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to Andrei Mironov on his 60's birthday

A new kind of anomaly: on W-constraints for GKM

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Abstract

We look for the origins of the single equation, which is a peculiar combination of W-constrains, which provides the non-abelian W-representation for generalized Kontsevich model (GKM), i.e. is enough to fix the partition function unambiguously. Namely we compare it with the scalar projection of the matrix Ward identity. It turns out that, though similar, the two equations do not coincide, moreover, the latter one is non-polynomial in time-variables. This discrepancy disappears for the cubic model if partition function is reduced to depend on odd times (belong to KdV sub-hierarchy of KP), but in general such reduction is not enough. We consider the failure of such direct interpretation of the "single equation" as a new kind of anomaly, which should be explained and eliminated in the future analysis of GKM.

1 Introduction

This is going to be a rather technical paper, targeted at clarification of the long-standing puzzles of Generalized Kontsevich model (GKM) [1]–[8]. It is not fully successful, still it can attract attention to potentially important aspects of the story. No doubt, at technical level these observations are well known to people who worked with GKM, but we make an attempt to summarize them and promote to a more conceptual level. This is needed because of the new accents introduced into the GKM theory quite recently, in [9] and [10]–[12], and the need to explain the origins and the form of the "single equation" [11] and the character expansion in terms of Hall-Littlewood polynomials [9]. We do not achieve these goals in the present paper, but we try to better explain the difficulties of one particularly promising suggestion from [12] – with the hope that it gains attention and will be somehow resolved in the near future.

GKM [4] is an eigenvalue matrix model [6] with the partition function

$$\mathcal{Z}_V[L] = \int_{N\times N} dX e^{-\mathrm{tr}\,(V(X)+LX)} = \frac{e^{\mathrm{tr}\,MV'(M)-V(M)}}{\sqrt{\det V''(M)}} \cdot Z_V\{p_k\} := \mathcal{Z}_V^{cl}[M]\cdot Z_V\{p_k\} \qquad (1)$$

with $V'(M) = L$, which depends on the matrix variable L and satisfies the obvious matrix-valued Ward identity [2, 13]

$$\left\{V'\left(\frac{\partial}{\partial L^{tr}}\right) - L\right\}\mathcal{Z}_V[L] = 0 \qquad (2)$$

We further restrict attention to the monomial case, $V(X) = \frac{X^{r+1}}{r+1}$ and label $\mathcal{Z}_r$ and Z_r by integer r. Qualitatively the properties of monomial GKM are well known [4, 6]:

1) the "quantum" pieces Z_r are KP τ-functions of the "time variables" $p_k = \mathrm{tr}\,M^{-k}$ (which are r-dependent in terms of L, $p_k = \mathrm{tr}\,L^{-k/r}$),

2) they are independent of all p_{kr} and belong to the r-reduction of KP [14],

3) the *shape* of $Z_r\{p\}$ is independent of the size N of the matrix M, only the *locus* $p_k = \operatorname{tr} M^{-k}$ where the particular integral is actually defined, depends on N,

4) the "classical" pieces $\mathcal{Z}_r^{cl}[M]$ also can be expressed through p_k, but *their* shapes do depend on N – this was the reason why these formulas are not very popular, and we discuss them in a special section 2 below,

5) Ward identities (2) can be rewritten as an infinite set of W-constraints on $Z_r\{p_k\}$ [4,6,14],

$$\hat{W}^{(i+1)}_{nr-r+i} \cdot Z_r = 0, \qquad i = 1, \ldots, r-1, \quad n \geq 1 \tag{3}$$

and, as established recently,

6) Z_r has a peculiar non-Abelian W-representation [12,16], i.e. can be unambiguously described by a *single* combination of W-constraints,

$$\textbf{Single Equation (SE)}: \qquad \boxed{\sum_{i=1}^{r-1} (-)^i \sum_n p_{nr-r+i} \hat{W}^{(i+1)}_{n-i-1} \cdot Z_r\{p\} = 0} \tag{4}$$

nicknamed "single equation" (SE) in what follows,

7) Z_r possesses character expansion in terms of Hall-Littlewood polynomials [9].

While rather well established in the case of the ordinary cubic ($r = 2$) Kontsevich model [1], these issues are quite difficult to address for $r > 2$. It is the purpose of this paper to make one more technical step in this direction.

Namely, we study a *scalar* implication of *matrix* Gross-Newmann equation (2),

$$\textbf{Main Equation (ME)}: \qquad \boxed{\operatorname{tr}\left(M \left\{ \left(\frac{\partial}{\partial L^{tr}} \right)^r - L \right\} \right) \mathcal{Z}_r[L] = 0} \tag{5}$$

and the suggestion of [12] to use it as SE – a basic equation which provides Z_r as a unique solution in the form of the non-Abelian W-representation. In this paper we call it "the main equation", or just ME to simplify the reference. In other words, the main question of the present paper is if

$$\textbf{ME} \stackrel{?}{=} \textbf{SE} \tag{6}$$

and, if not, what is the difference.

Substitution of a badly controlled system of matrix equations (2), which is *believed* to be equivalent to an infinite set of W-constraints (3), by a single equation SE [11] is a big simplification – surprisingly this is possible without a loss of information. However, to make it fully satisfactory, we need a maximally simple origin if this SE – and ME would be just a dream. Unfortunately, as anticipated in [12], the story is not just so simple – and details, though seemingly technical, are quite interesting. The fact that the simplest Ward identity ME is not quite the same as SE, which controls the solution, is an interesting twist of the story and this is what we call *anomaly* in the title of this paper:

$$\textbf{SE} = \textbf{ME} \text{ mod } \textit{anomaly} \tag{7}$$

The actual calculation consists of three steps. First one needs to express matrix derivatives through eigenvalues, this is discussed in a separate section 3). Then one needs to act on the product $\mathcal{Z}_r[L] = \mathcal{Z}_r^{cl}[M] \cdot Z_r\{p\}$ and convert the equation w.r.t. eigenvalues into the one for the "quantum" $Z_r\{p\}$, depending on time variables. And afterwards one should interpret the results. We demonstrate that "anomaly" has *two* origins. The first is that the coefficients of the terms with derivatives over p_{kr} are ugly and, actually, non-polynomial in time variables. This can serve as a possible interpretation of the need for the r-reduction, i.e. the need for these derivatives to vanish – what looks particularly convincing in the case of cubic ($r = 2$) model, when this is the only manifestation of the anomaly. Unfortunately, for $r > 2$ the situation gets more obscure. The second phenomenon is that for $r > 2$ this non-polynomiality shows up also in the coefficients of other derivatives – and ME is *not* sufficient to explain the vanishing of these unwanted contributions. Of course, other constituents of (3) should imply this nullification, but this brings us back to the complicated form of (3) and SE (4).

2 Strong dependence on N: the classical piece of partition function

Usually in discussion of GKM we emphasize the remarkable property 3) from above list – that the essential ("quantum") part of partition function depends on the matrix size N only through the choice of the *locus* $\Big\{p_k = \operatorname{tr} M^{-k}\Big\}$ – an N-dimensional non-linear subspace in the infinite-dimensional space of time-variables $\{p_k\}$. The *shape* of $Z\{p_k\}$ is, however, independent of N, and in this sense the N-dependence of $Z\{p\}$ on N is *weak*.

In this paper we switch the accent to another side of the story: to the "classical" part of partition function, which is much simpler, but depends on N much stronger – and this will have a serious impact on the Ward identities (3) and (4), making their simplest treatment through the otherwise appealing "main equation" (5) less straightforward – if not totally meaningless.

For monomial potential $V(M) = \frac{M^{r+1}}{r+1}$ the classical part of partition function can also be easily expressed through the time variables $p_k = \operatorname{tr} M^{-k}$, though expressions are a little lengthy. They are naturally written through the Schur functions $S_R\{p\}$, where R denotes the Young diagrams (for example, $S_{[1]} = p_1$, $S_{[2]} = \frac{p_2+p_1^2}{2}$, $S_{[1,1]} = \frac{-p_2+p_1^2}{2}$ an so on). Most important, these formulas have strong and explicit dependence on N:

$$\mathcal{Z}_r^{cl} := \frac{e^{\operatorname{tr} MV'(M)-V(M)}}{\sqrt{\det V''(M)}} = \frac{e^{\frac{r}{r+1}\sum_{i=1}^N \mu_i^{r+1}}}{\prod_{i=1}^N \mu_i^{\frac{r-1}{2}} \prod_{i<j}^N \frac{\mu_i^r-\mu_j^r}{\mu_i-\mu_j}} = \tag{8}$$

$$= \frac{S_{[1,1,\ldots,1]}^{\frac{(2N-1)(r-1)}{2}}}{S_{[(N-1)(r-1),\ldots,2(r-1),r-1]}} \exp\left(\frac{r}{r+1}\frac{S_{[r+1,r+1,r+1,\ldots,r+1]} - S_{[r+1,\ldots,r+1,r,1]} + S_{[r+1,\ldots,r+1,r,r,2]} - \ldots \pm S_{[r,r,\ldots,r,r,N-1]}}{S_{[1,1,\ldots,1]}^{r+1}}\right)$$

Two Young diagrams in Schur functions have $N-1$ columns, $S_{[\underbrace{r+1,r+1,r+1,\ldots,r+1}_{N-1}]}$ in the exponent and $S_{[\underbrace{(N-1)(r-1),\ldots,2(r-1),r-1}_{N-1}]}$ in the denominator. All the rest have N columns: from $S_{[\underbrace{r+1,\ldots,r+1,r,1}_{N}]}$ to $S_{[\underbrace{r,\ldots,r,r,N-1}_{N}]}$ in the exponent and also $S_{[\underbrace{1,\ldots,1}_{N}]} = \prod_{i=1}^N \mu_i^{-1}$. For $N=2$ eq.(8) becomes just

$$\frac{e^{\frac{r}{r+1}(\mu_1^{r+1}+\mu_2^{r+1})}}{(\mu_1\mu_2)^{\frac{r-1}{2}}\frac{\mu_1^r-\mu_1^r}{\mu_1-\mu_2}} = \frac{S_{[1,1]}^{\frac{3(r-1)}{2}}}{S_{[r-1]}} \exp\left(\frac{r}{r+1}\frac{S_{[r+1]}-S_{[r,1]}}{S_{[1,1]}^{r+1}}\right) \tag{9}$$

while for $N=3$ and $N=4$ it is

$$\frac{e^{\frac{r}{r+1}(\mu_1^{r+1}+\mu_2^{r+1}+\mu_3^{r+1})}}{(\mu_1\mu_2\mu_3)^{\frac{r-1}{2}}\frac{\mu_1^r-\mu_1^r}{\mu_1-\mu_2}\frac{\mu_1^r-\mu_3^r}{\mu_1-\mu_3}\frac{\mu_2^r-\mu_3^r}{\mu_2-\mu_3}} = \frac{S_{[1,1,1]}^{\frac{5(r-1)}{2}}}{S_{[r-1,2r-2]}} \exp\left(\frac{r}{r+1}\frac{S_{[r+1,r+1]}-S_{[r+1,r,1]}+S_{[r,r,2]}}{S_{[1,1,1]}^{r+1}}\right) \tag{10}$$

$$\frac{e^{\frac{r}{r+1}(\mu_1^{r+1}+\mu_2^{r+1}+\mu_3^{r+1}+\mu_4^{r+1})}}{(\mu_1\mu_2\mu_3\mu_4)^{\frac{r-1}{2}}\frac{\mu_1^r-\mu_1^r}{\mu_1-\mu_2}\frac{\mu_1^r-\mu_3^r}{\mu_1-\mu_3}\frac{\mu_1^r-\mu_4^r}{\mu_1-\mu_4}\frac{\mu_2^r-\mu_3^r}{\mu_2-\mu_3}\frac{\mu_2^r-\mu_4^r}{\mu_2-\mu_4}\frac{\mu_3^r-\mu_4^r}{\mu_3-\mu_4}} =$$

$$= \frac{S_{[1,1,1,1]}^{\frac{7(r-1)}{2}}}{S_{[r-1,2r-2,3r-3]}} \exp\left(\frac{r}{r+1}\frac{S_{[r+1,r+1,r+1]}-S_{[r+1,r+1,r,1]}+S_{[r+1,r,r,2]}-S_{[r,r,r,3]}}{S_{[1,1,1,1]}^{r+1}}\right)$$

As we will see, non-trivial Schur functions in the pre-exponent survive in the main equation (5) and make it non-polynomial in time variables. The only case when this does not matter at all, is $r=1$, which we will briefly mention in section 4 below. In conventional cubic model at $r=2$ the non-polynomiality can be eliminated by r-reduction (from KP to KdV in this case) – this we will see in s.5. Starting from $r=3$, however, the problem (anomaly) is far more difficult to cure, and the corrected form of **ME** – and thus the simple derivation of **SE** – still needs to be found.

3 From matrices to eigenvalues

As already mentioned, GKM (1) is an eigenvalue model, the integral is reduced to eigenvalues of X and the answer depends on the eigenvalues of $L = M^r$. Still the reason for the special properties of GKM is that originally it depends on the matrix variable, and the natural Ward identities [17]) are matrix-valued – given by (2). Since they contain matrix derivatives, it is separate exercise to convert them to the eigenvalue form. What we need are diagonal elements of $\frac{\partial^r \mathcal{Z}[L]}{\partial L_{tr}^r}$, evaluated at diagonal matrix $L = \mathrm{diag}(\lambda_i) = \mathrm{diag}(\mu_i^r)$. They do not arise from just a substitution of diagonal L into $\mathcal{Z}[L]$. Still the answer is well known from perturbation theory in quantum mechanics [18] (where one diagonalizes the Hamiltonian and obtains corrections to the wave functions): according to [2],

$$\left(\frac{\partial^r F}{\partial L_{tr}^r}\right)_{ii} = \sum_{j_1,\ldots,j_{r-1}} \left(\sum_{\text{permutations of } i,j_1,\ldots,j_{r-1}} \frac{\frac{\partial F}{\partial \lambda_i}}{\prod_{s=1}^{r-1}(\lambda_i - \lambda_{j_s})} \right) \tag{11}$$

Note that λ_{j_s} can coincide, also with λ_i – then one should apply the l'Hopitale rule, and this gives rise to more sophisticated structures. In [19] a special technique was developed on this occasion. We, however, just work with explicit formulas, without going into details of the derivations. In particular,

$$\begin{aligned}
\left(\frac{\partial F}{\partial L_{tr}}\right)_{ii} &= \frac{\partial F}{\partial \lambda_i},\\
\left(\frac{\partial^2 F}{\partial L_{tr}^2}\right)_{ii} &= \sum_j \frac{\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j}}{\lambda_i - \lambda_j} = \sum_{j\neq i} \frac{\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j}}{\lambda_i - \lambda_j} + \frac{\partial^2 F}{\partial \lambda_i^2},\\
\left(\frac{\partial^3 F}{\partial L_{tr}^3}\right)_{ii} &= \sum_{j,k} \frac{\frac{\partial F}{\partial \lambda_i}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} + \frac{\frac{\partial F}{\partial \lambda_j}}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)} + \frac{\frac{\partial F}{\partial \lambda_k}}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)} =\\
= \sum_{k\neq j\neq i}^N &\left(\frac{\frac{\partial F}{\partial \lambda_i}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} + \frac{\frac{\partial F}{\partial \lambda_j}}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)} + \frac{\frac{\partial F}{\partial \lambda_k}}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)} \right) - \sum_{j\neq i}^N \frac{\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j}}{(\lambda_i-\lambda_j)^2} +\\
&+ \sum_{j\neq i}^N \frac{2\frac{\partial^2 F}{\partial \lambda_i^2} - \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} - \frac{\partial^2 F}{\partial \lambda_j^2}}{\lambda_i - \lambda_j} + \frac{\partial^3 F}{\partial \lambda_i^3},\\
&\ldots
\end{aligned} \tag{12}$$

For example, at $N = 2$, for a function $F(\lambda_1,\lambda_2) := F\left(\frac{L_{11}+L_{22}\pm\sqrt{(L_{11}-L_{22})^2+4L_{12}L_{21}}}{2}\right)$ one can explicitly check, that

$$\left\{ \left(\frac{\partial^3 F}{\partial Ł_{\mathrm{tr}}^3}\right)_{11} := \frac{\partial^3 F}{\partial Ł_{11}^3} + 2\frac{\partial^3 F}{\partial Ł_{11}\partial L_{12}\partial L_{21}} + \frac{\partial^3 F}{\partial Ł_{12}\partial L_{21}\partial L_{22}} \right\}\Bigg|_{L=\mathrm{diag}(\lambda_1,\lambda_2)} = -\frac{F_{,1} - F_{,2}}{(\lambda_1-\lambda_2)^2} + \frac{2F_{,11} - F_{,12} - F_{,22}}{\lambda_1 - \lambda_2} + F_{1,1,1}$$

in accordance with this general prescription.

4 A toy example at $r = 1$

This is a special case, where equation (2) has a "wrong" power of $\partial/\partial L$. It was used as a training example in [2]. For $Z\{p_k\} = Z\left\{\sum_{i=1}^N \lambda_i^{-k}\right\}$ we have:

$$\begin{aligned}
\sum_{i=1}^N \left(\frac{\partial^2 \mathcal{Z}}{\partial L_{tr}^2}\right)_{ii} &= \sum_{i\neq j}^N \frac{\frac{\partial \mathcal{Z}}{\partial \lambda_i} - \frac{\partial \mathcal{Z}}{\partial \lambda_j}}{\lambda_i - \lambda_j} + \sum_{i=1}^N \frac{\partial^2 \mathcal{Z}}{\partial \lambda_i^2} = \sum_{n=1}^\infty \sum_{a=1}^{n+1} n p_a p_{n+2-a} \frac{\partial Z}{\partial p_n} + \sum_{n_1,n_2=1}^\infty n_1 n_2 p_{n_1+n_2+2} \frac{\partial^2 Z}{\partial p_{n_1}\partial p_{n_2}} =\\
&= \sum_{n=-1}^\infty p_{n+2}\left(\sum (k+n)p_k \frac{\partial Z}{\partial p_{k+n}} + \sum_{a+b=n} ab \frac{\partial^2 Z}{\partial p_a p_b} \right) = \sum_{n=1}^\infty p_n \hat{L}_{n-2}^{(1)} Z
\end{aligned} \tag{13}$$

where

$$\hat{L}_n^{(1)} = \sum_{k=1}^\infty (k+n)p_k \frac{\partial Z}{\partial p_{k+n}} + \sum_{a+b=n} ab \frac{\partial^2 Z}{\partial p_a p_b} \tag{14}$$

and superscript label (1) refers to $r=1$. This is the ordinary Virasoro operator, which defines Virasoro constraints in Hermitian matrix model, and it appears here because this model can be also treated as GKM with additional insertion of a power of $\det X$ in the integral, what causes also an increase of r by one [6,20].

5 Original cubic ($r=2$) Kontsevich model

5.1 Implication from the known Z_2

We now proceed to the study of the true main equation (5), beginning from the first case of cubic Kontsevich model. What we need is to substitute

$$\mathcal{Z}_2 = \overbrace{\frac{e^{\frac{2}{3}\sum\mu_i^3}}{\sqrt{\prod_i \mu_i}\prod_{i<j}(\mu_i+\mu_j)}}^{\mathcal{Z}_2^{cl}} \cdot Z_2\{p_k\} \tag{15}$$

into (5):

$$\frac{1}{\mathcal{Z}_2^{cl}}\sum_{i=1}^N\sqrt{\lambda_i}\Big(\overbrace{\frac{\partial^2\mathcal{Z}_2}{\partial\lambda_i^2}+\sum_{j\neq i}^N\frac{\frac{\partial\mathcal{Z}_2}{\partial\lambda_i}-\frac{\partial\mathcal{Z}_2}{\partial\lambda_j}}{\lambda_i-\lambda_j}}^{(\partial^2\mathcal{Z}_2/\partial L_{tr}^2)_{ii}}-\lambda_i\mathcal{Z}_2\Big)=\sum_{i=1}^N\sqrt{\lambda_i}\left(\frac{\partial^2 Z_2}{\partial\lambda_i^2}+\sum_{j\neq i}^N\frac{\frac{\partial Z_2}{\partial\lambda_i}-\frac{\partial Z_2}{\partial\lambda_j}}{\lambda_i-\lambda_j}\right)+$$

$$+\sum_{i=1}^N\sqrt{\lambda_i}\left(\frac{\partial^2\log\mathcal{Z}_2^{cl}}{\partial\lambda_i^2}+\left(\frac{\partial\log\mathcal{Z}_2^{cl}}{\partial\lambda_i}\right)^2+\sum_{j\neq i}\frac{\frac{\partial\log\mathcal{Z}_2^{cl}}{\partial\lambda_i}-\frac{\partial\log\mathcal{Z}_2^{cl}}{\partial\lambda_j}}{\lambda_i-\lambda_j}-\lambda_i\right)Z_2+2\sum_{i=1}^N\sqrt{\lambda_i}\frac{\partial\log\mathcal{Z}_2^{cl}}{\partial\lambda_i}\frac{\partial Z_2}{\partial\lambda_i} \tag{16}$$

First of all we can substitute the known series for $Z_2\{p_k\}$ (last time cited in the Appendix to [12]),

$$Z_2\{p_k\}=1+\left(\frac{p_2p_1^2}{6}+\frac{p_4}{36}\right)+\left(\frac{13}{216}p_4p_2p_1^2+\frac{13}{2592}p_4^2-\frac{1}{216}p_2^4+\frac{1}{72}p_2^2p_1^4+\frac{1}{27}p_5p_1^3+\frac{1}{27}p_7p_1\right)+\ldots \tag{17}$$

and we expect to get zero. It is instructive to see how this really works. If we substitute instead of Z_2 just 1 – the first term in the series,– we get a polynomial of grading 3: $\frac{p_3+4p_1^3}{16}$, if $1+\left(\frac{p_2p_1^2}{6}+\frac{p_4}{36}\right)$, then a polynomial of grading 6 and so on: the more gradings we include into $Z_2\{p\}$, the higher is the grading of (21): if gradings up to $3m$ are included into $Z_2\{p\}$, then (21) is of grading $3(m+1)$. Thus we obtain zero for (21) in the sense that every particular grading vanishes is we include appropriately many terms into $Z_2\{p\}$. Also at every stage the answer is not just of definite grading, it is actually a *polynomial* in time variables.

One can wonder, what happens to $S_{[1]}$ factor in (9) – why does not it produce non-polynomial contributions? It turns out to be a rather delicate adjustment. Already the quadratic singularity $S_{[1]}^{-2}=(\mu_1+\mu_2)^{-2}$ drops out from the sum $\sum_i\mu_i\frac{\partial^2(\mu_1+\mu_2)^\alpha}{\partial\lambda_i^2}$ because of the peculiar property $\frac{\mu_1}{\mu_1^2}+\frac{\mu_2}{\mu_2^2}\sim\mu_1+\mu_2$. The linear singularity is even more miraculous: potentially relevant terms in (9) are

$$\sum_i\mu_i\left\{\frac{\partial^2\log\frac{1}{S_{[1]}}}{\partial\lambda_i^2}+\left(\frac{\partial\left(\log\frac{S_{[1,1]}^{\frac{3\beta}{2}}}{S_{[1]}}+\frac{2\alpha(S_{[3]}-S_{[2,1]})}{3S_{[1,1]}^3}\right)}{\partial\lambda_i}\right)^2+\gamma\sum_{j\neq i}\frac{\frac{\partial\log\frac{1}{S_{[1]}}}{\partial\lambda_i}-\frac{\partial\log\frac{1}{S_{[1]}}}{\partial\lambda_j}}{\lambda_i-\lambda_j}\right\} \tag{18}$$

and the term $\frac{1}{\mu_1+\mu_2}$ is independent of α, but depends in β. It vanishes when $\gamma=3\beta-2$, what includes the true values $\beta=\gamma=1$, but clearly demonstrates the delicate balance between various contributions. Therefore it is not surprising that thus balance will be often violated – the surprise is that it continues to hold (anomaly is lacking) for $r=2$ for arbitrary N, and also for the coefficients of odd derivatives $\frac{\partial Z}{\partial p_{2n+1}}$ – as we will explain in the next sections.

5.2 $r = 2$, all times

The next exercise is to convert (21) into an equation for $Z_2\{p_k\}$, similar to what we considered above in s.4. Assume first that $Z_2\{p_k\}$ depends on all the time-variables p_k. Then, once again substituting (15) into (21) we get:

$$0 \stackrel{(5)}{=} \frac{1}{\mathcal{Z}_2^{cl}} \sum_{i=1}^N \sqrt{\lambda_i} \Big(\overbrace{\frac{\partial^2 \mathcal{Z}_2}{\partial \lambda_i^2} + \sum_{j\neq i}^N \frac{\frac{\partial \mathcal{Z}_2}{\partial \lambda_i} - \frac{\partial \mathcal{Z}_2}{\partial \lambda_j}}{\lambda_i - \lambda_j}}^{(\partial^2 \mathcal{Z}_2/\partial L_{tr}^2)_{ii}} - \lambda_i \mathcal{Z}_2 \Big) = \qquad (19)$$

$$= \frac{1}{4} \sum_{n_1,n_2=1}^\infty n_1 n_2 p_{n_1+n_2+3} \frac{\partial^2 Z}{\partial p_{n_1} \partial p_{n_2}} + \sum_{n=1}^\infty n \left(-p_n + \frac{1}{2} \sum_{a=1}^{\frac{n-1}{2}+2} p_{2a-1} p_{n+4-2a} + \underline{\xi_n^{(N)} \cdot \frac{\text{frac}\left(\frac{n-1}{2}\right)}{2}} \right) \frac{\partial Z}{\partial p_n} + \frac{p_3 + 4p_1^3}{16} Z$$

The last term is of course the same as we got from substitution of 1 into (21). However, with the first derivatives of Z there is a trouble (underlined): for derivatives w.r.t. even p_{2k} the coefficients are not polynomial in p. Moreover, they depend on N, with somewhat sophisticated self-consistency/reduction relations between different N. In terms of Schur functions S_R

$$\xi_n^{(2)} = \frac{p_{n+4} + (p_1^2 - p_2) p_{n+2}}{p_1} = \frac{S_{[n+4]} + 2S_{[n+3,1]} - 3S_{[n+2,2]}}{S_{[1]}}$$

$$\xi_n^{(3)} = \frac{S_{[n+5,1]} + 2S_{[n+4,2]} + 2S_{[n+4,1,1]} - 3S_{[n+3,3]} - S_{[n+2,3,1]} - 2S_{[n+2,2,2]} + 5S_{[n+1,3,2]}}{2S_{[2,1]}}$$

$$\xi_n^{(4)} = \frac{S_{[n+6,2,1]} + 2S_{[n+5,3,1]} + 2S_{[n+5,2,2]} + 2S_{[n+5,2,1,1]} - 3S_{[n+4,4,1]} - 2S_{[n+3,4,2]} - 2S_{[n+3,3,3]} - 2S_{[n+3,2,2,2]} - 2S_{[n+3,4,1,1]} + 5S_{[n+2,4,3]}}{S_{[3,2,1]}} +$$

$$+ \frac{1}{S_{[3,2,1]}} \cdot \begin{cases} 9S_{[n+1,3,3,2]} & \text{for} \quad n = 2 \\ 2S_{[n+1,3,3,2]} + 2S_{[n+1,4,3,1]} + 2S_{[n+1,4,2,2]} - 7S_{[n,4,3,2]} & \text{for} \quad n \geq 4 \end{cases}$$

$$\ldots \qquad (20)$$

i.e. denominator is equal to $S_{[N-1,\ldots,3,2,1]}$.

This problem of non-polynomiality is cured (the underlined terms are absent) in the action on functions $Z_2\{p_1, p_3, p_5, \ldots\}$, which depend only on odd times p_{2k-1}.

5.3 $r = 2$, odd times = cubic Kontsevich model

Assume now that $Z\{p_k\}$ depends on all the odd time-variables p_{2k-1}. Then the terms with $\xi^{(N)}$ drop out of (19) and we get a differential equation for $Z - 2\{p\}$ with polynomial coefficients in p:

$$0 \stackrel{(5)}{=} \sum_{i=1}^N \lambda_i^{1/r} \Big(\overbrace{\sum_{j\neq i}^N \frac{\frac{\partial Z}{\partial \lambda_i} - \frac{\partial Z}{\partial \lambda_j}}{\lambda_i - \lambda_j} + \frac{\partial^2 Z}{\partial \lambda_i^2}}^{(\partial^2 \mathcal{Z}/\partial L_{tr}^2)_{ii}} - \lambda_i Z \Big) =$$

$$= \frac{1}{4} \sum_{n_1,n_2=1}^\infty (2n_1 - 1)(2n_2 - 1) p_{2n_1+2n_2+1} \frac{\partial^2 Z}{\partial p_{2n_1-1} \partial p_{2n_2-1}} + \sum_{n=1}^\infty (2n-1) \left(-p_{2n-1} + \frac{1}{2} \sum_{a=1}^n p_{2a-1} p_{2n+3-2a} \right) \frac{\partial Z}{\partial p_{2n-1}} + \frac{p_3 + 4p_1^3}{16} Z =$$

$$= -\hat{l}_0 Z + \sum_{n=1}^\infty p_{2n-1} \hat{L}_{n-2}^{(2)} Z \qquad (21)$$

where

$$\hat{L}_n^{(2)} = \frac{1}{4} \delta_{n,0} + \frac{p_1^2}{16} \delta_{n,-1} + \frac{1}{2} \sum_{k=1}^\infty (2k + 2n - 1) p_{2k-1} \frac{\partial}{\partial p_{2k+2n-1}} + \frac{1}{4} \sum_{a+b=2n} (2a-1)(2b-1) \frac{\partial^2}{\partial p_{2a-1} \partial p_{2b-1}} \qquad (22)$$

and the grading-counting operator

$$\hat{l}_0 = \sum_{n=1}^{\infty}(2n-1)p_{2n-1}\frac{\partial}{\partial p_{2n-1}} \tag{23}$$

Note that elimination of derivatives $\frac{\partial Z}{\partial p_{2k}}$ automatically eliminates all even times p_{2k} from the coefficients of (21): **for $r = 2$ the r-reduction is necessary and sufficient for ME to reproduce SE**.

6 The first non-standard case: quartic model $r + 1 = 4$

6.1 Solution to projected Ward identity

Now we can repeat all the same steps in the first non-trivial case of quartic GKM with $r = 3$. We will see that the non-polynomiality gets now even more pronounced.

In terms of μ-variables ($\lambda_i = \mu_i^r$) the *main equation* in this case looks as follows:

$$\sum_{i=1}^{N}\lambda_i^{1/3}\left\{\sum_{k\neq j\neq i}^{N}\left(\frac{\frac{\partial \mathcal{Z}}{\partial\lambda_i}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}+\frac{\frac{\partial \mathcal{Z}}{\partial\lambda_j}}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}+\frac{\frac{\partial \mathcal{Z}}{\partial\lambda_k}}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)}\right)+\right.$$
$$\left.+\sum_{j\neq i}^{N}\left(-\frac{\frac{\partial \mathcal{Z}}{\partial\lambda_i}-\frac{\partial \mathcal{Z}}{\partial\lambda_j}}{(\lambda_i-\lambda_j)^2}+\frac{2\frac{\partial^2 \mathcal{Z}}{\partial\lambda_i^2}-\frac{\partial^2 \mathcal{Z}}{\partial\lambda_i\partial\lambda_j}-\frac{\partial^2 \mathcal{Z}}{\partial\lambda_j^2}}{\lambda_i-\lambda_j}\right)+\frac{\partial^3\mathcal{Z}}{\partial\lambda_i^3}-\lambda_i\mathcal{Z}\right\} \overset{(5)}{=} 0 \tag{24}$$

Conversion to μ-variables ($\lambda = \mu_i^r$) is easy: $\frac{\partial}{\partial\lambda_i} = \frac{1}{r\mu_j^{r-1}}\frac{\partial}{\partial\mu_i}$. For $r > 3$ denominators get larger and degenerations provide higher derivatives of $\mathcal{Z}$.

The next step is to substitute

$$\mathcal{Z}_3 = \frac{e^{\frac{3}{4}\operatorname{tr}M^4}}{\sqrt{\det\left(M^2\otimes 1+M\otimes M+1\otimes M^2\right)}}\cdot Z_3\{p_k\} = \frac{e^{\frac{3}{4}\sum_i\mu_i^4}}{\prod_i\mu_i\prod_{i<j}(\mu_i^2+\mu_i\mu_j+\mu_j^2)}\cdot Z_3\{p_k\} \tag{25}$$

and obtain an equation ME (5) for Z_3 with time-derivatives instead of the μ-ones.

Then we can compare it to SE (4), which in the case of $r = 3$ involves two operators [12]:

$$\hat{W}_n^{(2)} = \frac{1}{3}\sum_{k=1}^{\infty}(k+3n)P_k\frac{\partial}{\partial p_{k+3n}}+\frac{1}{6}\sum_{a+b=3n}ab\frac{\partial^2}{\partial p_a\partial p_b}+\frac{p_1p_2}{3}\delta_{n,-1}+\frac{1}{9}\delta_{n,0} \tag{26}$$

$$\hat{W}_n^{(3)} = \frac{1}{9}\sum_{k,l=1}^{\infty}(k+l+3n)P_kP_l\frac{\partial}{\partial p_{k+l+3n}}+\frac{1}{9}\sum_{k=1}^{\infty}\sum_{a+b=k+3n}abP_k\frac{\partial^2}{\partial p_a\partial p_b}+\frac{1}{27}\sum_{a+b+c=3n}abc\frac{\partial^3}{\partial p_a\partial p_b\partial p_c}+\frac{1}{27}\sum_{a+b+c=-3n}P_aP_bP_c$$

with $P_k = p_k - 3\delta_{k,4}$ and a, b, c, k, l not divisible by 3. Note that the sums are restricted more than it would follow from omission of derivatives w.r.t. p_{3k}, for example, there are no terms $\frac{\partial^3}{\partial p_1^2\partial p_2}$ and $\frac{\partial^3}{\partial p_1\partial p_2^2}$ in $\hat{W}^{(3)}$, only $\frac{\partial^3}{\partial p_1^3}$ and $\frac{\partial^3}{\partial p_2^3}$. This will be one of the apparent differences from the ME, which contains third derivatives of all the four kinds.

6.2 Non-trivial denominators

The other striking difference will be non-polynomiality. In fact, one can observe it at the very early stage. For $r > 2$ it is enough to look at the derivative-free term in ME. Namely, if $Z_3 = 1$ the l.h.s. of (24) is non-vanishing, but contains contributions of just two ($r-1$) gradings: -4 and 0. In the simplest case of $N = 2$

$$Z = 1 \quad \overset{(24)}{\Longrightarrow} \quad \frac{7S_{[4]}+5S_{[3,1]}}{9} - \frac{4\left(7S_{[10]}+7S_{[9,1]}+10S_{[8,2]}\right)}{27S_{[2]}} \tag{27}$$

The first is polynomial in times, the second is not. In other words we observe the same phenomenon as in (19), but now it is present already for the item Z, without derivatives.

Adding appropriate p-dependent pieces to $Z_3\{p\}$ [12] preserves the pattern – just shifts it to higher and higher gradings:

$$Z=1+\left(\frac{p_4}{36}+\frac{p_2p_1^2}{6}\right)\quad\Longrightarrow\quad\frac{35p_4\big(11S_{[4]}+13S_{[3,1]}\big)}{324}-\frac{35\big(32S_{[13,1]}+38S_{[12,2]}+S_{[2]}(22S_{[12]}-22S_{[11,1]}+35S_{[9,3]}-36S_{[8,4]}+35S_{[6,6]})\big)}{243S_{[2]}}$$

and so on. For generic N denominator becomes $S_{[2N-2,\ldots,6,4,2]}$:

$$Z_3=1\quad\stackrel{(24)}{\Longrightarrow}\quad\frac{\overbrace{7S_{[4]}+5S_{[3,1]}-5S_{[2,1,1]}-7S_{[1,1,1,1]}}^{\frac{p_4+6p_2p_1^2}{9}}}{9}-\tag{28}$$

$$-\begin{cases}\frac{4\left(7S_{[10]}+7S_{[9,1]}+10S_{[8,2]}\right)}{27S_{[2]}}=\frac{1}{27}\left((p_2+p_1^2)(4p_2^3+21p_2^2p_1^2-12p_2p_1^4+p_1^6)-\frac{12p[2]S_{[1,1]}^4}{S_{[2]}}\right) & N=2\\ \frac{1}{27S_{[4,2]}}\Big(28S_{[12,2]}+28S_{[11,3]}+28S_{[11,2,1]}+40S_{[10,4]}+18S_{[10,3,1]}+40S_{[10,2,2]}+30S_{[9,4,1]}+30S_{[9,3,2]}- & \\ -16S_{[8,5,1]}+42S_{[8,4,2]}-16S_{[8,3,3]}+3S_{[7,6,1]}-13S_{[7,5,2]}-22S_{[7,4,3]}+3S_{[6,6,2]}-19S_{[6,5,3]}-6S_{[6,4,4]}+12S_{[5,5,4]}\Big) & N=3\\ \frac{1}{27S_{[6,4,2]}}\Big(\ldots\Big) & N=4\\ \ldots & \end{cases}$$

While the first polynomial piece is stabilized and does not vary anymore for $N>r$, the shape of non-polynomial terms is not stable – it varies with N.

Building up the true $Z_3\{p\}$ results into the shift of the two non-vanishing pieces to infinite gradings p_∞, p^∞ – and in this sense the answer, understood as the contributions at every particular grading, gets vanishing.

The moral is that now the non-polynomiality is less related to r-reduction: a function can be independent of p_{rk} (like $Z_3=1$), still (24) does *not* convert it into a *polynomial* – non-trivial denominators occur. However, the *proper* $Z_3\{p\}$ is converted to zero. Together with occurrence of the underlined term in (27) this implies that at least some terms in the *main* equation with derivatives of $Z_3\{p_k\}$ w.r.t. p_k should be non-polynomial, even if k is *not* divisible by r. Since such non-polynomiality does not appear in the highest-derivative terms $\frac{\partial^r Z_r}{\partial p_{i_1}\ldots\partial p_{i_r}}$, the natural guess after that is that these additional terms are made from the lower W-constraints, i.e. from the complements of the main equation (5) – the other corollaries of the matrix Ward identity (2).

6.3 ME for $r=3$, all times

As we already know from the previous subsection, there will be problems with relating SE to ME. In addition to the two nice terms at the r.h.s. of

$$0\stackrel{(5)}{=}\sum_{i=1}^N\lambda_i^{1/r}\Big(\overbrace{\sum_{k\neq j\neq i}^N\frac{\frac{\partial Z}{\partial\lambda_i}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}-\sum_{j\neq i}^N\frac{\frac{\partial^2Z}{\partial\lambda_i^2}}{\lambda_i-\lambda_j}+\frac{\partial^3Z}{\partial\lambda_i^3}}^{(\partial^3\mathcal{Z}/\partial L_{tr}^3)_{ii}}-\lambda_iZ\Big)=$$

$$=\frac{1}{27}\sum_{n_1,n_2,n_3=1}^\infty n_1n_2n_3p_{n_1+n_2+n_3+8}\frac{\partial^3Z}{\partial p_{n_1}\partial p_{n_2}\partial p_{n_3}}+\frac{1}{3}\sum_{n_1,n_2}^\infty n_1n_2p_{n_1+n_2+4}\frac{\partial^2Z}{\partial p_{n_1}\partial p_{n_2}}+\ldots\tag{29}$$

the non-polynomial terms will appear, which depend on N. In the simplest case of $N=2$ the full expression is:

$$0 \overset{(5)}{=} -\sum_{n_1,n_2,n_3=1}^{\infty} \frac{n_1n_2n_3p_{n_1+n_2+n_3+8}}{27}\frac{\partial^3 Z}{\partial p_{n_1}\partial p_{n_2}\partial p_{n_3}}+$$

$$+\sum_{n_1,n_2=1}^{\infty} n_1n_2\left(\frac{p_{n_1+n_2+4}}{3}-\frac{(n_1+n_2+8)p_{n_1+n_2+8}}{18}-\frac{2p_{n_1+n_2+6}S_{[1,1]}+4p_{n_1+n_2+4}S_{[2,2]}+2p_{n_1+n_2}S_{[4,4]}+4p_{n_1+n_2-2}S_{[5,5]}}{18}+\right.$$

$$+\frac{S^7_{[1,1]}\cdot\left(S_{[n_1+n_2-4]}+2S_{[n_1+n_2-5,1]}+S_{[n_1-2,n_2-2]}\right)-\frac{S([1,1])^{n_2+5}}{2}p_1S_{[n_1-n_2-1]}+S^6_{[1,1]}S_{[n_1-1]}\delta_{n_2,1}}{9S_{[2]}}-$$

$$\left.-\frac{2S^7_{[1,1]}\cdot(\delta_{n_1,3}\delta_{n_2,1}+\delta_{n_1,2}\delta_{n_2,2})+S^6_{[1,1]}\left(2p_1\delta_{n_1,2}+\delta_{n_1,1}\right)\delta_{n_2,1}}{9S_{[2]}}\right)\frac{\partial^2 Z}{\partial p_{n_1}\partial p_{n_2}}+$$

$$+\sum_n n\left(\frac{(n+4)\left(S_{[n+6]}-S_{[n+3,2]}\right)+3\left(S_{[n+5,1]}+S_{[n+4,2]}\right)}{3S_{[2]}}-\right.$$

$$\left.-\frac{(n^2+12n+39)S_{[n+10]}+3(n+7)S_{[n+9,1]}+6(n+6)S_{[n+8,2]}-(n^2+12n+27)S_{[n+7,3]}-3(1-\delta_{n,1})S_{[n+4,6]}}{27S_{[2]}}\right)\frac{\partial Z}{\partial p_n}-$$

$$-np_n\frac{\partial Z}{\partial p_n}+\left(\frac{7S_{[4]}+5S_{[3,1]}}{9}-\frac{4\left(7S_{[10]}+7S_{[9,1]}+10S_{[8,2]}\right)}{27S_{[2]}}\right)Z$$

6.4 ME versus SE

This should be compared to the operator in (4), which after substitution of (3) becomes

$$-\sum_{a+b+c=3n-9}\frac{abcp_{a+b+c+8}}{27}\frac{\partial^3 Z}{\partial p_a\partial p_b\partial p_c}+\left(\frac{1}{3}\sum_{a+b=3n-5}+\frac{1}{6}\sum_{a+b=3n-6}\right)abp_{a+b+4}\frac{\partial^2 Z}{\partial p_a\partial p_b}-\sum_{a+b=3n-1}\frac{abp_kp_{a+b-k+8}}{9}\frac{\partial^2 Z}{\partial p_a\partial p_b}+$$

$$+\frac{2}{3}p_{3n-1}(k+3n-5)p_k\frac{\partial Z}{\partial p_{k+3n-5}}+\frac{1}{3}p_{3n-2}(k+3n-6)p_k\frac{\partial Z}{\partial p_{k+3n-6}}-\frac{1}{9}p_{3n-1}(k+l+3n-9)p_kp_l\frac{\partial Z}{\partial p_{k+l+3n-9}}-$$

$$-np_n\frac{\partial Z}{\partial p_n}+\left(\frac{p_4+6p_2p_1^2}{9}-\frac{p_5p_1^3+3p_4p_2p_1^2+p_2^4}{27}\right)Z \overset{(4)}{=} 0 \tag{30}$$

Like it was for $r=2$, in (29) there are items with the derivatives $\partial/\partial p_{3k}$, which are absent in (30). They can be eliminated by asking Z_3 to belong to the 3-reduction – exactly like it happened in the previous section for $r=2$. This is the positive part of the story: **cancellation of anomaly requires the r-reduction**. But is the r-reduction sufficient for deriving **SE** from **ME**?

Unfortunately, the answer is "no": now there are a few more striking differences between (29) and (30), e.g.

i) already in (29) there are items with p_{3k} in the sums, which are absent in (30),

ii) the full expression at the r.h.s. of (29) contains non-polynomial terms with denominators $S_{[\ldots,2]}$,

iii) terms like $\partial^3_{112}Z$, $\partial^3_{122}Z$ and $\partial^2_{11}Z$, $\partial^2_{22}Z$ are present already in (29), while only $\partial^3_{111}Z$, $\partial^3_{222}Z$ and $\partial^2_{12}Z$ are allowed in (30).

These are the *qualitative* deviations, as to the quantitative *details* of the two formulas, they look even more different. Still both are true. The only way out of this apparent discrepancy is that the *anomalous* difference between the two formulas is made from some other W-constraints (3), not incorporated into the simple equation (30). This would mean that literally **SE** $\neq$ **ME** even for r-reduced Z_r, still the anomaly is canceled by r-reduction *plus* some additional information – superficial for the scalar projection **ME** of the Ward identity (2), still implied by the entire (2). This is indeed the case, but it is quite difficult to see. We show how it works for contributions from a few lowest gradings $4m$ to $Z_4=\sum_{m=0}^{\infty}z_{4m}$.

In grading four we have exact matching: both (29) and (30) contribute

$$\underline{-np_n\frac{\partial z_4}{\partial p_n}}+\frac{p_4+6p_2p_1^2}{9}z_0=\left(-\frac{4}{4}+1\right)\cdot\frac{p_4+6p_2p_1^2}{9}=0 \tag{31}$$

In grading eight:

$$\mathbf{SE}: \qquad \underline{-np_n\frac{\partial z_8}{\partial p_n}} + \frac{p_4+6p_2p_1^2}{9}z_4 - \frac{p_5p_1^3+3p_4p_2p_1^2+p_2^4}{27}z_0 +$$

$$+\sum_n \sum_{k\neq 0 \mod 3}\left(\frac{2}{3}p_{3n-1}(k+3n-5)p_k\frac{\partial z_4}{\partial p_{k+3n-5}} + \frac{1}{3}p_{3n-2}(k+3n-6)p_k\frac{\partial z_4}{\partial p_{k+3n-6}}\right)+$$

$$+\Big(\underbrace{\frac{1}{3}\sum_{a+b=3n-5}}_{\text{does not contribute}} + \frac{1}{6}\sum_{a+b=3n-6}\Big)\, abp_{a+b+4}\frac{\partial^2 z_4}{\partial p_a\partial p_b} = 0 \tag{32}$$

while (for $N=2$)

$$\mathbf{ME}: \qquad \underline{-np_n\frac{\partial z_8}{\partial p_n}} + \frac{p_4+6p_2p_1^2}{9}z_4 - \frac{4(7S_{[10]}+7S_{[9,1]}+10S_{[8,2]})}{27S_{[2]}}z_0 + \tag{33}$$

$$+\frac{1}{S_{[2]}}\sum_n n\left(\frac{n+4}{3}(S_{[n+6]}-S_{[n+3,3]}) + (S_{[n+5,1]}+S_{[n+4,2]})\right)\frac{\partial z_4}{\partial p_n} + \frac{1}{3}\sum_{n_1,n_2} n_1n_2p_{n_1+n_2+4}\frac{\partial^2 z_4}{\partial p_{n_1}\partial p_{n_2}} = 0$$

Already at this level the difference between the two correct formulas looks quite pronounced – and it only increases at the next levels. Some new ideas are needed to express the (vanishing) difference in terms of the W-constraints (3) and, hopefully, find a concise and universal expression for this discrepancy. Since it relates two clearly distinguished quantities – the **SE** which is a single polynomial equation, which defines $Z\{p_k\}$, and **ME** which is the distinguished scalar projection of the fundamental matrix Ward-identity (2) – there *should* be some simple relation between them. We see that the hope of [12], that this relation is just an identity, fails. But in the simplest cases (like the basic Kontsevich model $r=2$) it is true – and thus the discrepancy is an *anomaly*, in the sense which still remains to be formulated. Anyhow, so far anomalies were always comprehensible – hopefully this will be the case with this new one as well.

7 Conclusion

In this paper we studied the properties of the *main equation* (5) from [12]. This is important because this equation seems to somehow accumulate the power of all the W-constraints in monomial GKM and fully define the time dependence of its partition function $Z_r\{p\}$. In particular it should imply that this partition function is independent of p_{rk} (of time-variables with the numbers divisible by r). We demonstrated that it does so in an elegant way: if there was a p_{rk}-dependence in Z_r, we would not get an equation for it, which is *polynomial* in time-variables p_k. Since Z_r is known from [2,4] to be a KP τ-function (this is relatively simple to demonstrate), independence of p_{rk} means that it belongs to the r-reduction of KP hierarchy. In fact, one can consider our calculation as a new kind of a proof of this statement (that Z_r is an *r-reduced* τ-function), but still a rather sophisticated and undirect one. A concise, clear and direct proof remains highly desirable.

Also desirable is a direct relation of our calculation with the elegant description [14] of the W-constraints for r-reductions as a normal ordering of "circular formula" $\prod_{m=1}^r J\left(z\cdot e^{2\pi i m/r}\right)$. There are now few doubts that the W-constraints are implied by the single main equation – but the way it works remains unclear. One can only hope that if this is clarified, the constraints will also come in some clever form – probably, provided by the circular formula.

Our main result is that the *main* equation (5), directly following from the matrix Ward identity (2), is not exactly the same as the "*single* equation" (SE) of [11,12], but differs from it by additional non-polynomial (!) terms, which presumably are proportional to

(a) some lower W-constraints and

(b) the terms with derivatives over p_{kr}, which do not contribute for r-reduced partition functions:
for a matrix $M=diag(\mu_1,\ldots,\mu_N)$ of the size N and on the locus $p_k=\sum_{i=1}^N\mu_i^{-k}$

$$\boxed{\overbrace{\mathrm{tr}\left(M\left\{\left(\frac{\partial}{\partial L^{tr}}\right)^r - L\right\}\right)}^{ME} = \overbrace{-\hat{l}_0 + \sum_{i=1}^{r-1}\sum_{n=1}^{\infty}p_{rn-i}\hat{W}^{r+1-i}_{n-r-1+i}}^{SE} + \frac{O\left(\hat{W}^{(2)},\ldots,\hat{W}^{(r-1)},\frac{\partial}{\partial p_{kr}}\right)}{S_{[(r-1),2(r-1),\ldots,(N-1)(r-1)]}}} \tag{34}$$

The fact that some other W constraints emerge in addition to the *single* equation in the truly-first-principle approach (based on [13]) can be important for better understanding of its surprising predictive power – a possibility for a single equation to substitute the entire set of the W-constraints (which has more than one generator: already two, L_{-1} and L_2, for $r = 2$). As a byproduct of our calculation we found an amusing structure of non-polynomial terms, with a peculiar embedded dependence on N. Since non-polynomial terms are coefficients of $\partial Z_r/\partial p_{kr}$ which actually vanish for the GKM partition function, the true significance of these formulas, at least in the case (b), remain unclear – still they look interesting by themselves and can show up in some other contexts.

The observation of "anomaly" $\mathbf{SE} \neq \mathbf{ME}$ even for r-reduced τ-functions leaves the puzzle of W-constraints and the origin of W-representation for GKM with $r > 2$ [12] unsolved. This adds to the equally puzzling complication of superintegrability formulas and character calculus for $r > 2$: at least the appropriate basis of Q-functions [9] remains unknown. It is unclear if there is a direct connection between these two complications – anyhow, the story of GKM is still incomplete and at least one additional idea is still lacking. Of course the previous ideas, like "circular formula" [14] and non-abelian W-representation [11, 12], also need to be polished and brought to the same level of clarity as determinantal representation and KP integrability [4], – but this is hard to do before the "anomaly" issue is fixed, which controls the puzzle of r-reduction and the very origin of sophisticated W-constraints and the way they follow from the apparent original Ward identity [13]. If superintegrability and character expansion will also get clarified by the resolution of this puzzle, or need to wait for additional insights, remains to be seen.

Last but not the least – all the formulas in this paper are obtained for particular low values of r and N, what is enough to reveal the emerging structures and phenomena. Still general consideration and proofs remain to be given. They can also bring new ideas and further develop and clarify the theory of GKM, which remains mysteriously complicated and transcendent – perhaps a little less now, but still far from simplicity and transparency achieved for the other eigenvalue matrix models (including the cubic GKM).

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