be sliced into a series of permutohedra. We also show how to select a feasible schedule to enforce different priority or fairness criteria.

A. Supermodular Functions and Permutohedra

The polymatroid theory was first developed in [6] and an extensive survey can be found in [7]. The standard introduction usually starts with submodular functions and polymatroids, but tailored for our application, we will center on supermodular functions and permutohedra instead.²

Definition 14: $\chi : 2^{\Omega} \to \mathbb{N}$ is a supermodular function over Ω if $\chi(\phi) = 0$, and for all $\Gamma, \Gamma' \subseteq \Omega$,

$$\chi(\Gamma) + \chi(\Gamma') \le \chi(\Gamma + \Gamma') + \chi(\Gamma\Gamma').$$
(61)

If χ is supermodular, let

 $\mathbb{P}(\chi) := \{ d^{[\Omega]} | d^{\langle \Omega \rangle} = \chi(\Omega) \text{ and } d^{\langle \Gamma \rangle} \ge \chi(\Gamma) \text{ for all } \Gamma \subseteq \Omega \},$ (62)

and we call $\mathbb{P}(\chi)$ the **permutohedron** generated by χ .

By definition, $\mathbb{P}(\chi)$ is an (n-1)-polytope potentially. To investigate its faces, for all $\{\phi, \Omega\} \subseteq S \subseteq 2^{\Omega}$, let

$$\mathbb{P}_{\mathcal{S}}(\chi) := \{ d^{[\Omega]} \in \mathbb{P}(\chi) | d^{\langle \Gamma \rangle} = \chi(\Gamma) \text{ for all } \Gamma \in \mathcal{S} \}.^3$$
 (63)

Each S identifies a potential face of $\mathbb{P}(\chi)$. There are totally 2^{2^n-2} such S's, but for the vast majority of them, $\mathbb{P}_S(\chi)$ is actually empty. So what makes it non-empty? To answer this question, we need the concept of chains. Given $\{\phi, \Omega\} \subseteq C \subseteq 2^{\Omega}$, we call C a *chain* if for all distinct $\Gamma, \Gamma' \in C, \Gamma \subset \Gamma'$ or $\Gamma' \subset \Gamma$. The next lemma, the proof for which can be found in the appendix, signifies the importance of this concept.

Lemma 15: For all $\{\phi, \Omega\} \subseteq S \subseteq 2^{\Omega}$, if $\mathbb{P}_{\mathcal{S}}(\chi)$ is nonempty, there must exist a chain, \mathcal{C} , such that $\mathbb{P}_{\mathcal{S}}(\chi) = \mathbb{P}_{\mathcal{C}}(\chi)$.

According to this lemma, each non-empty face of $\mathbb{P}(\chi)$ can be identified by a chain. Given a chain, C, if |C| = m + 1, $\mathbb{P}_{\mathcal{C}}(\chi)$ is a potential (n - m)-face of $\mathbb{P}(\chi)$. Furthermore, we call C **complete** if m = n, so each complete chain identifies a potential 0-face, which is of particular interests because a 0-face is but a vertex.

A complete chain, and thus a potential vertex, can in turn be identified by a permutation. A **permutation** over Ω is a bijective map, $\pi : \Omega \to \{1, 2, ..., n\} \in \Pi^{\Omega}$, where Π^{Ω} is the set of all such permutations. For all $0 \le i \le n$, let

$$\Gamma^{i}_{\pi} := \{ \omega \in \Omega | \pi(\omega) \le i \}.$$
(64)

²In Definition 14, replacing \leq by \geq in (61), we get the definition for a *submodular function*. If χ is submodular, let

$$\mathbb{M}(\chi) := \{ d^{[\Omega]} | d^{\langle \Gamma \rangle} \leq \chi(\Gamma) \text{ for all } \Gamma \subseteq \Omega \}$$

and we call $\mathbb{M}(\chi)$ the *polymatroid* generated by χ .

³If χ is supermodular, by definition, $d^{\langle \phi \rangle} = 0 = \chi(\phi)$, and if $d^{[\Omega]} \in \mathbb{P}(\chi)$, according to (62), $d^{\langle \Omega \rangle} = \chi(\Omega)$. Therefore, requiring $\{\phi, \Omega\} \subseteq S$ is not restrictive and we include it here only for completeness.



Fig. 5. Two low-dimension permutohedra in the cases of n = 3 and n = 4.

Denoting $\pi^{-1}(i)$ by ω^i_{π} , it is immediate that

$$\Gamma^{i}_{\pi} = \begin{cases} \phi & \text{if } i = 0\\ \{\omega^{1}_{\pi}, \omega^{2}_{\pi}, \dots, \omega^{i}_{\pi}\} & \text{if } 1 \le i \le n \end{cases}$$
(65)

On the one hand, this naturally leads to a complete chain,

$$\mathcal{C}_{\pi}: \phi = \Gamma^0_{\pi} \subset \Gamma^1_{\pi} \subset \dots \subset \Gamma^n_{\pi} = \Omega.$$
 (66)

On the other hand, let $v_{\pi}^{[\Omega]}(\chi)$ be the unique solution to the system of linear equations defined by

$$v_{\pi}^{\langle \Gamma_{\pi}^{i} \rangle}(\chi) = \chi(\Gamma_{\pi}^{i}) \text{ for all } 0 \le i \le n,$$
(67)

so that

$$v_{\pi}^{\omega_{\pi}^{i}}(\chi) = \chi(\Gamma_{\pi}^{i}) - \chi(\Gamma_{\pi}^{i-1}) \text{ for all } 1 \le i \le n.$$
 (68)

Then, according to (63), (66) and (67), either $\mathbb{P}_{\mathcal{C}_{\pi}}(\chi) = \phi$ or $\mathbb{P}_{\mathcal{C}_{\pi}}(\chi) = \{v_{\pi}^{[\Omega]}(\chi)\}$. The latter turns out to be the case because of the next lemma, the proof for which can also be found in the appendix.

Lemma 16: For all $\Gamma \subseteq \Omega$, $v_{\pi}^{\langle \Gamma \rangle}(\chi) \ge \chi(\Gamma)$. This lemma guarantees that $v_{\pi}^{\langle \Omega \rangle}(\chi)$ is indeed a vertex of $\mathbb{P}(\chi)$. Therefore, $\mathbb{P}(\chi)$ is completely determined by the n! vertices drawn from Π^{Ω} , which is where the name permutohedron comes from in the first place.

Now let us turn back to a general chain, C, and define

$$\Pi^{\Omega}_{\mathcal{C}} := \{ \pi \in \Pi^{\Omega} | \mathcal{C} \subseteq \mathcal{C}_{\pi} \}.$$
(69)

Then, for all $\pi \in \Pi^{\Omega}_{\mathcal{C}}$, according to (63), $\mathbb{P}_{\mathcal{C}_{\pi}}(\chi) \subseteq \mathbb{P}_{\mathcal{C}}(\chi)$. It follows that $v^{[\Omega]}_{\pi}(\chi)$ is a vertex of $\mathbb{P}_{\mathcal{C}}(\chi)$, so $\mathbb{P}_{\mathcal{C}}(\chi)$ is completely determined by the $|\Pi^{\Omega}_{\mathcal{L}}|$ vertices drawn from $\Pi^{\Omega}_{\mathcal{L}}$. A corollary is that $\mathbb{P}_{\mathcal{C}}(\chi)$ is non-empty, so each chain does identify a non-empty face of $\mathbb{P}(\chi)$. Two low-dimension permutohedra are illustrated in Figure 5. In the case of n = 3, it is a hexagon with 6 vertices and 6 edges. In the case of n = 4, it is a polytope with 24 vertices, 36 edges and 14 facets, among which 6 are rectangles and 8 are hexagons. In the figure, notice how each vertex is indexed by a permutation and how they are organized into each face according to these indexing permutations.

Let us turn to the topology of the face system of $\mathbb{P}(\chi)$. Given two chains, C and C', on the one hand, although C + \mathcal{C}' might not be a chain, in the case that it is, $\mathbb{P}_{\mathcal{C}}(\chi)$ and $\mathbb{P}_{\mathcal{C}'}(\chi)$ intersect at $\mathbb{P}_{\mathcal{C}+\mathcal{C}'}(\chi)$. On the other hand, $\mathcal{C}\mathcal{C}'$ is still a chain and $\mathbb{P}_{\mathcal{CC}'}(\chi)$ is the minimum face to which both $\mathbb{P}_{\mathcal{C}}(\chi)$ and $\mathbb{P}_{\mathcal{C}'}(\chi)$ belong. As an application of this principle, let

us consider the neighborhood of a vertex. Two vertices are neighbors if they belong to the same edge. An edge is but a 1-face, so for $v_{\pi}^{[\check{\Omega}]}(\chi)$ and $v_{\pi'}^{[\Omega]}(\chi)$ to be neighbors, there must be $|\mathcal{C}_{\pi}\mathcal{C}_{\pi'}| = n$. It is then easy to verify that there must exist $1 \leq i_* < n$ such that for all $1 \leq i \leq n$,

$$\pi'(\omega_{\pi}^{i}) = \begin{cases} i_{*} + 1 & \text{if } i = i_{*} \\ i_{*} & \text{if } i = i_{*} + 1 \\ i & \text{if } i \neq i_{*}, i_{*} + 1 \end{cases}$$

That is to say, the difference between π and π' is no more than a simple transposition, which is reflected by the arrangement of vertices in Figure 5.

B. The Baseline Function

If $\psi^{[\Omega]}$ is schedulable, given any $a^{[\Omega]}$, how to identify at least one $d^{[\Omega]}$ that is a feasible schedule? According to I2 in Theorem 11, if $d^{[\Omega]}$ is a feasible schedule, it must induce $\dot{\psi}^{[\Omega]}$ such that $\dot{\lambda}_{ij}^{(\Omega)} \leq c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$. In the case that i = 0, according to (41), this implies that, for all $\Gamma \subseteq \Omega$, $c_{1,j+1} \ge \dot{\lambda}_{0j}^{\langle \Omega \rangle} \ge \dot{\lambda}_{0j}^{\langle \Gamma \rangle} = \sum_{\omega \in \Gamma} (\hat{\lambda}_{0,j+1}^{\omega} - d^{\omega})^+ \ge \hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} - d^{\langle \Gamma \rangle},$

which in turn implies that

$$d^{\langle \Gamma \rangle} \ge \alpha(\Gamma) := \max_{j \in \mathbb{N}} (\hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} - c_{1,j+1}), \tag{70}$$

the multiple-flow extension of (60).

In the multiple-flow case, however, we can do better than (70) because other flows do exist. Let us use $\overline{\Gamma}$ to denote $\Omega \setminus \Gamma$. Then, according to (49), $\dot{\lambda}_{0j}^{\langle \overline{\Gamma} \rangle} \geq \hat{\lambda}_{1,j+1}^{\langle \overline{\Gamma} \rangle}$, the lower bound of which is achieved when $d^{\omega} = q^{\omega}$ for all $\omega \in \overline{\Gamma}$, that is, when all other flows' buffers are emptied. Therefore, $\dot{\lambda}_{0j}^{\langle\Omega
angle} \leq c_{1,j+1}$ implies that

$$c_{1,j+1} \ge \dot{\lambda}_{0j}^{\langle \Omega \rangle} = \dot{\lambda}_{0j}^{\langle \Gamma \rangle} + \dot{\lambda}_{0j}^{\langle \overline{\Gamma} \rangle} \ge \hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} - d^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \overline{\Gamma} \rangle},$$

which in turn implies that

$$d^{\langle \Gamma \rangle} \ge \beta(\Gamma) := \max_{j \in \mathbb{N}} (\hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \overline{\Gamma} \rangle} - c_{1,j+1}).$$
(71)

Comparing this to (70), it is immediate that $\beta \geq \alpha$, i.e., $\beta(\Gamma) > \alpha(\Gamma)$ for all $\Gamma \subseteq \Omega$ ⁴ Since (71) specifies the least number of tasks that must be served from any given subset of flows, we call it the *baseline constraint* and β the *baseline function*. Notice that β depends on $a^{[\Omega]}$ implicitly because, according to (40), $\hat{\lambda}_{ij}$ depends on q, which in turn depends on a.

Other formulations of β will be useful. For this purpose, let us introduce $p^{\omega} = [p_j^{\omega}]_{j \in \mathbb{N}} \in \mathbb{U}$, defined by

$$p_j^{\omega} := \hat{\lambda}_{0j}^{\omega} - \hat{\lambda}_{1j}^{\omega} \stackrel{(\dagger)}{=} \min\{\hat{\lambda}_{0j}^{\omega}, q^{\omega}\},\tag{72}$$

where (\dagger) holds because, according to (46),

$$\hat{\lambda}_{0j}^{\omega} - \hat{\lambda}_{1j}^{\omega} = \hat{\lambda}_{0j}^{\omega} - \max\{\hat{\lambda}_{0j}^{\omega} - q^{\omega}, 0\} = \min\{\hat{\lambda}_{0j}^{\omega}, q^{\omega}\}.$$

⁴Although we will not show it here, it can be shown that

$$\beta(\Gamma) = \max_{\Gamma' \subseteq \overline{\Gamma}} (\alpha(\Gamma + \Gamma') - q^{\langle \Gamma' \rangle}),$$

which relates α and β directly.

Here p^{ω} is a cumulative vector because, according to (43), $\hat{\lambda}_{0,i}^{\omega}$ is non-decreasing with respect to j. Also our usage of the same letter to denote scalar p^{ω} and vector p^{ω} here is justified by the fact that, using (72), (42) and (47), we have

$$p_1^{\omega} = \hat{\lambda}_{01}^{\omega} - \hat{\lambda}_{11}^{\omega} = \hat{\lambda}_{01}^{\omega} = p^{\omega}.$$
 (73)

Now, for all $\Gamma \subseteq \Omega$, it is immediate from (71) and (72) that

$$\beta(\Gamma) = \max_{j \in \mathbb{N}} \left(p_{j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1} \right)$$
(74)

$$=p_{j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma}+1}, \tag{75}$$

where

$$j_{\beta}^{\Gamma} := \operatorname*{arg\,max}_{j \in \mathbb{N}} (p_{j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1}), \tag{76}$$

with the understanding that j^{Γ}_{β} should be minimized whenever there are ties in comparisons. This allows us to rewrite (71) as

$$\beta(\Gamma) = \hat{\lambda}_{0,j_{\beta}^{\Gamma}+1}^{\langle\Gamma\rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle\overline{\Gamma}\rangle} - c_{1,j_{\beta}^{\Gamma}+1}.^{5}$$
(77)

A fundamental property of β is then in order. Theorem 17: If $\psi^{[\Omega]}$ is schedulable, β is a supermodular function over Ω . According to Definition 14, that is to say, $\beta(\Gamma) \geq 0$ for all $\Gamma \subseteq \Omega$, $\beta(\phi) = 0$, and for all $\Gamma, \Gamma' \subseteq \Omega$,

$$\beta(\Gamma) + \beta(\Gamma') \le \beta(\Gamma + \Gamma') + \beta(\Gamma\Gamma').$$
(78)

Proof: For all $\Gamma \subseteq \Omega$, using (71) and (40), we have

$$\beta(\Gamma) \ge (\hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \overline{\Gamma} \rangle} - c_{1,j+1})|_{j=0} = \hat{\lambda}_{01}^{\langle \Gamma \rangle} + \hat{\lambda}_{11}^{\langle \overline{\Gamma} \rangle} \ge 0.$$

A corollary is that $\beta(\phi) \ge 0$. But if $\psi^{[\Omega]}$ is schedulable, using (75), (48) and (50), we also have

$$\beta(\phi) = \hat{\lambda}_{1,j_{\beta}^{\phi}+1}^{\langle\Omega\rangle} - c_{1,j_{\beta}^{\phi}+1} \leq \lambda_{1,j_{\beta}^{\phi}+1}^{\langle\Omega\rangle} - c_{1,j_{\beta}^{\phi}+1} \leq 0$$

so $\beta(\phi) = 0$.

For all $\Gamma, \Gamma' \subseteq \Omega$, without loss of generality, assume that $j_{\beta}^{\Gamma} \geq j_{\beta}^{\Gamma'}$. It follows that

$$\begin{split} p_{j_{\beta}^{\Gamma}+1}^{\langle\Gamma\rangle} + p_{j_{\beta}^{\Gamma'}+1}^{\langle\Gamma\rangle} &= p_{j_{\beta}^{\Gamma}+1}^{\langle\Gamma\rangle} + p_{j_{\beta}^{\Gamma'}+1}^{\langle\overline{\Gamma}\Gamma'\rangle} + p_{j_{\beta}^{\Gamma'}+1}^{\langle\Gamma\Gamma'\rangle} \\ &\leq p_{j_{\beta}^{\Gamma}+1}^{\langle\Gamma\rangle} + p_{j_{\beta}^{\Gamma}+1}^{\langle\overline{\Gamma}\Gamma'\rangle} + p_{j_{\beta}^{\Gamma'}+1}^{\langle\Gamma\Gamma'\rangle} &= p_{j_{\beta}^{\Gamma}+1}^{\langle\Gamma+\Gamma'\rangle} + p_{j_{\beta}^{\Gamma'}+1}^{\langle\Gamma\Gamma'\rangle}. \end{split}$$

Then, according to (75) and (74),

$$\begin{split} &\beta(\Gamma) + \beta(\Gamma') \\ &= p_{j_{\beta}^{\Gamma}+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma}+1} + p_{j_{\beta}^{\Gamma'}+1}^{\langle \Gamma' \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma'}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma'}+1} \\ &\leq p_{j_{\beta}^{\Gamma}+1}^{\langle \Gamma+\Gamma' \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma}+1} + p_{j_{\beta}^{\Gamma'}+1}^{\langle \Gamma\Gamma' \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma'}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma'}+1} \\ &\leq \beta(\Gamma+\Gamma') + \beta(\Gamma\Gamma'). \end{split}$$

Two more important properties of β are listed below.

⁵It is also easy to derive two more formulations of β , i.e.,

$$\begin{split} \beta(\Gamma) &= \max_{j \in \mathbb{N}} (\hat{\lambda}_{0,j+1}^{\langle \Omega \rangle} - p_{j+1}^{\langle \overline{\Gamma} \rangle} - c_{1,j+1}) \\ &= \hat{\lambda}_{0,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - p_{j_{\beta}^{\Gamma}+1}^{\langle \overline{\Gamma} \rangle} - c_{1,j_{\beta}^{\Gamma}+1}. \end{split}$$

Theorem 18: For all $\Gamma \subseteq \Omega$ *and* $\Gamma' \subseteq \overline{\Gamma}$ *,*

$$\beta(\Gamma + \Gamma') - \beta(\Gamma') \le q^{\langle \Gamma \rangle}.$$
(79)

If $\boldsymbol{\psi}^{[\Omega]}$ is schedulable.

$$\beta(\Omega) \le c. \tag{80}$$

Proof: On the one hand, according to (75),

$$\beta(\Gamma+\Gamma') = p_{j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle \Gamma+\Gamma'\rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle \Omega\rangle} - c_{1,j_{\beta}^{\Gamma+\Gamma'}+1}.$$

On the other hand, according to (74),

$$\beta(\Gamma') \geq p_{j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle \Gamma'\rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle \Omega\rangle} - c_{1,j_{\beta}^{\Gamma+\Gamma'}+1}.$$

Therefore, if $\Gamma' \subseteq \overline{\Gamma}$, using (72), we have

$$\beta(\Gamma+\Gamma')-\beta(\Gamma') \leq p_{j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle\Gamma+\Gamma'\rangle} - p_{j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle\Gamma'\rangle} = p_{j_{\beta}^{\Gamma+\Gamma'}+1}^{\langle\Gamma\rangle} \leq q^{\langle\Gamma\rangle}.$$

If $\psi^{[\Omega]}$ is schedulable, using (77), (48) and (50), we also have

$$\begin{split} \beta(\Omega) &= \hat{\lambda}_{0,j_{\beta}^{\Omega}+1}^{\langle\Omega\rangle} - c_{1,j_{\beta}^{\Omega}+1} \leq \lambda_{0,j_{\beta}^{\Omega}+1}^{\langle\Omega\rangle} - c_{1,j_{\beta}^{\Omega}+1} \\ &\leq c_{0,j_{\beta}^{\Omega}+1} - c_{1,j_{\beta}^{\Omega}+1} = c. \end{split}$$

Finally let us introduce an interesting property of j_{β}^{Γ} . Theorem 19: For all $\Gamma, \Gamma' \subseteq \Omega$, $j_{\beta}^{\Gamma'} \leq j_{\beta}^{\Gamma}$ if $\Gamma' \subseteq \Gamma$.

According to (74), to calculate $\beta(\Gamma)$, we need only identify j_{β}^{Γ} . Then, according to the above theorem, to calculate β systematically, we can proceed in the following way. Starting with j^{Ω}_{β} , we identify firstly all j^{Γ}_{β} 's with $|\Gamma| = n - 1$, secondly all those with $|\Gamma| = n - 2$, thirdly all those with $|\Gamma| = n - 3$, and so on. This guarantees that all supersets of Γ will be visited before Γ so that j_{β}^{Γ} is always thoroughly bounded. In particular, as we proceed, if we find $j_{\beta}^{\Gamma} = 0$, set $j_{\beta}^{\Gamma'} = 0$ for all $\Gamma' \subseteq \Gamma$.

Proof of Theorem 19: If $\Gamma' \subseteq \Gamma$, for all $j > j_{\beta}^{\Gamma}$,

$$\begin{split} p_{j+1}^{\langle \Gamma' \rangle} - p_{j_{\beta}+1}^{\langle \Gamma' \rangle} &\leq p_{j+1}^{\langle \Gamma' \rangle} - p_{j_{\beta}+1}^{\langle \Gamma' \rangle} + p_{j+1}^{\langle \Gamma\overline{\Gamma'} \rangle} - p_{j_{\beta}+1}^{\langle \Gamma\overline{\Gamma'} \rangle} \\ &= p_{j+1}^{\langle \Gamma \rangle} - p_{j_{\beta}+1}^{\langle \Gamma \rangle} \\ &\stackrel{(\dagger)}{\leq} (\hat{\lambda}_{1,j_{\beta}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}+1}) - (\hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1}), \end{split}$$

 $^{-1}$ where (†) holds because, according to (76),

$$p_{j_{\beta}^{\Gamma}+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma}+1} \ge p_{j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1}$$

Then, for all $j > j_{\beta}^{\Gamma}$,

$$p_{j_{\beta}^{\Gamma}+1}^{\langle \Gamma' \rangle} + \hat{\lambda}_{1,j_{\beta}^{\Gamma}+1}^{\langle \Omega \rangle} - c_{1,j_{\beta}^{\Gamma}+1} \ge p_{j+1}^{\langle \Gamma' \rangle} + \hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1},$$

so $j_{\beta}^{\Gamma'} \leq j_{\beta}^{\Gamma}$ because, in using (76) to identify $j_{\beta}^{\Gamma'}$, $j_{\beta}^{\Gamma'}$ should be minimized.

C. The Feasible Polytope and Feasible Permutohedra

The importance of the baseline constraint, (71), is signified by the next theorem.

Theorem 20: If $\psi^{[\Omega]}$ is schedulable, a valid schedule, $d^{[\Omega]}$, is a feasible schedule if and only if $d^{\langle \Gamma \rangle} \geq \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$.

Proof: The necessity of this condition follows directly from the derivation of (71). To show its sufficiency, according to I2 in Theorem 11, we need only show that it implies that $d^{[\Omega]} \ge p^{[\Omega]}$ and $d^{[\Omega]}$ induces $\dot{\psi}^{[\Omega]}$ such that $\dot{\lambda}_{ij}^{\langle\Omega\rangle} \le c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$. If $d^{\langle\Gamma\rangle} \ge \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, for all $\omega \in \Omega$, using (74), (42) and (73), we have

$$\begin{split} d^{\omega} &\geq \beta(\{\omega\}) \geq (p_{j+1}^{\omega} + \hat{\lambda}_{1,j+1}^{\langle \Omega \rangle} - c_{1,j+1})|_{j=0} = p_1^{\omega} + \hat{\lambda}_{11}^{\langle \Omega \rangle} = p^{\omega} \\ \text{i.e., } d^{[\Omega]} &\geq p^{[\Omega]}. \end{split}$$

Let us move on to $\dot{\lambda}_{ij}^{(\Omega)}$. In the case that i = 0, on the one hand, using (41) and repeatedly applying the max-plus distributive law, i.e., $\max\{x, y\} + z = \max\{x + z, y + z\}$, we have

$$\dot{\lambda}_{0j}^{\langle\Omega\rangle} = \sum_{\omega\in\Omega} \max\{\hat{\lambda}_{0,j+1}^{\omega} - d^{\omega}, 0\} = \max_{\Gamma\subseteq\Omega}(\hat{\lambda}_{0,j+1}^{\langle\Gamma\rangle} - d^{\langle\Gamma\rangle}).$$

On the other hand, if $d^{\langle \Gamma \rangle} \ge \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, using (71) and (40), we also have

$$d^{\langle \Gamma \rangle} \ge \beta(\Gamma) \ge \hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} + \hat{\lambda}_{1,j+1}^{\langle \overline{\Gamma} \rangle} - c_{1,j+1} \ge \hat{\lambda}_{0,j+1}^{\langle \Gamma \rangle} - c_{1,j+1}.$$

Therefore,

$$\dot{\lambda}_{0j}^{\langle\Omega
angle} = \max_{\Gamma\subseteq\Omega} (\hat{\lambda}_{0,j+1}^{\langle\Gamma
angle} - d^{\langle\Gamma
angle}) \le c_{1,j+1}.$$

In the case that i > 0, if $\psi^{[\Omega]}$ is schedulable, according to (41), (48) and (50),

$$\dot{\lambda}_{ij}^{\langle\Omega\rangle} = \hat{\lambda}_{i+1,j+1}^{\langle\Omega\rangle} \le \lambda_{i+1,j+1}^{\langle\Omega\rangle} \le c_{i+1,j+1}.$$

It follows that $\dot{\lambda}_{ij}^{\langle\Omega\rangle} \leq c_{i+1,j+1}$ for all $i, j \in \mathbb{N}$.

We use \mathbb{F} to denote the set of feasible schedules. If $\psi^{[\Omega]}$ is schedulable, according to the above theorem, \mathbb{F} is completely determined by three linear constraints, (4), (5) and (71). So it is an *n*-polytope in general, and we call it the *feasible polytope*. For the two-flow case illustrated in Figure 6, \mathbb{F} is the pentagon, *ABCDE*, for which, intuitively, the causality and capacity constraints enclose from above, while the baseline constraint encloses from below. For a three-flow case, it will look like a diamond. But in general, what is the structure of \mathbb{F} ?

Our hunch is that somehow \mathbb{F} should be related to permutohedra because β is supermodular. The problem is that $d^{\langle \Omega \rangle}$ should remain constant in a permutohedron, which is not the case for \mathbb{F} . This motivates us to intersect \mathbb{F} with a hyperplane,

$$\mathbb{H}_{\mu} := \{ d^{[\Omega]} | d^{\langle \Omega \rangle} = \mu \}.$$
(81)

The intersection, $\mathbb{F}_{\mu} := \mathbb{F} \cap \mathbb{H}_{\mu}$, turns out to be a permutohedron.

For clarity, if $d^{[\Omega]} \in \mathbb{H}_{\mu}$, we denote it by $d^{[\Omega]}_{\mu}$ so that by definition, $d^{(\Omega)}_{\mu} = \mu$. Now, if $d^{[\Omega]}_{\mu} \in \mathbb{F}_{\mu}$, it has to meet (4), (5) and (71). As a result, for \mathbb{F}_{μ} to be non-empty, there has to be

$$\beta(\Omega) \le \mu \le \min\{c, q^{\langle \Omega \rangle}\}.$$
(82)



Fig. 6. A feasible polytope in the two-flow case.

We call such a μ *feasible*. If $\psi^{[\Omega]}$ is schedulable, it is immediate from (80) and (79) that $\beta(\Omega) \leq c$ and $\beta(\Omega) \leq q^{\langle \Omega \rangle}$, so at least one feasible μ must exist. Also, if $d_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, since (71) and (4) imply that $d_{\mu}^{\langle \Gamma \rangle} \geq \beta(\Gamma)$ and $d_{\mu}^{\langle \Gamma \rangle} + q^{\langle \overline{\Gamma} \rangle} \geq d_{\mu}^{\langle \Omega \rangle} = \mu$, for all $\Gamma \subseteq \Omega$,

$$d_{\mu}^{\langle \Gamma \rangle} \ge \beta_{\mu}(\Gamma) := \max\{\beta(\Gamma), \mu - q^{\langle \overline{\Gamma} \rangle}\}.$$
 (83)

It is then fundamental that, like β , β_{μ} is supermodular.

Theorem 21: If $\psi^{[\Omega]}$ is schedulable and μ is feasible, β_{μ} is a supermodular function over Ω . According to Definition 14, that is to say, $\beta_{\mu}(\Gamma) \geq 0$ for all $\Gamma \subseteq \Omega$, $\beta_{\mu}(\phi) = 0$, and for all $\Gamma, \Gamma' \subseteq \Omega$,

$$\beta_{\mu}(\Gamma) + \beta_{\mu}(\Gamma') \le \beta_{\mu}(\Gamma + \Gamma') + \beta_{\mu}(\Gamma\Gamma').$$
(84)

Proof: If $\psi^{[\Omega]}$ is schedulable, according to Theorem 17, β is supermodular. For all $\Gamma \subseteq \Omega$, it is immediate from (83) that $\beta_{\mu}(\Gamma) \geq \beta(\Gamma) \geq 0$. In addition, $\beta(\phi) = 0$, while if μ is feasible, $\mu \leq q^{\langle \Omega \rangle}$. Therefore, according to (83),

$$\beta_{\mu}(\phi) = \max\{\beta(\phi), \mu - q^{\langle \Omega \rangle}\} = 0.$$

As to (84), we need only consider the following four cases: C1 if $\beta_{\mu}(\Gamma) = \beta(\Gamma)$ and $\beta_{\mu}(\Gamma') = \beta(\Gamma')$, (84) follows directly from (78);

C2 if
$$\beta_{\mu}(\Gamma) = \mu - q^{\langle \overline{\Gamma} \rangle}$$
 and $\beta_{\mu}(\Gamma') = \mu - q^{\langle \overline{\Gamma}' \rangle}$,

$$\beta_{\mu}(\Gamma) + \beta_{\mu}(\Gamma') = \mu - q^{\langle \Gamma' \rangle} + \mu - q^{\langle \Gamma' \rangle}$$
$$= \mu - q^{\overline{\langle \Gamma + \Gamma' \rangle}} + \mu - q^{\overline{\langle \Gamma \Gamma' \rangle}}$$
$$\leq \beta_{\mu}(\Gamma + \Gamma') + \beta_{\mu}(\Gamma \Gamma');$$

C3 if
$$\beta_{\mu}(\Gamma) = \mu - q^{\langle \Gamma \rangle}$$
 and $\beta_{\mu}(\Gamma') = \beta(\Gamma')$,
 $\beta_{\mu}(\Gamma) + \beta_{\mu}(\Gamma') = \mu - q^{\langle \overline{\Gamma} \rangle} + \beta(\Gamma')$
 $\stackrel{(\dagger)}{\leq} \mu - q^{\langle \overline{\Gamma} \rangle} + q^{\langle \overline{\Gamma} \Gamma' \rangle} + \beta(\Gamma\Gamma')$
 $= \mu - q^{\langle \overline{\Gamma} + \overline{\Gamma'} \rangle} + \beta(\Gamma\Gamma')$
 $\leq \beta_{\mu}(\Gamma + \Gamma') + \beta_{\mu}(\Gamma\Gamma')$,

where (†) holds because $\beta(\Gamma') - \beta(\Gamma\Gamma') \leq q^{\langle \overline{\Gamma}\Gamma' \rangle}$ according to (79); and finally,

C4 if $\beta_{\mu}(\Gamma) = \beta(\Gamma)$ and $\beta_{\mu}(\Gamma') = \mu - q^{\langle \overline{\Gamma'} \rangle}$, this is symmetrical to C3.

In all cases, (84) must hold.

If $d_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, (83) implies that

$$d_{\mu}^{\langle \Omega \rangle} = \beta_{\mu}(\Omega) \stackrel{\text{(†)}}{=} \mu \text{ and } d_{\mu}^{\langle \Gamma \rangle} \ge \beta_{\mu}(\Gamma) \text{ for all } \Gamma \subseteq \Omega, \quad (85)$$

where (†) holds because, according to (83) and (82),

$$\beta_{\mu}(\Omega) = \max\{\beta(\Omega), \mu\} = \mu.$$
(86)

That is to say, $d_{\mu}^{[\Omega]} \in \mathbb{P}(\beta_{\mu})$, so $\mathbb{F}_{\mu} \subseteq \mathbb{P}(\beta_{\mu})$. In fact, \mathbb{F}_{μ} is exactly $\mathbb{P}(\beta_{\mu})$ as shown by the next theorem.

Theorem 22: If $\psi^{[\Omega]}$ is schedulable and μ is feasible,

$$\mathbb{F}_{\mu} = \mathbb{P}(\beta_{\mu}). \tag{87}$$

Since $\mathbb{P}(\beta_{\mu})$ is non-empty, by proving this theorem, we will for the first time establish I3 in Theorem 11, that is, establish the existence of feasible schedules, and thus finally prove that theorem. We call $\mathbb{P}(\beta_{\mu})$ the *feasible permutohedron* under μ , which is line segment $\overline{A'B'}$ in Figure 6. Although each feasible permutohedron is but one slice of \mathbb{F} , by putting all of them together, we can reassemble the entire \mathbb{F} . Notice that $\mathbb{P}(\beta_{\mu})$ could be highly degenerate. For instance, in the case that $\mu = q^{\langle \Omega \rangle} \leq c$, it shrinks to a single point, $q^{[\Omega]}$.⁶

Proof of Theorem 22: We need only show that $\mathbb{P}(\beta_{\mu}) \subseteq \mathbb{F}_{\mu}$, so we need only show that (85) implies (4), (5) and (71), i.e., $d_{\mu}^{[\Omega]} \leq q^{[\Omega]}, d_{\mu}^{(\Omega)} \leq c$, and $d_{\mu}^{(\Gamma)} \geq \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$. Firstly, according to (85) and the definition of β_{μ} in (83), $d_{\mu}^{(\Gamma)} \geq \beta_{\mu}(\Gamma) \geq \beta(\Gamma)$. Secondly, according to (85) and (82), $d_{\mu}^{(\Omega)} = \mu \leq c$. Finally, for all $\omega \in \Omega$, using (85) and the definition of β_{μ} in (83), we have

$$\begin{split} d^{\omega}_{\mu} &= d^{\langle \Omega \rangle}_{\mu} - d^{\langle \overline{\{\omega\}} \rangle}_{\mu} \leq \mu - \beta_{\mu}(\overline{\{\omega\}}) \\ &= \mu - \max\{\beta(\overline{\{\omega\}}), \mu - q^{\omega}\} \leq q^{\omega}, \end{split}$$

i.e., $d^{[\Omega]}_{\mu} \leq q^{[\Omega]}.$

D. How to Select a Feasible Schedule?

According to Theorem 22, to select a feasible schedule, we can first select a feasible μ , which fixes the total service, and then select $d_{\mu}^{[\Omega]}$ from $\mathbb{P}(\beta_{\mu})$. But what does it mean by selecting different points from $\mathbb{P}(\beta_{\mu})$? Let us start with its vertices. Recall that, given $\pi \in \Pi^{\Omega}$, the vertex identified by it is $v_{\pi}^{[\Omega]}(\beta_{\mu})$. It is immediate from (67) and (85) that

$$v_{\pi}^{\langle \Gamma_{\pi}^{i} \rangle}(\beta_{\mu}) = \beta_{\mu}(\Gamma_{\pi}^{i}) = \min_{d_{\mu}^{[\Omega]} \in \mathbb{P}(\beta_{\mu})} d_{\mu}^{\langle \Gamma_{\pi}^{i} \rangle} \text{ for all } 0 \le i \le n.$$
(88)

This, according to (65), essentially says that, by selecting $v_{\pi}^{[\Omega]}(\beta_{\mu})$, firstly $d_{\mu}^{\omega_{\pi}^{1}}$ is minimized, secondly $d_{\mu}^{\omega_{\pi}^{1}} + d_{\mu}^{\omega_{\pi}^{2}}$ is minimized, thirdly $d_{\mu}^{\omega_{\pi}^{1}} + d_{\mu}^{\omega_{\pi}^{2}} + d_{\mu}^{\omega_{\pi}^{3}}$ is minimized, and so on. Then a flow-by-flow priority order is enforced. In particular,

 $\pi(\omega)$ can be interpreted as a priority index so that the larger it is, the higher priority flow ω enjoys. Priorities, however, may not be our only concerns. For instance, sometimes we may value fairness more dearly. In that case, let us consider the vertex centroid of $\mathbb{P}(\beta_{\mu})$, i.e.,

$$v_{\mathrm{F}}^{[\Omega]}(\beta_{\mu}) := \frac{1}{n!} \sum_{\pi \in \Pi^{\Omega}} v_{\pi}^{[\Omega]}(\beta_{\mu}). \tag{89}$$

Intuitively, $v_{\rm F}^{[\Omega]}(\beta_{\mu})$ is fair in that it gives each vertex, each enforcing a unique flow-by-flow priority order, an equal weight.⁷

Also worth noting is an alternative approach to selecting a feasible schedule. In the case that $\mu = \beta(\Omega)$, using (83), it is easy to verify that $\beta_{\mu} = \beta$, i.e., $\beta_{\mu}(\Gamma) = \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, because $\beta(\Omega) - \beta(\Gamma) \leq q^{\langle \overline{\Gamma} \rangle}$ according to (79). In this case, then, $\mathbb{P}(\beta_{\mu}) = \mathbb{P}(\beta)$, and we call $\mathbb{P}(\beta)$ the **baseline permutohedron**. It is the bottom face of \mathbb{F} , which is line segment \overline{AB} in Figure 6. Now, to select a feasible schedule, we can first select $d_*^{[\Omega]}$ from $\mathbb{P}(\beta)$, and then select $d^{[\Omega]}$ such that $d_*^{[\Omega]} \leq d^{[\Omega]} \leq q^{[\Omega]}$ and $d^{\langle \Omega \rangle} \leq c$. This second step requires nothing but a simple allocation of the excess capacity that amounts to $c - \beta(\Omega)$. For instance, the famous generalizedprocessor-sharing (GPS) policy, as discussed in [3], [4], can be applied here to allocate the excess capacity according to the ratios determined by $d_*^{[\Omega]}$. The validity of this approach is assured by the next theorem.

Theorem 23: If $\psi^{[\Omega]}$ is schedulable, a valid schedule, $d^{[\Omega]}$, is a feasible schedule if and only if there exists $d_*^{[\Omega]} \in \mathbb{P}(\beta)$ such that $d^{[\Omega]} \geq d_*^{[\Omega]}$.

Proof: If there exists $d_*^{[\Omega]} \in \mathbb{P}(\beta)$ such that $d^{[\Omega]} \ge d_*^{[\Omega]}$, by definition, $d^{\langle \Gamma \rangle} \ge d_*^{\langle \Gamma \rangle} \ge \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$. Therefore, according to Theorem 20, this condition is sufficient. To show its necessity, we will show that if $d^{\langle \Gamma \rangle} \ge \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, there is always a direction for $d^{[\Omega]}$ to descend until it lands on $\mathbb{P}(\beta)$.

If $d^{\langle \Gamma \rangle} \geq \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, notice that in the case that $d^{\langle \Gamma \rangle} = \beta(\Gamma)$ and $d^{\langle \Gamma' \rangle} = \beta(\Gamma')$, according to (78),

$$\begin{split} d^{\langle \Gamma + \Gamma' \rangle} + d^{\langle \Gamma \Gamma' \rangle} &= d^{\langle \Gamma \rangle} + d^{\langle \Gamma' \rangle} \\ &= \beta(\Gamma) + \beta(\Gamma') \\ &\leq \beta(\Gamma + \Gamma') + \beta(\Gamma \Gamma') \leq d^{\langle \Gamma + \Gamma' \rangle} + d^{\langle \Gamma \Gamma' \rangle}. \end{split}$$

Since this is impossible unless $d^{\langle \Gamma+\Gamma'\rangle} = \beta(\Gamma+\Gamma')$ and $d^{\langle \Gamma\Gamma'\rangle} = \beta(\Gamma\Gamma')$, $d^{\langle \Gamma\rangle} = \beta(\Gamma)$ and $d^{\langle \Gamma'\rangle} = \beta(\Gamma')$ imply that $d^{\langle \Gamma+\Gamma'\rangle} = \beta(\Gamma+\Gamma')$. Let Γ_* be the union of all $\Gamma \subseteq \Omega$ such that $d^{\langle \Gamma\rangle} = \beta(\Gamma)$. Then there has to be $d^{\langle \Gamma_*\rangle} = \beta(\Gamma_*)$. If $d^{\langle \Omega\rangle} > \beta(\Omega)$, $\Gamma_* \subset \Omega$, and at least one $\omega_* \notin \Gamma_*$ must exist. By construction, $d^{\langle \Gamma\rangle} > \beta(\Gamma)$ for all $\Gamma \ni \omega_*$, so d^{ω_*} can be reduced until $d^{\langle \Gamma\rangle} = \beta(\Gamma)$ for some $\Gamma \ni \omega_*$. If there is still $d^{\langle \Omega\rangle} > \beta(\Omega)$, we can repeat the above procedure to find another direction, ω_{**} , for $d^{[\Omega]}$ to descend. This descending, while keeping $d^{\langle \Gamma\rangle} \ge \beta(\Gamma)$ for all $\Gamma \subseteq \Omega$, can continue until

⁷The definition of $v_{\rm F}^{[\Omega]}(\beta_{\mu})$ coincides with that of the Shapley value from cooperative game theory. In particular, for all $\omega \in \Omega$, it is easy to verify that

$$v_{\rm F}^{\omega}(\beta_{\mu}) = \sum_{\Gamma \subseteq \overline{\{\omega\}}} \frac{|\Gamma|!(n-|\Gamma|-1)!}{n!} (\beta_{\mu}(\Gamma+\{\omega\}) - \beta_{\mu}(\Gamma)),$$

which is the classical formulation of the Shapley value.

⁶In general, $\mathbb{P}(\beta_{\mu})$ is *degenerate* if not all its *n*! vertices are distinct from each other. Although we will not show it here, it can be shown that, due to the constant-capacity nature of our capacity model, $\mathbb{P}(\beta_{\mu})$ is almost always degenerate to a rather significant degree.

 $d^{\langle \Omega \rangle} = \beta(\Omega)$, which, by definition, guarantees $d^{[\Omega]}$ to land on some $d_*^{[\Omega]} \in \mathbb{P}(\beta)$ eventually.

VI. MAX-SLACK SCHEDULES

So far we have completed the introduction of our general framework. A downside to its generality is its complexity. One source of this complexity comes from the fact that, given a feasible μ , to fully exploit the flexibility of selecting any $d_{\mu}^{[\Omega]}$ from $\mathbb{P}(\beta_{\mu})$, for instance, of selecting $v_{\rm F}^{[\Omega]}(\beta_{\mu})$ as defined in (89), we need to calculate all 2^n values of β_{μ} , which is a daunting task to say the least and actually combinatorially forbidden in the higher dimensional case. Much of this calculation, however, can be avoided by introducing max-slack schedules and their generalization.

According to I2 in Theorem 11, a feasible schedule has to keep $\dot{\lambda}_{ij}^{\langle\Omega\rangle} \leq \dot{c}_{ij}$ for all $i, j \in \mathbb{N}$. This motivates the following question:

Q1 Among all valid schedules in \mathbb{H}_{μ} , is there one that minimizes $\dot{\lambda}_{ij}^{(\Omega)}$ for all $i, j \in \mathbb{N}$ simultaneously?

Such a minimizer would be as good a candidate for a feasible schedule as any, but does it exist? On the surface, given the dimensionality involved, there seems to be no reason to believe its existence. Nonetheless we will show its existence by explicitly constructing a max-slack schedule. We will also show it to be a feasible schedule if \mathbb{F}_{μ} is non-empty. Since \mathbb{F}_{μ} is non-empty if μ is feasible, a max-slack schedule is feasible if μ is feasible.

In constructing a max-slack schedule, as we will see soon, no calculation of β_{μ} is required. But by thus avoiding all calculation of β_{μ} , we are also stuck with a very special feasible schedule and lose all the flexibility promised by our general framework. So is there a way to explore the full flexibilityefficiency continuum instead of dwelling on either pole of it?

We start by aggregating flows into classes. Let \mathcal{P} be the partition of Ω according to which flows are aggregated. When it is used as an index set, following our convention, we denote the ensemble vector of all x^{Γ} 's by $x^{[\mathcal{P}]}$, and the sum, $\sum_{\Gamma \in \mathcal{P}} x^{\Gamma}$, by $x^{\langle \mathcal{P} \rangle}$. Now, for each $\Gamma \in \mathcal{P}$, let ν^{Γ} be the total service to all flows in class Γ so that, given $\nu^{[\mathcal{P}]}$, schedules must only be selected from

$$\mathbb{H}(\nu^{[\mathcal{P}]}) := \{ d^{[\Omega]} | d^{\langle \Gamma \rangle} = \nu^{\Gamma} \text{ for all } \Gamma \in \mathcal{P} \}.$$
(90)

We call $\nu^{[\mathcal{P}]}$ a *class schedule*. Let $\mu = \nu^{\langle \mathcal{P} \rangle}$, and for clarity, we denote $\nu^{[\mathcal{P}]}$ by $\nu^{[\mathcal{P}]}_{\mu}$. It is immediate that $\mathbb{H}(\nu^{[\mathcal{P}]}_{\mu}) \subseteq \mathbb{H}_{\mu}$. A new question is then raised:

Q2 Among all valid schedules in $\mathbb{H}(\nu_{\mu}^{[\mathcal{P}]})$, is there one that minimizes $\dot{\lambda}_{ij}^{\langle\Omega\rangle}$ for all $i, j \in \mathbb{N}$ simultaneously?

By extending the answer to Q1, we will construct this minimizer by generalizing the max-slack schedule so that, intraclass, flows are max-slack scheduled. We will also show it to be a feasible schedule if $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]}) := \mathbb{F} \cap \mathbb{H}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty. But:

Q3 What makes $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ non-empty?

To answer Q3, let us introduce

$$\beta_{\mu}^{\mathcal{P}}(\mathcal{S}) := \beta_{\mu}(\Sigma(\mathcal{S})) \text{ for all } \mathcal{S} \subseteq \mathcal{P}, \tag{91}$$

where $\Sigma(S)$ is the short-hand for $\bigcup_{\Gamma \in S} \Gamma$. By definition, $\beta_{\mu}^{\mathcal{P}}$ is the sampling of β_{μ} on the σ -algebra generated by \mathcal{P} . It is easy to verify that, like β_{μ} , $\beta_{\mu}^{\mathcal{P}}$ is supermodular, and it turns out that $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty if and only if $\nu_{\mu}^{[\mathcal{P}]} \in \mathbb{P}(\beta_{\mu}^{\mathcal{P}})$. Since $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$ is a lower dimensional projection of $\mathbb{P}(\beta_{\mu})$, by judiciously partitioning Ω to adjust the resolution of this projection, we will be able to trade off flexibility for efficiency intermediately.

In the rest of this section, we answer Q1, Q2 and Q3 by enriching the above outline with rigor and details. We also interpret max-slack schedules in terms of worst-case deadlines. This leads us to compare them with EDF schedules, which in turn motivates the introduction of deadline-rigid services.

A. Answering Q1

To answer Q1, we need only focus on the case that i = 0, because in the case that i > 0, according to (41), $\dot{\lambda}_{ij}^{\langle \Omega \rangle}$ is constant with respect to $d_{\mu}^{[\Omega]}$. In the case that i = 0, it is immediate from (41) that

$$\dot{\lambda}_{0j}^{(\Omega)}(d_{\mu}^{[\Omega]}) = \sum_{\omega \in \Omega} (\hat{\lambda}_{0,j+1}^{\omega} - d_{\mu}^{\omega})^+ \ge (\hat{\lambda}_{0,j+1}^{(\Omega)} - \mu)^+, \quad (92)$$

where we spell $\dot{\lambda}_{ij}^{\langle\Omega\rangle}$ as an explicit function of $d_{\mu}^{[\Omega]}$. It is easy to verify that the above lower bound can be achieved when for all $\omega \in \Omega$ and $j \in \mathbb{N}$,

$$\begin{cases} \hat{\lambda}_{0,j+1}^{\omega} \leq d_{\mu}^{\omega} & \text{if } \hat{\lambda}_{0,j+1}^{\langle \Omega \rangle} \leq \mu \\ \hat{\lambda}_{0,j+1}^{\omega} \geq d_{\mu}^{\omega} & \text{if } \hat{\lambda}_{0,j+1}^{\langle \Omega \rangle} > \mu \end{cases},$$
(93)

because when this is satisfied, both ends of (92) equal 0 if $\hat{\lambda}_{0,j+1}^{\langle\Omega\rangle} \leq \mu$, and both equal $\hat{\lambda}_{0,j+1}^{\langle\Omega\rangle} - \mu$ if $\hat{\lambda}_{0,j+1}^{\langle\Omega\rangle} > \mu$. But unfortunately, such a minimizer might not be valid because there is no guarantee that $d_{\mu}^{[\Omega]} \leq q^{[\Omega]}$.

To limit our scope to valid schedules alone, we need to improve the lower bound in (92). Using (49), (41) and (72), we have

$$\begin{split} \dot{\lambda}_{0j}^{\omega}(d_{\mu}^{\omega}) &= \hat{\lambda}_{1,j+1}^{\omega} + (\dot{\lambda}_{0j}^{\omega}(d_{\mu}^{\omega}) - \hat{\lambda}_{1,j+1}^{\omega})^{+} \\ &= \hat{\lambda}_{1,j+1}^{\omega} + (\hat{\lambda}_{0,j+1}^{\omega} - d_{\mu}^{\omega} - \hat{\lambda}_{1,j+1}^{\omega})^{+} \\ &= \hat{\lambda}_{1,j+1}^{\omega} + (p_{j+1}^{\omega} - d_{\mu}^{\omega})^{+}. \end{split}$$

It follows that

$$\dot{\lambda}_{0j}^{(\Omega)}(d_{\mu}^{[\Omega]}) = \hat{\lambda}_{1,j+1}^{(\Omega)} + \sum_{\omega \in \Omega} (p_{j+1}^{\omega} - d_{\mu}^{\omega})^+ \ge \hat{\lambda}_{1,j+1}^{(\Omega)} + (p_{j+1}^{(\Omega)} - \mu)^+$$
(94)

In the same spirit of (93), this improved lower bound can be achieved if there exists $e_{\mu}^{[\Omega]}$ such that, for all $\omega \in \Omega$ and $j \in \mathbb{N}$,

$$\begin{cases} p_{j+1}^{\omega} \leq e_{\mu}^{\omega} & \text{if } p_{j+1}^{\langle \Omega \rangle} \leq \mu \\ p_{j+1}^{\omega} \geq e_{\mu}^{\omega} & \text{if } p_{j+1}^{\langle \Omega \rangle} > \mu \end{cases},$$
(95)

because when $d_{\mu}^{[\Omega]} = e_{\mu}^{[\Omega]}$, both ends of (94) equal $\hat{\lambda}_{1,j+1}^{\langle\Omega\rangle}$ if $p_{j+1}^{\langle\Omega\rangle} \leq \mu$, and both equal $\hat{\lambda}_{1,j+1}^{\langle\Omega\rangle} + p_{j+1}^{\langle\Omega\rangle} - \mu$ if $p_{j+1}^{\langle\Omega\rangle} > \mu$. This leads to the following definition.

Definition 24: We call $e_{\mu}^{[\Omega]} \in \mathbb{H}_{\mu}$ the max-slack schedule under μ if for all $\omega \in \Omega$,

$$\begin{cases} p_{j_{\mu}}^{\omega} \le e_{\mu}^{\omega} \le p_{j_{\mu}+1}^{\omega} & \text{if } j_{\mu} < \infty \\ p_{\infty}^{\omega} \le e_{\mu}^{\omega} \le q^{\omega} & \text{if } j_{\mu} = \infty \end{cases},$$
(96)

where

$$j_{\mu} := \tau_{\mu+1}(\boldsymbol{p}^{\langle \Omega \rangle}). \tag{97}$$

According to (72), $p_{j_{\mu}+1}^{\omega} \leq q^{\omega}$, so (96), by default, requires that $e_{\mu}^{[\Omega]} \leq q^{[\Omega]}$. This is impossible unless $\mu = e_{\mu}^{\langle \Omega \rangle} \leq q^{\langle \Omega \rangle}$. If $\mu \leq q^{\langle \Omega \rangle}$, however, using (97) and (9), it is easy to verify that

$$\begin{cases} p_{j_{\mu}}^{\langle \Omega \rangle} \le \mu < p_{j_{\mu}+1}^{\langle \Omega \rangle} & \text{if } j_{\mu} < \infty \\ p_{\infty}^{\langle \Omega \rangle} \le \mu \le q^{\langle \Omega \rangle} & \text{if } j_{\mu} = \infty \end{cases}$$

$$\tag{98}$$

Comparing this to (96), it is immediate that the existence of $e_{\mu}^{[\Omega]}$ is guaranteed if $\mu \leq q^{\langle \Omega \rangle}$. Also guaranteed is (95), because according to (96) and (98), $p_{j+1}^{\omega} \leq e_{\mu}^{\omega}$ and $p_{j+1}^{\langle \Omega \rangle} \leq \mu$ if $j+1 \leq j_{\mu}$, while $p_{j+1}^{\omega} \geq e_{\mu}^{\omega}$ and $p_{j+1}^{\langle \Omega \rangle} > \mu$ if $j+1 > j_{\mu}$. This prepares for us the next fundamental theorem. *Theorem 25:* If \mathbb{F}_{μ} is non-empty, $e_{\mu}^{[\Omega]}$ exists, $e_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, and

$$\dot{\lambda}_{ij}^{\langle\Omega\rangle}(e_{\mu}^{[\Omega]}) = \min_{d_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}} \dot{\lambda}_{ij}^{\langle\Omega\rangle}(d_{\mu}^{[\Omega]}) \text{ for all } i, j \in \mathbb{N}.$$
(99)

Proof: If \mathbb{F}_{μ} is non-empty, there has to be $\mu \leq q^{\langle \Omega \rangle}$ and $\mu \leq c$. This not only guarantees the existence of $e_{\mu}^{[\Omega]}$, but also guarantees it to be valid, because by default, $e_{\mu}^{[\Omega]} \leq q^{[\Omega]}$ and $e_{\mu}^{\langle \Omega \rangle} = \mu \leq c$. To show that $e_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, then, according to I2 in Theorem 11, we need only show that $e_{\mu}^{[\Omega]} \ge p^{[\Omega]}$ and $\dot{\lambda}_{ij}^{\langle\Omega\rangle}(e_{\mu}^{[\Omega]}) \leq \dot{c}_{ij} \text{ for all } i, j \in \mathbb{N}.$

If \mathbb{F}_{μ} is non-empty, given any $d_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, since it is a feasible schedule, $\mu = d_{\mu}^{\langle \Omega \rangle} \ge p^{\langle \Omega \rangle} = p_1^{\langle \Omega \rangle}$ according to (73). This, according to (97) and (9), implies that $j_{\mu} \geq 1$. So, according to (96) and (73), $e_{\mu}^{\omega} \ge p_{1}^{\omega} = p^{\omega}$ for all $\omega \in \Omega$, i.e., $e_{\mu}^{[\Omega]} \ge p^{[\Omega]}$. Let us move on to $\dot{\lambda}_{ij}^{\langle\Omega\rangle}(e_{\mu}^{[\Omega]})$. Again, since $d_{\mu}^{[\Omega]}$ is a feasible schedule, $\dot{\lambda}_{ij}^{\langle\Omega\rangle}(d_{\mu}^{[\Omega]}) \le \dot{c}_{ij}$ for all $i, j \in \mathbb{N}$. In the case that i = 0, since we have shown that (95) is guaranteed by $e_{\mu}^{[\Omega]}$, the lower bound of (94) can be achieved by $e_{\mu}^{[\Omega]}$, implying that $\dot{\lambda}_{0j}^{(\Omega)}(e_{\mu}^{[\Omega]}) \leq \dot{\lambda}_{0j}^{(\Omega)}(d_{\mu}^{[\Omega]})$. In the case that i > 0, according to (41), $\dot{\lambda}_{ij}^{(\Omega)}(e_{\mu}^{[\Omega]}) = \dot{\lambda}_{ij}^{(\Omega)}(d_{\mu}^{[\Omega]})$. It follows that $\dot{\lambda}_{ij}^{(\Omega)}(e_{\mu}^{[\Omega]}) \leq \dot{\lambda}_{ij}^{(\Omega)}(e_{\mu}^{[\Omega]}) = \dot{\lambda}_{ij}^{(\Omega)}(e_{\mu}^{[\Omega]})$. $\dot{\lambda}_{ii}^{(\Omega)}(d_{\mu}^{[\Omega]}) \leq \dot{c}_{ij}$ for all $i, j \in \mathbb{N}$. Therefore, $e_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$, and (99) must be true.

It follows from the above theorem and Theorem 22 that $e^{[\Omega]}_{\mu}$ is a feasible schedule if $\psi^{[\Omega]}$ is schedulable and μ is feasible. In fact, to identify such a feasible schedule, all we need to know is that \mathbb{F}_{μ} is non-empty, while the knowledge of $\mathbb{F}_{\mu} = \mathbb{P}(\beta_{\mu})$ is utterly irrelevant.⁸ According to Definition 24, $e_{\mu}^{[\Omega]}$ is confined to a hypercuboid. Since this hypercuboid can be easily determined by μ , $p^{[\Omega]}$ and $q^{[\Omega]}$ alone, to construct $e^{[\Omega]}_{\mu}$, no calculation of β_{μ} is required. Of course, according to (82), we still need to calculate $\beta(\Omega)$ to ensure μ to be feasible. But even this can be saved if we let $\mu = \min\{c, q^{\langle \Omega \rangle}\}$, that

is, let the server be work-conserving so that it always serves as many tasks as possible.

In addition, (99) also establishes that $e_{\mu}^{[\Omega]}$ indeed maximizes the server's capacity slack in the next slot, which is where the name max-slack schedule comes from in the first place. An implication is that to select any non-max-slack schedule, there will be a price to pay in terms of less capacity slack in the future, which should be carefully weighed against any potential benefit of such a selection.

B. Answering Q2, and Q3 in a Special Case

Aggregating flows into classes according to \mathcal{P} , a partition of Ω , we can extend Definition 24 and Theorem 25 in the following way.

Definition 26: We call $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]}) \in \mathbb{H}(\nu_{\mu}^{[\mathcal{P}]})$ the intra-class max-slack schedule under $\nu_{\mu}^{[\mathcal{P}]}$ if for all $\omega \in \Gamma \in \mathcal{P}$,

$$\begin{cases} p_{j_{\mu}^{\omega}}^{\omega} \leq e^{\omega}(\nu_{\mu}^{[\mathcal{P}]}) \leq p_{j_{\mu}^{\Gamma}+1}^{\omega} & \text{if } j_{\mu}^{\Gamma} < \infty \\ p_{\infty}^{\omega} \leq e^{\omega}(\nu_{\mu}^{[\mathcal{P}]}) \leq q^{\omega} & \text{if } j_{\mu}^{\Gamma} = \infty \end{cases},$$
(100)

where

$$j_{\mu}^{\Gamma} := \tau_{\nu_{\mu}^{\Gamma}+1}(\boldsymbol{p}^{\langle \Gamma \rangle}). \tag{101}$$

Theorem 27: If $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty, $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]})$ exists, $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]}) \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$, and for all $\Gamma \in \mathcal{P}$,

$$\dot{\lambda}_{ij}^{(\Gamma)}(e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]})) = \min_{d_{\mu}^{[\Omega]} \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})} \dot{\lambda}_{ij}^{(\Gamma)}(d_{\mu}^{[\Omega]}) \text{ for all } i, j \in \mathbb{N}.$$
(102)

We can prove this theorem in the exactly same way that Theorem 25 was proved. The actual proof, however, will be omitted here to avoid repetitions. We call $\nu_{\mu}^{[\mathcal{P}]}$ *feasible* if $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty. Then, a feasible $\nu_{\mu}^{[\mathcal{P}]}$ leads to a feasible intra-class max-slack schedule, but what makes $\nu_{\mu}^{[\mathcal{P}]}$ feasible?

By definition, $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]}) \subseteq \mathbb{F}_{\mu}$, while $\mathbb{F}_{\mu} = \mathbb{P}(\beta_{\mu})$ according to Theorem 22. Therefore, if $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty, given any $d_{\mu}^{[\Omega]} \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]}), d_{\mu}^{[\Omega]} \in \mathbb{P}(\beta_{\mu})$, implying that it has to satisfy (85). For all $S \subseteq \mathcal{P}$, we use short-hand $x^{\langle S \rangle}$ to denote $\sum_{\Gamma \in S} x^{\Gamma}$. Then, using (90) and (85), we have

$$\nu_{\mu}^{\langle \mathcal{S} \rangle} = \sum_{\Gamma \in \mathcal{S}} d_{\mu}^{\langle \Gamma \rangle} = d_{\mu}^{\langle \Sigma(\mathcal{S}) \rangle} \geq \beta_{\mu}(\Sigma(\mathcal{S})),$$

and

$$\nu_{\mu}^{\langle \mathcal{P} \rangle} = d_{\mu}^{\langle \Omega \rangle} = \beta_{\mu}(\Omega) = \mu.$$

So, according to (91),

$$\nu_{\mu}^{\langle \mathcal{P} \rangle} = \beta_{\mu}^{\mathcal{P}}(\mathcal{P}) = \mu \text{ and } \nu_{\mu}^{\langle \mathcal{S} \rangle} \ge \beta_{\mu}^{\mathcal{P}}(\mathcal{S}) \text{ for all } \mathcal{S} \subseteq \mathcal{P}.$$
(103)

That is to say, a necessary condition for $\nu_{\mu}^{[\mathcal{P}]}$ to be feasible is that $\nu_{\mu}^{[\mathcal{P}]} \in \mathbb{P}(\beta_{\mu}^{\mathcal{P}})$, which turns out to be sufficient too.

We start with the special case that $\nu_{\mu}^{[\mathcal{P}]}$ is a vertex of $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$. Recall that each vertex of $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$ can be identified by a permutation over \mathcal{P} . Let $m = |\mathcal{P}|$. Then, given $\sigma : \mathcal{P} \to$ $\{1, 2, \ldots, m\} \in \Pi^{\mathcal{P}}$, where $\Pi^{\mathcal{P}}$ is the set of all permutations over \mathcal{P} , the vertex identified by it, $v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})$, is the unique solution to the system of linear equations defined by

$$\varphi_{\sigma}^{\langle \mathcal{S}_{\sigma}^{j} \rangle}(\beta_{\mu}^{\mathcal{P}}) = \beta_{\mu}^{\mathcal{P}}(\mathcal{S}_{\sigma}^{j}) \text{ for all } 0 \le j \le m,$$
(104)

⁸A related observation is that by definition, $e_{\mu}^{[\Omega]}$ is independent of c, so it will remain the same as c changes. In contrast, according to (83) and (71), β_{μ} is dependent on c, so $\mathbb{P}(\beta_{\mu})$ will expand as c increases and shrink as c decreases.

where

$$\mathcal{S}^{j}_{\sigma} := \{ \Gamma \in \mathcal{P} | \sigma(\Gamma) \le j \}.$$
(105)

The next theorem shows $\mathbb{F}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}))$ to be a non-empty face of $\mathbb{P}(\beta_{\mu})$.

Theorem 28: If $\psi^{[\Omega]}$ is schedulable and μ is feasible, for all $\sigma \in \Pi^{\mathcal{P}}$,

$$\mathbb{F}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})) = \mathbb{P}_{\mathcal{C}_{\sigma}^{\mathcal{P}}}(\beta_{\mu}), \qquad (106)$$

where $\mathbb{P}_{\mathcal{C}_{\mathcal{L}}^{\mathcal{P}}}(\beta_{\mu})$ is the face of $\mathbb{P}(\beta_{\mu})$ identified by a chain,

$$\mathcal{C}^{\mathcal{P}}_{\sigma}: \phi = \Sigma(\mathcal{S}^0_{\sigma}) \subset \Sigma(\mathcal{S}^1_{\sigma}) \subset \dots \subset \Sigma(\mathcal{S}^m_{\sigma}) = \Omega.$$
(107)

Proof: According to Theorem 22, by definition,

$$\mathbb{F}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})) = \mathbb{P}(\beta_{\mu}) \cap \mathbb{H}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})).$$

Notice that using (90), (105), (104), (91) and (107) consecutively, we have

$$\begin{split} \mathbb{H}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})) &= \{d_{\mu}^{[\Omega]} | d_{\mu}^{(\Gamma)} = v_{\sigma}^{\Gamma}(\beta_{\mu}^{\mathcal{P}}) \text{ for all } \Gamma \in \mathcal{P} \} \\ \stackrel{(\dagger)}{=} \{d_{\mu}^{[\Omega]} | d_{\mu}^{\langle \Sigma(S_{\sigma}^{j}) \rangle} = v_{\sigma}^{\langle S_{\sigma}^{j} \rangle}(\beta_{\mu}^{\mathcal{P}}) \text{ for all } 0 \leq j \leq m \} \\ &= \{d_{\mu}^{[\Omega]} | d_{\mu}^{\langle \Sigma(S_{\sigma}^{j}) \rangle} = \beta_{\mu}^{\mathcal{P}}(S_{\sigma}^{j}) \text{ for all } 0 \leq j \leq m \} \\ &= \{d_{\mu}^{[\Omega]} | d_{\mu}^{\langle \Sigma(S_{\sigma}^{j}) \rangle} = \beta_{\mu}(\Sigma(S_{\sigma}^{j})) \text{ for all } 0 \leq j \leq m \} \\ &= \{d_{\mu}^{[\Omega]} | d_{\mu}^{\langle \Gamma \rangle} = \beta_{\mu}(\Gamma) \text{ for all } \Gamma \in \mathcal{C}_{\sigma}^{\mathcal{P}} \}, \end{split}$$

where (†) holds because the linear transformation involved is invertible. Then, according to (62) and (63), $\mathbb{H}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}))$ is exactly what differs $\mathbb{P}_{\mathcal{C}_{\sigma}^{\mathcal{P}}}(\beta_{\mu})$ from $\mathbb{P}(\beta_{\mu})$, so (106) must be true.

C. Answering Q3 in the General Case

Theorem 28 prepares us for the general case that $\nu_{\mu}^{[\mathcal{P}]}$ can be anywhere in $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$.

Theorem 29: If $\psi^{[\Omega]}$ is schedulable and μ is feasible, $\nu^{[\mathcal{P}]}_{\mu}$ is feasible if and only if $\nu^{[\mathcal{P}]}_{\mu} \in \mathbb{P}(\beta^{\mathcal{P}}_{\mu})$.

Proof: ⁹ The necessity of this condition follows directly from its derivation, so we need only show its sufficiency. If $\nu_{\mu}^{[\mathcal{P}]} \in \mathbb{P}(\beta_{\mu}^{\mathcal{P}})$, it has to be a convex combination of all vertices of $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$, which is to say, there must exist $0 \leq w_{\sigma} \leq 1$ such that

$$\sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} = 1 \text{ and } \nu_{\mu}^{[\mathcal{P}]} = \sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}).$$

For each $\sigma \in \Pi^{\mathcal{P}}$, select any $d_{\sigma}^{[\Omega]} \in \mathbb{P}_{\mathcal{C}_{\sigma}^{\mathcal{P}}}(\beta_{\mu})$. It is immediate from (106) that $d_{\sigma}^{[\Omega]} \in \mathbb{F}_{\mu}$ and $d_{\sigma}^{[\Omega]} \in \mathbb{H}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}))$. Now let us construct

$$d^{[\Omega]}_{\mu} = \sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} d^{[\Omega]}_{\sigma}$$

 9 An alternative proof is to apply Frank's sandwich theorem, for which the readers are referred to [8] for the original result and to [7] (p. 799) for a comprehensive introduction. In particular, let us introduce

$$\gamma_{\mu}(\Gamma) := \min_{\mathcal{S} \subseteq \mathcal{P} \text{ and } \Sigma(\mathcal{S}) \supseteq \Gamma} \nu_{\mu}^{\langle \mathcal{S} \rangle} \text{ for all } \Gamma \subseteq \Omega$$

It can be verified that γ_{μ} is a submodular function, that $\gamma_{\mu} \geq \beta_{\mu}$, and that $d_{\mu}^{[\Omega]} \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ if and only if $\gamma_{\mu}(\Gamma) \geq d_{\mu}^{(\Gamma)} \geq \beta_{\mu}(\Gamma)$ for all $\Gamma \subseteq \Omega$. These facts, according to Frank's theorem, guarantee $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ to be non-empty. In comparison, our treatment in the main text not only is self-contained but also reveals more structural information than the mere non-emptiness of $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$.

Then, on the one hand, since $d_{\sigma}^{[\Omega]} \in \mathbb{F}_{\mu}$, $d_{\mu}^{[\Omega]} \in \mathbb{F}_{\mu}$. On the other hand, since $d_{\sigma}^{[\Omega]} \in \mathbb{H}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}))$, for all $\Gamma \in \mathcal{P}$, using (90), we have

$$d_{\mu}^{\langle \Gamma \rangle} = \sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} d_{\sigma}^{\langle \Gamma \rangle} = \sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} v_{\sigma}^{\Gamma} (\beta_{\mu}^{\mathcal{P}}).$$

But by the very definition of w_{σ} , we also have

$$\nu_{\mu}^{\Gamma} = \sum_{\sigma \in \Pi^{\mathcal{P}}} w_{\sigma} v_{\sigma}^{\Gamma}(\beta_{\mu}^{\mathcal{P}}),$$

so $d_{\mu}^{\langle\Gamma\rangle} = \nu_{\mu}^{\Gamma}$ for all $\Gamma \in \mathcal{P}$, i.e., $d_{\mu}^{[\Omega]} \in \mathbb{H}(\nu_{\mu}^{[\mathcal{P}]})$. It follows that $d_{\mu}^{[\Omega]} \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$, $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty and $\nu_{\mu}^{[\mathcal{P}]}$ must be feasible.

One subtlety needs to be addressed here. As a close examination of the above proof would reveal, what we have proved is that $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty in \mathbb{R}^n , but we have not proved it to contain any integral point. This, however, does not constitute an obstacle to the joint application of the above theorem and Theorem 27. Reviewing our entire proof of Theorem 25, nowhere was the integrality of \mathbb{F}_{μ} used. Similarly, in proving Theorem 27 as its extension, nowhere will the integrality of $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ be used, implying that the theorem will hold as long as $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty in \mathbb{R}^{n} .¹⁰

Then, according to Theorems 29 and 27, to select an intraclass max-slack schedule, we can first select a feasible μ , then select a feasible $\nu_{\mu}^{[\mathcal{P}]}$ from $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$, and finally construct $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]})$, which, according to Definition 26, is confined to a hypercuboid that can be easily determined. This makes possible intermediate tradeoffs of flexibility and efficiency. Given $m = |\mathcal{P}|$, to fully determine $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$, according to (91), 2^m valuations of β_{μ} are required. As m decreases, $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$ becomes a coarser projection of $\mathbb{P}(\beta_{\mu})$ and easier to determine, but we lose flexibility. In contrast, as m increases, $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$ becomes a finer projection of $\mathbb{P}(\beta_{\mu})$ and harder to determine, but we gain flexibility.

Recall that, by selecting different points from $\mathbb{P}(\beta_{\mu})$, we can serve all flows according to different priority or fairness criteria. In the same spirit, we can serve all classes according to different criteria by selecting different points from $\mathbb{P}(\beta_{\mu}^{\mathcal{P}})$. For instance, by selecting vertex $v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})$, and thus

$$e_{\sigma}^{[\Omega]}(\beta_{\mu}^{\mathcal{P}}) := e^{[\Omega]}(v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})), \tag{108}$$

a priority order is enforced so that the larger $\sigma(\Gamma)$ is, the higher priority class Γ enjoys. In contrast, by selecting the vertex centroid,

$$v_{\mathrm{F}}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}) = \frac{1}{m!} \sum_{\sigma \in \Pi^{\mathcal{P}}} v_{\sigma}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}}), \tag{109}$$

and thus

$$e_{\mathrm{F}}^{[\Omega]}(\beta_{\mu}^{\mathcal{P}}) := e^{[\Omega]}(v_{\mathrm{F}}^{[\mathcal{P}]}(\beta_{\mu}^{\mathcal{P}})), \qquad (110)$$

¹⁰Actually it can rather work the other way around. If $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ is non-empty in \mathbb{R}^n , Theorem 27 guarantees that $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]}) \in \mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$. But $e^{[\Omega]}(\nu_{\mu}^{[\mathcal{P}]})$ is integral by definition, so $\mathbb{F}(\nu_{\mu}^{[\mathcal{P}]})$ has to contain at least one integral point. Of course this can also be established directly. Should we choose to apply Frank's sandwich theorem as suggested in the preceding footnote, it would follow directly from the discrete part of that theorem. a fair criterion is enforced.

Example 30: In the simple case that $\mathcal{P} = \{\Gamma, \overline{\Gamma}\}, \mathbb{P}(\beta_{\mu}^{\mathcal{P}})$ *is but a line segment with two vertices, that is, two endpoints. At one endpoint, it is easy to verify that*

$$(\nu_{\mu}^{\Gamma}, \nu_{\mu}^{\overline{\Gamma}}) = (\mu - \beta_{\mu}(\overline{\Gamma}), \beta_{\mu}(\overline{\Gamma}))$$

which gives class Γ the highest priority possible against class $\overline{\Gamma}$. At the other endpoint,

$$(\nu_{\mu}^{\Gamma}, \nu_{\mu}^{\overline{\Gamma}}) = (\beta_{\mu}(\Gamma), \mu - \beta_{\mu}(\Gamma)),$$

which reveres the priority order to favor $\overline{\Gamma}$. To be fair, the vertex centroid, that is, the midpoint, should be selected. Of course, to fine-tune the balance, any point along the line segment could be considered.

D. Worst-Case Deadlines

All our discussion so far is flow-centric in that it has always focused on how many tasks of each flow should be served. For max-slack schedules, it is also enlightening to take a taskcentric view. Let us consider the *h*th task of a generic flow. According to (9), $\tau_h(\mathbf{p})$ is non-decreasing with respect to *h*. So, using (96), it is easy to verify that

$$\begin{cases} \tau_h(\boldsymbol{p}) \le j_\mu & \text{if } h \le e_\mu \\ \tau_h(\boldsymbol{p}) \ge j_\mu & \text{if } h > e_\mu \end{cases}.$$
(111)

That is to say, if e_{μ} tasks are served, the *h*th task must be served if $\tau_h(\mathbf{p}) < j_{\mu}$, but it cannot be served if $\tau_h(\mathbf{p}) > j_{\mu}$. Since j_{μ} is a constant across all flows, a max-slack schedule has to serve all tasks in a non-decreasing order of $\tau_h(\mathbf{p})$. Conversely, it can also be verified that serving in such an order will always result in a max-slack schedule. By the same reasoning, intra-class, the service order of an intra-class maxslack schedule has to be dictated by $\tau_h(\mathbf{p})$ too. But what is the meaning of $\tau_h(\mathbf{p})$?

For all h > 0, according to (9) and (72),

$$\tau_h(\boldsymbol{p}) = \max\{j \in \mathbb{N} | p_j = \max\{\hat{\lambda}_{0j}, q\} < h\}$$

In the case that $h \leq q$, using (40) and (9), we then have

$$\tau_{h}(\boldsymbol{p}) = \max\{j \in \mathbb{N} | \lambda_{0j} = \max_{\boldsymbol{q} \in \mathbb{U} | q} \psi_{j}(\boldsymbol{q}) < h\}$$

$$= \max\{j \in \mathbb{N} | \psi_{j}(\boldsymbol{q}) < h \text{ for all } \boldsymbol{q} \in \mathbb{U} | q\}$$

$$\stackrel{(\dagger)}{=} \max\{j \in \mathbb{N} | j \leq \tau_{h}(\psi(\boldsymbol{q})) \text{ for all } \boldsymbol{q} \in \mathbb{U} | q\}$$

$$= \min_{\boldsymbol{q} \in \mathbb{U} | q} \tau_{h}(\psi(\boldsymbol{q})), \qquad (112)$$

where (†) holds because by definition, $\psi_j(\mathbf{q}) < h$ if and only if $j \leq \tau_h(\psi(\mathbf{q}))$. For the *h*th task, since it should be served no later than slot $t + \tau_h(\psi(\mathbf{q}))$ to ensure that $\mathbf{d} \geq \psi(\mathbf{q})$, it should be served no later than $t + \tau_h(\mathbf{p})$ to ensure that ψ can be guaranteed no matter which $\mathbf{q} \in \mathbb{U}|q$ will realize. We call $t + \tau_h(\mathbf{p})$ the **worst-case deadline** for the *h*th task. In light of this, (111) says that a max-slack schedule has to serve all tasks in a non-decreasing order of their respective worst-case deadlines.

Worst-case deadlines are dynamic. In general, during a task's stay in the queue, its worst-case deadline can only be relaxed because well, the worst case could not get worse. To

be specific, consider the case that $q \ge h > d \ge p$. In this case, the *h*th task in slot *t* becomes the (h-d)th task in t+1. Accordingly, its worst-case deadline becomes $t+1+\tau_{h-d}(\dot{\mathbf{p}})$. Using (112), (9) and (31), we have

$$\begin{aligned} &\tau_{h-d}(\dot{\boldsymbol{p}}) \\ &= \min_{\dot{\boldsymbol{q}} \in \mathbb{U} \mid \dot{q}} \tau_{h-d}(\dot{\boldsymbol{\psi}}(\dot{\boldsymbol{q}})) \\ &= \min_{\dot{\boldsymbol{q}} \in \mathbb{U} \mid \dot{q}} \max\{j \in \mathbb{N} \mid \dot{\psi}_{j}(\dot{\boldsymbol{q}}) < h-d\} \\ &\stackrel{(\dagger)}{=} \min_{\boldsymbol{q} \in \mathbb{U} \mid q \text{ and } q_{2} = \dot{q} + d} \max\{j \in \mathbb{N} \mid [\mathcal{R}^{-1}(\boldsymbol{\psi}(\boldsymbol{q}) - d\boldsymbol{\delta})^{+}]_{j} < h-d\}. \end{aligned}$$

where (†) holds because, according to (28) and (27), $\dot{\boldsymbol{q}} \in \mathbb{U}|\dot{q}$ is equivalent to $\boldsymbol{q} \in \mathbb{U}|q$ and $q_2 = \dot{q} + d$. Using (30) and (9), we then have

$$1 + \tau_{h-d}(\mathbf{\dot{p}})$$

$$= 1 + \min_{\mathbf{q} \in \mathbb{U}|q \text{ and } q_2 = \dot{q}+d} \max\{j \in \mathbb{N} | \psi_{j+1}(\mathbf{q}) - d < h - d\}$$

$$= \min_{\mathbf{q} \in \mathbb{U}|q \text{ and } q_2 = \dot{q}+d} \max\{j \in \mathbb{N} \setminus \{0\} | \psi_j(\mathbf{q}) < h\}$$

$$\stackrel{(\dagger)}{=} \min_{\mathbf{q} \in \mathbb{U}|q \text{ and } q_2 = \dot{q}+d} \tau_h(\psi(\mathbf{q})), \qquad (113)$$

where (†) holds because, according to (24), h > p implies that $\psi_1(q) < h$ for all $q \in \mathbb{U}|q$, which in turn implies that $j \in \mathbb{N} \setminus \{0\}$, instead of \mathbb{N} , is inconsequential. Comparing (113) to (112), it is immediate that $1 + \tau_{h-d}(\dot{p}) \ge \tau_h(p)$, so the task's worst-case deadline can only be relaxed.

E. Deadline-Rigid Services

The task-centric interpretation suggests a strong affinity of max-slack schedules to EDF schedules, according to which, each task, right upon its arrival, is assigned a static deadline that will remain constant during its entire stay and all tasks are then served in a non-decreasing order of these static deadlines. A comprehensive survey of EDF schedules can be found in [9]. To generate EDF schedules, the EDF scheduler sorts all tasks in a single queue according to their respective static deadlines, with the task having the earliest static deadline to be always at the head and served first. When a new task arrives, it is inserted into the queue according to its newly assigned static deadline, without disturbing the relative order of old tasks already there. This cannot be applied to a general worst-case system because, as we have seen, worst-case deadlines are dynamic, which deems it impossible to keep the relative order of old tasks stable. There is, however, a special class of worst-case services for which worst-case deadlines become static.

Definition 31: A worst-case service, ψ , is deadline-rigid if

$$\psi(q) \stackrel{h}{=} \psi(q')$$
 for all $q, q' \in \mathbb{U} b, h > 0$ and $q \stackrel{h}{=} q',$
(114)

where we use $\mathbf{x} \stackrel{h}{=} \mathbf{x'}$ to denote the relation that $\min{\{\mathbf{x}, h\}} = \min{\{\mathbf{x'}, h\}}$.

The intuition behind this definition is best illustrated by Figure 7, according to which, if q and q' are identical up to point A, $\psi(q)$ and $\psi(q')$ have to be identical up to B. Since causality would only require them to be identical up to C, a corollary is that deadline-rigid services are causal, but not vice versa. For instance, it is easy to verify that uniform-delay



Fig. 7. Properties of a deadline-rigid service.

services in Example 3 are deadline-rigid but uniform-backlog services in Example 2 are not, though both are causal.

If ψ is deadline-rigid, on the one hand, using (9) and (114), it is easy to verify that $\tau_h(\psi(q)) = \tau_h(\psi(q'))$ if $q \stackrel{h}{=} q'$, which is illustrated in Figure 7. On the other hand, for all $q \ge h > d$, by definition, $q \stackrel{h}{=} q\delta$ for all $q \in \mathbb{U}|q$. Therefore, according to (112) and (113), $1 + \tau_{h-d}(\dot{p}) = \tau_h(\psi(q\delta))$. That is to say, if ψ is deadline-rigid, worst-case deadlines do become static. Deadline-rigid services are also update invariant. Recall that the effect of the update rule can be explained by a translated coordinate frame. This is illustrated in Figure 7, where O is translated to O'. Since such a translation does not change the relative positions of points A and B, it should preserve the deadline-rigidness of ψ .

A worst-case system is *deadline-rigid* if every worst-case service in it is deadline-rigid. In such a system, all worstcase deadlines become static. Then the EDF scheduler can be applied, and according to the task-centric interpretation, the EDF schedules so generated are automatically max-slack schedules. According to Theorem 25, to further ensure these schedules to be feasible, we need only ensure μ , the total service, to be feasible. In light of this, an additional corollary of Theorem 25 is that, if any scheduler can meet all static deadlines, so can the EDF scheduler, which is a well-known fact.¹¹ We can also generalize the EDF scheduler to generate intra-class max-slack schedules. Such a scheduler keeps a separate queue for each class and, intra-class, serves tasks earliest-deadline-first. Of course, to ensure the schedules so generated to be feasible, we need to first select a feasible class schedule.

VII. MIN-PLUS SERVICES

Another source of our framework's complexity comes from the fact that a general worst-case service, as an uncountably infinite full-blown map between cumulative vectors, is difficult to specify and update. For state-based scheduling, on the surface, it seems that we need only keep track of its spectrum, to admit new service requests, and its conditional spectrum, to identify feasible schedules, both of which are countably infinite. But these are not self-contained entities because, unless a worst-case service is uniquely identified by its spectrum, which is not the case in general, its conditional spectrum cannot be identified by its spectrum alone. So are there worst-case services that can be uniquely identified by their spectra?

We start with the observation that a worst-case service is upper bounded by its spectrum. To see this, notice that, according to (33), $\psi_i(\mathbf{q}) - q_i \leq \lambda_{ij}(\boldsymbol{\psi})$ for all $i, j \in \mathbb{N}$, so

$$\psi_j(\boldsymbol{q}) \le \min_{i \in \mathbb{N}} (q_i + \lambda_{ij}(\boldsymbol{\psi})). \tag{115}$$

This can be rewritten concisely using the *min-plus algebra*, in which operators min and + replace, respectively, + and \times in the standard algebra. Let us use \otimes to denote the minplus matrix multiplication and arrange all $\lambda_{ij}(\psi)$'s into the following matrix,

$$\Lambda(\boldsymbol{\psi}) = [\lambda_{ij}(\boldsymbol{\psi})]_{i,j\in\mathbb{N}} = \begin{bmatrix} \lambda_{00}(\boldsymbol{\psi}) & \lambda_{01}(\boldsymbol{\psi}) & \lambda_{02}(\boldsymbol{\psi}) & \cdots \\ \lambda_{10}(\boldsymbol{\psi}) & \lambda_{11}(\boldsymbol{\psi}) & \lambda_{12}(\boldsymbol{\psi}) & \cdots \\ \lambda_{20}(\boldsymbol{\psi}) & \lambda_{21}(\boldsymbol{\psi}) & \lambda_{22}(\boldsymbol{\psi}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ & & (116) \end{bmatrix}$$

Then, applying the min-plus matrix multiplication rule, (115) can be rewritten in the matrix form as

$$\psi(q) \le q \otimes \Lambda(\psi).$$
 (117)

It is exactly the formulation of this upper bound that motivates our definition for min-plus services. Interestingly, min-plus services so defined in turn make this upper bound achievable.

In the rest of this section, we first use spectral matrices to define min-plus services. We then show that these services are uniquely identified by their spectra, and that each schedulable non-min-plus system can be upgraded to a schedulable minplus system. We also use cumulative matrices to provide a second definition for min-plus services. This, among other things, allows us to introduce the composing rule for min-plus services.

A. A First Definition through Spectral Matrices

Definition 32: A semi-infinite matrix, $S = [s_{ij}]_{i,j \in \mathbb{N}}$, is a spectral matrix if for all $i, j \in \mathbb{N}$,

$$s_{ij} = 0 \quad \text{if} \ i \ge j, \tag{118}$$

$$s_{ij} \le s_{i,j+1},\tag{119}$$

$$s_{ij} \ge s_{i+1,j},\tag{120}$$

and

$$s_{ij} \le (s_{0j} - b\delta_i)^+.$$
 (121)

We call ψ^S the **min-plus service** identified by S if

$$\boldsymbol{\psi}^{S}(\boldsymbol{q}) := \boldsymbol{q} \otimes S \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b.$$
 (122)

According to this definition, a spectral matrix is triangular in that all elements in its lower triangle and along its diagonal

¹¹An independent proof of this fact can be found in [10]. The key idea is that when two tasks are not served in the order dictated by their static deadlines, the order can always be reversed without violating any static deadline. For a deadline-rigid system, this also provides the basis for an alternative proof of Theorem 25.

are 0. Also it has non-decreasing rows and non-increasing columns. For all $j \in \mathbb{N}$, using (122), (118) and the min-plus matrix multiplication rule, we have

$$\psi_j^S(\boldsymbol{q}) = [\boldsymbol{q} \otimes S]_j = \min_{i \in \mathbb{N}} (q_i + s_{ij})$$
(123)
$$= \min\left\{\min_{i < j} (q_i + s_{ij}), \min_{i \ge j} q_i\right\}$$
$$= \min\left\{\min_{i < j} (q_i + s_{ij}), q_j\right\}$$
$$= \min_{i \le j} (q_i + s_{ij}).$$
(124)

According to (123) and (119), $\psi_j^S(\mathbf{q}) \leq \psi_{j+1}^S(\mathbf{q})$, so $\psi^S(\mathbf{q})$ is a cumulative vector. According to the derivation of (124), $\psi_j^S(\mathbf{q}) \leq q_j$, so $\psi^S(\mathbf{q}) \leq \mathbf{q}$. Then, according to Definition 1, ψ^S is indeed a worst-case service. In addition, min-plus services are also deadline-rigid because, using (123), we have

$$\min\{\psi_j^S(\boldsymbol{q}), h\} = \min_{i \in \mathbb{N}} \min\{q_i + s_{ij}, h\}$$
$$= \min_{i \in \mathbb{N}} \min\{\min\{q_i, h\} + s_{ij}, h\},$$

implying that $\psi^{S}(q) \stackrel{h}{=} \psi^{S}(q')$ if $q \stackrel{h}{=} q'$.

Since (118), (119), (120) and (121) are direct counterparts of (34), (35), (36) and (37) respectively, s_{ij} has to behave very much like λ_{ij} , the spectral value of some worst-case service. It is fundamental that this some worst-case service could be ψ^{S} itself.

Theorem 33: Given a spectral matrix, S, for all $i, j \in \mathbb{N}$,

$$\lambda_{ij}(\boldsymbol{\psi}^S) = s_{ij}.\tag{125}$$

To prove this and the next theorem, let us introduce $\varepsilon = [\varepsilon_j]_{j \in \mathbb{N}} := [0, \infty, \infty, \ldots] \in \mathbb{U}$, i.e.,

$$\varepsilon_j := \begin{cases} 0 & \text{if } j = 0\\ \infty & \text{if } j > 0 \end{cases} .$$
(126)

Intuitively ε models an infinite traffic burst in slot t. Then, according to (123) and (120),

$$\psi_j^{S}(\mathcal{R}^i \boldsymbol{\varepsilon} + q\boldsymbol{\delta}) = \min_{k \in \mathbb{N}} ([\mathcal{R}^i \boldsymbol{\varepsilon}]_k + q\delta_k + s_{kj})$$

$$\stackrel{(\dagger)}{=} \min_{k \leq i} (q\delta_k + s_{kj})$$

$$= \min\{s_{0j}, q\delta_i + s_{ij}\}, \qquad (127)$$

where (†) holds because by definition,

$$[\mathcal{R}^{i}\varepsilon]_{k} = \begin{cases} 0 & \text{if } k \leq i \\ \infty & \text{if } k > i \end{cases}$$
(128)

Proof of Theorem 33: On the one hand, since by default, $\mathcal{R}^i \varepsilon + b \delta \in \mathbb{U} \rangle b$, using (33), (127), (128) and (121), we have

$$\lambda_{ij}(\boldsymbol{\psi}^{S}) \geq (\psi_{j}^{S}(\boldsymbol{q}) - q_{i})^{+}|_{\boldsymbol{q} = \mathcal{R}^{i}\boldsymbol{\varepsilon} + b\boldsymbol{\delta}}$$

$$= (\psi_{j}^{S}(\mathcal{R}^{i}\boldsymbol{\varepsilon} + b\boldsymbol{\delta}) - [\mathcal{R}^{i}\boldsymbol{\varepsilon}]_{i} - b\delta_{i})^{+}$$

$$= (\min\{s_{0j}, b\delta_{i} + s_{ij}\} - b\delta_{i})^{+}$$

$$= \min\{(s_{0j} - b\delta_{i})^{+}, s_{ij}\} = s_{ij}.$$

On the one hand, using (33) and (123), we have

$$\lambda_{ij}(\boldsymbol{\psi}^S) = \max_{\boldsymbol{q} \in \mathbb{U} \rangle b} \left(\min_{k \in \mathbb{N}} (q_k + s_{kj}) - q_i \right)^+$$
$$\leq \max_{\boldsymbol{q} \in \mathbb{U} \rangle b} \left(q_i + s_{ij} - q_i \right)^+ \qquad = s_{ij}.$$

It follows that $\lambda_{ij}(\boldsymbol{\psi}^S) = s_{ij}$.

According to the above theorem, s_{ij} and $\lambda_{ij}(\psi^S)$ are synonymous, which is where the name spectral matrix comes from in the first place. A corollary is that a min-plus service, through its spectral matrix, is uniquely identified by its spectrum. So its conditional spectrum is also uniquely identified by its spectrum. We will denote $\lambda_{ij}(\psi^S|q)$ by \hat{s}_{ij} when no confusion can be introduced. The next theorem shows how to identify \hat{s}_{ij} through s_{ij} .

Theorem 34: Given a spectral matrix, S, for all $i, j \in \mathbb{N}$,

$$\hat{s}_{ij} = \begin{cases} \min\{s_{0j}, q + s_{1j}\} & \text{if } i = 0\\ \min\{(s_{0j} - q)^+, s_{ij}\} & \text{if } i > 0 \end{cases} .$$
(129)

Proof: In the case that i = 0, on the one hand, since by default, $\mathcal{R}\varepsilon + q\delta \in \mathbb{U}|q$, using (40) and(127), we have

$$\hat{s}_{0j} \ge \psi_j^S(\boldsymbol{q})|_{\boldsymbol{q}=\mathcal{R}\boldsymbol{\varepsilon}+q\boldsymbol{\delta}} = \psi_j^S(\mathcal{R}\boldsymbol{\varepsilon}+q\boldsymbol{\delta}) = \min\{s_{0j}, q+s_{1j}\}.$$

On the other hand, using (40) and (123), we have

$$\hat{s}_{0j} = \max_{q \in \mathbb{U}|q} \min_{k \in \mathbb{N}} (q_k + s_{kj}) \le \max_{q \in \mathbb{U}|q} \min\{s_{0j}, q_1 + s_{1j}\} = \min\{s_{0j}, q + s_{1j}\}.$$

It follows that the first half of (129) must be true.

In the case that i > 0, on the one hand, since by default, $\mathcal{R}^i \varepsilon + q \delta \in \mathbb{U}|q$, using (40), (127) and (128), we have

$$\begin{aligned} \hat{s}_{ij} &\geq (\psi_j^S(\boldsymbol{q}) - q_i)^+ |_{\boldsymbol{q} = \mathcal{R}^i \boldsymbol{\varepsilon} + q \boldsymbol{\delta}} \\ &= (\psi_j^S(\mathcal{R}^i \boldsymbol{\varepsilon} + q \boldsymbol{\delta}) - [\mathcal{R}^i \boldsymbol{\varepsilon}]_i - q)^+ \\ &= (\min\{s_{0j}, q + s_{ij}\} - q)^+ \qquad = \min\{(s_{0j} - q)^+, s_{ij}\}.\end{aligned}$$

On the other hand, using (40) and (123), we have

$$\begin{aligned} \hat{s}_{ij} &= \max_{q \in \mathbb{U}|q} \left(\min_{k \in \mathbb{N}} (q_k + s_{kj}) - q_i \right)^+ \\ &\leq \max_{q \in \mathbb{U}|q} \left(\min\{s_{0j}, q_i + s_{ij}\} - q_i \right)^+ \\ &= \max_{q \in \mathbb{U}|q} \min\{(s_{0j} - q_i)^+, s_{ij}\} \quad \stackrel{(\dagger)}{\leq} \min\{(s_{0j} - q)^+, s_{ij}\} \end{aligned}$$

where (†) holds because by definition, $q_i \ge q_1 = q$ for all $q \in \mathbb{U}|q$. It follows that the second half of (129) must be true.

Finally the next theorem shows that min-plus services are update invariant.

Theorem 35: In Theorem 4, let ψ be a min-plus service, identified by a spectral matrix, S, i.e., $\psi = \psi^S$. Then $\dot{\psi}$ is also a min-plus service that can be identified by a spectral matrix, $\dot{S} = [\dot{s}_{ij}]_{i,j \in \mathbb{N}}$, i.e., $\dot{\psi} = \psi^{\dot{S}}$, where

$$\dot{s}_{ij} = \begin{cases} (\min\{s_{0,j+1}, q + s_{1,j+1}\} - d)^+ & \text{if } i = 0\\ \min\{(s_{0,j+1} - q)^+, s_{i+1,j+1}\} & \text{if } i > 0 \end{cases} .$$
(130)

The proof for this theorem will be skipped here because we will prove its alternative formulation later. It is, however, worth noting that, comparing (130) to (129), it is immediate that

$$\dot{s}_{ij} = \begin{cases} (\hat{s}_{0,j+1} - d)^+ & \text{if } i = 0\\ \hat{s}_{i+1,j+1} & \text{if } i > 0 \end{cases} .$$
(131)

Now this relation between \dot{s}_{ij} and \hat{s}_{ij} exactly replicates that between $\dot{\lambda}_{ij}$ and $\hat{\lambda}_{ij}$ in (41), which is not a coincidence, because according to Theorem 33, $\dot{s}_{ij} = \lambda_{ij}(\psi^{\dot{S}})$.

B. The Spectral Hull

A *min-plus system* is a worst-case system in which every worst-case service is a min-plus service. For all $\omega \in \Omega$, let the min-plus service guaranteed to flow ω be identified by a spectral matrix, S^{ω} , and we denote the system by $S^{[\Omega]}$. Quite impressive is its efficiency in comparison to a general worst-case system. Besides that $S^{[\Omega]}$ can be more efficiently identified, according to Theorem 35, it can also be more efficiently updated. In addition, according to Theorems 33 and 34, it is straightforward to keep track of all spectra and conditional spectra for $S^{[\Omega]}$. In particular, for $S^{[\Omega]}$, the schedulability condition, (50), can be rewritten in the matrix form as

$$S^{\langle \Omega \rangle} \le C = [c_{ij}]_{i,j \in \mathbb{N}} = [(j-i)^+ c]_{i,j \in \mathbb{N}}.$$
(132)

A worst-case system, $\psi^{[\Omega]}$, is *dominated* by another, $\bar{\psi}^{[\Omega]}$, if for all $\omega \in \Omega$, $\bar{\psi}^{\omega} \geq \psi^{\omega}$, i.e., $\bar{\psi}^{\omega}(q^{\omega}) \geq \psi^{\omega}(q^{\omega})$ for all $q^{\omega} \in \mathbb{U} \rangle b^{\omega}$. If both are also schedulable, we can then, by upgrading $\psi^{[\Omega]}$ to $\bar{\psi}^{[\Omega]}$, improve the service to each flow, and yet, preserve schedulability. Within this broader context, as we will see, the efficiency of min-plus systems is no longer of isolated interests only, but assumes a broader significance in that any schedulable non-min-plus system can be upgraded to a schedulable min-plus system.

Given a worst-case service, ψ , let $S = \Lambda(\psi)$, where $\Lambda(\psi)$ is specified by (116). Then S is a spectral matrix because (118), (119), (120) and (121) are direct counterparts of (34), (35), (36) and (37) respectively. Also, according to (125), $\Lambda(\psi^S) = S$, so according to (122), $\psi^S(q) = q \otimes \Lambda(\psi^S)$. Comparing this to (117), it is immediate that $\psi^S \ge \psi$. That is to say, among all worst-case services that share the same spectrum, there is a min-plus service that turns out to be the maximum. Notice that \ge only defines a partial order among worst-case services, so the existence of such a maximizer is not self-evident.

Given a worst-case system, $\psi^{[\Omega]}$, let us construct a min-plus system, $S^{[\Omega]}$, such that for all $\omega \in \Omega$, $S^{\omega} = \Lambda(\psi^{\omega})$. It follows from our reasoning above that $S^{[\Omega]}$ dominates $\psi^{[\Omega]}$, and that they share the same system of spectra, implying that if $\psi^{[\Omega]}$ is schedulable, so is $S^{[\Omega]}$. We call $S^{[\Omega]}$ the *spectral hull* of $\psi^{[\Omega]}$. Then we can always upgrade a schedulable non-minplus system to its spectral hull. A corollary is that, in Figure 4, the Pareto frontier encircling the schedulable region must consist of min-plus systems exclusively.

Example 36: For $\psi^{[\Omega]}$ in Example 12, according to (56), $S^{[\Omega]}$ is its spectral hull if for all $\omega \in \Omega$ and $i, j \in \mathbb{N}$,

$$s_{ij}^{\omega} = \begin{cases} 1 & \text{if } j - i > \bar{\theta}^{\omega} \\ 0 & \text{if } j - i \le \bar{\theta}^{\omega} \end{cases} .$$
(133)

Then, using (123), it can be verified that for all $q^{\omega} \in \mathbb{U}$,

$$\boldsymbol{\psi}^{S^{\omega}}(\boldsymbol{q}^{\omega}) = \begin{cases} \mathcal{R}^{i+\bar{\theta}^{\omega}}\boldsymbol{\delta} & \text{if } \mathcal{R}^{i}\boldsymbol{\delta} \leq \boldsymbol{q}^{\omega} \leq \mathcal{R}^{i}\boldsymbol{\varepsilon} \text{ for some } i \in \mathbb{N} \\ \boldsymbol{0} & \text{if } \boldsymbol{q}^{\omega} = \boldsymbol{0} \end{cases}$$
(134)

which says that the first task of flow ω will be served with a delay no more than $\overline{\theta}^{\omega}$. Comparing this to (55), it is immediate that $\psi^{S^{\omega}} \geq \psi^{\omega}$, so the guarantees of $S^{[\Omega]}$ are much stronger

than those of $\psi^{[\Omega]}$, though they do share the same system of spectra. ¹²

By upgrading $\psi^{[\Omega]}$ to its spectral hull, $S^{[\Omega]}$, since they share the same system of spectra, the server's capacity slack in the current slot is kept intact. There is, however, still a price to pay. Imagine running $\psi^{[\Omega]}$ and $S^{[\Omega]}$ side by side. For all $\omega \in \Omega$ and $i, j \in \mathbb{N}$, using (48), (46) and (45), it is easy to verify that

$$\hat{\lambda}_{ij}^{\omega} \leq \begin{cases} \min\{\lambda_{0j}^{\omega}, q^{\omega} + \lambda_{1j}^{\omega}\} & \text{if } i = 0\\ \min\{(\lambda_{0j}^{\omega} - q^{\omega})^+, \lambda_{ij}^{\omega}\} & \text{if } i > 0 \end{cases},$$
(135)

where λ_{ij}^{ω} and $\hat{\lambda}_{ij}^{\omega}$ are the short-hands for $\lambda_{ij}(\psi^{\omega})$ and $\lambda_{ij}(\psi^{\omega}|q^{\omega})$ respectively. Comparing this to (129), it is immediate that $\hat{\lambda}_{ij}^{\omega} \leq \hat{s}_{ij}^{\omega}$ because, by construction, $\lambda_{ij}^{\omega} = s_{ij}^{\omega}$. So, according to (71), $\beta^{S} \geq \beta^{\psi}$, i.e., $\beta^{S}(\Gamma) \geq \beta^{\psi}(\Gamma)$ for all $\Gamma \subseteq \Omega$, where we use β^{S} and β^{ψ} to denote the baseline functions for $S^{[\Omega]}$ and $\psi^{[\Omega]}$ respectively. This, according to Theorem 20, in turn implies that $\mathbb{F}^{S} \subseteq \mathbb{F}^{\psi}$, where we use \mathbb{F}^{S} and \mathbb{F}^{ψ} to denote the feasible polytopes for $S^{[\Omega]}$ and $\psi^{[\Omega]}$ respectively. Therefore, by upgrading $\psi^{[\Omega]}$ to $S^{[\Omega]}$, we might reduce the feasible polytope and thus reduce our flexibility to select any feasible schedule from it.

Furthermore, imagine selecting the same feasible schedule, $d^{[\Omega]} \in \mathbb{F}^S \subseteq \mathbb{F}^{\psi}$, for both $\psi^{[\Omega]}$ and $S^{[\Omega]}$. Comparing (41) to (131), it is immediate that $\dot{\lambda}_{ij}^{\omega} \leq \dot{s}_{ij}^{\omega}$, where $\dot{\lambda}_{ij}^{\omega}$ is the short-hand for $\lambda_{ij}(\dot{\psi}^{\omega})$, because, as we have seen, $\hat{\lambda}_{ij}^{\omega} \leq \hat{s}_{ij}^{\omega}$. That is to say, although $\dot{S}^{[\Omega]}$ still dominates $\dot{\psi}^{[\Omega]}$, they might no longer share the same system of spectra. Therefore, by upgrading $\psi^{[\Omega]}$ to $S^{[\Omega]}$, we might reduce the server's capacity slack in the future.

Example 37: For $\psi^{[\Omega]}$ in Example 12, consider the case that $\mathbf{q}^{\omega} = 2\boldsymbol{\delta}$ for all $\omega \in \Omega$. According to (55), no task need be served in this case. In contrast, for the spectral hull of $\psi^{[\Omega]}$, $S^{[\Omega]}$, in Example 36, according to (134), the first task of each flow needs to be served either immediately, to reduce the feasible polytope, or later, to reduce the server's capacity slack in the future.

C. A Second Definition through Cumulative Matrices

Examining our derivation of (123) and (124), nowhere were (120) and (121) used, so neither is necessary for ψ^S to be a worst-case service. This motivates a second definition for minplus services.

Definition 38: A semi-infinite matrix, $M = [m_{ij}]_{i,j \in \mathbb{N}}$, is a cumulative matrix if for all $i, j \in \mathbb{N}$,

$$m_{ij} = 0 \quad \text{if} \ i \ge j, \tag{136}$$

and

$$m_{ij} \le m_{i,j+1}.\tag{137}$$

¹²According to (55), ψ^{ω} is clearly not causal. Then, to guarantee ψ^{ω} , a causal scheduler has to guarantee ψ^{C} , where, using (22), it can be verified that for all $q^{\omega} \in \mathbb{U}$,

$$\psi^{\mathsf{C}}(\boldsymbol{q}^{\omega}) = \begin{cases} \mathcal{R}^{i+\bar{\theta}^{\omega}}\boldsymbol{\delta} & \text{if } \mathcal{R}^{i}\boldsymbol{\delta} \leq \boldsymbol{q}^{\omega} \leq \mathcal{R}^{i}\boldsymbol{\delta} + \mathcal{R}^{i+\bar{\theta}^{\omega}+1}\boldsymbol{\varepsilon} \text{ for some } i \in \mathbb{N} \\ \boldsymbol{0} & \text{otherwise} \end{cases}$$

Comparing this to (134), it is immediate that $\psi^{S^{\omega}} \ge \psi^{C}$, so the guarantee of $\psi^{S^{\omega}}$ is not only stronger than that of ψ^{ω} but also stronger than that of ψ^{C} .

We call ψ^M the **min-plus service** identified by M if

$$\mathcal{B}^{M}(\boldsymbol{q}) := \boldsymbol{q} \otimes M \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b.$$
 (138)

Then, for all $j \in \mathbb{N}$, we can replicate (123) and (124) with

$$\psi_j^M(\boldsymbol{q}) = [\boldsymbol{q} \otimes M]_j = \min_{i \in \mathbb{N}} (q_i + m_{ij}) \tag{139}$$

$$= \min_{i \le j} (q_i + m_{ij}),$$
 (140)

because no counterpart of (120) or (121) is needed here. But by leaving out the counterparts of (120) and (121) in the above definition, would not we unduly enlarge the concept for min-plus services? This worry turns out to be unwarranted according to the next theorem.

Theorem 39: Given cumulative matrix M, let us introduce a semi-infinite matrix, $S = [s_{ij}]_{i,j \in \mathbb{N}}$, such that

$$s_{ij} = \min\left\{ (m_{0j} - b\delta_i)^+, \min_{k \le i} m_{kj} \right\}.$$
 (141)

Then S is a spectral matrix and $\psi^S = \psi^M$.

Proof: It is easy to verify that S is indeed a spectral matrix. Then, according to (123) and (141),

$$\psi_j^S(\boldsymbol{q}) = \min\left\{\min_{i\in\mathbb{N}}[q_i + (m_{0j} - b\delta_i)^+], \min_{i\in\mathbb{N}}\min_{k\leq i}(q_i + m_{kj})\right\}$$

On the one hand, for the first term above, using (11) and (139), we have

$$\min_{i \in \mathbb{N}} [q_i + (m_{0j} - b\delta_i)^+] \ge \min_{i \in \mathbb{N}} (q_i - b\delta_i) + m_{0j}$$
$$= \min_{i \in \mathbb{N}} a_i + m_{0j}$$
$$= m_{0j} \qquad \stackrel{(\dagger)}{\ge} \psi_j^M(\boldsymbol{q}),$$

where (†) holds because $m_{0j} = q_0 + m_{0j} \ge \psi_j^M(\mathbf{q})$. On the other hand, for the second term above, changing the order of the two min operators and then using (139), we have

$$\min_{i \in \mathbb{N}} \min_{k \leq i} (q_i + m_{kj}) = \min_{k \in \mathbb{N}} \min_{i \geq k} (q_i + m_{kj})$$
$$\stackrel{(\dagger)}{=} \min_{k \in \mathbb{N}} (q_k + m_{kj}) \qquad = \psi_j^M(\boldsymbol{q}),$$

where (†) holds because $q_i \ge q_k$ for all $i \ge k$. It follows that $\psi_j^S(\boldsymbol{q}) = \psi_j^M(\boldsymbol{q})$ for all $\boldsymbol{q} \in \mathbb{U} > b$, i.e., $\boldsymbol{\psi}^S = \boldsymbol{\psi}^M$.

According to Theorem 33, each unique spectral matrix identifies a unique min-plus service with a unique spectrum. In contrast, according to the above theorem, the same min-plus service can be identified by different cumulative matrices. Is not this redundant? But it is exactly this redundancy that affords us additional maneuvering room in treating several subjects subsequently. Let us start with an alternative formulation of Theorem 35.

Theorem 40: In Theorem 4, let ψ be a min-plus service, identified by a cumulative matrix, M, i.e., $\psi = \psi^M$. Then $\dot{\psi}$ is also a min-plus service that can be identified by a cumulative matrix, $\dot{M} = [\dot{m}_{ij}]_{i,j\in\mathbb{N}}$, i.e., $\dot{\psi} = \psi^M$, where

$$\dot{m}_{ij} = \begin{cases} (\min\{m_{0,j+1}, q + m_{1,j+1}\} - d)^+ & \text{if } i = 0\\ m_{i+1,j+1} & \text{if } i > 0 \end{cases}$$
(142)

Once this theorem is proved, Theorem 35 is but a corollary. In particular, to derive (130), we can first treat S as a general

cumulative matrix and then use (142) to find \dot{M} . Although the \dot{M} so found might not be a spectral matrix, we can always use (141) to turn it into \dot{S} . This is quite straightforward conceptually, but it does require a bit of derivation. There is actually a shortcut. Once the update invariance of min-plus services is established, according to Theorem 33, $\dot{s}_{ij} = \lambda_{ij}(\psi^{\dot{S}})$. Then we can replicate (41) with (131), which, through (129), leads to (130) directly. Also observe that the second half of (142) is much simpler than that of (130), which prepares us for the introduction of dual-curve services later.

Proof of Theorem 40: We first show \dot{M} to be a cumulative matrix. Using (142), (137) and (136), it is easy to verify that $\dot{m}_{ij} \leq \dot{m}_{i,j+1}$, and $\dot{m}_{ij} = 0$ if $i \geq j > 0$. Then it all hinges on whether $\dot{m}_{00} = 0$. Since $d \geq p$, using (24), (140) and (136), we have

$$d \ge \max_{q \in \mathbb{U}|q} \psi_1^M(q) = \max_{q \in \mathbb{U}|q} \min\{m_{01}, q_1 + m_{11}\} = \min\{m_{01}, q\}.$$

So, according to (142) and (136),

$$\dot{m}_{00} = (\min\{m_{01}, q + m_{11}\} - d)^+ = (\min\{m_{01}, q\} - d)^+ = 0.$$

It follows that \dot{M} is indeed a cumulative matrix.

It remains for us to show that $\dot{\psi}$ can be identified by \dot{M} . For all $\dot{q} \in \mathbb{U} \mid \dot{b}$ and $j \in \mathbb{N}$, using (31), (30), (139), (27) and (142) consecutively, we have

$$\begin{split} \dot{\psi}_{j}(\dot{q}) &= [\mathcal{R}^{-1}(\psi^{M}(q) - d\delta)^{+}]_{j} \\ &= (\mathcal{W}_{j+1}^{M}(q) - d)^{+} \\ &= \left(\min_{i \in \mathbb{N}} (q_{i} + m_{i,j+1}) - d\right)^{+} \\ &= \left(\min\left\{m_{0,j+1}, q_{1} + m_{1,j+1}, \min_{i>1} (q_{i} + m_{i,j+1})\right\} - d\right)^{+} \\ &= \left(\min\left\{m_{0,j+1}, q_{1} + m_{1,j+1}, \min_{i>0} (q_{i+1} + m_{i+1,j+1})\right\} - d\right)^{+} \\ &= \left(\min\left\{m_{0,j+1}, q + m_{1,j+1}, \min_{i>0} (\dot{q}_{i} + d + m_{i+1,j+1})\right\} - d\right)^{+} \\ &= \min\left\{(\min\{m_{0,j+1}, q + m_{1,j+1}\} - d)^{+}, \min_{i>0} (\dot{q}_{i} + m_{i+1,j+1})\right\} \\ &= \min_{i \in \mathbb{N}} (\dot{q}_{i} + \dot{m}_{ij}). \end{split}$$

That is to say,
$$\dot{\psi}(\dot{q}) = \dot{q} \otimes \dot{M}$$
 for all $\dot{q} \in \mathbb{U} \rangle \dot{b}$, i.e., $\dot{\psi} = \psi^{\dot{M}}$.

D. Monotonicity and Composability

A worst-case service, ψ , is *monotone* if $q \ge q'$ implies that $\psi(q) \ge \psi(q')$. Intuitively it says that there should be no punishment for feeding the server too many tasks. The next theorem shows that monotone services are composable.

Theorem 41: As illustrated in Figure 8, when a flow is guaranteed monotone services, ψ^{I} and ψ^{II} , by two servers in tandem, the overall effect can be modeled by a single server, with $\mathbf{a} = \mathbf{a}^{I}$, $b = b^{I} + b^{II}$ and $\mathbf{d} = \mathbf{d}^{II}$, that guarantees the flow a monotone service, $\psi = \psi^{II} \star \psi^{I}$, defined by

$$\psi(\boldsymbol{q}) = \boldsymbol{\psi}^{\mathrm{II}} \star \boldsymbol{\psi}^{\mathrm{I}}(\boldsymbol{q}) := \boldsymbol{\psi}^{\mathrm{II}}(\boldsymbol{\psi}^{\mathrm{II}}(\boldsymbol{q} - b^{\mathrm{II}}\boldsymbol{\delta}) + b^{\mathrm{II}}\boldsymbol{\delta}) \text{ for all } \boldsymbol{q} \in \mathbb{U} \rangle b.$$
(143)



Fig. 8. Composing two monotone services into one.

Proof: Since ψ^{II} is monotone, using (12) twice, we have $\psi^{II}(\psi^{I}(\boldsymbol{q} - b^{II}\boldsymbol{\delta}) + b^{II}\boldsymbol{\delta}) \leq \psi^{II}(\boldsymbol{q} - b^{II}\boldsymbol{\delta} + b^{II}\boldsymbol{\delta}) = \psi^{II}(\boldsymbol{q}) \leq \boldsymbol{q}.$

That is to say, $\psi(q) \leq q$, so according to Definition 1, ψ is a worst-case service. If $q \geq q'$, since both ψ^{I} and ψ^{II} are monotone, it is immediate that

$$\psi^{\mathrm{II}}(\psi^{\mathrm{I}}(\boldsymbol{q}-b^{\mathrm{II}}\boldsymbol{\delta})+b^{\mathrm{II}}\boldsymbol{\delta}) \geq \psi^{\mathrm{II}}(\psi^{\mathrm{I}}(\boldsymbol{q'}-b^{\mathrm{II}}\boldsymbol{\delta})+b^{\mathrm{II}}\boldsymbol{\delta}).$$

That is to say, $\psi(q) \ge \psi(q')$ if $q \ge q'$, so ψ is also monotone.

To show that the flow is guaranteed ψ , it is convenient to recast a worst-case service as a map of *a* instead of *q*. To be specific, let us introduce φ^{I} and φ^{II} such that

$$\boldsymbol{\varphi}^{\mathrm{I}}(\boldsymbol{a}^{\mathrm{I}}) := \boldsymbol{\psi}^{\mathrm{I}}(\boldsymbol{q}^{\mathrm{I}}) = \boldsymbol{\psi}^{\mathrm{I}}(\boldsymbol{a}^{\mathrm{I}} + b^{\mathrm{I}}\boldsymbol{\delta}) \text{ for all } \boldsymbol{a}^{\mathrm{I}} \in \mathbb{U},$$

and

$$\boldsymbol{\varphi}^{\mathrm{II}}(\boldsymbol{a}^{\mathrm{II}}) := \boldsymbol{\psi}^{\mathrm{II}}(\boldsymbol{q}^{\mathrm{II}}) = \boldsymbol{\psi}^{\mathrm{I}}(\boldsymbol{a}^{\mathrm{II}} + b^{\mathrm{II}}\boldsymbol{\delta}) \ \text{ for all } \boldsymbol{a}^{\mathrm{II}} \in \mathbb{U}.$$

Since φ^{II} is also monotone,

$$oldsymbol{d} = oldsymbol{d}^{\mathrm{II}} \geq oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}^{\mathrm{II}}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{d}^{\mathrm{II}}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a})) = oldsymbol{arphi}^{\mathrm{II}}(oldsymbol{a}) = oldsymbol{arphi}^{\mathrm{II}}$$

But by definition,

$$\varphi^{\mathrm{II}}(\varphi^{\mathrm{I}}(a)) = \psi^{\mathrm{II}}(\psi^{\mathrm{I}}(a^{\mathrm{I}} + b^{\mathrm{I}}\delta) + b^{\mathrm{II}}\delta) \stackrel{(\dagger)}{=} \psi^{\mathrm{II}}(\psi^{\mathrm{I}}(q - b^{\mathrm{II}}\delta) + b^{\mathrm{II}}\delta),$$

where (†) holds because, according to (11),

$$q = a + b\delta = a^{\mathrm{I}} + b^{\mathrm{I}}\delta + b^{\mathrm{II}}\delta$$

Therefore, according to (143), $d \ge \psi(q)$ for all $q \in \mathbb{U} \rangle b$, and ψ must be guaranteed.

Using (139), it is easy to verify that min-plus services are monotone. The next theorem shows that they are also composable.

Theorem 42: In Theorem 41, let ψ^{I} and ψ^{II} both be minplus services, identified by cumulative matrices, M^{I} and M^{II} , respectively, i.e., $\psi^{I} = \psi^{M^{I}}$ and $\psi^{II} = \psi^{M^{II}}$. Then $\psi = \psi^{II} \star \psi^{I}$ is also a min-plus service that can be identified by a cumulative matrix,

$$M = (M^{I} + b^{II}\Delta) \otimes M^{II}, \qquad (144)$$

where $\Delta = [\delta_{ij}]_{i,j \in \mathbb{N}}$ is a cumulative matrix defined by

$$\delta_{ij} := (\delta_j - \delta_i)^+ = \begin{cases} 1 & \text{if } i = 0 \text{ and } j > 0\\ 0 & \text{otherwise} \end{cases}$$
(145)

According to this theorem, if a flow passes through a network of servers and is guaranteed a min-plus service by each server along its path, the end-to-end service to it can be modeled by a single min-plus service. This makes minplus services amenable to network performance analysis. In addition, notice that, in (144), even if M^{I} and M^{II} are both spectral matrices, M is not guaranteed to be one. Of course, we can always use (141) to turn M into S, but we will thus lose the simplicity of (144).

Proof of Theorem 42: For all
$$\boldsymbol{q} \in \mathbb{U} \rangle b$$
 and $j > 0$,

$$[(\boldsymbol{q} - b^{\mathrm{II}} \boldsymbol{\delta}) \otimes M^{\mathrm{I}} + b^{\mathrm{II}} \boldsymbol{\delta}]_{j} = \min_{i \in \mathbb{N}} (q_{i} - b^{\mathrm{II}} \delta_{i} + m^{\mathrm{I}}_{ij}) + b^{\mathrm{II}} \delta_{j}$$

$$\stackrel{(\dagger)}{=} \min_{i \in \mathbb{N}} (q_{i} + m^{\mathrm{I}}_{ij} + b^{\mathrm{II}} \delta_{ij})$$

$$= [\boldsymbol{q} \otimes (M^{\mathrm{I}} + b^{\mathrm{II}} \Delta)]_{j},$$

where (†) holds because, according to (145), $\delta_{ij} = \delta_j - \delta_i$ if j > 0. While by default, both ends above equal 0 if j = 0, so

$$(\boldsymbol{q} - b^{\mathrm{II}}\boldsymbol{\delta}) \otimes M^{\mathrm{I}} + b^{\mathrm{II}}\boldsymbol{\delta} = \boldsymbol{q} \otimes (M^{\mathrm{I}} + b^{\mathrm{II}}\Delta).$$

Then, using (143), (138), (144) and the associativity of \otimes , we have

$$\begin{split} \boldsymbol{\psi}(\boldsymbol{q}) &= \boldsymbol{\psi}^{M^{\mathrm{II}}} \star \boldsymbol{\psi}^{M^{\mathrm{I}}}(\boldsymbol{q}) \\ &= [(\boldsymbol{q} - b^{\mathrm{II}}\boldsymbol{\delta}) \otimes M^{\mathrm{I}} + b^{\mathrm{II}}\boldsymbol{\delta}] \otimes M^{\mathrm{II}} \\ &= \boldsymbol{q} \otimes (M^{\mathrm{I}} + b^{\mathrm{II}}\Delta) \otimes M^{\mathrm{II}} \qquad = \boldsymbol{q} \otimes M, \end{split}$$

i.e., $\boldsymbol{\psi} &= \boldsymbol{\psi}^{M}. \end{split}$

VIII. DUAL-CURVE SERVICES

A key observation from (142) is that if $m_{ij} = m_{i+1,j+1}$ for all i > 0, $\dot{m}_{ij} = m_{ij}$ for all i > 0. In this case, while the 0th row of M, by definition, is a cumulative vector that is dynamic, the part of M below the 0th row is static. Since the elements in this static part remain constant along its diagonal and along each off-diagonal, they can be compressed into a cumulative vector that is also static. Such is the motivation for dual-curve services. Their relation to some other services that we have investigated is illustrated in Figure 9. Being the most specialized, these services exhibit the greatest efficiency, which verges on practical viability.

In the rest of this section, we summarize main results regarding dual-curve services and then relate these results to service curves. We also comment on practical issues regarding implementation.

A. A Summary

Definition 43: Given a cumulative matrix, M, if there exist $u, v \in \mathbb{U}$ such that for all $i, j \in \mathbb{N}$,

$$m_{ij} = \begin{cases} u_j & \text{if } i = 0\\ v_{(j-i)^+} & \text{if } i > 0 \end{cases},$$
(146)

we call ψ^M the **dual-curve service** identified by the pair, (u, v), and denote it by $\psi^{(u,v)}$.

Most results regarding dual-curve services are straightforward specializations of those regarding min-plus services. Here we will sample some important ones without derivation. For all $q \in \mathbb{U} \rangle b$ and $j \in \mathbb{N}$, using (140), it is easy to verify that

$$\psi_j^{(\boldsymbol{u},\boldsymbol{v})}(\boldsymbol{q}) = \min\left\{u_j, \min_{0 < i \le j} (q_i + v_{j-i})\right\}.$$
 (147)



Fig. 9. Relating dual-curve services to some other services.

Dual-curve services are update invariant. In Theorem 40, if M can be identified by $(\boldsymbol{u}, \boldsymbol{v})$, it can be shown that \dot{M} can be identified $(\dot{\boldsymbol{u}}, \boldsymbol{v})$ so that \boldsymbol{v} , being static, remains the same, and \boldsymbol{u} , being dynamic, is updated to $\dot{\boldsymbol{u}} = [\dot{u}_j]_{j \in \mathbb{N}} \in \mathbb{U}$, where

$$\dot{u}_j = (\min\{u_{j+1}, q + v_j\} - d)^+,$$
 (148)

or in the vector form,

$$\dot{\boldsymbol{u}} = \mathcal{R}^{-1}(\min\{\boldsymbol{u}, q\boldsymbol{\delta} + \mathcal{R}\boldsymbol{v}\} - d\boldsymbol{\delta})^+.$$
(149)

Dual-curve services are also composable. In Theorem 42, if M^{I} and M^{II} can be identified by $(\boldsymbol{u}^{\text{I}}, \boldsymbol{v}^{\text{I}})$ and $(\boldsymbol{u}^{\text{II}}, \boldsymbol{v}^{\text{II}})$ respectively, it can be shown that M can be identified by $(\boldsymbol{u}, \boldsymbol{v})$ such that for all $j \in \mathbb{N}$,

$$\begin{cases} u_{j} = \min\left\{\min_{0 < i \le j} (u_{i}^{\mathrm{I}} + b^{\mathrm{II}} + v_{j-i}^{\mathrm{II}}), u_{j}^{\mathrm{II}}\right\} \\ v_{j} = \min_{0 \le i \le j} (v_{i}^{\mathrm{I}} + v_{j-i}^{\mathrm{II}}) \end{cases}$$
(150)

Let us move on to the spectrum. For all $i, j \in \mathbb{N}$, according to (146), and Theorems 39 and 33,

$$\lambda_{ij}^{(\boldsymbol{u},\boldsymbol{v})} = \begin{cases} u_j & \text{if } i = 0\\ \min\left\{(u_j - b)^+, v_{(j-i)^+}\right\} & \text{if } i > 0 \end{cases}, \quad (151)$$

where $\lambda_{ij}^{(u,v)}$ is the short-hand for $\lambda_{ij}(\psi^{(u,v)})$. As to the conditional spectrum, according to (146), and Theorems 39 and 34,

$$\hat{\lambda}_{ij}^{(\boldsymbol{u},\boldsymbol{v})} = \begin{cases} \min\{u_j, q + v_{(j-1)^+}\} & \text{if } i = 0\\ \min\{(u_j - q)^+, v_{(j-i)^+}\} & \text{if } i > 0 \end{cases}, \quad (152)$$

where $\hat{\lambda}_{ij}^{(\boldsymbol{u},\boldsymbol{v})}$ is the short-hand for $\lambda_{ij}(\boldsymbol{\psi}^{(\boldsymbol{u},\boldsymbol{v})}|q)$.

A *dual-curve system* is a min-plus system in which every min-plus service is a dual-curve service. For all $\omega \in \Omega$, let the dual-curve service guaranteed to flow ω be identified by $(\boldsymbol{u}^{\omega}, \boldsymbol{v}^{\omega})$, and we denote the system by $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$. If it is schedulable, according to (151), the schedulability condition, (50), requires that for all $j \in \mathbb{N}$,

$$\begin{cases} u_j^{\langle \Omega \rangle} \le jc \\ \sum_{\omega \in \Omega} \min\left\{ (u_{\infty}^{\omega} - b^{\omega})^+, v_j^{\omega} \right\} \le jc \end{cases}$$
(153)

We call $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ non-degenerate if for all $\omega \in \Omega$, $u_{\infty}^{\omega} = \infty$ and $v_{\infty}^{\omega} = \infty$, in which case, (153) can be rewritten in the vector form as

$$\begin{cases} \boldsymbol{u}^{\langle \Omega \rangle} \leq \boldsymbol{c} \\ \boldsymbol{u}^{\langle \Omega \rangle} \leq \boldsymbol{c} \end{cases}, \tag{154}$$

where

$$\boldsymbol{c} = [c_j]_{j \in \mathbb{N}} := [jc]_{j \in \mathbb{N}}.$$
(155)

Notice that, for (154) to hold, $v_{\infty}^{\omega} = \infty$ is in fact not necessary. The reason that it is required here is to ensure that if $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ is non-degenerate, so is $(\boldsymbol{\dot{u}}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$.

For all $j \in \mathbb{N}$, it is immediate from (72) and (152) that

$$p_j^{\omega} = \min\{u_j^{\omega}, q^{\omega}\},\tag{156}$$

or in the vector form,

$$\boldsymbol{p}^{\omega} = \min\{\boldsymbol{u}^{\omega}, q^{\omega}\boldsymbol{\delta}\}.$$
 (157)

Then, for all $\Gamma \subseteq \Omega$, according to (74), (156) and (152),

$$\beta(\Gamma) = \max_{j \in \mathbb{N}} \left\{ \sum_{\substack{\omega \in \Gamma \\ w \in \Omega}} \min\{u_{j+1}^{\omega}, q^{\omega}\} + \sum_{\substack{\omega \in \Omega \\ w \in \Omega}} \min\{(u_{j+1}^{\omega} - q^{\omega})^{+}, v_{j}^{\omega}\} - jc \right\}.$$
(158)

The above results allow us to check schedulability, determine the feasible polytope, and identify max-slack schedules as well as intra-class max-slack schedules.

B. Relation to Service Curves

In the case that b = 0, if u = v, $\psi^{(v,v)}$ is equivalent to the service curve specified by v, because in this case, it is easy to verify that (147) and (150) recover the well-known min-plus convolution rules of service curves, for which a comprehensive survey can be found in [11]. Since neither b = 0 nor u = v can be preserved by, respectively, (3) or (149), service curves are not update invariant, and thus only by introducing dual-curve services, they are extended to their dynamic closure. In light of this, what we have done with dual-curve services is to incorporate the case that b > 0, but such an extension is essential for state-based scheduling because we cannot update a state to something that has not been defined.

According to [12], service curves can be viewed as timeinvariant min-plus filters, whose time-invariant nature is reflected by the fact that, in the case that b = 0, according to (147),

$$\psi^{(\boldsymbol{v},\boldsymbol{v})}(\mathcal{R}^{i}\boldsymbol{\varepsilon}) = \mathcal{R}^{i}\boldsymbol{v} \text{ for all } i \in \mathbb{N}.$$
 (159)

This leads to the introduction of time-varying min-plus filters in [13]. The relation of these filters to min-plus services exactly parallels that of service curves to dual-curve services. In particular, in the case that b = 0, min-plus services are equivalent to these filters, whose time-varying nature is reflected by the fact that, in this case, according to (123),

$$\boldsymbol{\psi}^{S}(\mathcal{R}^{i}\boldsymbol{\varepsilon}) = S_{i}. \text{ for all } i \in \mathbb{N},$$
(160)

where S_i is the *i*th row of S.

Since min-plus services are deadline-rigid, so are dual-curve services and thus service curves. By taking advantage of this fact, the service-curve-earliest-deadline (SCED) scheduler was proposed in [14] to guarantee service curves. At its core, it is but a work-conserving EDF scheduler that automatically generates feasible max-slack schedules. Interestingly, it was realized in [15] that such a scheduler need not be always work-conserving. Recall that being work-conserving implies that $\mu = \min\{c, q^{\langle \Omega \rangle}\}$. Instead, it was proposed in [15], to rephrase in our terminology, to estimate $\beta(\Omega)$, set $\mu = \beta(\Omega)$, and then allocate the excess capacity, up to $c - \beta(\Omega)$, in the spirit of the GPS policy. Also worth noting in [14], [15] is that some dynamic features of dual-curve services were explored implicitly, though service specification there was still tied to the case that b = 0.

C. Practical Issues

We expect a state-based scheduler, if ever implemented, to be first implemented for a dual-curve system because of its efficiency, which is on display in every preceding result regarding dual-curve services. We further speculate that, for early adoptions, logistic systems and data centers would be particularly promising areas. In the former case, since a natural slot in typical logistic systems can often last for hours, days, or even weeks, there should be enough time to carry out a significant amount of computation during each slot. In the latter case, since modern data centers concentrate huge capacities for data storage, processing and distribution, there should exist strong incentives to improve capacity utilization even by a tiny fraction.

As we have seen, a schedulable worst-case system, $\psi^{[\Omega]}$, can always be upgraded to its spectral hull, $S^{[\Omega]}$. Now $S^{[\Omega]}$ can in turn be dominated by a dual-curve system, $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$, if for all $\omega \in \Omega$ and $j \in \mathbb{N}$,

$$\begin{cases} u_{j}^{\omega} = s_{0j}^{\omega} \\ v_{j}^{\omega} = \max_{i>0} s_{i,i+j}^{\omega} \end{cases}$$
(161)

We call $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ the *dual-curve hull* of $S^{[\Omega]}$. Notice that it might not be schedulable even if $S^{[\Omega]}$ is schedulable. But when it is, $S^{[\Omega]}$ can be upgraded to its dual-curve hull, which endows the efficiency of dual-curve systems with a broader significance.

It is nonetheless impossible to implement a state-based scheduler for a dual-curve system in its most general form despite its efficiency, because well, countably infinite is still infinite. We need to further reduce the system's dimensionality. One approach is to restrict dual-curve services to be piece-linear. According to (149), if both u and v are piece-linear, so is \dot{u} . As time goes by, however, u could still grow unwieldy. So how to contain this growth? In [14], [15], problems of a similar nature were encountered and additional restrictions were imposed on service curves to cap the number of their linear pieces. We can certainly borrow the technique, but we will not delve into the details here. Instead, let us outline a couple of techniques unique to our framework.¹³

The idea is to take advantage of the dynamic nature of statebased scheduling. On the one hand, given that $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ is schedulable, we can upgrade $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ to $(\bar{\boldsymbol{u}}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ if there exists $\bar{\boldsymbol{u}}^{[\Omega]} \geq \boldsymbol{u}^{[\Omega]}$ such that $(\bar{\boldsymbol{u}}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ is also schedulable, implying that $\bar{\boldsymbol{u}}^{\langle\Omega\rangle} \leq \boldsymbol{c}$ if $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ is nondegenerate. Of course, this upgrade could reduce the server's capacity slack, but it may be well worth it if the number of linear pieces can become much smaller. On the other hand, according to (149), the larger d^{ω} is, the simpler $\dot{\boldsymbol{u}}^{\omega}$ tends to be, because more linear pieces of \boldsymbol{u}^{ω} tend to be buried below 0. Then, in selecting a feasible schedule, the more unwieldy \boldsymbol{u}^{ω} is, the higher priority should be given to flow ω .

Example 44: The above techniques can be applied to any dual-curve system, not necessarily piece-linear. Given any $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ that is schedulable, when $d^{[\Omega]} = q^{[\Omega]}$ so that $\dot{\boldsymbol{b}}^{\omega} = 0$ for all $\omega \in \Omega$, according to (149), $\dot{\boldsymbol{u}}^{[\Omega]} \leq \boldsymbol{v}^{[\Omega]}$. So $(\dot{\boldsymbol{u}}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ can be upgraded to $(\boldsymbol{v}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ if $(\boldsymbol{u}^{[\Omega]}, \boldsymbol{v}^{[\Omega]})$ is non-degenerate. That is to say, whenever all buffers are emptied, a non-degenerate dual-curve system can be upgraded to a system of non-degenerate service curves. A corollary is that, if we start with a system of non-degenerate service curves, we can always return to it whenever all buffers are emptied, which is a well-known fact in [14].

Another approach to reduce the system's dimensionality is to restrict services to be finite. This is only of limited applicability, because although in reality all flow lifetimes are finite, in most cases, they are long enough to be taken as infinite. That being said, there are still cases for which flow lifetimes become meaningfully finite, for instance, file transfers through a communication link or sporadic jobs in a real-time system. In these cases, their finiteness can be taken advantage of.

In general, a worst-case service, ψ , is **finite** if there exists $g \in \mathbb{N}$ such that $\psi_j(q) = \psi_q(q)$ for all $q \in \mathbb{U} \mid b$ and $j \geq g$, thus limiting its guarantee to period [t, t + g). A min-plus service, ψ^{M} , is then finite if $m_{ij} = m_{ig}$ for all $j \geq g$, implying that $m_{ij} = 0$ for all $i \ge g$. Therefore, what really matters is $M_{[0 \le i, j \le g]}$, the $(g+1) \times (g+1)$ submatrix at the upper-left corner of M. But unless v = 0, a dual-curve service, $\psi^{(u,v)}$, cannot be finite because, according to (146), there simply cannot be $m_{ij} = 0$ for all $i \ge g$. To get around this difficulty, we can limit the applicable range of (146) to $0 \leq i, j \leq g$ so that $M_{[0 \leq i, j \leq q]}$ alone can be identified by a pair of finite vectors, $(u_{[0 \le j \le g]}, v_{[0 \le j \le g-1]})$. More results along this line, paralleling those regarding original dual-curve services, can be developed, and we can even develop the finite counterparts of piece-linear dual-curve services, though we will not delve into the details here.

IX. CONCLUDING REMARKS

In this paper, we have shed new light on the classical problem of using short-run scheduling decisions to provide long-run service guarantees. We have drawn our inspiration from cumulative vectors, the state-space approach, the polymatroid theory, EDF schedules, the min-plus algebra, and service curves. Although individually none of these is new, it is our contribution to weave all of them into a general framework of worst-case services and state-based scheduling that provides novel solutions to the classical problem.

Among possible extensions, the most interesting ones, in our view, are suggested by generalizing the capacity model.

¹³These techniques, we believe, can make piece-linear dual-curve systems manageable in most practical cases. Of course, the final arbiter of this matter can only be simulation and implementation, which are out of the scope of this paper.

For instance, comparing the definition of c_{ij} in (50) to that of c in (155), it is immediate that

$$c_{ij} = (c_j - c_i)^+ \text{ for all } i, j \in \mathbb{N}.$$
(162)

Now, with this formulation, if we allow c to be an arbitrary cumulative vector, then, the server's capacity is $c_{j+1} - c_j$ in slot t + j, and we obtain a *time-varying* capacity model. It is straightforward to extend our results to this model with minimal modifications.

Alternatively, the capacity constraint, (5), can be generalized to $d^{[\Omega]} \in \mathbb{C}$. To circumvent the nuances caused by integrality, let us also switch to a continuous traffic model so that $d^{[\Omega]}$ takes its value in \mathbb{R}^n and \mathbb{C} is a subset of \mathbb{R}^n .¹⁴ In a basic case, let

$$\sum_{\omega \in \Omega} \frac{d^{\omega}}{c^{\omega}} \le 1, \tag{163}$$

and we obtain a *linear* capacity model. It is easy to extend our results to this model by scaling the worst-case service guaranteed to each flow accordingly. For instance, the schedulability condition, (50), in this case morphs into

$$\sum_{\omega \in \Omega} \frac{\lambda_{ij}^{\omega}}{c^{\omega}} \le (j-i)^+ \text{ for all } i, j \in \mathbb{N}.$$
 (164)

Any violation of linearity in the specification of \mathbb{C} , however, will pose a great challenge. Even in the simple case that \mathbb{C} is determined by only two linear constraints, which, for instance, is the case for a switching network with two effective bottlenecks and no internal buffer or delay, it is not clear how our results can be extended.¹⁵

The heart of the problem lies in the fact that in the nonlinear case, there seems to be no obvious counterpart of the spectrum, which, as we have seen in the linear case, helps us distill all essential information inherent in a worst-case service that is relevant to schedulability. Therefore, a key bridge to schedulability is lost. This situation is not unlike that in the traditional system theory, where frequency-domain analysis, though powerful in the linear case, has no obvious counterpart in the non-linear case. We leave further study of this issue to future work.

APPENDIX

In this appendix, we prove two lemmas from Section V-A.

Proof of Lemma 15: We denote $\mathbb{P}_{\mathcal{S}}(\chi)$ by $\mathbb{P}_{\mathcal{S}}$. If $\mathbb{P}_{\mathcal{S}}$ is non-empty, given any $d^{[\Omega]} \in \mathbb{P}_{\mathcal{S}}$, for all $\Gamma, \Gamma' \in \mathcal{S}$, using (63) and (62), we have

$$\begin{split} \chi(\Gamma) + \chi(\Gamma') &= d^{\langle \Gamma \rangle} + d^{\langle \Gamma' \rangle} \\ &= d^{\langle \Gamma + \Gamma' \rangle} + d^{\langle \Gamma \Gamma' \rangle} \geq \chi(\Gamma + \Gamma') + \chi(\Gamma \Gamma'). \end{split}$$

But χ is supermodular, so (61) must hold with equality. This equality embodies a special relation between Γ and Γ' , and

 14 We have got a flavor of these nuances in our discussion following the proof of Theorem 29. As shown in that case, they could be subtle, but not insurmountable.

¹⁵This case can be viewed as the intersection of two linear models. Then a naive guess is that it would be schedulable should (164) hold for both models. This is certainly necessary, but it is not perpetuatable because there is no guarantee that the two feasible polytopes thus determined would always intersect. we denote it by $\Gamma \simeq \Gamma'$. Then $\mathbb{P}_{\mathcal{S}}$'s non-emptiness implies that $\Gamma \simeq \Gamma'$ for all $\Gamma, \Gamma' \in \mathcal{S}$. By default, $\Gamma \simeq \Gamma'$ if $\Gamma \subseteq \Gamma'$ or $\Gamma' \subseteq \Gamma$. Conversely, in the case that $\Gamma \subseteq \Gamma'$ or $\Gamma' \subseteq \Gamma$ if $\Gamma \simeq \Gamma'$, we call χ strictly supermodular. In this case, $\mathbb{P}_{\mathcal{S}}$'s non-emptiness implies that \mathcal{S} has to be a chain.

Let us move on to the case that χ is not strictly supermodular.¹⁶ We use $\Gamma \sim \Gamma'$ to denote the relation that $\Gamma \simeq \Gamma'$ but $\Gamma \nsubseteq \Gamma'$ and $\Gamma' \nsubseteq \Gamma$. If χ is not strictly supermodular, then, there might exist $\Gamma, \Gamma' \in S$ such that $\Gamma \sim \Gamma'$. In this case, we can replace Γ and Γ' by $\Gamma + \Gamma'$ and $\Gamma\Gamma'$ to get S', and it turns out that $\mathbb{P}_{S'} = \mathbb{P}_S$. To see this, notice that since $\Gamma \sim \Gamma'$, for all $d^{[\Omega]} \in \mathbb{P}_S$, using (63), we have

$$\begin{aligned} d^{\langle \Gamma + \Gamma' \rangle} + d^{\langle \Gamma \Gamma' \rangle} &= d^{\langle \Gamma \rangle} + d^{\langle \Gamma' \rangle} \\ &= \chi(\Gamma) + \chi(\Gamma') = \chi(\Gamma + \Gamma') + \chi(\Gamma \Gamma'). \end{aligned}$$

But according to (62), $d^{\langle \Gamma+\Gamma'\rangle} \geq \chi(\Gamma+\Gamma')$ and $d^{\langle \Gamma\Gamma'\rangle} \geq \chi(\Gamma\Gamma')$, so the equalities must hold in both cases, implying that $d^{[\Omega]} \in \mathbb{P}_{\mathcal{S}'}$. Conversely, for all $d^{[\Omega]} \in \mathbb{P}_{\mathcal{S}'}$, by the same logic, $d^{[\Omega]} \in \mathbb{P}_{\mathcal{S}}$, because

$$d^{\langle \Gamma \rangle} + d^{\langle \Gamma' \rangle} = d^{\langle \Gamma + \Gamma' \rangle} + d^{\langle \Gamma \Gamma' \rangle} = \chi(\Gamma + \Gamma') + \chi(\Gamma \Gamma') = \chi(\Gamma) + \chi(\Gamma').$$

If S' still contains \sim relations, it can be replaced by S''in the same way that S is replaced by S', and so on. The question is whether through this process, all \sim relations can be eliminated to arrive at a chain. The answer is yes, because S' is guaranteed to contain less \sim relations than S. To see this, on the one hand, notice that at least one \sim relation, $\Gamma \sim \Gamma'$, is eliminated when S is replaced by S'. On the other hand, for all $\Gamma'' \in S$ with $\Gamma'' \neq \Gamma, \Gamma'$, we need only consider the following four cases:

- C1 if $\Gamma'' \subset \Gamma$ and $\Gamma'' \subset \Gamma'$, there must be $\Gamma'' \subset \Gamma + \Gamma'$ and $\Gamma'' \subseteq \Gamma\Gamma'$;
- C2 if $\Gamma \subset \Gamma''$ and $\Gamma' \subset \Gamma''$, there must be $\Gamma + \Gamma' \subseteq \Gamma''$ and $\Gamma\Gamma' \subset \Gamma''$;
- C3 if $\Gamma'' \subset \Gamma$ and $\Gamma'' \sim \Gamma'$, or if $\Gamma'' \sim \Gamma$ and $\Gamma'' \subset \Gamma'$, there must be $\Gamma'' \subset \Gamma + \Gamma'$; and finally,
- C4 if $\Gamma \subset \Gamma''$ and $\Gamma' \sim \Gamma''$, or if $\Gamma \sim \Gamma''$ and $\Gamma' \subset \Gamma''$, there must be $\Gamma\Gamma' \subset \Gamma''$.

In all cases, the number of \sim relations cannot increase. It follows that S' must contain at least one less \sim relation than S.

Proof of Lemma 16: For all $\Gamma \subseteq \Omega$ and $1 \le i \le n$, it is immediate from (64) that

$$\Gamma\Gamma^{i}_{\pi}\Gamma^{i-1}_{\pi} = \Gamma\Gamma^{i-1}_{\pi}.$$

If $\omega_{\pi}^{i} \in \Gamma$, using (65), it is also easy to verify that

$$\Gamma\Gamma^{i}_{\pi} + \Gamma^{i-1}_{\pi} = \Gamma^{i}_{\pi}.$$

so according to (61),

$$\chi(\Gamma\Gamma_{\pi}^{i}) + \chi(\Gamma_{\pi}^{i-1}) \le \chi(\Gamma_{\pi}^{i}) + \chi(\Gamma\Gamma_{\pi}^{i-1})$$

¹⁶Although we will not show it here, it can be shown that $\mathbb{P}(\chi)$ is degenerate if and only if χ is not strictly supermodular.

Let $\Gamma = \{\omega_{\pi}^{i_1}, \omega_{\pi}^{i_2}, \dots, \omega_{\pi}^{i_l}\}$, with $i_1 < i_2 < \dots < i_l$. Then, on the one hand, using (68) and the above inequality, we have

$$v_{\pi}^{(\Gamma)}(\chi) = \sum_{k=1}^{l} (\chi(\Gamma_{\pi}^{i_{k}}) - \chi(\Gamma_{\pi}^{i_{k}-1})))$$

$$\geq \sum_{k=1}^{l} (\chi(\Gamma\Gamma_{\pi}^{i_{k}}) - \chi(\Gamma\Gamma_{\pi}^{i_{k}-1})))$$

$$= \chi(\Gamma\Gamma_{\pi}^{i_{l}}) + \sum_{k=1}^{l-1} (\chi(\Gamma\Gamma_{\pi}^{i_{k}}) - \chi(\Gamma\Gamma_{\pi}^{i_{k+1}-1})) - \chi(\Gamma\Gamma_{\pi}^{i_{1}-1}).$$

On the other hand, using (65), it is also easy to verify that

$$\Gamma\Gamma_{\pi}^{i_{l}} = \{\omega_{\pi}^{i_{1}}, \omega_{\pi}^{i_{2}}, \dots, \omega_{\pi}^{i_{l}}\} = \Gamma,$$

$$\Gamma\Gamma_{\pi}^{i_{k}} = \{\omega_{\pi}^{i_{1}}, \omega_{\pi}^{i_{2}}, \dots, \omega_{\pi}^{i_{k}}\} = \Gamma\Gamma_{\pi}^{i_{k+1}-1},$$

and

$$\Gamma\Gamma_{\pi}^{i_1-1} = \phi.$$

It follows that $v_{\pi}^{\langle \Gamma \rangle}(\chi) \geq \chi(\Gamma)$.

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