# AMBITROPICAL GEOMETRY, HYPERCONVEXITY AND ZERO-SUM GAMES

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Abstract. Shapley operators of undiscounted zero-sum two-player games are order-preserving maps that commute with the addition of a constant. We characterize the fixed point sets of Shapley operators, in finite dimension (i.e., for games with a finite state space). Some of these characterizations are of a lattice theoretical nature, whereas some other rely on metric or tropical geometry. More precisely, we show that fixed point sets of Shapley operators are special instances of hyperconvex spaces: they are sup-norm non-expansive retracts of $\mathbb{R}^n$, and also lattices in the induced partial order. Moreover, they retain properties of convex sets, with a notion of "convex hull" defined only up to isomorphism. This provides an effective construction of the injective hull or tight span, in the case of additive cones. For deterministic games with finite action spaces, these fixed point sets are supports of polyhedral complexes, with a cell decomposition attached to stationary strategies of the players, in which each cell is an alcoved polyhedron of $A_n$ type. We finally provide an explicit local representation of the latter fixed point sets, as polyhedral fans canonically associated to lattices included in the Boolean hypercube.

## 1. Introduction

1.1. **Motivation.** Shapley operators play a fundamental role in the study of zero-sum repeated games, see [Sha53, Ney03, MSZ15a]. For infinite horizon problems, including the cases of a discounted payoff, of a total payoff up to a stopping time, or of a mean payoff (in which the payoff is given by a time average), the fixed point of a suitable Shapley operator determines the value of the game as well as optimal stationary strategies. The mean payoff problem is somehow the most difficult one. Then, the notion of fixed point is defined in a projective sense (up the action of additive constants), see [Mou76, Th. VI.1]. The existence and uniqueness of such fixed points have been investigated in the setting of non-linear Perron-Frobenius theory [GG04, LN12, AGH20]. The absence of fixed points is tied to the time-dependent nature of the nearly optimal strategies [BF68, BK76], whereas multiple fixed points yield multiple optimal stationary strategies [KY92].

The structure of the fixed point set of Shapley operators has been studied especially in the deterministic one player case, as part of tropical spectral theory [BCOQ92, KM97a, AGW09], or in the setting of viscosity solutions of ergodic Hamilton-Jacobi partial differential equations and weak-KAM theory, see [FS05, IM07, Fat08]. The results there show in particular that fixed points are uniquely determined by their restriction to a suitable subset or "boundary" of the state space, in a way somehow analogous to the Poisson-Martin representation of harmonic functions [Dyn69]. For one player deterministic problems, the role of the "boundary" is played either by a distinguished subset of the state space ("critical nodes" [BCOQ92], "projected Aubry set" [FS05]) or by a metric boundary (the horoboundary) of the state space [IM07, AGW09]. Fixed point sets appear to be wilder objects in the two-player case, even when the space space is finite. The geometry of these fixed point sets is the main subject of this paper.

A further motivation arises from algebra in characteristic one [CC19] and tropical geometry [MS15]. In this setting, one needs to work in categories in which the objects are tropical analogues of linear spaces, and the arrows are linear maps. However, tropical duality results require to consider at the same time properties of linearity in a primal and in a dual sense, and sometimes to compose maps that are linear in each of these senses (e.g., compositions of min-plus and max-plus linear maps). This is the case, for instance, of projections onto linear spaces, which are generally non-linear [CGQ96, CGQ04], but which are still Shapley operators. Hence, it may be desirable to develop a broader, "self-dual", framework, in which the objects include both tropical (max-plus) linear spaces and their duals, and the arrows are Shapley operators.

Thinking of Shapley operators in abstract terms leads to a metric geometry approach, exploiting a connection with the theory of nonexpansive mappings. Recall that a self-map $T$ of a metric space

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$(X, d)$ is *nonexpansive* if $d(T(x), T(y)) \leqslant d(x, y)$. For undiscounted games with state space $[n]$, the Shapley operators are precisely the self-maps $T$ of $\mathbb{R}^n$ that are nonexpansive with respect to the metric associated with the sup-norm and that commute with the action of additive constants [CT80, KM97a]. As shown in [GK95], this is equivalent to $T$ being nonexpansive in the weak (non-symmetric) metric $d(x, y) = \mathsf{t}(y - x)$ where $\mathsf{t}(z) := \max_{i \in n} z_i$. The map $\mathsf{t}(\cdot)$ here is an example of *weak Minkowski norm* or *hemi-norm*, and it is a special case of the local norm associated to the Finsler structure of the "Funk metric", studied in Hilbert's geometry, see [PT09, Wal18, GV12].

From this perspective, the study of fixed point sets of Shapley operators becomes tied to the classical topic of fixed point sets of nonexpansive mappings and nonexpansive retracts (i.e., fixed points of idempotent nonexpansive maps) in Banach spaces, see [KR07] for background. Recall that a nonexpansive retract $C$ of a Banach space $X$ that satisfies a technical condition (the so called *fixed point property for spheres*) valid in particular in finite dimension, must be *metrically convex* [Bru73], meaning that for all $x, y \in C$ and for all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = 1$, there must exist a point $z$ in $C$ such that $d(x, z) = \alpha d(x, y)$ and $d(z, y) = \beta d(x, y)$. In particular, nonexpansive retracts of strictly convex Banach spaces are closed and convex. Conversely, any closed and convex subset $C$ of a Hilbert space $X$ is a nonexpansive retract of this space, and indeed, a retraction is given by the best approximation map, which associates to a point of $X$ the nearest point in $C$, see e.g. [GR84, Th. 3.6]. Moreover, if $X$ is a Banach space of dimension at least 3, it is known that all the closed and convex subsets of $X$ are nonexpansive retracts if and only if $X$ is a Hilbert space, see [Rei77, Prop. 2.2], and also [Kak39, Kle60, Bru74] for earlier results of this nature.

In this way, classical convexity appears to be linked to the geometry of Euclidean retractions. Then, one may wonder whether other types of (weak) metric spaces are tied to interesting convexity theories. In particular, we may ask whether fixed point sets of Shapley operators may be thought of as "convex sets" in a useful sense. We give here a positive answer to this question, by establishing links between the theory of fixed point sets of Shapley operators, tropical geometry, order preserving retracts of lattices, and hyperconvexity. We focus on the finite dimensional case.

1.2. **Summary of results.** We define a subset $C$ of $\mathbb{R}^n$ to be an *ambitropical* cone if $C$ is invariant by translation by constant vectors, and if $C$ is a lattice in the order induced by the standard partial order of $\mathbb{R}^n$ (but not necessarily a sublattice of $\mathbb{R}^n$). This includes as special cases the max-plus and min-plus convex cones arising in idempotent analysis [LMS01] and tropical geometry [CGQ04, DS04]. In contrast with the classes of max-plus and min-plus cones, the class of ambitropical cones is self-dual since it is invariant by the "flip" (change of sign) operation. Hence, we use the name *ambitropical*, as it includes both tropical convexity and its dual. We call *Shapley retract* the image of an idempotent Shapley operator.

Theorem 3.8 below shows that Shapley retracts are precisely closed ambitropical cones. Further, we show that there are canonical retractions on a closed ambitropical cone $C$, characterized as the composition of nearest point projection mappings on the tropical convex cone and dual tropical convex cone generated by $C$, see Theorem 4.7. This leads to a further characterization of ambitropical cones, by an analogue of the "best co-approximation property" arising in the theory of Banach spaces [PS79], see Theorem 4.12 below. We also show that the class of ambitropical cones admits an analogue of the "convex hull" operation: although the intersection of ambitropical cones may not be ambitropical, there is a well defined notion of ambitropical hull, the minimal closed ambitropical cone containing a given set, which is unique up to isomorphism (the morphisms being Shapley operators). One main result, Theorem 5.6, shows that ambitropical cones are precisely hyperconvex sets that are additive cones. Since we provide an effective construction of the ambitropical hull, in terms of the range of a tropical Petrov-Galerkin projector (Theorem 4.17), this leads to an explicit construction of the hyperconvex hull of an additive cone.

We subsequently study ambitropical cones with a semilinear structure, which we call *ambitropical polyhedra*. The building blocks of ambitropical polyhedra are the alcoved polyhedra of the root system $A_n$, studied in [LP07], which include as special cases Stanley's order polyhedra [Sta86]. Ambitropical polyhedra are defined as ambitropical cones that are finite unions of alcoved polyhedra. Theorem 8.8 shows that ambitropical polyhedra coincide with fixed point sets of Shapley operators of deterministic games with finite action spaces. Further, we show that ambitropical polyhedra are polyhedral complexes whose cells are associated to pairs of stationary policies of both players that are optimal in the mean payoff problem, see Theorem 8.3. We study, in particular, the case of *homogeneous* ambitropical polyhedra, i.e., ambitropical polyhedra that are invariant by the multiplicative action of positive scalars. We show that homogeneous ambitropical polyhedra arise when considering tangent cones of ambitropical polyhedra,

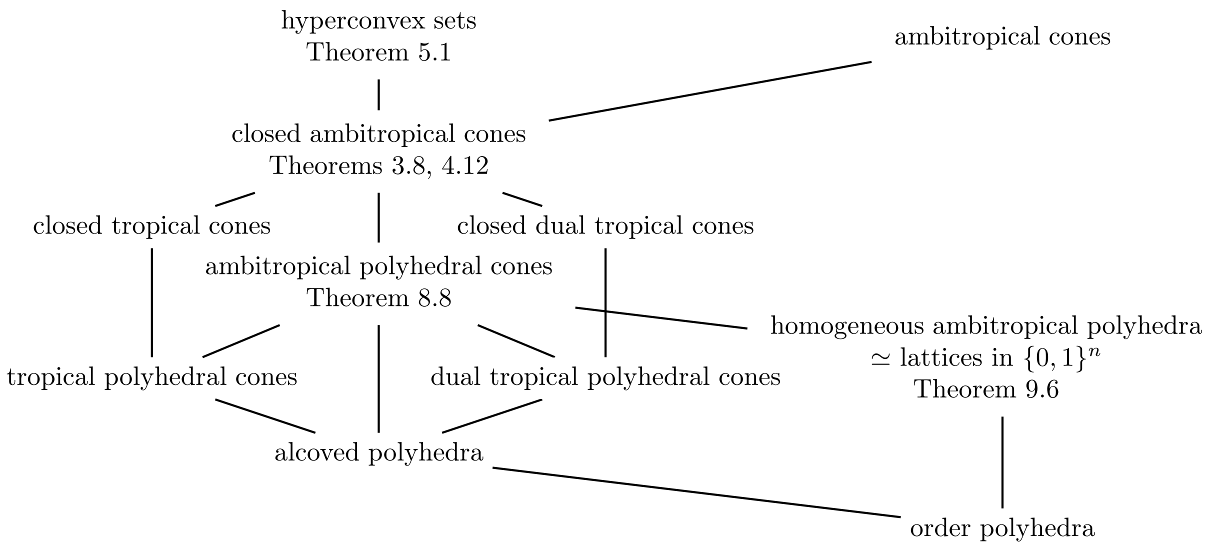


Table 1. The hierarchy of ambitropical cones

so, they provide a *local* description of these sets. Theorem 9.6 provides a characterization of homogeneous ambitropical polyhedra, as polyhedral fans associated to subsets of $\{0,1\}^n$ that are lattices in the induced order.

The hierarchy of classes of sets considered in this paper is presented on Table 1.

1.3. **Related work.** As indicated in §1.1, the present results are related to several series of works. A first source of inspiration is the theory of "best approximation" in tropical geometry [CGQ04, AGNS11], in which (non-linear) projectors onto max-plus / min-plus spaces have been studied. In particular, the canonical retractions on ambitropical cones turn out to coincide with the tropical analogue of Petrov-Galerkin projectors, introduced in [CGQ96] and applied in [AGL08] to the numerical solution of optimal control problems.

The representation of ambitropical polyhedra (Theorem 8.3) as the support of a polyhedral complex is somehow inspired by the characterization of Develin and Sturmfels [DS04] of polyhedral complexes arising from tropical polyhedral cones, in terms of the duals to regular subdivisions of the product of two simplices. The latter property makes use of (classical) convex duality, and so this does not carry over to the ambitropical case given its "minimax" nature. However, we still get a complete combinatorial characterization in the special case of *homogeneous* ambitropical polyhedra, see Theorem 9.6, showing these are equivalent to lattices included in $\{0,1\}^n$ with the induced order. Such lattices are precisely the order preserving retracts of $\{0,1\}^n$ studied by Crapo [Cra82].

Moreover, sup-norm nonexpansive retracts (not necessarily order preserving) have been studied in the setting of *hyperconvexity*, a notion introduced by Aronszajn and Panitchpakdi [AP56], see [Bai88, EK01] for more information. Hyperconvex spaces are metrically convex spaces in which the collection of closed balls has Helly number two. It follows from [AP56, Th. 9] that the sup-norm nonexpansive retracts of $\mathbb{R}^n$ are precisely the closed subsets of $\mathbb{R}^n$ that are hyperconvex. Hence, closed ambitropical cones are special instances of closed hyperconvex sets, with an additional structure induced by the order and the additive homogeneity. Furthermore, an equivalence holds, namely that an additive cone of $\mathbb{R}^n$ is a closed ambitropical cone if and only if it is hyperconvex for the sup-norm metric. This result allows us to compute the hyperconvex hull of an additive cone. The problem of computing the hyperconvex hull has received much attention, Isbell and Dress [Isb64, Dre84] shows that this hull can be realized as a "tight-span". Even in dimension 2, computations of hyperconvex hull are difficult [KcK16]. Our results solve this problem for the special case of additive cones.

As mentioned above, it is an open problem to understand what Fathi's characterization of weak-KAM solutions, as spaces of Lipchitz functions on the "projected Aubry set" [FS05, Fat08], becomes in the "two-player" case. Our results answers the analogue of this question in the discrete, finite dimensional case: whereas one-player solution spaces are alcoved polyhedra, in the two-player case, we show that the solution spaces are precisely ambitropical sets, which in the finite action case are obtained by gluing alcoved polyhedra. This interpretation is elaborated in Theorem 8.3.

Finally, our treatment of Shapley operators is inspired by the "operator approach" of zero-sum games, early work in these directions include [Eve57, Koh74, BK76], see [RS01a, Ren11, BGV15, Zil16] for more recent developments.

## 2. Preliminary results

In this section, we establish, or recall, basic results concerning tropical cones.

2.1. **Additive cones, tropical cones and semimodules.** The tropical semifield, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $(x, y) \mapsto x \vee y := \max(x, y)$ and with the multiplication $(x, y) \mapsto x + y$. It admits a zero element, equal to $-\infty$, and a unit element, equal to 0. We shall also use the min-plus version of the tropical semifield, $\mathbb{R}_{\min}$, which is the set $\mathbb{R} \cup \{+\infty\}$ equipped with the addition $(x, y) \mapsto x \wedge y := \min(x, y)$ and with the multiplication $(x, y) \mapsto x + y$. The semifields $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are isomorphic. We denote by $\leqslant$ the partial coordinatewise order of $(\mathbb{R} \cup \{\pm\infty\})^n$, and we use the notation $\lambda + x := (\lambda + x_i)_{i\in[n]}$, for all $\lambda \in \mathbb{R}_{\max}$ and $x \in (\mathbb{R}_{\max})^n$, and similarly for $\lambda \in \mathbb{R}_{\min}$ and $x \in (\mathbb{R}_{\min})^n$. We extend the notation $\vee$ to denote the supremum of vectors of $(\mathbb{R} \cup \{\pm\infty\})^n$. Similarly, $\wedge$ denote the infimum of vectors.

**Definition 2.1.** An *additive cone* of $\mathbb{R}^n$ is a subset $C$ of $\mathbb{R}^n$ such that

(1) $x \in C, \lambda \in \mathbb{R} \implies \lambda + x \in C$.

A *tropical cone* is an additive cone $C$ such that:

(2) $x, y \in C \implies x \vee y \in C$.

A *dual tropical cone* is defined similarly, by requiring that $x, y \in C \implies x \wedge y \in C$, instead of (2).

Tropical cones are, essentially, special cases of semimodules over the tropical semifield. Recall that semimodules (modules over semirings) are defined in a way similar to modules over rings, see [CGQ04]. In particular, semimodules over idempotent semirings have been studied under the name of *idempotent spaces* in [LMS01]. A simple example of semimodule over $\mathbb{R}_{\max}$ is the $n$-fold Cartesian product of $\mathbb{R}_{\max}$, $(\mathbb{R}_{\max})^n$; the internal law is $(x, y) \mapsto x \vee y := (x_i \vee y_i)_{i\in[n]}$, for $x, y \in (\mathbb{R}_{\max})^n$, and the action of $\mathbb{R}_{\max}$ on $(\mathbb{R}_{\max})^n$ is defined by $(\lambda, x) \mapsto \lambda + x$. This yields a free, finitely generated semimodule. If $C \subset \mathbb{R}^n$ is a tropical cone, then $C \cup \{(-\infty, \dots, -\infty)\}$ is a subsemimodule of $(\mathbb{R}_{\max})^n$, and vice versa.

We shall consider, in particular, tropical cones satisfying a topological assumption. We equip $\mathbb{R}_{\max}$ with the topology defined by the metric $(a, b) \mapsto |e^a - e^b|$. The semimodule $(\mathbb{R}_{\max})^n$, equipped with the topology of the metric $d_\infty(x, y) = \max_{i\in[n]} |e^{x_i} - e^{y_i}|$ is a topological semimodule (meaning that the structure laws are continuous). Observe that the induced topology on $\mathbb{R}^n \subset (\mathbb{R}_{\max})^n$ is the Euclidean topology. Dual considerations apply to $(\mathbb{R}_{\min})^n$.

**Definition 2.2.** Given a subset $C \subset \mathbb{R}^n$, we define the *lower closure* of $C$, $\mathrm{clo}^{\downarrow} C \subset (\mathbb{R}_{\max})^n$, to be the set of limits of nonincreasing sequences of elements of $C$. Similarly, we define the *upper closure* $\mathrm{clo}^{\uparrow} C \subset (\mathbb{R}_{\min})^n$ to be the set of limits of nondecreasing sequences of elements of $C$.

For instance, if $C = \{x \in \mathbb{R}^2 \mid |x_1 - x_2| \leqslant 1\}$, $\mathrm{clo}^{\downarrow} C = C \cup \{(-\infty, -\infty)\}$, whereas if $C = \{x \in \mathbb{R}^2 \mid x_1 \geqslant x_2\}$, $\mathrm{clo}^{\downarrow} C = \{x \in (\mathbb{R}_{\max})^2 \mid x_1 \geqslant x_2\}$. The following proposition shows a correspondence between closed tropical cones and a class of closed tropical subsemimodules.

**Proposition 2.1.** *The map $C \mapsto V := \mathrm{clo}^{\downarrow} C$ establishes a bijective correspondence between the nonempty closed tropical cones $C \subset \mathbb{R}^n$ and the closed subsemimodules $V$ of $(\mathbb{R}_{\max})^n$ such that $V \cap \mathbb{R}^n \neq \emptyset$. The inverse map is given by $V \mapsto V \cap \mathbb{R}^n$.*

The dual property, concerning $\mathrm{clo}^{\uparrow} C$ and $(\mathbb{R}_{\min})^n$ instead of $\mathrm{clo}^{\downarrow} C$ and $(\mathbb{R}_{\max})^n$, also holds. We denote by $\mathrm{supp}\, y := \{i \in [n] \mid y_i > -\infty\}$ the *support* of a vector $y \in (\mathbb{R}_{\max})^n$.

*Proof.* Suppose $C \subset \mathbb{R}^n$ is a tropical cone. Since $(\mathbb{R}_{\max})^n$ is a topological semimodule, $\mathrm{clo}^{\downarrow} C$ is a semimodule over $\mathbb{R}_{\max}$. We next show that $\mathrm{clo}^{\downarrow} C$ is closed. Let $x^k$ denote a sequence of elements of $\mathrm{clo}^{\downarrow} C$ converging to an element $x \in (\mathbb{R}_{\max})^n$. Let $\epsilon_k$ denote an arbitrary sequence of positive numbers decreasing to 0. Let $y^k := x^k + \epsilon_k e \in \mathrm{clo}^{\downarrow} C$. Observe that $\eta_k := d_\infty(x^k, y^k) = (e^{\epsilon_k} - 1) \max_{i\in[n]} e^{x_i^k} \to 0$ as $k \to \infty$. Moreover, by definition of $\mathrm{clo}^{\downarrow} C$, we can find $z^k \in C$ such that $y^k \leqslant z^k$ and $d_\infty(y^k, z^k) \leqslant \eta_k$. It follows that the sequence $z^k$ also converges to $x$. We claim that there is a subsequence $z^{n_k}$ of $z^k$ that is nonincreasing. Indeed, suppose by induction that $z^{n_1} \geqslant \dots \geqslant z^{n_k}$ has already been selected. Let $I := \mathrm{supp}\, x$. Since $z^l \to x$ as $l \to \infty$, we get $z_i^l \to x_i \in \mathbb{R}$ for $i \in [n]$, and $z_i^l \to -\infty$ for $i \in [n] \setminus I$.

However, by construction, $z^{n_k} \geqslant \epsilon_{n_k} e + x$, and so $z_i^{n_k} \geqslant \epsilon_{n_k} + x_i$ for all $i \in I$. We deduce that there is an index $l$ such that $z^l \leqslant z^{n_k}$, and we set $n_{k+1} := l$. This shows that $x \in \mathrm{clo}^{\downarrow} C$, and so, $\mathrm{clo}^{\downarrow} C$ is closed.

Conversely, suppose that $V$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. Then, it is immediate that $V \cap \mathbb{R}^n$ is a closed tropical cone. It remains to show that the correspondence is bijective. We have trivially $\mathrm{clo}^{\downarrow} C \cap \mathbb{R}^n = C$ for all nonempty closed tropical cones $C \subset \mathbb{R}^n$. Conversely, if $V$ is a closed tropical subsemimodule of $(\mathbb{R}_{\max})^n$ such that $V \cap \mathbb{R}^n$ is nonempty, we must show that $V = \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$. let $x \in V$, and let us choose an arbitrary element $y \in V \cap \mathbb{R}^n$. Then, the path $\lambda \mapsto \gamma(\lambda) := x \vee (\lambda + y)$, defined for $\lambda \in (-\infty, 0]$ is such that $\gamma(\lambda) \in V \cap \mathbb{R}^n$, and $\lim_{\lambda \to -\infty} \gamma(\lambda) = x$. It follows that $x \in \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$, hence, $V \subset \mathrm{clo}^{\downarrow}(V \cap \mathbb{R}^n)$. The other inclusion is immediate. □

Recall that an element $u$ of a tropical subsemimodule $V \subset (\mathbb{R}_{\max})^n$ is an *extreme generator* of $V$ if $u = v \vee w$ with $v, w \in V$ implies that $u = v$ or $u = w$. A *tropical linear combination* of elements of $V$ is a vector of the form $\vee_{i \in I}(\lambda_i + a_i)$ where $(\lambda_i)_{i \in I} \subset \mathbb{R}_{\max}$ and $(a_i)_{i \in I} \subset V$ are finite families. We say that $G \subset V$ is a *tropical generating set* if every element of $V$ is a tropical linear combination of a family of elements of $G$. We say also that two vectors are *tropically proportional* if they differ by an additive constant. The next result summarizes results from [GK07, BSS07]; it shows that a closed tropical subsemimodule of $(\mathbb{R}_{\max})^n$ is generated by its extreme rays.

**Theorem 2.2** (See Theorem 3.1 in [GK07] or Theorem 14 in [BSS07])**.** *Suppose that $V$ is a closed tropical subsemimodule of $(\mathbb{R}_{\max})^n$. Then, every element of $V$ is a tropical linear combination of at most $n$ extreme generators of $V$. Moreover, these extreme generators are characterized as follows. For all $i \in [n]$, let $V_i := \{x \in V \mid x_i = 0\}$, and let $\operatorname{Min} V_i$ denote the set of minimal elements of $V_i$. Then, every extreme generator of $V$ is tropically proportional to an element of $\cup_{i \in [n]} \operatorname{Min} V_i$.*

This is reminiscent of the classical Carathéodory theorem for closed convex pointed cones.

2.2. **Alcoved polyhedra and metric closures.** An important class of tropical cones and dual tropical cones consists of *alcoved polyhedra*. The latter were introduced in [LP07]: in general, an alcoved polyhedron associated to a root system is a polyhedron whose facets have normals that are proportional to vectors of this root system. Here, the root system is $A_n$, the collection of vectors $\{e_i - e_j \mid i, j \in [n], i \neq j\}$, where $e_i$ denotes the $i$th vector of the canonical basis of $\mathbb{R}^n$.

**Definition 2.3.** An *alcoved polyhedron* [LP07] is a polyhedron of the form

$$\mathcal{A}(M) = \{x \in \mathbb{R}^n \mid x_i \geqslant M_{ij} + x_j, \quad \forall 1 \leqslant i, j \leqslant n\} \tag{1}$$

for some matrix $M = (M_{ij}) \in (\mathbb{R}_{\max})^{n \times n}$.

*Order polyhedra* are remarkable examples of alcoved polyhedra. They are of the form $\{x \in \mathbb{R}^n \mid x_i \geqslant x_j \text{ if } (i,j) \in E\}$ where $E \subset [n] \times [n]$ is a partial order relation on the set $[n]$. Intersection of order polyhedra with the hypercube $[0,1]^n$ are known as *order polytopes*, they were studied by Stanley [Sta86].

We shall denote by $\vee$ the tropical addition of matrices, so that, for all $A, B \in (\mathbb{R}_{\max})^{m \times n}$, $(A_{ij}) \vee (B_{ij}) := (A_{ij} \vee B_{ij}) \in (\mathbb{R}_{\max})^{m \times n}$. The tropical multiplication of matrices will be denoted by concatenation, i.e, for $A \in (\mathbb{R}_{\max})^{m \times n}$ and $B \in (\mathbb{R}_{\max})^{n \times p}$, $AB \in (\mathbb{R}_{\max})^{m \times p}$ is the matrix with $(i,j)$-entry $\vee_{k \in [n]}(A_{ik} + B_{kj})$. Then, when $m = n$, for all $r$, the *rth tropical power* of $A$ is denoted by $A^r := A \cdots A$ ($A$ is repeated $r$ times).

There are well known relations between alcoved polyhedra and operations of metric closures which we next recall. The *tropical Kleene star* of $M$ is defined by $M^* := I \vee M \vee M^2 \vee \cdots$. This supremum may be infinite, i.e., in general $(M^*)_{ij}$ may take the value $+\infty$. Recall that to the matrix $M$ is associated a digraph with set of nodes $[n]$ and an arc $i \to j$ of weight $M_{ij}$ whenever $M_{ij} > -\infty$. Then, $(M^k)_{ij}$ yields the maximal weight of a path of length $k$ from $i$ to $j$, and $(M^*)_{ij}$ yields the supremum of the weights of paths from $i$ to $j$, of arbitrary length. We have $(M^*)_{ij} < +\infty$ for all $i, j$ if and only if there is no circuit with positive weight in the digraph of $M$. Then, $M^* = M^0 \vee \cdots \vee M^{n-1}$, see e.g. Prop 2.2 of [AGW05].

The *critical circuits* of the matrix $M^*$ are the circuits in the digraph of $M^*$ with weight 0. The union of the critical circuits constitutes the *critical digraph*. The following result is well known: the first statements follow readily from the results recalled above, whereas the characterization of generators is a special case of the characterization of tropical eigenspaces, see e.g. [BCOQ92, Th. 3.100], Theorem 6.4 of [AGW05], or [But10, Th. 4.4.5].

**Lemma 2.3.** *The polyhedron $\mathcal{A}(M)$ is non-empty if and only there is no circuit of positive weight in the digraph of $M$. Then,*

$$\mathcal{A}(M) = \{x \in \mathbb{R}^n \mid M^* x \leqslant x\} = \{x \in \mathbb{R}^n \mid M^* x = x\} = \{M^* y \mid y \in \mathbb{R}^n\} \ . \tag{2}$$

*Moreover, a tropical generating set of* $\mathrm{clo}^{\downarrow}\mathcal{A}(M)$ *is obtained as follows: denote by* $C_1,\dots,C_s$ *the strongly connected components of the critical digraph of* $M^*$*, and select indices* $i_1\in C_1,\dots,i_s\in C_s$ *in an arbitrary manner. Then, the set of columns of* $M^*$ *indexed by* $i_1,\dots,i_s$ *tropically generates* $\mathrm{clo}^{\downarrow}\mathcal{A}(M)$*, and every generating set of* $\mathrm{clo}^{\downarrow}\mathcal{A}(M)$ *contains at least one scalar multiple of each of these columns.* □

*Remark* 2.4. It follows from Lemma 2.3 that the minimum number of elements of a tropical generating family of the tropical semimodule $\mathrm{clo}^{\downarrow}\mathcal{A}(M)\subset(\mathbb{R}_{\max})^n$ never exceeds $n$. The same observation applies to dual tropical generating families of $\mathrm{clo}^{\uparrow}\mathcal{A}(M)$. In contrast, there are finitely generated tropical subsemimodules of $(\mathbb{R}_{\max})^n$ with an arbitrarily large number of generators, see e.g. [AGG13].

## 3. Shapley operators and Ambitropical Cones

3.1. **Abstract Shapley Operators.** Shapley operators are dynamic programming operators allowing one to compute the value function of zero-sum games. The typical example of Shapley operator $T:\mathbb{R}^n\to\mathbb{R}^n$ is of the form

$$T_i(x)=\inf_{a\in A_i}\sup_{b\in B_i}\big(r_i^{ab}+\sum_{j\in[n]}P_{ij}^{ab}x_j\big)\ , \tag{3}$$

where $[n]=\{1,\dots,n\}$ is the state space, $A_i,B_i$ are the sets of actions available in state $i$, of the two players, called "Min" and "Max", $r_i^{ab}$ is a payment made by by Player Min to Player Max at a given stage, assuming that Min selected action $a$ and that Max selected action $b$, and $P_{ij}^{ab}\geqslant 0$ is the probability of transition from $i$ to $j$, so that $\sum_j P_{ij}^{ab}=1$. Such operators capture zero-sum perfect or "turned based" information games, without discount. In the original model considered by Shapley, the two players play simultaneously and the actions are randomized, which can be cast as (3), in which $A_i$ and $B_i$ are simplices [RS01a, Ney03]. Actually, many variants of Shapley operators (depending on the nature of the turns and on the information structure) can be considered, and so, it will be convenient to introduce a general definition.

**Definition 3.1** (Shapley operator)**.** An (abstract) *Shapley operator* is a map $T:\mathbb{R}^n\to\mathbb{R}^p$ with the following properties:

(1) $x\leqslant y$ implies $T(x)\leqslant T(y)$, for all $x,y\in\mathbb{R}^n$;
(2) $T(\lambda+x)=\lambda+T(x)$, for all $x\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$.

The example (3) of Shapley operator obviously satisfies these properties. Conversely, Kolokoltsov showed that every abstract Shapley operator $\mathbb{R}^n\to\mathbb{R}^n$ can be written as (3) (see [KM97b]) and Singer and Rubinov [RS01b] showed that the transition probabilities can even be chosen to be 0/1.

Note that, taking into account the semimodule structure of $\mathbb{R}^n$ and $\mathbb{R}^p$, the canonical choice of morphisms to consider would be tropically linear maps (maps that commute with the supremum and with the addition of a constant). *Shapley operators* constitute a larger class of maps and they are precisely canonical morphisms of additive cones (see Def. 3.1).

In the square case, i.e., when $n=p$, Shapley operators arise as dynamic programming operators of two-player zero-sum games, see e.g. [RS01a, Ney03]. It is known that Shapley operators are nonexpansive in the sup-norm; this observation plays a key role in the "operator approach" of zero-sum games [CT80, RS01a, Ney03, GV12]. Furthermore, the following observation, made in [GK95, Prop. 1.1], shows that Shapley operators are characterized by a nonexpansiveness property. Recall that $\mathbf{t}(x):=\max_{i\in[n]}x_i$ denotes the "top" hemi-norm of a vector $x\in\mathbb{R}^n$. It will also be convenient to use the notation $\mathbf{b}(x):=-\mathbf{t}(-x)=\min_{i\in[n]}x_i$.

**Proposition 3.1** ([GK95, Prop. 1.1])**.** *Let* $T:\mathbb{R}^n\to\mathbb{R}^p$*. The following assertions are equivalent:*

*(1)* $T$ *is a Shapley operator;*
*(2)* $\mathbf{t}(T(x)-T(y))\leqslant\mathbf{t}(x-y)$ *for all* $x,y\in\mathbb{R}^n$*;*
*(3)* $\mathbf{b}(T(x)-T(y))\geqslant\mathbf{b}(x-y)$ *for all* $x,y\in\mathbb{R}^n$*.*

The following result is a consequence of a general result on [BNS03] concerning the continuous extension of positive homogeneous maps defined on the interior of polyhedral cones.

**Proposition 3.2** (Corollary of [BNS03, Theorem 3.10])**.** *A Shapley operator* $T:\mathbb{R}^n\to\mathbb{R}^p$ *admits a unique continuous extension* $T_-:(\mathbb{R}_{\max})^n\to(\mathbb{R}_{\max})^p$*, given by*

$$T_-(x)=\inf\{T(y)\mid y\geqslant x,\ y\in\mathbb{R}^n\}\ .$$

*Similarly, $T$ has a unique continuous extension $(\mathbb{R}_{\min})^n \to (\mathbb{R}_{\min})^p$, given by*

$$T_+(x) = \sup\{T(y) \mid y \leqslant x,\ y \in \mathbb{R}^n\}\ .$$

*Remark* 3.3. Proposition 3.2 provides only one-sided extensions of $T$, either to $(\mathbb{R} \cup \{-\infty\})^n$ or to $(\mathbb{R} \cup \{+\infty\})^n$. A Shapley operator defined on $\mathbb{R}^n$ generally does not extend canonically to $(\mathbb{R} \cup \{\pm\infty\})^n$ (consider $n = 2$ and $T_i(x) = ((x_1 + x_2)/2)$ for $i = 1, 2$).

A Shapley operator is said to be *tropically linear* if $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in \mathbb{R}^n$. It is said to be *dual tropically linear* if $T(x \wedge y) = T(x) \wedge T(y)$. Denoting by $e_j^{\text{trop}} := (-\infty, \dots, -\infty, 0, -\infty, \dots, -\infty)$ (with 0 in the $j$th position) the $j$th vector of the tropical canonical basis of $(\mathbb{R}_{\max})^n$, and $M_{ij} := (T_-(e_j^{\text{trop}}))_i$, we see that if $T$ is tropically linear, then

$$(T(x))_i = \vee_{j \in [n]}(M_{ij} + x_j)\ \ ,$$

i.e., $T$ is represented by a matrix product. A similar representation holds for dual tropically linear Shapley operators.

3.2. **Ambitropical cones.** We next introduce our main object of study: ambitropical cones. We are looking for a class of objects which includes tropical cones and their duals. This leads to the following definitions. We recall that $\mathbb{R}^n$ is equipped with the standard partial order.

**Definition 3.2.** An *ambitropical cone* is a non-empty additive cone $C$ of $\mathbb{R}^n$ such that $C$ is a lattice in the induced order of $(\mathbb{R}^n, \leqslant)$.

Recall that $C$ being a lattice means that every two elements $x, y$ of $C$ have a *least upper bound*

$$x \vee^C y := \min\{z \in C \mid z \geqslant x,\ z \geqslant y\},$$

and a *greatest lower bound*

$$x \wedge^C y = \max\{z \in C \mid z \leqslant x,\ z \leqslant y\},$$

where the symbols "max" and "min" indicate the greatest and smallest elements of a set. Similarly, we will use the notation $\sup^C X$ and $\inf^C X$ for the least upper bound and greatest lower bound in $C$ of a subset $X \subset C$, when it exists. In particular, $x \vee^c y = \sup^C\{x, y\}$ and $x \wedge^c y = \inf^C\{x, y\}$. The operations $\sup^C$ and $\inf^C$ defined for subsets of $C$ should not be confused with the restriction of the operations $\sup = \sup^{\mathbb{R}^n}$ and $\inf = \inf^{\mathbb{R}^n}$ defined for subsets of $\mathbb{R}^n$: indeed, for all $X \subset C$ that has a least upper bound in $C$, $\sup^C X \geqslant \sup X$, and similarly $\inf^C X \leqslant \inf X$ if $X$ has a greatest lower bound in $C$. In other words, an ambitropical cone is a lattice but it *may not be a sublattice* of $\mathbb{R}^n$.

We shall especially consider ambitropical cones satisfying the following property.

**Definition 3.3** (Conditionally complete lattice)**.** A lattice $L$ is said *conditionally complete* if every nonempty subset of $L$ that has an upper bound has a join (a least upper bound), and if every nonempty subset of $L$ that has a lower bound has a meet (a greatest lower bound).

The following observation is elementary.

**Lemma 3.4.** *Let $C$ be an ambitropical cone. A subset of $\mathbb{R}^n$ is bounded from above by an element of $\mathbb{R}^n$ if and only if it is bounded from above by an element of $C$. The dual statement holds for subsets bounded from below.*

*Proof.* Let $X \subset \mathbb{R}^n$ and suppose that there exists $u \in \mathbb{R}^n$ such that $x \leqslant u$ holds for all $x \in X$. Let $y \in C$. Then, $u \leqslant z := y + \mathbf{t}(u - y)$. It follows that $x \leqslant z \in C$ holds for all $x \in X$. □

**Proposition 3.5.** *An ambitropical cone is a conditionally complete lattice if and only if it is closed in the Euclidean topology.*

*Proof.* Suppose that the ambitropical cone $C$ is closed in the Euclidean topology, and let $X \subset C$ be a nonempty set bounded above by some element $y \in C$. Let $\mathcal{P}_f(X)$ denote the set of nonempty finite subsets of $X$. For all $F \in \mathcal{P}_f(X)$, let $u_F := \sup^C F$. Then, $(u_F)_{F \in \mathcal{P}_f(X)}$ is a nondecreasing net of elements of $C$, bounded above by $y$. Since $C$ is closed in the Euclidean topology, the limit of a net of elements of $C$ belongs to $C$, and so $u := \lim_F u_F \in C$. By construction, $u \geqslant x$ holds for all $x \in X$. Moreover, if $z \in C$ is an upper bound of $X$, we get $z \geqslant u_F$ for all $F \in \mathcal{P}_f(X)$, and so $z \geqslant u$. This shows that $u$ is the least upper bound of $X$. A dual argument works for greatest lower bounds. Hence, $C$ is a conditionally complete lattice.

Conversely, suppose that $C$ is a conditionally complete lattice. Observe that for every bounded sequence $(x_k)$ of elements of $C$, the following "liminf" and "limsup" constructions both define elements that belong to $C$:

$$\operatorname{limsup}^C_{k\to\infty} x_k := \inf^C_{k\geqslant 1}\sup^C_{\ell\geqslant k} x_\ell , \qquad \operatorname{liminf}^C_{k\to\infty} x_k := \sup^C_{k\geqslant 1}\inf^C_{\ell\geqslant k} x_\ell \ .$$

We shall use the fact that $\operatorname{limsup}^C_{k\to\infty} x_k \geqslant \operatorname{liminf}^C_{k\to\infty} x_k$. This inequality, which is standard when $C=\mathbb{R}^n$, is still valid in general. Indeed, for all $k,m\geqslant 1$, we have $\sup^C_{\ell\geqslant k} x_\ell \geqslant \inf^C_{\ell\geqslant m} x_\ell$, and so $\sup^C_{\ell\geqslant k} x_\ell \geqslant \sup^C_{m\geqslant 1}\inf_{\ell\geqslant m} x_\ell = \operatorname{liminf}^C_{r\to\infty} x_r$. Hence,

$$\operatorname{limsup}^C_{k\to\infty} x_k = \inf^C_{k\geqslant 1}\sup^C_{\ell\geqslant k} x_\ell \geqslant \operatorname{liminf}^C_{r\to\infty} x_r \ . \tag{4}$$

Suppose that the sequence $(x_k)_{k\geqslant 1}$ of elements of $C$ converges to $x\in\mathbb{R}^n$. Then, for all $\epsilon>0$, there exists an index $m$ such that $\|x_\ell - x\|\leqslant\epsilon$ for all $\ell\geqslant m$. In particular, $x_\ell\leqslant \|x_\ell - x_m\| + x_m \leqslant 2\epsilon + x_m$. We deduce that $\operatorname{limsup}^C_{\ell\to\infty} x_\ell \leqslant 2\epsilon + x_m \leqslant 3\epsilon + x$. Since the latter inequality holds for all $\epsilon>0$, we deduce that $\operatorname{limsup}^C_{\ell\to\infty} x_\ell\leqslant x$. A dual argument sows that $\operatorname{liminf}^C_{\ell\to\infty} x_\ell \geqslant x$. Using (4), we conclude that $x = \operatorname{limsup}^C_{\ell\to\infty} x_\ell = \operatorname{liminf}^C_{\ell\to\infty} x_\ell \in C$, showing that $C$ is closed in the Euclidean topology. □

In the sequel, when writing that an ambitropical is *closed*, we shall always refer to the Euclidean topology.

We define, for all closed ambitropical cones $C$, and for all $x\in\mathbb{R}^n$, the following canonical retractions:

$$Q^-_C(x) := \sup{}^C\{y\in C \mid y\leqslant x\} \ , \qquad Q^+_C(x) := \inf{}^C\{y\in C\mid y\geqslant x\} \ . \tag{5}$$

We shall denote by $\operatorname{Im} f := \{f(x)\mid x\in X\}$ the *image* or *range* of a map $f: X\to Y$. Recall that a *retraction* onto $C$ is a map $P:\mathbb{R}^n\to C$ that is a continuous, idempotent map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with range $C$. Then, $C$ is a retract of $\mathbb{R}^n$. The following result shows that any closed ambitropical cone is a Shapley retract, i.e., the image of a retraction that is a Shapley operator.

**Proposition 3.6.** *Suppose that $C$ is a closed ambitropical cone of $\mathbb{R}^n$. Then, $Q^-_C$ is an idempotent Shapley operator, i.e., $(Q^-_C)^2 = Q^-_C$, and the range of $Q^-_C$ is $C$. The same is true for $Q^+_C$.*

*Proof.* Since $C$ is conditionally complete, for all $x\in\mathbb{R}^n$, $Q^-_C(x)$ is well defined and $Q^-_C(x)\in C$. Moreover, $Q^-_C$ trivially fixes $C$, implying that $\operatorname{Im} Q^-_C = C$ and $(Q^-_C)^2 = Q^-_C$. We also have, for all $x\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, $Q^-_C(\lambda+x) = \sup^C\{y\in C\mid y\leqslant\lambda+x\} = \sup^C\{y\in C\mid -\lambda+y\leqslant x\} = \sup^C\{\lambda+z\in C\mid z\leqslant x\} = \lambda + Q^-_C(x)$. The operator $Q^-_C$ is trivially order preserving, hence it is a Shapley operator. Dual arguments apply to $Q^+_C$. □

**Theorem 3.7.** *Let $C$ be a closed ambitropical cone contained in $\mathbb{R}^n$. The set of Shapley retractions onto $C$ (i. e. idempotent Shapley operators with range $C$) constitutes a complete lattice, with bottom element $Q^-_C$ and top element $Q^+_C$.*

*Proof.* First of all, we shall prove that for any Shapley retraction $P$ onto $C$ and for every $x\in\mathbb{R}^n$ $Q^-_C(x)\leqslant P(x)\leqslant Q^-_C(x)$ . Let $y\in C$ such that $y\leqslant x$, we get $y=P(y)\leqslant P(x)\in C$, and so $Q^-_C(x) = \sup^C\{y\in C\mid y\leqslant x\}\leqslant P(x)$. Similarly, $P(x)\leqslant Q^+_C(x)$. Let $(Q_\alpha)_{\alpha\in A}:\mathbb{R}^n\to\mathbb{R}^n$ be a collection of Shapley retractions onto $C$. Since $Q^-_C(x)\leqslant Q_\alpha(x)\leqslant Q^+_C(x)$ for all $x\in\mathbb{R}^n$, the family

$$(Q_\alpha(x))_{\alpha\in A}\subset C$$

is bounded from above and from below. By Proposition 3.5 $C$ is conditionally complete, which allows us to define

$$Q(x) = \sup{}^C\{Q_\alpha(x)\mid \alpha\in A\}.$$

It is immediate that $Q$ is a Shapley operator that fixes $C$ and that $C$ is its range. So, $Q$ is the least upper bound of the family $(Q_\alpha)_{\alpha\in A}$ in the set of Shapley retractions. □

Proposition 3.6 shows that closed ambitropical cone are Shapley retracts of $\mathbb{R}^n$. The following result shows that we have actually an equivalence.

**Theorem 3.8.** *Let $E$ be a subset of $\mathbb{R}^n$. The following assertions are equivalent*

*(1) $E$ is a closed ambitropical cone;*
*(2) $E$ is a Shapley retract of $\mathbb{R}^n$;*
*(3) $E$ is the fixed point set of a Shapley operator;*

*Proof.* *(1) ⇒ (2).* If $E$ is a closed ambitropical cone, then, by Proposition 3.6, $E$ is the image of $Q^-_E$ and $Q^-_E$ is an idempotent Shapley operator.

*(2) ⇒ (3)* is trivial.

*(3) ⇒ (1).* Since $E$ is the fixed point set of a Shapley operator $T$, $E$ is a closed cone. We shall now show that it is also a lattice in the induced order. Let $x, y \in E$ and we claim that $x \vee_E y = \lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y)$. We have that $x, y \leqslant x \vee_{\mathbb{R}^n} y$, so $x = T(x) \leqslant T(x \vee_{\mathbb{R}^n} y)$ and $y = T(y) \leqslant T(x \vee_{\mathbb{R}^n} y)$. Applying again $T$ to these inequalities and passing to the limit, we obtain that $\lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y)$ is an upper bound of $x$ and $y$. Let $z \in E$ such that $x, y \leqslant z$. Then $x \vee_{\mathbb{R}^n} y \leqslant z$ and $T(x \vee_{\mathbb{R}^n} y) \leqslant T(z) = z$; applying again $T$ to this inequality and passing to the limit, we obtain that $\lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y) \leqslant z$. We have now to prove that $\lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y) \in E$. $T(\lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y)) = \lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y)$ by definition. So, $\lim_{k\to\infty} T^k(x \vee_{\mathbb{R}^n} y)$ is a fixed point of $T$ and it belongs to $E$. The case of inf is dual. □

## 4. From tropical hulls to ambitropical hulls

4.1. **Decomposition of canonical retractions in terms of projections on tropical cones.** Appropriate tropical analogues of Hilbert's spaces are obtained by considering spaces that are closed by taking suprema [LMS01, CGQ04]. Considering suprema of bounded sets leads to the notion of *b-complete idempotent spaces* in [LMS01], whereas allowing unconditional suprema leads to the notion of *complete semimodules* [CGQ04].

Hence, we shall perform a (one sided) conditional completion. If $E$ is a nonempty subset of $\mathbb{R}^n$, we shall denote by $E^{\max}$ the subset of $\mathbb{R}^n$ consisting of tropical linear combinations of *possibly infinite* families of elements of $E$, i.e., the set of elements of the form

$$\sup\{\lambda_f + f \mid f \in E\} \tag{6}$$

where the $\lambda_f \in \mathbb{R}_{\max}$ are such that the family of elements $(\lambda_f + f)_{f\in E}$ is bounded from above and the $\lambda_f$ are not identically $-\infty$. Up to the adjunction of a bottom element, the set $E^{\max}$ is the b-complete idempotent space generated by $E$ in the sense of [LMS01].

We shall also need to consider the $\mathbb{R}_{\max}$-semimodule obtained by taking the *lower closure* of $E^{\max}$, a notion already introduced in Definition 2.2:

$$\bar{E}^{\max} := \operatorname{clo}^{\downarrow} E^{\max} \enspace .$$

Similarly, we shall denote by $E^{\min}$ the set of elements of the form $\inf\{\lambda_f + f \mid f \in E\}$ where the $\lambda_f \in \mathbb{R}_{\min}$ are such that the family of elements $(\lambda_f + f)_{f\in E}$ is bounded from below, and the $\lambda_f$ are not identically $+\infty$. We also set $\bar{E}^{\min} := \operatorname{clo}^{\uparrow} E^{\min}$.

**Proposition 4.1.** *Let $C \subset \mathbb{R}^n$ be an additive cone. Then, the following statements are equivalent:*

*(1) $C$ is closed;*
*(2) $C$ is stable by limits of bounded nondecreasing sequences;*
*(3) $C$ is stable by limits of bounded nonincreasing sequences.*

*Proof.* The implication (1)⇒(2) is trivial.

We next show that (2)⇒(3). Let $x_k$ be a bounded nonincreasing sequence of elements of $C$ converging to $x \in \mathbb{R}^n$. Consider the sequence $y_k := x_k - 2\|x - x_k\|_\infty e \in C$. We have $y_k \leqslant -\|x - x_k\|_\infty e + x$. Moreover, $y_k$ also converges to $x$. It follows that for all $k$, we can find an index $l \geqslant k$ such that $y_l \geqslant y_k$. Hence, we can construct a nondecreasing subsequence $y_{n_k}$ converging to $x$. Applying (2), we conclude that $x \in C$.

We finally show that (3)⇒(1). Suppose $x_k$ is a sequence of elements of $C$ converging to $x \in \mathbb{R}^n$. Consider now $y_k := 2\|x - x_k\|_\infty + x_k$. Then, arguing as in the proof of the previous implication, we deduce that we can construct a nonincreasing subsequence $y_{n_k}$ still converging to $x$. Applying (3), we conclude that $x \in C$. □

**Corollary 4.2.** *Let $E$ denote a non-empty subset of $\mathbb{R}^n$. Then, $E^{\max}$ is a closed tropical cone. Similarly, $E^{\min}$ is a closed dual tropical cone.*

*Proof.* By definition, $E^{\max}$ is a tropical cone. Let us consider a bounded nondecreasing sequence $x_k \in E^{\max}$. We can write $x_k = \sup\{\lambda^k_f + f \mid f \in E\}$ where for each $k$, the family $(\lambda^k_f)_{f\in F}$ is not identically $-\infty$. Let $x := \lim_k x_k = \sup_k x_k \in \mathbb{R}^n$. From $\lambda^k_f + f \leqslant x_k \leqslant x$, we deduce that $\lambda^k_f \leqslant \mathbf{b}(x - f)$. So, the sequence $(\lambda_f)_{k\geqslant 1}$ is bounded from above. It follows that $\lambda_f := \sup\{\lambda^k_f \mid k \geqslant 1\} < +\infty$. Moreover, using the associativity of the supremum operation, we get $x = \sup x_k = \sup\{\lambda_f + f \mid f \in E\} \in E^{\max}$. Hence,

$E^{\max}$ is stable by limits of nondecreasing sequences. It follows from Proposition 4.1 that $E^{\max}$ is closed in the Euclidean topology. A dual argument applies to $E^{\min}$. $\square$

If $C$ is a closed tropical cone, then $C$ is closed by tropical linear combinations, and it is also closed by taking the supremum of nondecreasing sequences, it follows that the supremum $\sup^C$ relative to $C$ coincides with the supremum of $\mathbb{R}^n$. We deduce the following result.

**Lemma 4.3.** *If $C$ is a closed tropical cone, then $Q_C^- \leqslant I$. Similarly, if $C$ is a closed dual tropical cone, then $Q_C^+ \geqslant I$.* $\square$

**Corollary 4.4.** *Let $E$ denote a non-empty subset of $\mathbb{R}^n$. Then, $\bar{E}^{\max}$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. Similarly, $\bar{E}^{\min}$ is a closed subsemimodule of $(\mathbb{R}_{\min})^n$*

*Proof.* By definition, $E^{\max}$ is a tropical cone, and by Corollary 4.2, it is a closed subset of $\mathbb{R}^n$. Then, it follows from Proposition 2.1 that $\bar{E}^{\max} = \mathrm{clo}^{\downarrow} E^{\max}$ is a closed subsemimodule of $(\mathbb{R}_{\max})^n$. $\square$

For all nonempty subsets $E$ of $\mathbb{R}^n$, and for all $x \in \mathbb{R}^n$, we define the tropical projections

$$P_E^{\max}(x) := \sup\{y \in E^{\max} \mid y \leqslant x\} \enspace, \qquad P_E^{\min}(x) := \inf\{y \in E^{\min} \mid y \geqslant x\} \enspace .$$

This is a specialization of the notion of projectors $Q_C^-$ and $Q_C^+$ to $C = E^{\max}$ or $C = E^{\min}$, introduced in (5). Indeed, if $C = E^{\max}$, the operation $\sup^C$ coincides with the ordinary supremum sup of $\mathbb{R}^n$. The dual property holds for $C = E^{\min}$. If $G$ is any tropical generating set of $E^{\max}$, then, we have the explicit representation

$$P_E^{\max}(x) = \sup_{g \in G} \Big( \mathbf{b}(x-g) + g \Big) \enspace, \qquad \forall x \in \mathbb{R}^n \enspace , \tag{7}$$

see [CGQ04, Th. 5], and a dual formula applies to $P_E^{\min}$. The next proposition tabulates elementary properties of these projectors.

**Proposition 4.5.** *Let $E$ be a nonempty subset of $\mathbb{R}^n$. Then, $P_E^{\max}$ and $P_E^{\min}$ are Shapley operators from $\mathbb{R}^n \to \mathbb{R}^n$ such that:*

$$P_E^{\max} \leqslant I, \qquad P_E^{\min} \geqslant I \enspace . \tag{8}$$

$$\mathrm{Im}\, P_E^{\max} = E^{\max}, \qquad \mathrm{Im}\, P_E^{\min} = E^{\min} \enspace . \tag{9}$$

$$P_E^{\max} = (P_E^{\max})^2, \qquad P_E^{\min} = (P_E^{\min})^2 \enspace . \tag{10}$$

*Proof.* The inequalities (8) follow from Lemma 4.3. By definition, $P_E^{\max}$ fixes $E^{\max}$, and $P_E^{\max}(\mathbb{R}^n) \subset E^{\max}$, so $P_E^{\max} = (P_E^{\max})^2$. The same property holds for $P^{\min}$, showing (10). The last claim follows from the fact that $P_E^{\max}$ fixes $E^{\max} \supset E$, and from the same property for $P_E^{\min}$. $\square$

The maps $\bar{Q}_E^-$ and $\bar{Q}_E^+$ defined in the next proposition will play a key role.

**Proposition 4.6.** *Let $E$ be a nonempty subset of $\mathbb{R}^n$. Then, the maps*

$$\bar{Q}_E^- := P_E^{\min} \circ P_E^{\max}, \quad \textit{and} \quad \bar{Q}_E^+ := P_E^{\max} \circ P_E^{\min}$$

*satisfy the following properties*

*(i)* $\bar{Q}_E^-$; $\bar{Q}_E^+$ *are Shapley operators;*
*(ii)* $\bar{Q}_E^-$ *and* $\bar{Q}_E^+$ *fix* $E$*;*
*(iii)* $(\bar{Q}_E^-)^2 = \bar{Q}_E^-$; $(\bar{Q}_E^+)^2 = \bar{Q}_E^+$*;*
*(iv)* $\bar{Q}_E^+ \circ \bar{Q}_E^- \circ \bar{Q}_E^+ = \bar{Q}_E^+$; $\bar{Q}_E^- \circ \bar{Q}_E^+ \circ \bar{Q}_E^- = \bar{Q}_E^-$*;*
*(v)* $\bar{Q}_E^+ \leqslant \bar{Q}_E^- \circ \bar{Q}_E^+$; $\bar{Q}_E^- \geqslant \bar{Q}_E^+ \circ \bar{Q}_E^-$.

*Proof. (i).* We showed in Proposition 4.5 that $P_E^{\min}$ and $P_E^{\max}$ are both Shapley operators. The collection of Shapley operators is stable by composition.

*(ii).* This follows from the fact that $P_E^{\min}$ and $P_E^{\max}$ fix $E^{\max}$ and $E^{\min}$, respectively (Proposition 4.5), which both contain $E$.

*(iii).* Using the second inequality in (8), and the first equality in (10), we get $(\bar{Q}_E^-)^2 = P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \geqslant P_E^{\min} \circ P_E^{\max} \circ P_E^{\max} = P_E^{\min} \circ P_E^{\max}$. Using now the first inequality in (8), and the second equality in (10), we get $P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \leqslant P_E^{\min} \circ P_E^{\min} \circ P_E^{\max} = P_E^{\min} \circ P_E^{\max}$, showing that $(\bar{Q}_E^-)^2 = \bar{Q}_E^-$. The second property in *(iii)* is dual.

*(iv).* Using (10), we get $\bar{Q}_E^+ \circ \bar{Q}_E^- \circ \bar{Q}_E^+ = P_E^{\max} \circ P_E^{\min} \circ P_E^{\min} \circ P_E^{\max} \circ P_E^{\max} \circ P_E^{\min} = P_E^{\max} \circ P_E^{\min} \circ P_E^{\max} \circ P_E^{\min} = (\bar{Q}_E^+)^2 = \bar{Q}_E^+$, by *(iii)*. The second property is dual.

*(v).* We have that $\bar{Q}^-_E \circ \bar{Q}^+_E = P^{\min}_E \circ P^{\max}_E \circ P^{\max}_E \circ P^{\min}_E = P^{\min}_E \circ P^{\max}_E \circ P^{\min}_E \geqslant P^{\max}_E \circ P^{\min}_E(z) = \bar{Q}^+_E$, using $P^{\min}_E \geqslant I$. The second inequality is dual. □

The next theorem motivates the introduction of the operators $\bar{Q}^-_E$ and $\bar{Q}^+_E$ above, it deals with the situation in which $E$ is a closed ambitropical cone.

**Theorem 4.7** (Factorizations of canonical retractions on closed ambitropical cones)**.** *For all closed ambitropical cones $C$, we have*

$$Q^-_C = \bar{Q}^-_C = P^{\min}_C \circ P^{\max}_C \quad \textit{and} \quad Q^+_C = \bar{Q}^+_C = P^{\max}_C \circ P^{\min}_C$$

*Proof.* Suppose $C$ is a closed ambitropical cone. Observe first that for all $z$ in $\mathbb{R}^n$,

$$P^{\max}_C(z) = \sup\{x \in C \mid x \leqslant z\} \ . \tag{11}$$

Indeed, $P^{\max}_C(z) = \sup\{x \in C^{\max} \mid x \leqslant z\} \geqslant \sup\{x \in C \mid x \leqslant z\}$. However, an element $u \leqslant z$ of $C^{\max}$ can be written as $u = \vee_{y \in C}(\lambda_y + y)$ with $\lambda_y \in \mathbb{R}_{\max}$, and $\lambda_y + y \leqslant z$. So, $\lambda_y + y \leqslant \sup\{x \in C \mid x \leqslant z\}$, and so, $u \leqslant \sup\{x \in C \mid x \leqslant z\}$, which entails (11). Dually, $P^{\min}_C(z) = \inf\{x \in C \mid x \geqslant z\}$.

Using (11), and the dual property, we get

$$\begin{aligned} \bar{Q}^-_C(x) = P^{\min}_C \circ P^{\max}_C(x) &= \inf\{y \in C \mid y \geqslant P^{\max}_C(x)\} \\ &= \inf\{y \in C \mid y \geqslant \sup\{z \in C \mid z \leqslant x\}\} \\ &= \inf\{y \in C \mid (z \leqslant x, z \in C) \implies z \leqslant y\} \\ &= \sup{}^C\{z \in C \mid z \leqslant x\} = Q^-_C(x) \ . \end{aligned}$$

The proof that $\bar{Q}^+_C = Q^+_C$ is dual. □

For closed tropical cones $C$, the projection $Q^-_C = P^{\max}_C$ has the property that $Q^-_C(x)$ is a point of $C$ with minimal distance to $C$ with respect to Hilbert's seminorm, see [CGQ04], hence, it is a tropical analogue of the "nearest point projection" arising in Euclidean spaces. A basic property of this projection on a closed convex set of an Euclidean space is that it is sunny. Recall that a retraction $F$ from $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$ is *sunny* if, given $x$, $y \in \mathbb{R}^n$, for every $z$ in the segment $[x, y]$, $F(x) = y \implies F(z) = y$. The following result shows that the canonical projections on closed tropical or dual tropical cone are also sunny.

**Proposition 4.8.** *If $C$ is a closed tropical cone, then, the projection $Q^-_C = P^{\max}_C$ is sunny. Similarly, if $C$ is a closed dual tropical cone, then, the projection $Q^+_C = P^{\min}_C$ is sunny.*

*Proof.* It is trivial that $Q^-_C = P^{\min}_C \circ P^{\max}_C = P^{\max}_C$. Let $y := P^{\max}_C(x)$, so by (8) $y \leqslant x$. Then any point $z = (1-t)y + tx$ with $0 < t < 1$ satisfies $y \leqslant z \leqslant x$, and so, $y = P^{\max}_C(y) \leqslant P^{\max}_C(z) \leqslant P^{\max}(x) = y$, implying that $P^{\max}_C$ is a sunny retraction. □

However, the following example shows that Proposition 4.8 does not carry over to closed ambitropical cones, the canonical retractions $Q^-_C = P^{\min}_C \circ P^{\max}_C$ and $Q^+_C = P^{\max}_C \circ P^{\min}_C$ are generally not sunny.

*Example* 4.9. Consider the Shapley operator

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \max(x_1, x_3) \\ \max(x_2, x_3) \\ -\frac{1}{2} + \frac{x_1 + x_2}{2} \end{pmatrix}$$

The fixed point set of $T$ is the closed ambitropical cone $E = \{x \in \mathbb{R}^3 \mid \frac{x_1+x_2}{2} = x_3 + \frac{1}{2}, x_1 \geqslant x_3, x_2 \geqslant x_3\}$ shown on Figure 1. The cross section of $E$ in $\mathbb{R}^2$ defined by $x_3 = 0$ is displayed by the black segment, as well as the cross section of $E^{\max}$ (triangle in light gray above this segment) and the cross section of $E^{\min}$ (triangle in dark gray below this segment). In this case $P^{\min}_E \circ P^{\max}_E$ is not a sunny retraction. Indeed, consider the point $x = (2, \alpha, 0)$, with $0 < \alpha < 1$. We have $P^{\max}_E(x) = u := (1, \alpha, 0)$, and $P^{\min}_E(u) = y := (1 - \alpha/2, \alpha/2, 0)$. However, consider the mid-point $x' := (y + x)/2 = (3/2 - \alpha/4, 3\alpha/4, 0)$. We have $u' := P^{\max}_E(x') = (1, 3\alpha/4, 0)$ and $y' := P^{\min}_E(u') = (1 - 3\alpha/8, 3\alpha/8, 0) \neq y$ showing that $P^{\min}_E \circ P^{\max}_E$ is not sunny. This is illustrated in the figure.

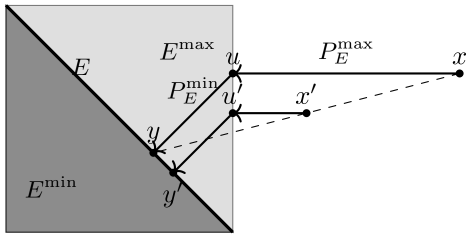


FIGURE 1. An ambitropical cone such that the canonical retractions $Q_E^{\pm}$ are not sunny.

4.2. **Characterization of ambitropical cones in terms of best co-approximation.** We saw in Theorem 4.7 that the two canonical retractions $Q_C^-$ and $Q_C^+$ on a closed ambitropical cone can be decomposed in terms of the projection operators $P_C^{\max}$ and $P_C^{\min}$. In a perhaps surprising way, we shall see that this leads to an order-theoretical analogue of a notion of best co-approximation, introduced by Papini and Singer [PS79] to characterize nonexpansive retracts. If $E$ is a subset of a Banach space $(X, \|\cdot\|)$, $E$ is said to be a *set of existence of best co-approximation* if, for all $z \in X$, the set

$$B_E^{\|\cdot\|}(z) := \{x \in X \mid \|y - x\| \leqslant \|y - z\|,\ \forall y \in E\}$$

contains an element of $E$. It is immediate that if $E$ is a nonexpansive retract of $X$, then $E$ is a set of existence of best co-approximation. The converse is known to hold in $L^p$ spaces with $1 \leqslant p < \infty$, see [Wes92] and the references therein. Here we shall be interested in Shapley retracts of $\mathbb{R}^n$. By Proposition 3.1, these are precisely the images of $\mathbb{R}^n$ by idempotent maps that are nonexpansive in the "top" hemi-metric $\mathbf{t}x = \max_{i\in[n]} x_i$. In view of this property, we introduce the following analogue of the set $B_E^{\|\cdot\|}(z)$.

**Definition 4.1.** Let $E$ be a subset of $\mathbb{R}^n$. For any $y, z \in \mathbb{R}^n$ we define $B(y,z) = \{x \in \mathbb{R}^n \mid y + \mathbf{b}(z-y) \leqslant x \leqslant y + \mathbf{t}(z-y)\}$ where $\mathbf{t}(x) = \max_{i\in[n]} x_i$ and $\mathbf{b}(x) = \min_{i\in[n]} x_i$ for any $x \in \mathbb{R}^n$. Then, we define $B_E(z) = \cap_{y\in E} B(y,z)$.

**Definition 4.2.** Let $E$ be a subset of $\mathbb{R}^n$, we say that $E$ is a *set of existence of best tropical co-approximation* if $B_E(z) \cap E \neq \emptyset$ for every $z \in \mathbb{R}^n$.

**Lemma 4.10.** *We have* $B_E(z) = \{x \in \mathbb{R}^n \mid P_E^{\max}(z) \leqslant x \leqslant P_E^{\min}(z)\}$.

*Proof.* We shall prove that $\sup_{y\in E}(\mathbf{b}(z-y)+y) = P_E^{\max}(z) = \sup\{x \in E^{\max} \mid x \leqslant z\}$. Let $x \in E^{\max}$, $x = \sup_{y\in E}(\lambda + y)$. Suppose $x \leqslant z$, then we have that $\lambda \leqslant \mathbf{b}(x-y) \leqslant \mathbf{b}(z-y)$. In a similar way, we see that $P_E^{\min}(z) = \inf_{y\in E} \mathbf{t}(z-y) + y$. □

**Lemma 4.11.** *Let $E$ be a subset of $\mathbb{R}^n$. If $E$ is a set of existence of best tropical co-approximation, then $\bar{B}_E(z) := [\bar{Q}_E^-(z), \bar{Q}_E^+(z)] \cap E \neq \emptyset$ for any $z \in \mathbb{R}^n$.*

*Proof.* We know that, if $E$ is a set of best tropical co-approximation then for any $z \in \mathbb{R}^n$, there exists $u \in E$ such that $P_E^{\max}(z) \leqslant u \leqslant P_E^{\min}(z)$. Composing the first inequality by $P_E^{\min}$, and composing the second inequality by $P_E^{\max}$, we get $\bar{Q}_E^-(z) = P_E^{\min} \circ P_E^{\max}(z) \leqslant P_E^{\min}(u) = u$, and $u = P_E^{\max}(u) \leqslant P_E^{\max} \circ P_E^{\min}(z) = \bar{Q}_E^+(z)$. □

The following result completes Theorem 3.8, it characterizes ambitropical cones in terms of best co-approximation and of the projections $P_E^{\max}$ and $P_E^{\min}$.

**Theorem 4.12.** *Let $E$ be a subset of $\mathbb{R}^n$. The following assertions are equivalent*

*(1) $E$ is a closed ambitropical cone of $\mathbb{R}^n$;*
*(2) $E$ is a set of existence of best tropical co-approximation;*
*(3) for all $z \in \mathbb{R}^n$, $[P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset$;*
*(4) $P_E^{\min}(z) \in E$ holds for all $z \in E^{\max}$;*
*(5) $P_E^{\max}(z) \in E$ holds for all $z \in E^{\min}$;*
*(6) $E$ is the fixed point set of the operator $\bar{Q}_E^+ = P_E^{\max} \circ P_E^{\min}$;*
*(7) $E$ is the fixed point set of the operator $\bar{Q}_E^- = P_E^{\min} \circ P_E^{\max}$.*

*Proof. (1) ⇒ (2).* By Theorem 3.8, we have that $E = P(\mathbb{R}^n)$ where $P = P^2$ is a Shapley operator. Then, for all $y \in E$, and for all $z \in \mathbb{R}^n$, $\mathbf{t}(P(z) - y) = \mathbf{t}(P(z) - P(y)) \leqslant \mathbf{t}(z-y)$, i.e., $P(z) \leqslant \mathbf{t}(z-y) + y$, and dually, $P(z) \geqslant \mathbf{b}(z-y) + y$, showing that $P(z) \in E \cap B_E(z)$.

*(2) ⇒ (3).* This follows from Lemma 4.10.

*(3) ⇒ (4).* We first observe that

$$\forall z \in \mathbb{R}^n,\ [P_E^{\max}(z), P_E^{\min}(z)] \cap E \neq \emptyset \Rightarrow \forall z \in E^{\max}, [z, P_E^{\min}(z)] \cap E \neq \emptyset\ . \tag{12}$$

Now, the condition $[z, P_E^{\min}(z)] \cap E \neq \emptyset$ is equivalent to: $\exists u \in E$ such that $z \leqslant u \leqslant P_E^{\min}(z)$. However, $P_E^{\min}(z)$ is the minimal vector $v \in E^{\min}$ such that $v \geqslant z$, it follows that $P_E^{\min}(z) \leqslant u$, and so $P_E^{\min}(z) = u \in E$.

*(4) ⇒ (5).* By hypothesis, we have that for any $z \in E^{\max}$, $P_E^{\min}(z) \in E$, so in particular $[z; P_E^{\min}(z)] \cap E \neq \emptyset$. Consider an arbitrary $z \in \mathbb{R}^n$. Then, we have that $P_E^{\max}(z) \in E^{\max}$ and, consequently $[P_E^{\max}(z), P_E^{\min}(P_E^{\max}(z))] \cap E \neq \emptyset$. Recalling that $[P_E^{\max}(z), P_E^{\min}(P_E^{\max}(z))] \subseteq [P_E^{\max}(z), P_E^{\min}(z)]$ since $P_E^{\max} \leqslant I$, we have that for any $z \in \mathbb{R}^n$, $[P_E^{\max}(z); P_E^{\min}(z)] \cap E \neq \emptyset$. In particular, let $z \in E^{\min}$, then we have that $[P_E^{\max}(z), z] \cap E \neq \emptyset$ and by the dual argument of the previous implication we obtain that $P_E^{\max}(z) \in E$.

*(5) ⇒ (6).* We will denote with $\mathrm{Fix}(\bar{Q}_E^+)$ the fixed points set of $\bar{Q}_E^+$. Since any element of $E$ is fixed by $\bar{Q}_E^+$, $E \subseteq \mathrm{Fix}(\bar{Q}_E^+)$. We shall now prove the other inclusion. Let $z \in \mathbb{R}^n$ such that $P_E^{\max}(P_E^{\min}(z)) = z$. Since $P_E^{\min}(z) \in E^{\min}$, $z = P_E^{\max}(P_E^{\min}(z)) \in E$.

*(6) ⇒ (7).* By the idempotency $\bar{Q}_E^+$ we have that $P_E^{\max} \circ P_E^{\min}(y) \in E$ for every $y \in \mathbb{R}^n$, in particular for any $y \in E^{\min}$, $P_E^{\max}(y) \in E$. As in the previous implication we know that $E \subseteq \mathrm{Fix}(\bar{Q}_E^-)$ and we shall now prove the other inclusion. Let $z \in \mathbb{R}^n$ such that $P_E^{\min}(P_E^{\max}(z)) = z$, so $z \in E^{\min}$ and consequently $P_E^{\max}(z) \in E$. Since $E$ is fixed by $P_E^{\min}$, we have that $z = P_E^{\min}(P_E^{\max}(z)) \in E$.

*(7) ⇒ (1).* This follows from Proposition 4.6, *((iii))* and Theorem 3.8.

□

**Corollary 4.13.** *Let $E$ be a non-empty subset of $\mathbb{R}^n$, included in a closed ambitropical cone $F$. Then, $E^{\max} \cap E^{\min} \subset F$.*

*Proof.* The set $E^{\max} \cap E^{\min}$ is fixed both by $P_E^{\max}$ and $P_E^{\min}$. If $E \subset F$, the fixed point set of $P_E^{\max}$ is included in the fixed point set of $P_F^{\max}$. The same is true for $P_E^{\min}$ and $P_F^{\min}$. So, $E^{\max} \cap E^{\min}$ is included in the fixed point set of $P_F^{\min} \circ P_F^{\max}$, which by Theorem 4.12,*(7)*, coincides with $F$. □

**Corollary 4.14.** *Suppose $C$ is a closed ambitropical cone. Then $C = C^{\max} \cap C^{\min}$.*

*Proof.* The inclusion $C^{\max} \cap C^{\min} \subset C$ follows from Corollary 4.13. The other inclusion is trivial. □

*Example* 4.15. Conversely, given an additive cone $E \subset \mathbb{R}^n$, the condition that $E = E^{\max} \cap E^{\min}$ does not imply that $E$ is an ambitropical cone. Consider, for example, the set $E = (a + \mathbb{R}) \cup (b + \mathbb{R})$ where $a = (1, 0, 0)$ and $b = (0, 1, 0)$. This set, as well as the spaces $E^{\max}$ and $E^{\min}$, are shown on Figure 8. We see that $E = E^{\max} \cap E^{\min}$ but since $E$ is disconnected, it cannot be ambitropical.

4.3. **Ambitropical hull.** The intersection of ambitropical cones is generally not ambitropical, so the notion of ambitropical hull of a set $E$ cannot be defined in the naïve manner, as the intersection of ambitropical cones containing $E$. However, we shall see that there is a proper notion of ambitropical hull, unique up to isomorphism.

**Definition 4.3.** Let $E \subset \mathbb{R}^n$. We say that $\tilde{E} \subset \mathbb{R}^n$ is a *ambitropical hull* of $E$ if $\tilde{E}$ is a closed ambitropical cone which is a superset of $E$ and if it is minimal with respect to inclusion.

**Proposition 4.16.** *For each nonempty subset $E \subset \mathbb{R}^n$, the sets $\mathrm{Im}\,\bar{Q}_E^-$ and $\mathrm{Im}\,\bar{Q}_E^+$ are closed ambitropical cones containing $E$ that are isomorphic.*

*Proof.* If $E \subset \mathbb{R}^n$ is non-empty, the operator $\bar{Q}_E^-$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$, and it follows from its definition that it satisfies the axioms of Shapley operators (Definition 3.1). Moreover, we have seen in Proposition 4.6, *(iii)* that $\bar{Q}_E^-$ is idempotent. It follows that $\mathrm{Im}\,\bar{Q}_E^- = \bar{Q}_E^-(\mathbb{R}^n)$ is a Shapley retract, and so $\mathrm{Im}\,\bar{Q}_E^-$ is ambitropical. Moreover, $\mathrm{Im}\,\bar{Q}_E^- \supset E$. By duality, the same is true for $\mathrm{Im}\,\bar{Q}_E^+$. By Prop. 4.6, (iv), the map $\bar{Q}_E^+$ is a bijection from $\mathrm{Im}\,\bar{Q}_E^-$ to $\mathrm{Im}\,\bar{Q}_E^+$ with inverse map $\bar{Q}_E^-$. □

The next result shows that if $F$ is a closed ambitropical cone containing $E$, then, there is a Shapley operator which injects $\mathrm{Im}\,Q_E^-$ into $F$.

**Theorem 4.17** (Ambitropical hulls)**.** *Let $E$ be a non-empty subset of $\mathbb{R}^n$, and suppose that $F$ is a closed ambitropical cone containing $E$. Then,*

*(i)* $\bar{Q}_E^+ \circ \bar{Q}_F^- \circ \bar{Q}_E^+ = \bar{Q}_E^+$;

*(ii)* $\bar{Q}^-_E \circ \bar{Q}^+_F \circ \bar{Q}^-_E = \bar{Q}^-_E$;

*Moreover,* $\operatorname{Im} \bar{Q}^+_E$ *is isomorphic to a subset* $F^+$ *of* $F$*, analogously* $\operatorname{Im} \bar{Q}^-_E$ *is isomorphic to a subset* $F^-$ *of* $F$ *and* $F^+$ *is isomorphic to* $F^-$*. In particular, both* $\operatorname{Im} \bar{Q}^+_E$ *and* $\operatorname{Im} \bar{Q}^-_E$ *are ambitropical hulls of* $E$*, and all the ambitropical hulls of* $E$ *are isomorphic.*

*Proof.* Observe first that since $E \subset F$,

$$P^{\min}_E \circ P^{\min}_F = P^{\min}_E \tag{13}$$

Indeed, $E \subset F$ implies that $P^{\min}_F \leqslant P^{\min}_E$. Hence, $P^{\min}_E \circ P^{\min}_F \leqslant P^{\min}_E \circ P^{\min}_E = P^{\min}_E$. Moreover, since $P^{\min}_F \geqslant I$, $P^{\min}_E \circ P^{\min}_F \geqslant P^{\min}_E$, which shows (13). Since $E^{\max} \subset F^{\max}$, $P^{\max}_F$ which fixes $E^{\max} = \operatorname{Im} P^{\max}_E$, we have

$$P^{\max}_F \circ P^{\max}_E = P^{\max}_E \tag{14}$$

Using (13) and (14), we get $\bar{Q}^+_E \circ \bar{Q}^-_F \circ \bar{Q}^+_E = P^{\max}_E \circ P^{\min}_E \circ P^{\min}_F \circ P^{\max}_F \circ P^{\max}_E \circ P^{\min}_E = P^{\max}_E \circ P^{\min}_E \circ P^{\max}_E \circ P^{\min}_E = (\bar{Q}^+_E)^2 = \bar{Q}^+_E$, by Proposition 4.6, *(v)*. This shows *(i)*. The proof of *(ii)* is dual.

Let $F^+ = \bar{Q}^-_F(\operatorname{Im} \bar{Q}^+_E) \subseteq F$ and consider

$$\bar{Q}^-_F : \operatorname{Im} \bar{Q}^+_E \to F^+, \qquad \bar{Q}^+_E : F^+ \to \operatorname{Im} \bar{Q}^+_E \ ;$$

they are inverses to each other by *(i)*, so $\operatorname{Im} \bar{Q}^+_E$ and $F^+$ are isomorphic. Analogously, by *(i)*, $\operatorname{Im} \bar{Q}^-_E$ is isomorphic to the subset of $F$, $F^- = \bar{Q}^+_F(\operatorname{Im} \bar{Q}^-_E)$. Moreover, by Corollary 4.16, $\operatorname{Im} \bar{Q}^+_E \cong \operatorname{Im} \bar{Q}^-_E$ and consequently $F^- \cong F^+$. Finally, let $\tilde{E}$ be an ambitropical hull of $E$, then $\operatorname{Im} \bar{Q}^+_E$ is isomorphic to a subset $\tilde{E}^+$ of $\tilde{E}$ but, since $\tilde{E}$ is minimal for inclusion, we have that $\tilde{E} \cong \tilde{E}^+ \cong \operatorname{Im} \bar{Q}^+_E$. □

We next point out an elementary metric property of ambitropical cones. Recall that a *geodesic* between $x$ and $y$ with respect to a seminorm $\|\cdot\|$ on $\mathbb{R}^n$ is a map $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\|\gamma(t_2) - \gamma(t_1)\| + \|\gamma(t_3) - \gamma(t_2)\| = \|\gamma(t_3) - \gamma(t_1)\|$, for all $0 \leqslant t_1 < t_2 < t_3 \leqslant 1$.

**Proposition 4.18.** *If* $E$ *is a closed ambitropical cone of* $\mathbb{R}^n$*, then, for any two points* $x, y$ *of* $E$*, there is a curve connecting* $x$ *and* $y$ *and included in* $E$ *that is a geodesic both in Hilbert's seminorm and in the sup-norm.*

*Proof.* Since $E$ is closed and ambitropical, there is an idempotent Shapley operator $P$ such that $E = P(\mathbb{R}^n)$. Let $x, y \in E$, and consider the ordinary line segment, $\gamma(s) = x + s(y-x)$, for $s \in [0,1]$, which connects $x$ and $y$ in $\mathbb{R}^n$, and is a geodesic in any semi-norm, in particular, in Hilbert's semi-norm and in the sup-norm. Then, since $P$ is nonexpansive both in Hilbert's seminorm and in the sup-norm, the map $s \mapsto P(\gamma(s))$ is still a geodesic, both in Hilbert's seminorm and in the sup-norm, and it is included in $E$. □

## 5. Ambitropical convexity and hyperconvexity

We relate here the above notions of ambitropical cones and of ambitropical hull with hyperconvexity. This notion was introduced by Aronszajn and Panitchpadki [AP56], We refer the reader to [Isb64, Dre84, Bai88, EK01] for insights on hyperconvex spaces and on the related notions of injective metric spaces and tight-span.

**Definition 5.1.** A metric space $M$ is said to be *hyperconvex* if $\bigcap_{\alpha \in \Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$ for any collection of points $\{x_\alpha\}_{\alpha \in \Gamma}$ in $M$ and positive numbers $\{r_\alpha\}_{\alpha \in \Gamma}$ such that $d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta$ for any $\alpha$, $\beta \in \Gamma$.

We recall that a metric space $M$ is said to be *injective* if for every $X, Y$ metric spaces, where $Y$ is a subspace of $X$ and $f\colon Y \to M$ nonexpansive, there exists an extension of $f$, $f'\colon X \to M$, that is nonexpansive. (Note that injective metric spaces do not coincide with injective objects in the category of metric spaces with nonexpansive maps. Indeed, it is true that every injective object is an injective metric space but the converse does not hold since the inverse of a nonexpansive map need not be nonexpansive.) Hyperconvex spaces are complete metric spaces and they are exactly injective metric spaces [Isb64, Dre84].

Theorem 3.8 shows that closed ambitropical cones are Shapley retracts of $\mathbb{R}^n$. Therefore, they are particular cases of sup-norm nonexpansive retracts of $\mathbb{R}^n$, so we get by [AP56, Th. 9], that they are hyperconvex subsets of $\mathbb{R}^n$ with sup-norm metric, which are, in addition, additive cones. We next show that the converse implication holds.

**Theorem 5.1.** *Let* $E$ *be an additive cone of* $\mathbb{R}^n$*. The following assertions are equivalent*

(1) *$E$ is a closed ambitropical cone;*
(2) *$E$ is hyperconvex for the sup-norm metric.*

*Proof.* We only need to show that (2) $\implies$ (1). Suppose that $E$ is an additive cone which is hyperconvex in the sup-norm metric. Since an hyperconvex space is complete, and $E$ is a subspace of $\mathbb{R}^n$, $E$ must be closed in the Euclidean topology. By Definition 3.2, it is enough to prove that $E$ is a lattice in the induced order. Let $x, y \in E$ and let, for any $s \leqslant r \in \mathbb{R}$,

$$B_s^r(u) = \{x \in E \mid u + s \leqslant x \leqslant u + r\} = B(u + (s+r)/2, (r-s)/2) \tag{15}$$

Let $U = \{u \in E \mid u \leqslant x, \quad u \leqslant y\}$. We want to prove that there exists a maximal element $z$ of $U$, that is an element $z \in E$ such that $z \leqslant x$, $z \leqslant y$ and $u \leqslant z$ for every $u \in U$.

This is equivalent to $z \in B_{r_x}^0(x)$, $z \in B_{r_y}^0(y)$ for some $r_x$, $r_y < 0$ and, for every $u \in U$, $z \in B_0^{r_u}(u)$ for some $r_u > 0$. Let us choose $r_x$ and $r_y$ such that $x + r_x/2 \leqslant y$, $y + r_y/2 \leqslant x$, and $r_u$ such that, for all $u \in U$, $x \leqslant u + r_u$ and $y \leqslant u + r_u$. Then, we assume without loss of generality that $r_x \geqslant r_y$, and observe that $x + r_x/2 \in B_{r_x}^0(x) \cap B_{r_y}^0(y)$. Moreover, $x \in B_{r_x}^0(x) \cap B_0^{r_u}(u)$ holds for all $u \in U$. Hence, we have that the balls in the above family have pairwise nonempty intersections. Therefore, using (15), they can be rewritten as balls satisfying the conditions in Definition 5.1. So, using the hyperconvexity of $E$, we obtain an element $z \in E$ belonging to the intersection of all the balls, and such an element is the maximal element of $U$. □

**Definition 5.2.** Let $E$ be a metric space and let denote the metric by $d$. We call $E$ a *metric space with a real action* if there is an action of the additive group $(\mathbb{R}, +, 0)$ on $E$, denoted by $(\lambda, x) \in \mathbb{R} \times E \mapsto \lambda \cdot x \in E$, such that

$$d(\lambda \cdot x, \lambda' \cdot x) = |\lambda - \lambda'| \ , \tag{16a}$$

$$d(\lambda \cdot x, \lambda \cdot x') = d(x, x') \ , \tag{16b}$$

for any $x, x' \in E$ and any $\lambda, \lambda' \in \mathbb{R}$.

Let us consider the category of metric spaces with a real action, with the nonexpansive, action-preserving maps. In the same spirit as for metric spaces, we say that $C$ is an *injective metric space with a real action* if for any metric space with a real action $Y$ and any subset $X$ of $Y$ which is closed by the real action on $Y$, every nonexpansive, action-preserving map from $X$ to $C$ has an extension from $Y$ to $C$ that is nonexpansive and action-preserving. Note that, as for metric spaces, this definition does not coincide with the usual definition of injective objects in the above category.

A prominent example of hyperconvex space is given by the *tight span* of a metric space, which provides its hyperconvex or injective hull [Isb64, Dre84]. We now recall the definition and basic results regarding the tight span.

**Definition 5.3** (Tight span of a metric space [Isb64, Dre84])**.** Let $X$ be any metric space with a metric $d$. The *tight span* $T(X)$ is the set of functions $f : X \to \mathbb{R}_{\geqslant 0}$ satisfying one of the two equivalent properties:
(1) $f(x) = \sup_{y \in X}(d(x, y) - f(y))$, for all $x \in X$;
(2) $f(x) + f(y) \geqslant d(x, y)$, for all $x, y \in X$, and $f$ is minimal among the functions with this property.

The second property in Definition 5.3 was used in the original construction of Isbell, where a map $f$ satisfying this property is called an extremal function. The first property in Definition 5.3 was used by Dress, who noted the equivalence with the second one and established a number of additional properties [Dre84, Th. 3]. The injectivity hull property of the tight span and some other properties were proved in [Isb64, Section 2]. We gather below some of the properties established in [Dre84, Th. 3] or in [Isb64, Section2], that will be used in the sequel.

We equip $T(X)$ with the supremum distance

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)| \ .$$

**Theorem 5.2** ([Dre84, Th. 3] and [Isb64, Section 2])**.** *For any $x \in X$, we denote by $e(x) = d(x, \cdot)$ the map $y \in X \mapsto d(x, y)$. We have the following properties*
(1) *Any element $f$ of $T(X)$ is 1-Lipschitz continuous, that is $|f(x) - f(y)| \leqslant d(x, y)$.*
(2) *For all $x \in X$ and $f \in T(X)$, we have $d_\infty(e(x), f) = f(x)$.*
(3) *For all $x \in X$, $e(x) \in T(X)$, and the map $e : X \to T(X)$, $x \mapsto e(x)$ is an isometry. Then, $X$ is isometric to the range $e(X) \subset T(X)$ of $e$, and can be identified to a subset of $T(X)$.*

*(4) Any nonexpansive map $\phi$ from $T(X)$ to itself that fixes $e(X)$ is the identity map.*
*(5) $T(X)$ is an hyperconvex, or equivalently, injective metric space.*
*(6) $T(X)$ is the injective hull of $X$ in the category of metric spaces, meaning that for any injective space $Y$ such that $X \subset Y$, or such that there is an isometry $\iota$ from $X$ to $Y$, there exists an isometry $\varphi$ from $T(X)$ to $Y$ such that $\iota = \varphi \circ e$.*

Recall also that in Theorem 5.2, Item 6 follows from Item 5 and Item 4.

We now show that when $X$ is a metric space with a real action, the tight span $T(X)$ is canonically equipped with a structure of metric space with real action.

**Proposition 5.3.** *Assume that $X$ is a metric space with a real action $(\lambda, x) \in \mathbb{R} \times X \mapsto \lambda \cdot x \in X$. For all $f \in T(X)$ and $\lambda \in \mathbb{R}$, we set*

$$\lambda \cdot f : X \to \mathbb{R}, \quad y \in X \mapsto (\lambda \cdot f)(y) = f((-\lambda) \cdot y) \ .$$

*Then, $\lambda \cdot f \in T(X)$. Moreover, the map $(\lambda, f) \in \mathbb{R} \times T(X) \mapsto \lambda \cdot f \in T(X)$ yields a real action on $T(X)$, and, together with the supremum distance $d_\infty$, this equips $T(X)$ with a structure of metric space with real action. Moreover, the map $e : X \to T(X)$ of* (3) *of Theorem 5.2 is an action-preserving isometry.*

*Proof.* Using that (16b) holds for the metric and action of $X$, we shall deduce that $\lambda \cdot f \in T(X)$ holds for any $f \in T(X)$ and $\lambda \in \mathbb{R}$. Indeed, for any $x \in X$, we have

$$\begin{aligned}\sup_{y \in X}(d(x,y) - (\lambda \cdot f)(y)) &= \sup_{y \in X}(d(x,y) - f((-\lambda) \cdot y)) = \sup_{z \in X}(d(x, \lambda \cdot z) - f(z)) \\ &= \sup_{z \in X}(d((-\lambda) \cdot x, z) - f(z)) = f((-\lambda) \cdot x) = (\lambda \cdot f)(x) \ ,\end{aligned}$$

where the first equality in the second line follows from (16b), and the second one holds since $f \in T(X)$. So, by Definition 5.3 (first property), $\lambda \cdot f \in T(X)$. It follows that the map $(\lambda, f) \mapsto \lambda \cdot f$ defines an action of the group $(\mathbb{R}, +, 0)$ on $T(X)$.

We now show that the axioms (16) are satisfied, i.e., in the present setting:

$$d_\infty(\lambda \cdot f, \lambda' \cdot f) = |\lambda - \lambda'| \ , \tag{17a}$$

$$d_\infty(\lambda \cdot f, \lambda \cdot f') = d_\infty(f, f') \ , \tag{17b}$$

for all $f, f' \in T(X)$ and $\lambda, \lambda' \in \mathbb{R}$. Property (17b) follows from a trivial change of variable:

$$d_\infty(\lambda \cdot f, \lambda \cdot f') = \sup_{x \in X} |f((-\lambda) \cdot x) - f'((-\lambda) \cdot x)| = \sup_{x \in X} |f(x) - f'(x)| = d_\infty(f, f') \ .$$

It remains to show (17a). We have

$$d_\infty(\lambda \cdot f, \lambda' \cdot f) = \sup_{x \in X} |f((-\lambda) \cdot x) - f((-\lambda') \cdot x)| \leqslant \sup_{x \in X} d((-\lambda) \cdot x, (-\lambda') \cdot x) = |\lambda - \lambda'| \ , \tag{18}$$

where the inequality in (18) uses that any element of $T(X)$ is 1-Lipschitz continuous, see Item 1 of Theorem 5.2. If this inequality is strict for some $\lambda, \lambda' \in \mathbb{R}$ and $f \in T(X)$, then

$$\nu := \sup_{x \in X} |f((-\lambda) \cdot x) - f((-\lambda') \cdot x)| < |\lambda - \lambda'| \ .$$

Denote $\mu = \lambda' - \lambda$, then applying a change of variable, we obtain that

$$|f(\mu \cdot x) - f(x)| \leqslant \nu < |\mu|, \quad \text{for all } x \in X \ .$$

In particular, $f(\mu \cdot x) \leqslant f(x) + \nu$, and applying successively this inequality to the elements $(n\mu) \cdot x$ such that $n$ is an integer, we get $f((n\mu) \cdot x) \leqslant f(x) + n\nu$, for all $n \geqslant 1$. Since $f \in T(X)$, we also have $f((n\mu) \cdot x) + f(x) \geqslant d((n\mu) \cdot x, x) = n|\mu|$, where the last equation follows from (16a). We deduce that $n|\mu| \leqslant 2f(x) + n\nu$ for all $n \geqslant 1$, which is impossible since $\nu < |\mu|$. Therefore, the inequality in (18) is an equality, whih proves (17a).

By (3) of Theorem 5.2, the map $e : X \to T(X), x \mapsto d(x, \cdot)$ is an isometry. We also have that $e(\lambda \cdot x)(y) = d(\lambda \cdot x, y) = d(x, (-\lambda \cdot y)) = e(x)(-\lambda \cdot y) = (\lambda \cdot e(x))(y)$. Hence, $e(\lambda \cdot x) = \lambda \cdot e(x)$. This shows that $e$ is also action-preserving. □

**Theorem 5.4.** *Injective metric spaces with a real action are exactly hyperconvex spaces with a real action.*

*Proof.* To show that hyperconvex spaces with a real action are injective, we proceed as in the proof of Theorem 4.2 of [EK01], which shows the analogous property in the absence of a real action. In the following we will outline the parts which require to be adjusted.

Let $H$ be an hyperconvex metric space with a real action, we want to prove that $H$ is injective. We shall consider a metric space with real action $(M, d)$ and $D$, a subset of $M$ which is closed by the action. Let $(H, d')$ be a metric space with real action and $g\colon D \to H$ a nonexpansive, action preserving map. To prove that $H$ is injective we shall show that there exists a nonexpansive and action-preserving extension of $g$ from $M$ to $H$.

Let define $\mathcal{C}$ as follows:

$$\mathcal{C} = \{(T_F, F) \mid D \subseteq F \subseteq M,\ F \text{ is closed by the action of } M,\ T_F\colon F \to H,\ T_F \text{ preserves the action}\}$$

$\mathcal{C}$ is nonempty since it contains $(g, D)$. It is ordered by $(T_F, F) \leqslant (T_{F'}, F')$ if $F \subset F'$ and $T_{F'}$ is an extension of $T_F$ over $F'$. We need to show that any maximal element $(T_{F_1}, F_1)$ of $\mathcal{C}$ is such that $F_1 = M$. We proceed by contradiction, assuming that there exists $z \in M \setminus F_1$, and constructing $(T_F, F) \in \mathcal{C}$, such that $(T_{F_1}, F_1) < (T_F, F)$ and $z \in F$. Since the set $F$ must be close under the action, we take $F = F_1 \cup \{\lambda \cdot z \mid \lambda \in \mathbb{R}\}$. Also, the extended function $T_F$ need to preserve the action. So we need to choose $x \in H$ and define $T_F(\lambda \cdot z) = \lambda \cdot x$ for all $\lambda \in \mathbb{R}$. Since we are assuming that $z \notin F_1$, we have that for every $\lambda$, $\lambda \cdot z \notin F_1$, so we do not have ambiguities in the definition of $T_F$. Since $H$ is hyperconvex, the arguments of the proof of Theorem 4.2 of [EK01] shows that there exists $x \in H$ such that the map $T_F$ restricted to the set $F_1 \cup \{z\}$ is nonexpansive. We need to prove that $T_F$ is nonexpansive on all the metric space $F$. Since $T_F$ is already nonexpansive on $F_1$, we only need to check the following conditions:

$$d'(T_F(\lambda \cdot z), T_F(\lambda' \cdot z)) \leqslant d(\lambda \cdot z, \lambda' \cdot z)\ ,$$
$$d'(T_F(z_1), T_F(\lambda \cdot z) \leqslant d(z_1, \lambda \cdot z)\ ,$$

for all $z_1 \in F_1$ and $\lambda, \lambda' \in \mathbb{R}$. Since $T_F(\lambda \cdot z) = \lambda \cdot x$, the first inequality is an equality and follows from (16a). Since $T_F$ is preserving the action on $F$, $F_1$ is closed under the action, and $T_F$ is nonexpansive on $F_1 \cup \{z\}$, the first inequality follows from (16b).

To show the converse implication, we shall make use of the properties of the tight span.

Consider now a metric space with real action $X$ and assume that it is injective. Then, by (5) of Theorem 5.2 and Proposition 5.3, $T(X)$ is a hyperconvex set with a real action, and the map $e : X \to T(X), x \mapsto d(x, \cdot)$ is an action-preserving isometry. By injectivity of $X$ there exists an action-preserving nonexpansive map from $h\colon T(X) \to X$ such that $h \circ e$ is the identity. Indeed, we can consider the identity map on $X$ as the map which will be expanded. Then, the map $e \circ h : T(X) \to T(X)$ is a nonexpansive map which fixes $e(X)$.

By Item 4 of Theorem 5.2, we have that the map $e \circ h$ is the identity, so $X = T(X)$ and it is hyperconvex. □

We deduce the following result.

**Corollary 5.5.** *Assume that $X$ is a metric space with a real action. Then, $T(X)$ is the injective hull of $X$, meaning that for any injective metric space with a real action $Y$ such that $X \subset Y$, or such that there is an action preserving isometry $\iota$ from $X$ to $Y$, there exists an action preserving isometry $\varphi$ from $T(X)$ to $Y$ such that $\iota = \varphi \circ e$.*

*Proof.* Let $X$ is a metric space with a real action, and let $Y$ be an injective metric space with a real action such that there is an action preserving isometry $\iota$ from $X$ to $Y$. As above, by (5) of Theorem 5.2 and Proposition 5.3, $T(X)$ is a hyperconvex set with a real action, and the map $e : X \to T(X), x \mapsto d(x, \cdot)$ is an action-preserving isometry. Moreover, by Theorem 5.4, $T(X)$ is injective.

By injectivity of $Y$ there exists an action-preserving nonexpansive map from $g\colon T(X) \to Y$ such that $g \circ e = \iota$. By injectivity of $T(X)$ there exists an action-preserving nonexpansive map from $h\colon Y \to T(X)$ such that $h \circ \iota = e$. Then, the map $h \circ g : T(X) \to T(X)$ is a nonexpansive map which satisfies $h \circ g \circ e = h \circ \iota = e$, so it fixes $e(X)$.

By Item 4 of Theorem 5.2, we have that the map $h \circ g$ is the identity, so $g$ is an isometry such that $\iota = g \circ e$. □

**Theorem 5.6.** *Let $X \in \mathbb{R}^n$ be an additive cone. Then, the ambitropical hull of $X$ and the injective hull of $X$ are in bijective correspondence under an action-preserving isometry.*

*Proof.* Let $A \subset \mathbb{R}^n$ be an ambitropical hull of $X$, with $\imath : X \to A$ the canonical injection. By Theorem 5.1, $A$ is a hyperconvex additive cone, and by Theorem 5.4, it is an injective space with real action, and by Corollary 5.5, there is an action-preserving isometry $\varphi$ from $T(X)$ to a subset of $A$ and such that $\varphi \circ e = \imath$. Hence, $\varphi(T(X))$ is injective, and by Theorem 5.1 and Theorem 5.4, it is an ambitropical cone. By minimality of $A$, $A = \varphi(T(X))$ which entails that $T(X)$ and $A$ are in bijective correspondence under the action preserving isometry $\varphi$. □

In general, it is already a difficult problem to construct the tight span of finite metric spaces with more than a few points. A characterization of the tight span of arbitrary subsets of the plane with the maximum metric is given in [KcK16]. Thanks to the above theorem, we get a way to compute the tight span of an additive cone, since this last one is isometric to its ambitropical hull, see e.g. Figure 3 for an illustration.

## 6. Special classes of ambitropical cones

We next review several canonical classes of sets in tropical geometry, showing these are special cases of ambitropical cones, that can be characterized by suitable reinforcements of Theorem 3.8.

The simplest examples of ambitropical cones consist of alcoved polyhedra, discussed in Section 2. Actually, the following result shows that alcoved polyhedra are characterized by the property of being sublattices of $\mathbb{R}^n$. Observe that all properties but one do not assume polyhedrality, polyhedrality comes as a consequence of the other properties.

**Proposition 6.1.** *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

*(1) $C$ is an alcoved polyhedron;*
*(2) $C$ is a closed tropical cone and a closed dual tropical cone,*
*(3) $C$ is a closed ambitropical cone in which the infimum and supremum laws coincide with the ones of $\mathbb{R}^n$.*
*(4) There is a tropically linear Shapley operator $T$ such that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$.*
*(5) There is a dually tropically linear Shapley operator $T$ such that $C = \{x \in \mathbb{R}^n \mid T(x) \geqslant x\}$.*

*Proof of Proposition 6.1.* An alcoved polyhedron is stable by pointwise supremum and pointwise infimum of vectors, so (1) implies (2). Trivially, (2) implies (3).

Suppose now that (3) holds. Then, by Proposition 3.5, $C$ is a conditionally complete lattice, and the lattice operations of $C$ are the pointwise supremum and pointwise infimum of vectors. Define, for all $i, j \in [n]$,

$$M_{ij} := \sup\{\lambda \mid v_i \geqslant \lambda + v_j, \qquad \forall v \in C\}$$

The latter set is nonempty, it is closed and bounded from above, so that the supremum is achieved. In particular, we have $M_{ij} \in \mathbb{R}_{\max}$. By construction, $C \subset \mathcal{A}(M)$, and $M_{ij} = M^*_{ij}$. Observe also that the inequality $v_i \geqslant \lambda + v_j$, is equivalent to $w_i \geqslant \lambda$ where $w := v - v_j\delta_j \in C$ is such that $w_j = 0$, denoting by $\delta_j = (0, \dots, 0, 1, 0, \dots 0)$ the $j$-th vector of the canonical basis of $\mathbb{R}^n$. It follows that:

$$M_{ij} = \inf C_{ij}, \text{ where } C_{ij} := \{v_i \mid v \in C_j\} \text{ and } C_j := \{v \in C \mid v_j = 0\} \enspace .$$

Denoting by $u^j$ the $j$th column of the matrix $M$, we deduce that

$$u^j = \inf C_j \in \operatorname{clo}^{\downarrow} C \enspace .$$

Define, $A_j := \{v \in \operatorname{clo}^{\downarrow} C \mid v_j = 0\}$. Since $C$ is a conditionally complete lattice, the set $A_j \subset \operatorname{clo}^{\downarrow} C$ is stable by taking infima. Hence, the set $\operatorname{Min} A_j$ consists of a single point, $u^j$. By Theorem 2.2, every element of $C$ is a tropical linear combination of vectors $u^j$. This implies that $\mathcal{A}(M) = \{M^* y \mid y \in \mathbb{R}^n\} = C$. So (3) implies (1).

If $C = \mathcal{A}(M)$ is an alcoved polyhedron, it follows from (2) that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$ where $T(x) = M^* x$ is a Shapley operator. So (1) implies (4).

Conversely, if $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$ for some tropically linear Shapley operator, then, for all $x, y \in C$, since $T$ is order preserving, $T(x \wedge y) \leqslant T(x) \wedge T(y) \leqslant x \wedge y$, and since $T$ is tropically linear, $T(x \vee y) = T(x) \vee T(y) \leqslant x \vee y$, showing that $x \wedge y$ and $x \vee y$ belong to $C$. Moreover, $C$ is closed, since $T$ is continuous (in fact, $T$ is sup-norm nonexpansive). This shows that (4) implies (3).

(1) and (5) are equivalent. Indeed, observe that $C$ is of the form $\{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$ for some tropically linear Shapley operator iff $-C$ is of the form $\{x \in \mathbb{R}^n \mid P(x) \geqslant x\}$ for some dually tropically linear Shapley operator (consider the involution $T \mapsto P$, $P(x) := -T(-x)$ on the space of Shapley operators). Moreover, $C$ is an alcoved polyhedron iff $-C$ is an alcoved polyhedron. Hence, the announced equivalence follows from the equivalence of (1) and (4), already established. □

Note that in the above proof we made use of a particular instantiation of the "flip" application $X \to -X$ which sends ambitropical cones to ambitropical cones.

Tropical cones, and dual tropical cones, are also remarkable examples of ambitropical cones. The relation between these cones and sub or super-fixed point sets of Shapley operators was already noted in [AGG12].

**Proposition 6.2.** *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

*(1) $C$ is a closed tropical cone;*
*(2) $C$ is a closed ambitropical cone in which the supremum law coincides with the one of $\mathbb{R}^n$;*
*(3) there is a Shapley operator $T$ such that $C = \{x \in \mathbb{R}^n \mid T(x) \geqslant x\}$.*

*Proof.* (1)⇒(2). Suppose that $C$ is a closed tropical cone, and let $X$ denote a non-empty subset of $C$ bounded from above by an element of $\mathbb{R}^n$. Then, for all finite subsets $F \in \mathcal{P}_f(X)$, $\sup F$ belongs to $C$, because $C$ is stable by supremum, and $\sup X = \lim_{F\in\mathcal{P}_f(X)} \sup F \in C$ because $C$ is closed. It follows that $X$ has a supremum in $C$ which coincides with its supremum in $\mathbb{R}^n$. Suppose now that $X$ is bounded from below by an element $z$ of $\mathbb{R}^n$. Consider $Y := \{y \in C \mid y \leqslant x, \forall x \in X\}$. Then, $Y$ is non-empty and it is bounded from above. It follows from the previous observation that $\sup Y$ is the supremum of $Y$, in $C$. Moreover, $\sup Y$ is precisely the infimum of $X$ in $C$, showing that $C$ is ambitropical.

(2)⇒(3). Since $C$ is a closed ambitropical cone, then, it is the fixed point set of $z \mapsto Q^-_C(z) = \sup\{x \in C \mid x \leqslant z\}$, and $Q^-_C \leqslant I$. So, $C = \{z \in \mathbb{R}^n \mid Q^-_C(z) = z\} = \{z \in \mathbb{R}^n \mid Q^-_C(z) \geqslant z\}$.

(3)⇒(1). Suppose that $C = \{x \in \mathbb{R}^n \mid T(x) \geqslant x\}$. Since $T$ is continuous, $C$ is closed. Moreover, since $T$ is order preserving, for all $x, y \in C$, $T(x \vee y) \geqslant T(x) \vee T(y) \geqslant x \vee y$, showing that $x \vee y \in C$. □

We state the following dual version of Proposition 6.2.

**Proposition 6.3.** *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

*(1) $C$ is a closed dual tropical cone;*
*(2) $C$ is an ambitropical cone in which the infimum law coincides with the one of $\mathbb{R}^n$;*
*(3) there is a Shapley operator $T$ such that $C = \{x \in \mathbb{R}^n \mid T(x) \leqslant x\}$.* □

We next define the subclass of *homogeneous* tropical cones – which will arise as tangent spaces or recession sets of ordinary tropical cones.

**Definition 6.1.** An ambitropical cone $C$ is *homogeneous* if for all $\alpha > 0$ and for all $x \in C$, $\alpha x \in C$. Recall that a Shapley operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is *homogeneous* if $T(\alpha x) = \alpha T(x)$ holds for all $\alpha > 0$ and for all $x \in \mathbb{R}^n$.

**Proposition 6.4.** *Let $C \subset \mathbb{R}^n$. The following statements are equivalent:*

*(1) $C$ is a closed homogeneous ambitropical cone;*
*(2) there is an idempotent homogeneous Shapley operator whose fixed point set is $C$;*
*(3) there is a homogeneous Shapley operator whose fixed point set is $C$.*

*Proof.* The implication *(2)⇒(3)* is immediate. If $C$ is the fixed point set of an homogeneous Shapley operator $T$, then, we know from Theorem 3.8 that $C$ is an ambitropical cone, and it follows from the homogeneity of $T$ that $C$ is homogeneous. This shows the implication *(3)⇒(1)*. If $C$ is a closed homogeneous ambitropical cone, we know from Theorem 3.8 that $C$ is the range of the idempotent Shapley operator $\bar{Q}^-_C = Q^-_C$. Observe that, for all $\alpha > 0$ and $x \in \mathbb{R}^n$, $Q^-_C(\alpha x) = \sup^C\{y \in C \mid y \leqslant \alpha x\} = \sup^C\{\alpha\alpha^{-1} y \mid y \in C,\ \alpha^{-1} y \leqslant x\} = \sup^C\{\alpha z \mid z \in C,\ z \leqslant x\} = \alpha Q^-_C(x)$ since $y \mapsto \alpha^{-1} y$ is a bijection from $C$ to $C$, which is order preserving and whose inverse also preserves the order. This shows the implication *(1)⇒(2)*. □

## 7. Correspondence between fixed points of Shapley operators and calibrated policies

### 7.1. **Finitely generated Shapley operators.** The following definition is taken from [CTGG99].

**Definition 7.1.** A *min-max function* in the variables $x_1, \dots, x_n$ is a map $f : \mathbb{R}^n \to \mathbb{R}$, defined by a term in the context-free grammar $X \to x_1, \dots, x_n, X \vee X, X \wedge X, X + c$ where $c$ stands for any real constant.

For instance, $f(x_1, x_2, x_3) = ((x_1 \vee (x_2 + 3)) \wedge ((x_3 - 1) \vee x_1)) \vee x_2$ is a min-max function. It follows readily from the definition that the set of min-max functions is stable by the operations $\vee$, $\wedge$, and by the translation by a constant. Using the distributivity law, we may always rewrite a min-max function in conjunctive normal form, i.e.,

$$f(x) = \wedge_{k\in[K]} \vee_{i\in[n]} (c_{ki} + x_i) \tag{19}$$

for some integer $K$ and coefficients $c_{ki} \in \mathbb{R}_{\max}$, such that for all $k \in [K]$, $c_{ki} \neq -\infty$ for some $i \in [n]$. Similarly, we may rewrite $f$ in disjunctive normal form:

$$f(x) = \vee_{k \in [K']} \wedge_{i \in [n]} (c'_{ki} + x_i) \tag{20}$$

for some integer $K'$ and coefficients $c'_{ki} \in \mathbb{R}_{\min}$, such that for all $k \in [K']$, $c'_{ki} \neq +\infty$ for some $i \in [n]$. *Monotone Boolean functions* are special cases of min-max functions, they are obtained by restricting the above grammar rule to exclude the derivation $X \mapsto X + c$. For instance, $(x_1 \vee x_2) \wedge x_3$ is a monotone Boolean function.

**Definition 7.2.** A Shapley operator is *finitely generated* if its coordinates are min-max functions.

It will be convenient to write finitely generated Shapley operators in an algebraic way, along the lines of [AGG12], making apparent the game interpretation. Let $A \in (\mathbb{R}_{\max})^{m \times p}$. The *adjoint* of the tropically linear map associated with the matrix $A$ is the dual tropically linear map

$$y \mapsto A^\sharp y, \qquad (A^\sharp y)_k = \wedge_{i \in [m]} (-A_{ik} + y_i), \qquad k \in [p] \ ,$$

which sends $\mathbb{R}^m \to \mathbb{R}^p$ if $A$ has no identically $-\infty$ column. Observe that, for $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^m$,

$$Ax \leqslant y \iff x \leqslant A^\sharp y \ .$$

If $B \in (\mathbb{R}_{\max})^{m \times n}$ The tropically linear map

$$x \mapsto Bx, \qquad (Bx)_i = \vee_{j \in [n]} (B_{ij} + x_j), \qquad i \in [m]$$

sends $\mathbb{R}^n$ to $\mathbb{R}^m$ if $B$ has not identically $-\infty$ row. This motivates the following definition.

**Definition 7.3.** We say that a pair of matrices $A \in (\mathbb{R}_{\max})^{m \times p}, B \in (\mathbb{R}_{\max})^{m \times n}$ is *proper* if $A$ has no identically $-\infty$ column, and $B$ has no identically $-\infty$ row.

Given a proper pair of matrices $A \in (\mathbb{R}_{\max})^{m \times p}, B \in (\mathbb{R}_{\max})^{m \times n}$, we consider the operator $T = A^\sharp \circ B$, with coordinates

$$T_i(x) = \wedge_{j \in [m]} \Big( -A_{ji} + \vee_{k \in [n]} (B_{jk} + x_k) \Big), \qquad i \in [p] \ . \tag{21}$$

This is a finitely generated Shapley operator from $\mathbb{R}^n \to \mathbb{R}^p$.

**Proposition 7.1.** *Every finitely generated Shapley operator can be written as $T = A^\sharp \circ B$ for some proper pair of matrices $A, B$.*

*Proof.* Every coordinate of $T$ can be represented by a min-max function in disjunctive normal form, and this representation is of the form (21) (with $A_{ij} \in \{0, -\infty\}$). ☐

Recall that a map $T : \mathbb{R}^n \to \mathbb{R}^p$ is (positively) *homogeneous* (of degree 1) if $T(\alpha x) = \alpha T(x)$ holds for all $\alpha > 0$ and $x \in \mathbb{R}^n$.

**Proposition 7.2.** *If $T$ is a finitely generated homogeneous Shapley operator, then the coordinates of $T$ are given by monotone Boolean functions, and all the finitely generated homogeneous Shapley operators arise is this way.*

*Proof.* Suppose that $T = A^\sharp \circ B$ as above. Then, $T_i(x) = \alpha^{-1} T_i(\alpha x)$, and so,

$$\begin{aligned} T_i(x) &= \lim_{\alpha \to \infty} \wedge_{j \in [m]} \Big( -\alpha^{-1} A_{ji} + \vee_{k \in [n]} (\alpha^{-1} B_{jk} + x_k) \Big) \\ &= \wedge_{j \in [m],\ A_{ij} \neq -\infty} \big( \vee_{k \in [n],\ B_{jk} \neq -\infty}\ x_k \big) \end{aligned}$$

which is a monotone Boolean function. ☐

**Proposition 7.3.** *Let $T_1$, $T_2$ be two finitely generated Shapley operators $\mathbb{R}^n \to \mathbb{R}^p$. Then the operators $T_1 \vee T_2$ and $T_1 \wedge T_2$ are finitely generated. Suppose now that $T_2$ is a finitely generated Shapley operator from $\mathbb{R}^q \to \mathbb{R}^n$. Then, $T_1 \circ T_2$ is a finitely generated Shapley operator.*

*Proof.* By definition, min-max functions are stable by the laws $\vee$ and $\wedge$. They are also stable by composition, meaning that if $f(x_1, \dots, x_n)$ is a min-max function in the variables $x_1, \dots, x_n$, and if for all $i \in [n]$, $g_i(y_1, \dots, y_q)$ is a min-max function in the variables $y_1, \dots, y_q$, then $f(g_1(y_1, \dots, y_q), \dots, g_n(y_1, \dots, y_q))$ is a min-max function in the same variables. This implies the announced properties. ☐

7.2. **Eigenvectors of Shapley operators and Deterministic Mean Payoff Games.** We next recall that the operators of the form (21) are precisely the dynamic programming operators of *deterministic* zero-sum games without discount, referring the reader to [AGG12] for background. We shall also present an equivalence between the fixed points of the Shapley operator and a remarkable class of optimal positional policies, *calibrated* policies.

We associate to the proper pair of matrices $(A, B)$ a game in which two players, Max and Min, move alternatively a token in a digraph. The set of states is the disjoint union of the sets $[n]$ and $[m]$. If the token is in a state $i \in [n]$, Player Min chooses a new state $j \in [m]$ such that $A_{ji}$ is finite, moves the token to this state, and receives a payment $A_{ji}$ from Max. If the token is in a state $j \in [m]$, Player Max chooses a new state $k \in [n]$ such that $B_{jk}$ is finite, moves the token to this state, and receives a payment $B_{jk}$ from Min. The assumption that the pair $(A, B)$ is proper guarantees that each player has always at least one available action. Given an initial state $i \in [n]$, and an integer $k$, one can consider the *game in horizon* $k$, in which each the two players makes $k$ moves, alternatively, so that the total payment received by Player Max is

$$R^k = -A_{j_1 i_0} + B_{j_1 i_1} - A_{j_2 i_1} + B_{j_2 i_2} + \cdots - A_{j_k i_{k-1}} + B_{j_k i_k}$$

where $i_0, j_1, i_1, j_2, \ldots, j_k, i_k$ is the sequence of states that are visited and $i_0 = i$ is the initial state. A *history* of the game, at a given stage, consists of the sequence of visited states, so if $l$ turns have been played, and if it Min's turn to play, the history is $i_0, j_1, i_1, j_2, \ldots, j_l, i_l$ whereas if it is Max's turn to play, the history is $i_0, j_1, i_1, j_2, \ldots, j_l, i_l, j_{l+1}$. We assume that the game is in perfect information, meaning that the two players observe the history. A (pure) *strategy* of a player is a map which associates an action to each history. Therefore, the total payment is a function of the strategies $\sigma$ and $\pi$ of the two players and of the initial state $i$, i.e., $R^k = R_i^k(\sigma, \pi)$. It follows from dynamic programming arguments, that the game in horizon $k$ with initial state $i$ has a value, $v_i^k$, and that there are associated optimal strategies $\sigma^*$ and $\pi^*$, meaning that the following saddle point property holds:

$$R_i^k(\sigma^*, \pi) \leqslant v_i^k \leqslant R_i^k(\sigma, \pi^*)$$

for all strategies $\sigma$ of Player Min and $\pi$ of Player Max, which entails in particular that $v_i^k = R_i^k(\sigma^*, \pi^*)$. Indeed, the *value vector* $v^k = (v_i^k)_{i \in [n]} \in \mathbb{R}^n$ satisfies the dynamic programming equation

$$v^0 = 0, \qquad v^k = T(v^{k-1}) \ ,$$

and the actions that achieve the min or max in the expression $T_i(v^{k-1})$ provide optimal decisions for the two players, depending only on the current state and on the time remaining to play. We refer the reader to [MSZ15b] for background on game theory and on the dynamic programming approach.

A *deterministic policy* of Player Min (resp. Max) is a map $\sigma : [n] \to [m]$ (resp. $\pi : [m] \to [n]$). A deterministic policy of Player Min induces a stationary positional strategy, obtained by moving to state $\sigma(i)$ when in state $i \in [n]$, and similarly for Player Max, the state being now $j \in [m]$.

We are interested in the *mean payoff game*, in which Player Min wants to minimize the mean payment per time unit made to Player Max, and Player Max wants to maximize it. Liggett and Lippman [LL69], and Ehrenfeucht and Mycielski [EM79] showed that there exists a vector $\chi = (\chi_i)_{i \in [n]} \in \mathbb{R}^n$ and deterministic policies $\sigma^*$ and $\pi^*$, of Player Min and Max, respectively, such that, for all strategies $\sigma$ and $\pi$ of these two players, and for all initial states $i \in [n]$,

$$\limsup_{k \to \infty} k^{-1} R_i^k(\sigma^*, \pi) \leqslant \chi_i \leqslant \liminf_{k \to \infty} k^{-1} R_i^k(\sigma, \pi^*) \ . \tag{22}$$

The number $\chi_i$ is known as the *value* of the mean payoff game with initial state $i$. Moreover, the vector $\chi$ coincides with the limit $\lim_{k \to \infty} v^k / k$.

A remarkable case arises when the non-linear eigenproblem

$$T(u) = \lambda + u, \qquad u \in \mathbb{R}^n, \lambda \in \mathbb{R}$$

is solvable. Then, the value of the mean payoff game $\chi_i$ is independent of the initial state $i$, and optimal deterministic policies can be obtained from the eigenvector $u$. To explain this relation, it will be convenient to extend the notion of policy as follows: a *nondeterministic policy* of Player Min (resp. Max) is a map $\sigma : [n] \to \mathcal{P}([m]) \setminus \emptyset$, such that $\sigma(i) \subset \{j \in [m] \mid A_{ji} \text{ finite}\}$ for all $i \in [n]$. Similarly, a *nondeterministic policy* of Player Max is a map $\pi : [m] \to \mathcal{P}([n]) \setminus \emptyset$, such that $\pi(j) \subset \{i \in [n] \mid B_{ji} \text{ finite}\}$ for all $j \in [m]$. A nondeterministic policy of Player Min *induces* a whole collection of strategies (not necessarily stationary or positional), obtained by restricting the moves of Min to the $i \to j \in \sigma(i)$. In particular, it induces *deterministic policies*, obtained by selecting, for all $i \in [n]$, a single element in $\sigma(i)$.

We denote by $\mathcal{S}$ the set of nondeterministic policies of Player Min, and by $\mathcal{P}$ the set of nondeterministic policies of Player Max.

The following definition extends to the two-player case the notion of *calibrated trajectory* introduced by Fathi in the context of weak-KAM theory, see [FS04, Fat08].

**Definition 7.4** (calibrated policies)**.** Given $u \in \mathbb{R}^n$, we say that a pair of non-deterministic policies $(\sigma^u, \pi^u)$ is *u-calibrated* if, for some $\lambda \in \mathbb{R}$,

i) By playing any strategy induced by $\sigma^u$, Player Min can guarantee that, whatever strategy Max chooses, and for all horizons $k$ and initial states $i_0$,

$$-A_{j_0 i_0} + B_{j_0 i_1} + \cdots - A_{j_{j-1} i_{k-1}} + B_{j_{k-1} i_k} \leqslant u_{i_0} - u_{i_k} + k\lambda \ , \tag{23}$$

ii) By playing any strategy induced by $\pi^u$, Player Max can guarantee that, whatever strategy Min chooses, and for all horizons $k$ and initial states $i_0$,

$$-A_{j_0 i_0} + B_{j_0 i_1} + \cdots - A_{j_{j-1} i_{k-1}} + B_{j_{k-1} i_k} \geqslant u_{i_0} - u_{i_k} + k\lambda \ ; \tag{24}$$

where $i_0, j_0, i_1, j_1 \ldots, i_k$ is the sequence of states that are visited.

In the special one-player case, assuming for instance that Player Max is a dummy (with only one possible action in each state), we can replace the inequality by an equality in (23), and then, we recover the original notion of calibrated trajectory. In particular, Fathi established a correspondence between the global viscosity solutions of the ergodic Hamilton-Jacobi PDE and calibrated trajectories, see [FS04, Prop. 3.5] and [Fat08, Prop 4.1.10]. The following elementary proposition states an analogous property in the two-player setting.

**Proposition 7.4.** *Suppose that* $T(u) = \lambda + u$*, define* $\pi^*(j) = \operatorname{argmax}_i B_{ji} + u_i$ *and* $\sigma^*(i) = \operatorname{argmin}_j -A_{ji} + (Bu)_j$*. Then, the pair of policies* $(\sigma^*, \pi^*)$ *is u-calibrated. Moreover, all pairs of u-calibrated policies arise in this way.*

To establish this result, it will be convenient to introduce, for all pair of deterministic policies $(\sigma, \pi)$ of the two players, the operators $T^\sigma$ and ${}^\pi T$, $\mathbb{R}^n \to \mathbb{R}^n$, such that

$$T_i^\sigma(x) = -A_{\sigma(i)i} + \vee_k (B_{\sigma(i)k} + x_k), \qquad {}^\pi T_i(x) = \wedge_j (-A_{ji} + B_{j\pi(j)} + x_{\pi(j)}) \ .$$

These operators represent the one-player games obtained by fixing a policy of one of the players.

*Proof.* If $T(u) = \lambda + u$, by definition of $\sigma^*$ and $\pi^*$, we have, for all deterministic policies $\sigma$ and $\pi$ compatible with $\sigma^*$ and $\pi^*$,

$$T^\sigma(u) = \lambda + u, \qquad {}^\pi T(u) = \lambda + u \ .$$

It follows that for all $k$, $(T^\sigma)^k(u) = k\lambda + u$, which implies that (23) holds. Similarly, $({}^\pi T)^k(u) = k\lambda + u$ yields (24).

Conversely, if $\sigma^*$ and $\pi^*$ are $u$-calibrated, by specializing (23) to $k = 1$, we deduce that $T(u) \leqslant \lambda + u$, and similarly, we deduce from (24) that $T(u) \geqslant \lambda + u$. □

Any deterministic policies $(\sigma^*, \pi^*)$ induced by a pair $(\sigma^u, \pi^u)$ of $u$-calibrated policies are optimal in the mean payoff game, meaning they satisfy the saddle point property (22). However, being $u$-calibrated is a finer property than being optimal in the mean payoff game, since it involves not only the mean payoff but also the deviation to the mean payoff. This is easily seen in the one-player case. Then, playing a deterministic policy induces state trajectories which ultimately cycle. As long as the the ultimate cycle reached from each initial state is unchanged, the mean-payoff optimality is preserved, but not the property of being calibrated.

*Example* 7.5. Consider the one player game shown in Figure 2, in which the Player is Max. This can be represented by the pair of matrices $(A, B)$ where $A$ is the tropical identity map, meaning that player Min is a dummy, and $B$ is the matrix of payments shown in the figure. The value of the mean payoff game is equal to 1, since it is optimal for Player Max to reach the cycle $4 \to 4$ which has the best weight-to-length ratio, equal to 1. The matrix $B$, and so, the operator $T = A^\sharp \circ B$, has only one eigenvector up to an additive constant, given by $u = (-2, -2, -1, 0)$, and so, by Proposition 7.4, there is only one $u$-calibrated strategy, namely $1 \to 3$, $3 \to 4$ and $2 \to 4$. However, the strategy $1 \to 2$, $3 \to 4$ and $2 \to 4$ is also optimal, since the same ultimate cycle is reached by any initial state, but it is not $u$-calibrated.

$$B = \begin{pmatrix} -\infty & -1 & 0 & -\infty \\ -\infty & -1 & -\infty & -1 \\ -\infty & -\infty & 0 & 0 \\ -\infty & -\infty & -\infty & 1 \end{pmatrix}$$

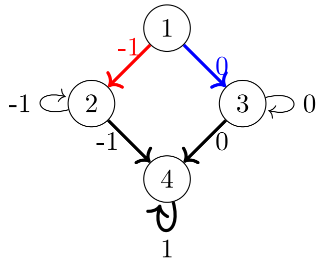


FIGURE 2. A one player game (maximizing the average cost), with two optimal policies, only one of which (going right, in blue) is calibrated.

*Example* 7.6. An example of calibrated policies arises from the notion of Blackwell optimality, originally introduced in the one player setting [Bla62]. Let $0 < \alpha < 1$ be a discount factor, and consider the value vector $v^\alpha$ of the discounted version of the game, so that $v^\alpha$ is the unique solution of the fixed point equation $v^\alpha = T(\alpha v^\alpha)$. It follows from the result of Kohlberg [Koh80] that, as soon as the mean payoff of the game is independent of the initial state, $v^\alpha$ has a Laurent series expansion satisfying in particular $v^\alpha = \lambda/(1-\alpha) + u + O(1-\alpha)$, where $\lambda \in \mathbb{R}$ is the mean payoff, as $\alpha \to 1^-$, and we have $T(u) = \lambda + u$. There are policies $\sigma^*, \pi^*$ such that $T(\alpha v^\alpha) = T^{\sigma^*}(\alpha v^\alpha) = {}^{\pi^*}T(\alpha v^\alpha)$ for all values of $\alpha$ close enough to 1, meaning that $\sigma^*, \pi^*$ are optimal in all the discounted games with a discount factor sufficiently close to one (see [AGGCG19, Th. 8] for a proof of this property in the two player setting). These policies are said to be Blackwell optimal. They are $u$-calibrated.

## 8. Ambitropical polyhedra

We now consider ambitropical cones with a polyhedral structure.

**Definition 8.1.** An *ambitropical polyhedron* is an ambitropical cone that is a finite union of alcoved polyhedra. An *ambitropical polytope* is an ambitropical polyhedron that is bounded in Hilbert's seminorm.

Observe that an ambitropical polyhedron is closed. We shall first show that fixed point sets of finitely generated Shapley operators are ambitropical polyhedra, and then, we will show that all ambitropical polyhedra arise in this way and provide a notion of generating family.

### 8.1. Polyhedral structure of the fixed point sets of finitely generated Shapley operators.

Recall that a *polyhedral complex* $\mathcal{K}$ is a set of polyhedra, called *cells*, that satisfies the following conditions: every face of a polyhedron from $\mathcal{K}$ is also in $\mathcal{K}$; the intersection of any two polyhedra $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both $\sigma_1$ and $\sigma_2$. A polyhedral complex is a *fan* if every cell is a cone. The *support* of a polyhedral complex is the union of its cells. (Here, cone is understood in the sense of convex analysis, not in the sense of ordered additive cones.)

Suppose $E$ is the fixed point set of a finitely generated Shapley operator. So $E = \{x \mid x = A^\sharp \circ Bx\}$, for some proper pair of matrices $A, B \in (\mathbb{R}_{\max})^{m\times n}$. We will show that $E$ is the support of a polyhedral complex, and that the cells of this complex correspond to *calibrated policies* of the two players.

It will be convenient to lift the ambitropical cone, considering

$$F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x = A^\sharp y,\ y = Bx\}$$

so that $E = \mathrm{proj}_x(F)$, where $\mathrm{proj}_x$ is the projection $(x, y) \mapsto x$ from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$.

Given any pair of nondeterministic policies $(\sigma, \pi) \in \mathcal{S} \times \mathcal{P}$ (see §7.2), we denote by $Z_{\sigma,\pi}$ the set of couples $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ verifying the following relations

$$x \leqslant A^\sharp y, \qquad x_i \geqslant -A_{ji} + y_j, \quad \forall i \in [n], \forall j \in \sigma(i) \tag{25a}$$

$$y \geqslant Bx \qquad y_j \leqslant B_{jk} + x_k, \quad \forall j \in [m], \forall k \in \pi(j) \ . \tag{25b}$$

Let us consider $\sigma^{-1}(j) = \{i \in [n] \mid j \in \sigma(i)\}$ and $\pi^{-1}(i) = \{j \in [m] \mid i \in \pi(j)\}$. Observe that $\cup_{j\in[m]}\sigma^{-1}(j) = [n]$ and $\cup_{i\in[n]}\pi^{-1}(i) = [m]$, because $\sigma(i)$ and $\pi(j)$ are non-empty, for all $i \in [n]$ and $j \in [m]$.

**Proposition 8.1.** *We have*

$$F = \bigcup_{(\sigma,\pi)\in\mathcal{S}\times\mathcal{P}} Z_{\sigma,\pi} \ . \tag{26}$$

*Proof.* The relations (25a) entail that $x = A^\sharp y$, and similarly (25b) entail that $y = Bx$. So, for all $(\sigma, \pi) \in \mathcal{S} \times \mathcal{P}$, $Z_{\sigma,\pi} \subset F$. Moreover, for all $(x, y) \in F$, taking $\sigma(i) := \operatorname{argmin}_{j\in[m]}(-A_{ji} + y_j)$, we can check that (25a) is satisfied by $x = A^\sharp y$. Similarly, for all $x$ in $\mathbb{R}^n$, taking $\pi(j) := \operatorname{argmax}_{i\in[n]}(B_{ji} + x_i)$, we can check that (25b) is satisfied by $y = Bx$. So, $(x, y) \in Z_{\sigma,\pi}$. □

**Proposition 8.2.** *The image $X_{\sigma,\pi} := \operatorname{proj}_x(Z_{\sigma,\pi})$ is characterized by the relations*

$$[(A \vee B)x]_j \leqslant B_{jk} + x_k, \qquad \forall j \in [m], \forall k \in \pi(j) \tag{27a}$$

$$[(A \vee B)x]_j \leqslant A_{ji} + x_i, \qquad \forall j \in [m], \forall i \in \sigma^{-1}(j) \ . \tag{27b}$$

*Moreover, both $Z_{\sigma\pi}$ and $X_{\sigma,\pi}$ are alcoved polyhedra.*

*Proof.* Let $(x, y) \in Z_{\sigma,\pi}$. Since $x \leqslant A^\# y$ is equivalent to $Ax \leqslant y$, we deduce that $(A \vee B)x \leqslant y$. Moreover, (25) entail that

$$A_{ji} + x_i \geqslant y_j, \quad \forall j \in [m], \forall i \in \sigma^{-1}(i) \ . \tag{28}$$

It follows that $x$ satisfies (27). Conversely, suppose that (27) holds, and let $y := Bx$. The inequalities in (27a) entail that $[Bx]_j \leqslant [(A \vee B)x]_j \leqslant B_{jk} + x_k \leqslant [Bx]_j$ for all $j \in [m]$ and $k \in \pi(j)$, so that $B_{jk} + x_k = y_j = [(A \vee B)x]_j$. Then, it follows from these inequalities that $Ax \leqslant y$, and so $x \leqslant A^\sharp y$. Then, we deduce from (27b) that $y_j \leqslant A_{ji} + x_i$ for all $j \in [n]$ and $i \in \sigma^{-1}(j)$, which can be rewritten as $-A_{ji} + y_j \leqslant x_i$ for all $i \in [n]$ and $j \in \sigma(i)$, and so $A^\sharp y \leqslant x$, which shows that $(x, y) \in Z_{\sigma,\pi}$.

It is immediate from the form of the constraints in (25) and (27) that $Z_{\sigma,\pi}$ and $X_{\sigma,\pi}$ are alcoved polyhedra. □

We define the *type* of a point $x \in \mathbb{R}^n$ to be the pair of partially defined maps $\tau := (\sigma, \pi)$ where for all $j \in [m]$, $\pi(j)$ denotes the set of $k \in [n]$ such that the relation (27a) holds, and $\sigma^{-1}(j)$ denotes the set of $i \in [n]$ such that (27b) holds. We say that a type is *proper* if both $\sigma$ and $\pi$ are policies (this requires the maps $\sigma$ and $\pi$ to be totally defined). We denote by $\mathcal{T}$ the set of proper types associated to points $x \in \mathbb{R}^n$.

**Theorem 8.3.** *Suppose $E$ is the fixed point set of the finitely generated Shapley operator. Then, the collection of alcoved polyhedra $(X_\tau)_{\tau\in\mathcal{T}}$ constitutes a polyhedral complex whose support is $E$. Moreover, the cell $X_\tau$ consists precisely of those fixed points $u$ such that the pair $\tau$ of nondeterministic policies in the mean payoff game associated to $T$ is $u$-calibrated.*

*Proof.* It follows from (26), $E = \operatorname{proj}_x(F)$ and Proposition 8.2 that $E = \bigcup_\tau X_\tau$. We have to show that the collection of polyhedra $\{X_\tau\}_{\tau\in\mathcal{T}}$ is a polyhedral complex.

Consider $\tau = (\sigma, \pi) \in \mathcal{T}$ and $\tau' = (\sigma', \pi') \in \mathcal{T}$. Let $\pi'' \in \mathcal{P}$ be such that $\pi''(j) = \pi(j) \cup \pi'(j)$ for all $j \in [m]$, and let $\sigma'' \in \mathcal{S}$ be such that $\sigma''(i) = \sigma(i) \cup \sigma'(i)$ for all $i \in [n]$. It is immediate from (27) that $X_\tau \cap X_{\tau'} = X_{\tau''}$. Let now $x$ be a point in the relative interior of $X_{\tau''}$. For $j \in [m]$, let $\pi'''(j)$ be defined as the set of $k$ such that $[(A \vee B)x)]_j \leqslant B_{jk} + x_k$. For $i \in [n]$, let $\sigma'''(i)$ be defined as the set of $j$ such that $[(A \vee B)x]_j \leqslant A_{ji} + x_i$, so that $\tau'''$ is the type of $x$. Observe that $\sigma'''(i) \supset \sigma''(i) \supset \sigma'(i) \neq \emptyset$, which entails that $\sigma'''$ is proper. Similarly, $\pi'''$ is proper. Moreover, the inclusion $X_\tau \cap X_{\tau'} \supset X_{\tau'''}$ is trivial. The reverse inclusion follows from the observation that $\operatorname{relint}(X_\tau \cap X_{\tau'}) \subset X_{\tau'''}$. This follows from the fact that $\tau'''$ is the type of any point $x \in \operatorname{relint}(X_\tau \cap X_{\tau'})$.

Finally, observing that a face $F'$ of $X_\tau$ is obtained by saturating some of the inequalities (27), and taking for $\tau'$ the type of an arbitrary point in the relative interior of this face, it is immediate that $F' = X_{\tau'}$.

Moreover, considering the proof of Proposition 7.4, we see that a pair of nondeterministic policies $\tau = (\sigma, \pi) \in \mathcal{T}$ is $u$-calibrated if and only if, for all deterministic policies $\sigma_{\rm d}$, $\pi_{\rm d}$ induced by $\sigma$ and $\pi$, we have $u = T(u) = T^{\sigma_{\rm d}}(u) = {}^{\pi_{\rm d}}T(u)$, and this means precisely that $u \in X_\tau$. □

*Remark* 8.4. The polyhedral complex of Theorem 8.3 generalizes the complex introduced by Develin and Sturmfels to represent tropical polyhedra [DS04]. The latter complex is recovered by considering the special case in which $A = B$, so that $T = B^\sharp \circ B$. Then, the range of $T$ is precisely the dual tropical cone generated by the opposite of the columns of $B$. By Proposition 8.2, the cell $X^{\sigma,\pi}$ is given by $\{x \mid (Bx)_j \leqslant B_{jk} + x_k,\ k \in \pi(j),\ (Bx)_j \leqslant B_{ji} + x_i,\ i \in \sigma^{-1}(j)\}$, and then, we see that this cell coincides with $X^{\bar\pi^{-1},\bar\pi}$, in which $\bar\pi$ is the nondeterministic policy whose graphs is the union of those of $\pi$ and $\sigma^{-1}$. The cells $X^{\bar\pi^{-1},\bar\pi}$ are precisely the ones that constitute the polyhedral complex of [DS04] and the policy $\bar\pi$ is equivalent to the *combinatorial type* defined there.

8.2. **Polyhedral complexes associated with ambitropical polyhedra.** Whereas ambitropical cones arise as fixed point sets of Shapley operators, we shall see that ambitropical polyhedra arise as fixed point sets of finitely generated Shapley operators.

In the special case of alcoved polyhedra, the following lemma shows that these Shapley operators are simple, and its proof shows that they are associated with one-player games.

**Lemma 8.5.** *Let $E \subset \mathbb{R}^n$ be an alcoved polyhedron, $Q_E^-(x) = \sup^E\{y \in E; y \leqslant x\}$ and $Q_E^+(x) = \inf^E\{y \in E; y \geqslant x\}$. Then $Q_E^-$ and $Q_E^+$ are finitely generated Shapley operators.*

*Proof.* Observe first that $\sup^E$ coincides with the sup law of $\mathbb{R}^n$ and that similarly $\inf^E$ coincides with the inf law of $\mathbb{R}^n$, because $E$ is an alcoved polyhedron (stable by these sup and inf laws). We can find a matrix $M \in (\mathbb{R}_{\max})^{n\times n}$ such that $E = \mathcal{A}(M)$, i.e., $E = \{x \in \mathbb{R}^n \mid x \geqslant Mx\}$. We claim that $Q_E^+(x) = M^*x$. Indeed, by Lemma 2.3, $M^*x \in \mathcal{A}(M)$. Moreover, since $M^* \geqslant I$, $M^*x \geqslant x$, and so $Q_E^+(x) \leqslant M^*x$. Now, if $z \geqslant x$ for some $z \in \mathcal{A}(M)$, we have $z = M^*z \geqslant M^*x$, showing that $M^*x \leqslant Q_E^+(x)$. The operator $x \mapsto M^*x$ is finitely generated. A dual argument shows that $Q_E^-$ which is also finitely generated. □

We now compute a finitely generated Shapley retraction on an ambitropical polyhedron represented as a union of alcoved polyhedra.

**Lemma 8.6.** *Suppose that $E$ is the union of a finite family of alcoved polyhedra $(E_k)_{k\in K}$. Then,*

$$P_E^{\max} = \sup_{l\in K} Q_{E_l}^-, \qquad P_E^{\min} = \inf_{l\in K} Q_{E_l}^+ \enspace .$$

*Proof.* By definition of $Q_{E_l}^-$, for all $x \in \mathbb{R}^n$, $E_l \ni Q_{E_l}^-(x) \leqslant x$. Since $E_l \subset E^{\max}$ and $E^{\max}$ is stable by supremum, we deduce that $E^{\max} \ni \sup_{l\in K} Q_{E_l}^-(x) \leqslant x$, and so, $P_E^{\max}(x) \geqslant \sup_{l\in K} Q_{E_l}^-(x)$. Moreover, any element $z$ of $E^{\max}$ can be written as $z = \sup_{l\in K} z^l$ for some $z^l \in E_l$. If $z \leqslant x$, it follows that $z^l \leqslant Q_{E_l}^-(x)$, from which we deduce that $z \leqslant \sup_{l\in K} Q_{E_l}^-(x)$. Since this holds for all $E^{\max} \ni z \leqslant x$, it follows that $P^{\max}(x) \leqslant \sup_{l\in K} Q_{E_l}^-(x)$. The proof of the characterization of $P_E^{\min}$ is dual. □

**Corollary 8.7.** *Let $E$ be the union of a finite family of alcoved polyhedra $(E_k)_{k\in K}$. Then*

$$Q_E^-(x) = \inf_{k\in K} Q_{E_k}^+(\sup_{l\in K} Q_{E_l}^-(x)), \qquad Q_E^+(x) = \sup_{k\in K} Q_{E_k}^-(\inf_{l\in K} Q_{E_l}^+(x)) \enspace . \tag{29}$$

*Proof.* This follows from Theorem 4.7 and Lemma 8.6. □

**Theorem 8.8.** *Let $E$ be a subset of $\mathbb{R}^n$, then the following are equivalent:*

*(1) $E$ is an ambitropical polyhedron*
*(2) There exists a finitely generated Shapley operator $P$ such that $P = P^2$ and $E = \{x \in \mathbb{R}^n | x = P(x)\}$.*
*(3) $E$ is the fixed point set of a finitely generated Shapley operator;*
*(4) $E$ is a closed ambitropical cone and $\bar{E}^{\max}$ and $\bar{E}^{\min}$ are finitely generated as modules over $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$, respectively.*

*Proof.* (1)⇒(2). The set $E$ is the range of the idempotent Shapley operator $\bar{Q}_E^-$, and it follows from Corollary 8.7 and Lemma 8.5 that this Shapley operator is finitely generated.

(2) ⇒ (3) is obvious.

(3) ⇒ (4). Since $E$ is the fixed point set of a Shapley operator, $E$ is a closed ambitropical cone. If this Shapley operator is finitely generated, then by Theorem 8.3, $E$ can be written as the union of a finite family of alcoved polyhedra $(X_\tau)_{\tau\in\mathcal{T}}$. Then, $\bar{E}^{\max} = \mathrm{clo}^{\downarrow} E^{\max}$ coincides with the union of the lower closures $\mathrm{clo}^{\downarrow} X_\tau$, for $\tau \in \mathcal{T}$. By Lemma 2.3, every $\mathrm{clo}^{\downarrow} X_\tau$ is a finitely generated $\mathbb{R}_{\max}$-semimodule. It follows that $\bar{E}^{\max}$ is a finitely generated $\mathbb{R}_{\max}$-semimodule.

(4) ⇒ (1). By Corollary 8.7, $E$ is the fixed point set of the operator $\bar{Q}_E^-$ given in (29). By Lemma 8.5, every operator $Q_{E_k}^{\pm}$ is finitely generated. Hence, $\bar{Q}_E^-$ is finitely generated. Then, by Theorem 8.3, $E$ is a finite union of alcoved polyhedra. □

The description of an ambitropical polyhedron as the fixed point set of a Shapley operator is analogous to the "external" description of a polyhedron. We next show that ambitropical polyhedra admits an alternative description, by generators.

**Definition 8.2.** A *description by generators* of a closed ambitropical cone $E$ consists of a pair $(U^{\max}, U^{\min})$ such that $U^{\max}$ is a tropical generating set of $\bar{E}^{\max}$ and $U^{\min}$ is a dual tropical generating set of $\bar{E}^{\min}$. We say that the description is *finite* if the sets $U^{\max}$ and $U^{\min}$ are finite.

The generating sets $U^{\max}$ and $U^{\min}$ uniquely determine $E^{\max}$ and $E^{\min}$, and so they uniquely determine the ambitropical cone $E$, which coincides with $\operatorname{Im} P_E^{\max} \circ P_E^{\min}$.

**Corollary 8.9.** *A closed ambitropical cone is an ambitropical polyhedron if and only if it has a finite description by generators. Moreover, it is an ambitropical polytope if and only if these generators belong to* $\mathbb{R}^n$*.*

*Proof.* If $E$ is an ambitropical polyhedron, it follows from Theorem 8.8, (4) that $E$ admits a finite description by generators.

Conversely, if $E$ is a closed ambitropical cone that admits a finite description by generators, $(U^{\max}, U^{\min})$, by (7) and its dual, $P_E^{\max}$ and $P_E^{\min}$ are finitely generated Shapley operators, and so, $E$ is the fixed point set of the finitely generated Shapley operator $P_E^{\max} \circ P_E^{\min}$. So, $E$ is an ambitropical polyhedron by Theorem 8.8 (2).

Now, if $E$ is bounded in Hilbert's seminorm, we have $\bar{E}^{\max} = E^{\max} \cup \{(-\infty, \dots, -\infty)\}$, and dually, $\bar{E}^{\min} = E^{\min} \cup \{(+\infty, \dots, +\infty)\}$, we get $U^{\max} \subset E^{\max}$ and $E^{\min} \subset U^{\min}$, showing that the generators belong to $\mathbb{R}^n$. Conversely, suppose that the elements of $U^{\max}$ and $U^{\min}$ belong to $\mathbb{R}^n$, and let $R$ denote the maximal Hilbert seminorm of these elements. We observe that balls in Hilbert's seminorm are invariant by the operations of suprema and infima, and so, considering the formula (7), we deduce that $E \subset \operatorname{Im} P_E^{\max}$ is included in the ball of radius $R$ in Hilbert's seminorm. □

When $E$ is an ambitropical polytope, for any description by generators $(U^{\max}, U^{\min})$, $U^{\max}$ and $U^{\min}$ are necessarily subsets of $E$, and so we actually get a proper notion of *internal* representation of $E$ by generators.

In particular, we have the following characterization of ambitropical polytopes.

**Corollary 8.10.** *Every finite subset of* $\mathbb{R}^n$ *admits an ambitropical hull that is an ambitropical polytope, and all the ambitropical polytopes arise in this way.*

*Proof.* By Theorem 4.17, an ambitropical hull of a finite subset $E = \{u^1, \dots, u^k\}$ of $\mathbb{R}^n$ is given by the range of the finitely generator operator $\bar{Q}_E^+ = P_E^{\max} \circ P_E^{\min}$, which, by Corollary 8.9, is bounded in Hilbert's seminorm, and so, it is an ambitropical polytope.

Conversely, suppose that $F$ is an ambitropical polytope. Then, $F$ is a finite union of alcoved polyhedra $F_k$ that are bounded in Hilbert seminorm. Each of these alcoved polyhedra $F_k$ has a finite set of tropical generators. By taking the union of these finite sets we obtain a (possibly redundant) finite set $F^+$ of tropical generators of $E^{\max}$. The dual constructions yields a finite set $F^-$ of dual tropical generators of $E^{\min}$. Observe that the explicit construction of the retraction $\bar{Q}_E^+$ given in Corollary 8.7 involves elementary operators $Q_{E_k}^{\pm}$ which only depend on the primal and dual tropical generators of $E_k$. Hence, by taking for $E$ the union of the two sets $F^{\pm}$, we get that $\bar{Q}_E^+ = \bar{Q}_F^+$, showing that $F$ is an ambitropical hull of the finite set $E$. □

*Example* 8.11. Corollary 8.10 is illustrated in Figure 3, showing an ambitropical hull of the points $a_1, \dots, a_9$ given by the columns of the matrix

$$\begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{array}$$
$$\begin{pmatrix} 4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 4 & 3 & 4 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 & 3 \end{pmatrix}$$

## 9. Homogeneous ambitropical polyhedra

We next study the class of ambitropical polyhedra that are *homogeneous* in the sense of Definition 6.1. We shall see that such polyhedra arise when studying "locally" ambitropical polyhedra, or when considering their behavior at infinity. Moreover, they admit a combinatorial characterization, in terms of posets.

**Definition 9.1.** Let $C$ be an ambitropical polyhedron in $\mathbb{R}^n$ and $u \in C$. The *tangent cone* of $C$ at point $u$, denoted by $\mathcal{T}_u(C)$, is the set of vectors $v$ such that $u + sv \in C$ holds for all $s \geqslant 0$ small enough.

Let us recall the following definition from variational analysis.

**Definition 9.2.** A function $T : \mathbb{R}^n \to \mathbb{R}^p$ is *semidifferentiable* at point $u \in \mathbb{R}^n$ if there exists a continuous map $T'_u$, (positively) homogeneous (so $T'_u(\alpha x) = \alpha T'_u(x)$ holds for all $\alpha > 0$ and $x \in \mathbb{R}^n$) such that

$$T(u + h) = T(u) + T'_u(h) + o(\|h\|) \ . \tag{30}$$

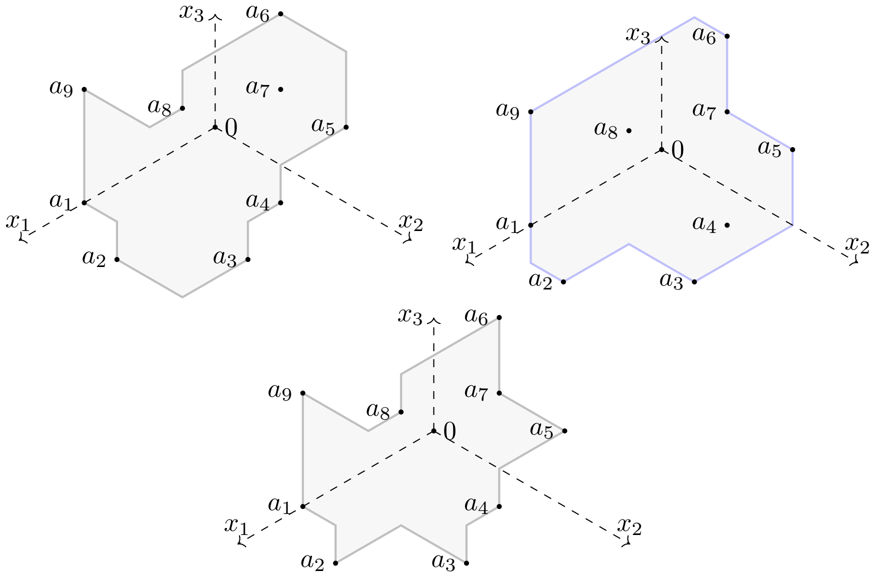


FIGURE 3. A finite collection of points $E = \{a_1, \dots, a_9\}$; the tropical cone $E^{\max}$ that it generates (left); the dual tropical cone $E^{\min}$ (right); and the ambitropical hull $\operatorname{Im} P_E^{\max} \circ P_E^{\min}$ (middle).

Then, $T'_u(h)$ must coincide with the one sided directional derivative:

$$T'_u(h) = \lim_{s \to 0^+} s^{-1}(T(u + sh) - T(u)) \enspace .$$

Conversely, if $T$ is Lipschitz continuous, an application of Ascoli's theorem shows that if this directional derivate exists for all $h \in \mathbb{R}^n$, then, $T$ is semidifferentiable at point $u$, see e.g. [AGN16, Lemma 3.2]. We shall be consider especially the situation in which $T$ is continuous and *piecewise linear*, meaning that $\mathbb{R}^n$ can be covered by finitely many polyhedra on each of which the restriction of $T$ is an affine map. Then, $T$ is automatically Lipschitz continuous, and the directional derivative always exists, showing that $T$ is semidifferentiable. In this case, the semiderivative $h \mapsto T'_u(h)$ is also piecewise linear, and this entails that the local expansion (30) is exact for $h$ small enough:

**Proposition 9.1.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be piecewise linear, and let $u \in \mathbb{R}^n$. Then, there exists a neighborhood $V$ of $0$ such that, for all $h \in V$,*

$$T(u + h) = T(u) + T'_u(h) \enspace . \tag{31}$$

□

The chain rule extends to semidifferentiable maps: if $f, g$ are Lipschitz continuous, if $f$ is semidifferentiable at point $u$, and if $g$ is semidifferentiable at point $f(u)$, then

$$(g \circ f)'_u = g'_{f(u)} \circ f'_u \enspace , \tag{32}$$

see Lemma 3.4 of [AGN16].

Let us also recall the rule of computation of directional derivatives of suprema and infima. If $f$ is a function $\mathbb{R}^n \to \mathbb{R}$ that can be written as a supremum of a finite family of functions $f = \max_{i \in I} f_i$ and if each $f_i$ has one sided directional derivatives at point $u$, then

$$f'_u(h) = \max_{i \in I^*(u)} (f'_i)_u(h), \qquad \text{where } I^*(u) = \{i \in I \mid f(u) = f_i(u)\} \enspace , \tag{33}$$

see [RW98, Exercise 10.27]. A dual rule applies to a function $f = \min_{i \in I} f_i$.

We call *homogeneous ambitropical polyhedron* an ambitropical polyhedron that is a homogeneous ambitropical cone.

**Proposition 9.2.** *Suppose $C$ is an ambitropical polyhedron, and let $u \in C$. Then, the tangent cone $\mathcal{T}_u(C)$ is a homogeneous ambitropical polyhedron.*

*Proof.* It follows from the definition of $\mathcal{T}_u(C)$ that if $v \in \mathcal{T}_u(C)$, then $sv \in \mathcal{T}_u(C)$ for all $s > 0$. Since $C$ is an ambitropical polyhedron, we have $C = \{x \in \mathbb{R}^n \mid T(x) = x\}$ where $T$ is a finitely generated Shapley operator. We claim that

$$\mathcal{T}_u(C) = \{h \in \mathbb{R}^n \mid T'_u(h) = h\} \ .$$

Let $h \in \mathbb{R}^n$ such that $T'_u(h) = h$. For $s > 0$ small enough such that $sh$ belongs to the neighborhood $V$ of Proposition 9.1, we have that $T(u + sh) = T(u) + T'_u(sh) = u + sT'_u(h) = u + sh$. It follows that $h \in \mathcal{T}_u(C)$. Conversely if $h \in \mathcal{T}_u(C)$, $u + sh \in C$ holds for all $s$ small enough, hence, $T(u + sh) = u + sh$ holds for all such $s$, and using (31), we deduce that $T'_u(h) = h$. Therefore, $\mathcal{T}_u(C)$ is the fixed point set of the homogeneous Shapley operator $T'_u$. Using the chain rule (32), the rule of semidifferentiation of suprema (33), and the dual rule of semidifferentiation of infima, we deduce that $T'_u$ is finitely generated. □

**Definition 9.3.** Suppose $C$ is a finite union of (ordinary) polyhedra. Then, the *recession cone* of $C$, $\hat{C}$, is the set of vectors $v$ such that there is a vector $x \in C$ such that $x + sv$ belongs to $C$ for all $s \geqslant 0$.

If $T$ is piecewise linear $\mathbb{R}^n \to \mathbb{R}^p$, an in particular if $T$ is finitely generated, then the *recession function*

$$\hat{T}(x) := \lim_{s\to\infty} s^{-1}T(sx)$$

is well defined. Observe that $\hat{T}$ is finitely generated as soon as $T$ is finitely generated (this follows from the proof of Proposition 7.2).

**Proposition 9.3.** *Suppose $C$ is an ambitropical polyhedron. Then, the recession cone $\hat{C}$ is a homogeneous ambitropical polyhedron.*

*Proof.* If $v \in \hat{C}$, then $\alpha v \in \hat{C}$ holds for all $\alpha > 0$, showing that $C$ is homogeneous. Suppose that $C = \{x \in \mathbb{R}^n \mid T(x) = x\}$, where $T$ is a finitely generated Shapley operator. We claim that $\hat{C} = \{v \in \mathbb{R}^n \mid \hat{T}(v) = v\}$. Let $v \in \hat{C}$. Then, the exists $y \in C$ such that $y + sv \in C$ holds for all $s \geqslant 0$. So $T(y + sv) = y + sv$. Dividing by $s$, using the nonexpansive character of $T$, and letting $s$ tend to infinity, we deduce that $\hat{T}(v) = v$. Conversely, suppose that $\hat{T}(v) = v$. Then, we can find a vector $y$ such that the ray $[0, \infty) \ni s \mapsto y + sv$ is included in a region in which $T$ is affine. Then, $T(y + sv) = w + sCv$ for some matrix $C$ and for some vector $w$. Specializing at $s = 0$, we deduce that $w = T(y)$. We also have $T(y + sv)/s \to Cv$ as $s \to \infty$, and so $\hat{T}(v) = Cv$. It follows that $T(y + sv) = T(y) + s\hat{T}(v)$. Hence, $T(y + sv) = y + sv$, showing that $y + sv \in C$, for all $s \geqslant 0$, and so $v \in \hat{C}$. Since $\hat{T}$ is a finitely generated homogeneous Shapley operator, this entails that $\hat{C}$ is a homogeneous ambitropical polyhedron. □

**Theorem 9.4.** *Let $E$ be a subset of $\mathbb{R}^n$, then the following are equivalent:*

*(1) $E$ is a homogeneous ambitropical polyhedron;*
*(2) There exists a homogeneous finitely generated Shapley operator such that $P = P^2$ and $E = \{x \in \mathbb{R}^n | x = P(x)\}$;*
*(3) There exists a homogeneous finitely generated Shapley operator $T$ such that $E = \{x \in \mathbb{R}^n | x = T(x)\}$.*

*Proof.* (1)⇒(2). If $E$ is a homogeneous ambitropical polyhedron, we can write $E$ as a finite union $\cup_k E_k$ where the $E_k$ are homogeneous alcoved polyhedra. Then, the operators $Q^{\pm}_{E_k}$ are homogeneous and finitely generated. We conclude as in the proof of the implication (1)⇒(2) of Theorem 8.8.

(2)⇒(3): trivial.

(3)⇒(1): by Theorem 8.8, $E$ is an ambitropical polyhedron. Since $E = \{x \in \mathbb{R}^n | x = T(x)\}$, and $T$ is homogeneous, $E$ is homogeneous. □

Given a homogeneous ambitropical polyhedron $C$ of $\mathbb{R}^n$, we define the *skeleton* of $C$, $\operatorname{Sk} C$, to be the intersection of $C$ with $\{0, 1\}^n$. Given an (ordered) partition $\mathcal{I} = (I_1, \dots, I_S)$ of $[n]$, we define the Weyl cell of $C$ to be

$$W^{\mathcal{I}} = \{x \in \mathbb{R}^n \mid (i \in I_r, j \in I_s, r \leqslant s) \implies x_i \leqslant x_j\} \ .$$

E.g., $\{x \in \mathbb{R}^4 \mid x_1 \leqslant x_2 = x_3 \leqslant x_4\}$ is the Weyl cell corresponding to the partition $(\{1\}, \{2, 3\}, \{4\})$ of the set $\{1, 2, 3, 4\}$. When each of the sets $I_1, \dots, I_S$ has exactly one element, $W^{\mathcal{I}}$ is a Weyl chamber of $A_n$ type, i.e., a set of the form $\{x \in \mathbb{R}^n \mid x_{\sigma(1)} \leqslant \dots \leqslant x_{\sigma(n)}\}$ for some permutation $\sigma$.

We shall need the following observation, which is a variation on the construction of the canonical triangulation of order polytopes by Stanley [Sta86, § 5].

**Lemma 9.5.** *Any homogeneous ambitropical polyhedron is a union of Weyl cells.*

*Proof.* Any ambitropical polyhedron is a finite union of alcoved polyhedra, and if this polyhedron is homogeneous, the alcoved polyhedra must be homogeneous. It suffices to show that a homogeneous alcoved polyhedron is a finite union of Weyl cells. A homogeneous alcoved polyhedron is of the form $E = \{x \mid x_i \leqslant x_j, \forall (i,j) \in L\}$ where $L$ is a subset of $[n] \times [n]$. We shall think of $L$ as a relation on the set $[n]$, and, since, $E$ is unchanged if $L$ is replaced by its reflexive and transitive closure, we assume that $L$ is a preorder. Then, we define the equivalence relation $\equiv_L$, on $[n]$, such that for any $i,j \in [n]$, $i \equiv_L j$ if and only if $(i,j) \in L$ and $(j,i) \in L$. The equivalence classes determined by this relation are nonempty subsets of $[n]$, that constitute a partition of $[n]$. The relation $L$ determines a partial order $\leqslant_L$ on the set of these equivalence classes, the order being defined by $I \leqslant_L J$ if $(i,j) \in L$ for all $i \in I$ and $j \in J$, and for all equivalence classes $I, J$. We choose a linear extension of this partial order, allowing us to write the equivalence classes as $I_1, \dots, I_S$, in such a way that $I_k \leqslant_L I_l \implies k \leqslant l$. Such a linear extension determines then a Weyl cell $W^{\mathcal{I}}$ with $\mathcal{I} = (I_1, \dots, I_S)$. By construction, $W^{\mathcal{I}} \subset E$. Moreover, if $x \in E$, then, taking $J_1 := \operatorname{argmin}_{i \in [n]} x_i$, we see that $J_1$ must be a union of equivalence classes $I_{i_1}, \dots, I_{i_{m_1}}$. Similarly, $J_2 = \operatorname{argmin}_{i \in [n] \setminus I_1} x_i$ must be a union of equivalence classes $I_{i_{m_1+1}}, \dots, I_{i_{m_2}}$. Continuing in this way, setting $J_k := \operatorname{argmin}_{i \in [n] \setminus I_{k-1}} x_i = I_{i_{m_{k-1}}} \cup \dots \cup I_{m_k}$until $J_1 \cup \dots \cup J_k = [n]$, we get that $x \in W^{\mathcal{I}}$ with $\mathcal{I} = (I_{i_1}, \dots, I_{i_s})$. This shows that $E$ is the union of the Weyl cells $W^{\mathcal{I}}$ arising from all the linear extensions of the order $\leqslant_L$. Note that different extensions give different Weyl cells. □

The following theorem characterizes the polyhedral complexes associated with homogeneous ambitropical polyhedra, showing that they are in bijection with lattices included in $\{0,1\}^n$. These lattices have been studied by Crapo [Cra82].

**Theorem 9.6** (Homogeneous ambitropical cones are equivalent to lattices in $\{0,1\}^n$)**.** *The map $C \mapsto \operatorname{Sk} C$ establishes a bijective correspondence between homogeneous ambitropical polyhedra of $\mathbb{R}^n$ and subsets of the partially ordered set $(\{0,1\}^n, \leqslant)$ that contain the bottom and the top element, and that are lattices in the induced order. Moreover, there is a one-to-one correspondence between the chains in $\operatorname{Sk} C$ and the Weyl cells included in $C$; the cardinality of each of these chains coincides with the dimension of the corresponding Weyl cell plus one unit; and the collection of these Weyl cells constitutes a polyhedral fan with support $C$.*

*Proof.* If $C$ is a homogeneous ambitropical polyhedron, then, by Theorem 9.4, there is a homogeneous finitely generated Shapley operator $P = P^2 : \mathbb{R}^n \to \mathbb{R}^n$ such that $C = \{x \in \mathbb{R}^n \mid x = P(x)\}$. The coordinates of a homogeneous finitely generated Shapley operator can be written as min-max formula without additive translations (i.e., as a monotone Boolean formula), and so, $P$ admits a restriction $\{0,1\}^n \to \{0,1\}^n$. Since $\operatorname{Sk}(C) = C \cap \{0,1\}^n$, this entails that $\operatorname{Sk}(C) = P(\{0,1\}^n)$ is an order preserving retract of $\{0,1\}^n$. Moreover, since $P$ is positive homogeneous, the identically zero vector is fixed by $P$, and since $P$ commutes with the addition of a constant vector, the unit vector is also fixed by $P$. This implies $P(\{0,1\}^n)$ is a lattice in the induced order of $\{0,1\}^n$, containing the bottom and top elements.

We now claim that for all permutations $\sigma$ of $[n]$, the action of $P$ on the chamber $W^\sigma := \{x \mid x_{\sigma(1)} \leqslant \dots \leqslant x_{\sigma(n)}\}$ is uniquely determined by its action on $\{0,1\}^n$. In fact, $P$ is linear on the chamber $W^\sigma$, and since this chamber is a cone with a generating family consisting of vectors in $\{0,1\}^n$, it follows that $P$ is uniquely determined by its restriction to $\{0,1\}^n$. In particular, $P$ fixes the full chamber $W^\sigma$ if and only if it fixes each generator of each chamber belonging to $\{0,1\}^n$. So, the fixed point set of $P$, which is $C$, is uniquely determined by the fixed point set of $P$ restricted to $\{0,1\}^n$, which is $\operatorname{Sk}(C)$. This shows that the correspondence between a homogeneous ambitropical polyhedron and its skeleton is bijective.

We now show the following claim: every subset $S$ of $\{0,1\}^n$ that is a lattice in the induced order and contains the bottom and top elements of $\{0,1\}^n$, can be realized as a skeleton of a homogeneous ambitropical polyhedron. The map $T(x) := \sup_S \{u \in \{0,1\}^n \mid u \leqslant x\}$ is an order preserving self-map of $\{0,1\}^n$ such that $S = \{x \in \{0,1\}^n \mid x = T(x)\}$.Now, by a standard result, any order preserving map $f$ from $\{0,1\}^n$ to $\{0,1\}$ such that $f(0,\dots,0) = 0$ and $f(1,\dots,1) = 1$ can be represented by a monotone Boolean function. Indeed, let $F := \{y \in \{0,1\}^n \mid f(y) = 1\}$, and consider the monotone Boolean function

$$g(x) := \bigvee_{y \in F} \bigwedge_{i \in [n],\, y_i = 1} x_i \quad .$$

We have that $f(x) = g(x)$ holds for all $x \in \{0,1\}^n$. This establishes the claim.

By Lemma 9.5, $C$ is a finite union of Weyl cells. We observe that the intersection of the Weyl cell $W^{\mathcal{I}}$ with $\{0,1\}^n$ is a chain. Indeed, the bottom element of $W^{\mathcal{I}}$ is the zero vector. The smallest non-zero element of $W^{\mathcal{I}}$ is the vector $x$ gotten by setting $x_i = 0$ for all $i \in \cup_{s<S} I_s$ and $x_i = 1$ for all $i \in I_S$. The smallest element of $W^{\mathcal{I}}$ greater than $x$ is the vector $y$ gotten by setting $y_i = 0$ for all $i \in \cup_{s<S-1} I_s$ and $y_i = 1$ for all $i \in \cup_{s\geqslant S-1} I_s$, etc. This yields a chain of length $S+1$. Moreover, consider the linear space $H^{\mathcal{I}} = \{x \in \mathbb{R}^n \mid x_i = x_j, \forall i,j \in I_s, \forall s \in [S]\}$. Every set $I_s$ yields $|I_s|-1$ independent linear relations, so, $H^{\mathcal{I}}$ is of dimension $n - (\sum_{s=1}^{S} |I_s| - 1) = n + S - \sum_{s=1}^{S} |I_s| = S$. We have $W^{\mathcal{I}} \subset H^{\mathcal{I}}$, and since for all $0 < \alpha_1 < \cdots < \alpha_S$, the vector $x$ such that $x_i = \alpha_s$ for all $i \in I_s$, we deduce that $W^{\mathcal{I}}$ is of dimension at least $S$. It follows that $W^{\mathcal{I}}$ is of dimension equal to $S$.

Finally, a face of the Weyl cell associated with an ordered partition $I_1, \dots, I_S$ is again a Weyl cell, associated with a new partition obtained by merging several consecutive sets of the partition in a single class (this corresponds to the operation of taking a subchain in the skeleton). Moreover, taking the intersection of Weyl cells corresponds to taking the intersection of the associated chains, which entails that the collection of these Weyl cells constitutes a polyhedral fan. □

We deduce that the class of ambitropical sets is not closed under projection:

*Example* 9.7. Consider the subset $L$ of $\{0,1\}^5$ given by bottom, top and the following elements $(0,1,0,0,1)$, $(0,0,1,0,1)$, $(0,1,1,1,0)$, $(1,1,1,0,1)$, with induced order. The set $L$ is a lattice, so we know by the previous result that it corresponds to an homogeneous ambitropical polyhedron $C$ of $\mathbb{R}^5$, of which it is the skeleton $L = C \cap \{0,1\}^5$. Let us consider now the projection $\mathrm{proj}(C)$ on $\mathbb{R}^4$ of $C$ which is obtained by taking the first 4 coordinates, and observe that it is a homogeneous polyedron. Assume that it is an ambitropical cone. Then, by Theorem 9.6 again, its skeleton would be a lattice. But the skeleton of $\mathrm{proj}(C)$ is the projection of the skeleton $L$ of $C$, which is not a lattice since the sup of the two elements $(0,1,0,0)$ and $(0,0,1,0)$ is not well defined, because the two elements $(0,1,1,1)$ and $(1,1,1,0)$ are minimal upper bounds. This shows that $\mathrm{proj}(C)$ is not an ambitropical cone.

*Example* 9.8. Consider the finitely generated Shapley operator $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T(y) = \big((x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3), x_2, x_3\big) \enspace .$$

The fixed point set $E$ of $T$ is the butterfly shaped polyhedral complex with two full dimensional cells $E_1$ and $E_2$, shown in Figure 4. Explicitly, $E_1 = \{x \in \mathbb{R}^3 \mid x_2 \geqslant x_1 \geqslant x_3\}$, $E_2 = \{x \in \mathbb{R}^3 \mid x_3 \geqslant x_1 \geqslant x_2\}$. Using Lemma 8.6, we get that the tropical projections are given by:

$$P_E^{\max}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (x_1 \wedge (x_2 \vee x_3)) \\ x_2 \\ x_3 \end{pmatrix} \text{ and } P_E^{\min}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (x_1 \vee (x_2 \wedge x_3)) \\ x_2 \\ x_3 \end{pmatrix}$$

*Example* 9.9. A union of Weyl cells that is not ambitropical is shown at the left of Figure 7. This union is not ambitropical because it does not contain the unique geodesic between two specific points of this union, contradicting the conclusion of Proposition 4.18.

*Example* 9.10. Consider the ambitropical polyhedron $E$ in Figure 6. We have that in this case

$$Q_E^{-}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (x_1 \wedge x_2 \wedge (1+x_3)) \vee (x_1 \wedge (1+x_2) \wedge x_3) \vee (x_2 \wedge x_3 \wedge (1+x_1)) \\ x_2 \wedge (1+x_1) \wedge (1+x_3) \\ x_3 \wedge (1+x_2) \wedge (1+x_1) \end{pmatrix}$$

The construction of the sets $E^{\max}$ and $E^{\min}$, as well as Theorem 4.12, showing that $P^{\max} \circ P^{\min}$ and $P^{\min} \circ P^{\max}$ are retractions on an ambitropical set $E$, are illustrated in the figure.

*Example* 9.11. The construction of the ambitropical hulls by means of Theorem 4.17 is illustrated in Figure 8. Here, $E = \{a, b\}$ where $a = (1,0,0)$ and $b = (0,1,0)$. In this special case, the sets $E^{\max}$ and $E^{\min}$ are the ranges of $\bar{Q}_E^{+}$ and $\bar{Q}_E^{-}$, respectively, and so, they provides ambitropical hulls of $\{a,b\}$. There is an infinite family of ambitropical hulls interpolating between $E^{\max}$ and $E^{\min}$. One element of this family is shown in black. It constitutes a polyhedral complex whose cells are cones, and whose rays are generated by the vectors $a, b$ and by the integer vectors $(i, 4-j, 0)$ for $i = 1,2,3$ and $(i-1, 4-j, 0)$ for $i = 1, \dots, 4$.

*Example* 9.12. We now give an example in dimension 4. Consider

$$E^I = \{x \in \mathbb{R}^4 \mid x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4\}$$

and for the circular permutation $\gamma$ with cycle $(1,2,3,4)$,

$$E^{\gamma} = \{x \in \mathbb{R}^4 \mid x_4 \leqslant x_1 \leqslant x_2 \leqslant x_3\} \enspace .$$

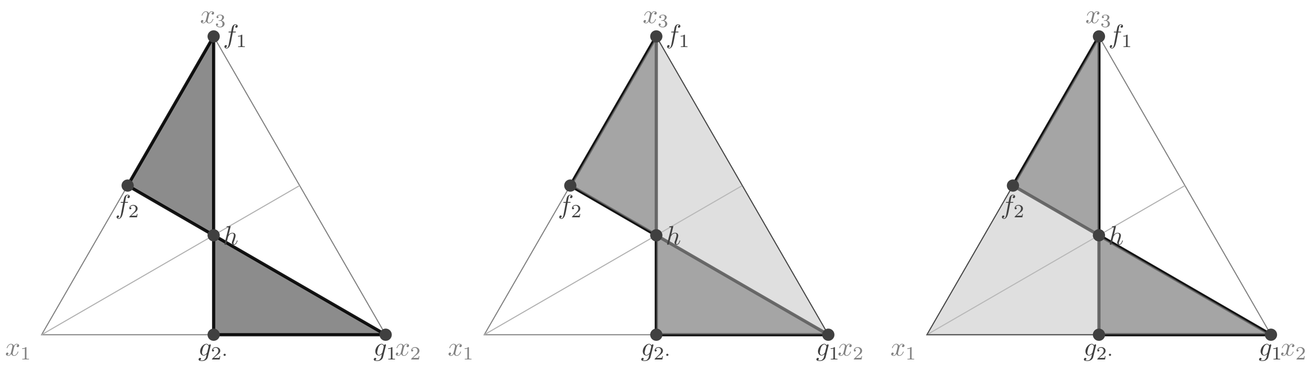


Figure 4. A homogeneous ambitropical polyhedron $E$ consisting of two unbounded alcoved polyhedra (left). The tropical polyhedral cones $E^{\max}$ (center) and $E^{\min}$ (right). See Example 9.8.

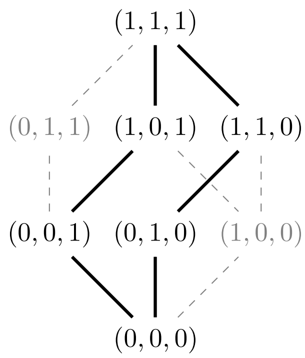


Figure 5. The skeleton (in bold) of the ambitropical polyhedron of Figure 4 (the two nodes $(1,0,0)$ and $(0,1,1)$ in gray do not belong to the skeleton). The two maximal chains (of length 4) yield the representation of $C$ as the union of two Weyl cells (each being of dimension 3).

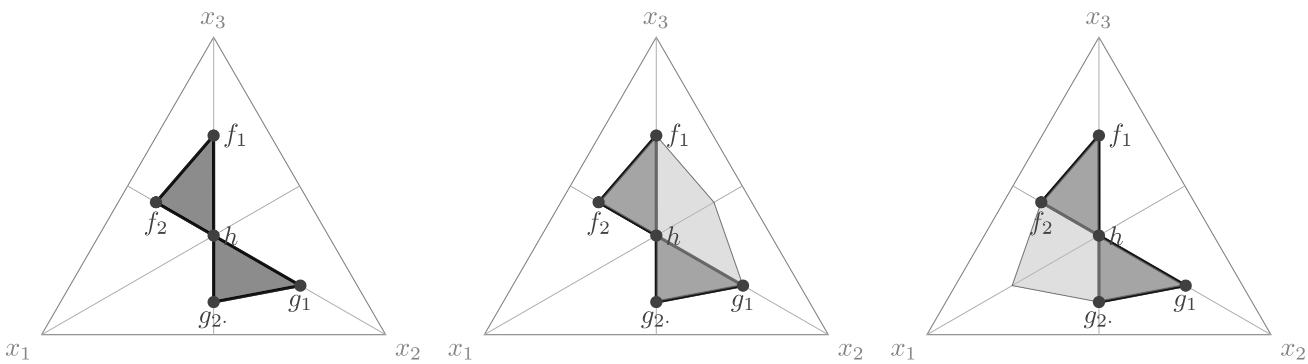


Figure 6. An ambitropical polyhedron $E$ consisting of two alcoved polyhedra (left). The tropical polyhedral cones $E^{\max}$ (center) and $E^{\min}$ (right). The homogeneous ambitropical polyhedron of Figure 4 is precisely the tangent cone $\mathcal{T}_{(0,0,0)}E$.

The union of the chambers $E^I$ and $E^\gamma$ is shown in Figure 9, as well as the range of $Q_E^-$, which is larger than this union, implying that $E$ is not ambitropical. AN example of non-trivial ambitropical cone in $\mathbb{R}^4$ is shown in Figure 10.

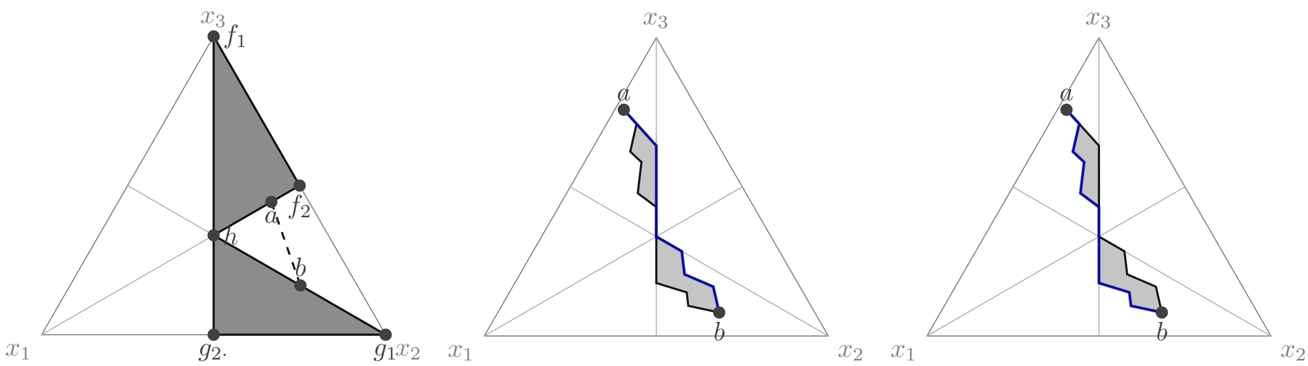


FIGURE 7. Illustrating the metric convexity property of ambitropical cones. A fan which is not ambitropical (left). The geodesic in Hilbert's projective metric connecting the two points $a$ and $b$ (dotted segment) is unique, but it is not included in the fan. An ambitropical fan (middle, and right). An example of geodesic connecting $a$ and $b$, included in the fan is shown (dark blue broken line, middle). Another example of such a geodesic is shown at right.

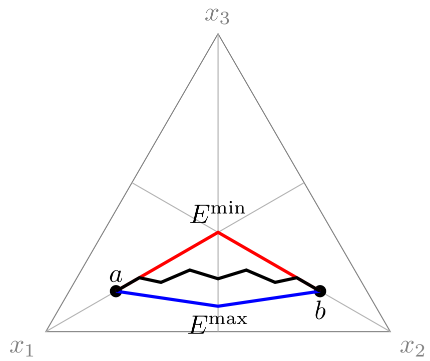


FIGURE 8. Three examples of ambitropical hulls of a set with two elements $E = \{a, b\}$. The range of $\bar{Q}_E^+$ is the tropical cone $E^{\max}$ shown in blue, whereas the range of $\bar{Q}_E^-$ is the dual tropical cone $E^{\min}$ shown in red. Another ambitropical hull is represented by the zigzag line (in black). By Theorem 4.17, all these ambitropical hulls are isomorphic.

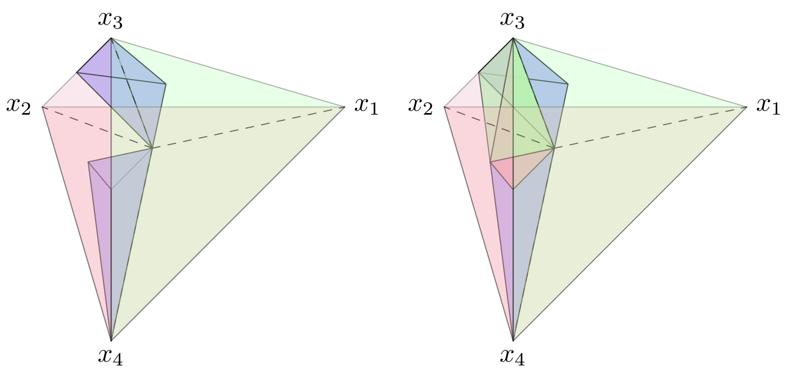


FIGURE 9. The union $E$ of the chambers $x_4 \geqslant x_3 \geqslant x_2 \geqslant x_1$ and $x_3 \geqslant x_2 \geqslant x_1 \geqslant x_4$ (left) is not ambitropical. Indeed, the fixed point set of $Q_E^-$ (middle) is the union of these chambers with the chambers $x_3 \geqslant x_4 \geqslant x_2 \geqslant x_1$ and $x_3 \geqslant x_2 \geqslant x_4 \geqslant x_1$, which coincides with the alcoved polyhedron defined by $x_3 \geqslant x_2 \geqslant x_1$. The latter coincides with the range of $P_E^{\max}$ and with the range of $P_E^{\min}$.

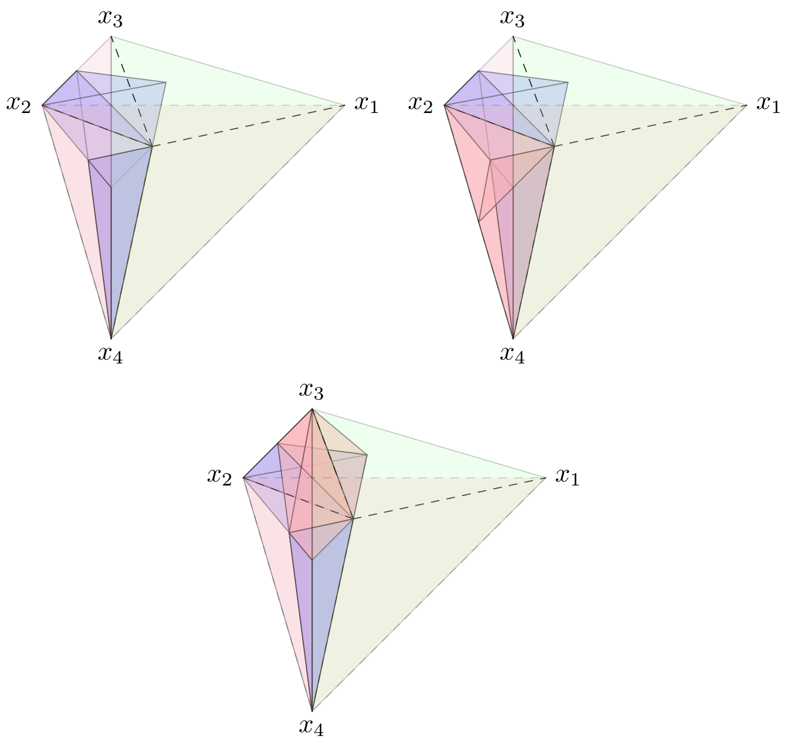


FIGURE 10. The union $E$ of the three chambers $\{x_4 \geqslant x_3 \geqslant x_2 \geqslant x_1\}$, $\{x_2 \geqslant x_3 \geqslant x_4 \geqslant x_1\}$ and $\{x_2 \geqslant x_3 \geqslant x_1 \geqslant x_4\}$ (left) is an ambitropical cone. Tropical cone $E^{\max}$ (middle) and $E^{\min}$ (right).

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