Quantitative Uniform Stability of the Iterative Proportional Fitting Procedure

George Deligiannidis*, Valentin De Bortoli[†], Arnaud Doucet[‡]

Department of Statistics, University of Oxford, UK

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Abstract

We establish that the iterates of the Iterative Proportional Fitting Procedure, also known as Sinkhorn's algorithm and commonly used to solve entropy-regularised Optimal Transport problems, are stable w.r.t. perturbations of the marginals, uniformly in time. Our result is quantitative and stated in terms of the 1-Wasserstein metric. As a corollary we establish a quantitative stability result for Schrödinger bridges.

Keywords— entropy regularized optimal transport; Schrödinger bridge; Iterative Proportional Fitting Procedure; Sinkhorn algorithm; particle filtering.

1 Introduction

The basic problem of Optimal Transport (OT) (see Villani (2009) for a broad overview), in its modern formulation introduced by Kantorovich (1942), is to find a *coupling* of two distributions μ, ν that minimises

$$\inf_{\pi \in \mathbf{T}(\mu,\nu)} \int \|x - y\|^2 d\pi(x,y),$$
 OT (μ,ν)

where $\mathbf{T}(\mu,\nu)$ denotes the collection of probability measures with marginals μ,ν and the Euclidean distance ||x-y|| may be replaced by any other metric or cost function c(x,y). OT provides a theoretical framework for analysis in the space of probability measures and has deep connections with many branches of mathematics including partial differential equations and probability. Beyond its intrinsic interest, OT has recently become an extremely important tool for data science and machine learning, finding numerous applications in fields as diverse as imaging, computer vision or natural language processing (Peyré and Cuturi, 2019).

This ubiquity of OT in modern applications is largely due to the computational tractability of the Entropy-Regularised Optimal Transport problem

$$\inf_{\pi \in \mathbf{T}(\mu,\nu)} \int \|x - y\|^2 d\pi(x,y) + \varepsilon \operatorname{KL}(\pi | \mu \otimes \nu), \qquad \mathsf{OT}_{\varepsilon}(\mu,\nu)$$

which is equivalent to the static Schrödinger bridge, a problem going back to Schrödinger (1931), see Equation (1) in Section 3. Here $KL(\pi|\rho)$ denotes the Kullback–Leibler divergence between the probability measures π and ρ , defined as

$$\mathrm{KL}(\pi|\rho) = \begin{cases} \int \log(\frac{\mathrm{d}\pi}{\mathrm{d}\rho}(x)) \mathrm{d}\pi(x), & \pi \ll \rho, \\ +\infty, & \text{otherwise.} \end{cases}$$

^{*}corresponding author: deligian@stats.ox.ac.uk

[†]valentin.debortoli@gmail.com

[‡]doucet@stats.ox.ac.uk

The great interest in $OT_{\varepsilon}(\mu,\nu)$, as explained in the seminal paper of Cuturi (2013), stems from its amenability to the *Iterative Proportional Fitting Procedure* (IPFP), see Equation (3) below. The theoretical properties of IPFP, also known as the Sinkhorn algorithm, have been investigated in numerous works, and are therefore fairly well understood.

Due to its computational tractability, $\mathsf{OT}_\varepsilon(\mu,\nu)$ has been used in applications as an approximation to $\mathsf{OT}(\mu,\nu)$. Rigorous justification of this approximation has been the subject of intense research recently. Indeed it has been established, see e.g. Cominetti and San Martín (1994); Mikami (2004); Léonard (2012); Carlier et al. (2017), that as the regularisation parameter $\varepsilon \to 0$, the solution of $\mathsf{OT}_\varepsilon(\mu,\nu)$ converges to that of $\mathsf{OT}(\mu,\nu)$.

More recently however, Schrödinger bridges and entropy-regularised OT are being studied for their own sake, finding applications in control, computational statistics and machine learning, see e.g. Bernton et al. (2019); Chen et al. (2021); Corenflos et al. (2021); De Bortoli et al. (2021); Huang et al. (2021); Li et al. (2020); Vargas et al. (2021). In these applications, the entropy regularisation may be a desirable feature rather than an approximation, and the main source of error is the fact that the marginal distributions are typically intractable and often approximated by empirical versions. It is then desirable that as the number of samples increases, this error vanishes. For example, a quantitative version of this statement, can then be used to establish that the differentiable particle filter proposed in Corenflos et al. (2021) converges as the sample size increases, for any $\varepsilon > 0$, thus strengthening the analysis of Corenflos et al. (2021) which requires $\varepsilon_N \to 0$ as $N \to \infty$ to ensure consistency towards the true optimal filter.

This is the question we study in this paper. In particular we establish the stability of the IPFP and of the solution of the corresponding Schrödinger bridge problem w.r.t. perturbations of the marginals.

For standard OT, a classical argument using compactness and cyclical monotonicity guarantees a qualitative version of this result, see e.g.(Villani, 2009, Theorem 5.23, Corollary 5.23). Quantitative versions of this result appeared much more recently, at least in the case of quadratic costs, in Mérigot et al. (2020), Li and Nochetto (2021), Delalande and Merigot (2021). In particular it is established that the optimal transport plans, or maps in the case of absolutely continuous measures, is Hölder continuous in the marginals, with exponent 1/2 w.r.t. the marginals. It is also known that the exponent 1/2 is the best possible, see Gigli (2011).

For entropy-regularised OT and the static Schrödinger bridge problem, the first qualitative result appeared very recently in Ghosal et al. (2021), based on a version of cyclical monotonicity for entropy-regularised OT introduced by Bernton et al. (2021). In the quantitative direction, Luise et al. (2019) establish Lipschitz continuity of the potentials w.r.t. the marginals, measured in the total variation metric, which is too strong to capture the situation where the marginals are being approximated by empirical versions. For smooth cost functions, Luise et al. (2019) also establish that the sample complexity of learning the potentials is of order n^2 , leveraging results from Genevay et al. (2019) on the regularity of potentials and the duality between Maximum Mean Discrepancy type metrics and Sobolev spaces. However, if one is interested in learning the Schrödinger bridge the situation is more complicated; the Wasserstein-1 distance between two couplings is lower bounded by the distance of the marginals and so the results by Fournier and Guillin (2015) imply that the sample complexity of learning the Schrödinger bridge must scale at least as n^d on \mathbb{R}^d .

We present here the first, to the best of our knowledge, quantitative stability result for entropy-regularised OT. In particular, this follows from a stronger result, namely the uniform in time stability of IPFP, that is the Sinkhorn iterates, w.r.t. the marginal distributions. We think this result is of particular importance for practical applications as IFPF is typically used for a finite number of iterations to approximate the Schrödinger bridge. One interesting fact is that in contrast to the standard OT problem, the solution of the entropy-regularised problem is Lipschitz continuous, in the Wasserstein metric, w.r.t. the marginals. However, as the regularisation parameter ε vanishes, the Lipschitz constant blows up as expected by the Hölder continuity of the OT plan.

The recent paper by Eckstein and Nutz (2021), which appeared a couple of months after the first version of the present manuscript, uses very interesting methods, completely different to the ones in the present paper, to establish the quantitative stability of the Schrödinger bridge w.r.t. the marginals measured in the Wasserstein distance. On the one hand, the setting of Eckstein and Nutz (2021) is more general than ours, as it does not require compactness. On the other hand, Eckstein and Nutz (2021) only establish the stability of the Schrödinger bridge instead of the full iterates of IPFP and Hölder continuity w.r.t. the marginals with exponent 1/2.

2 Notation

For a metric space $(\mathsf{Z}, d_\mathsf{Z})$, we write $\mathscr{B}(\mathsf{Z})$ for the Borel σ -algebra on Z and \mathfrak{d}_Z for the diameter of Z , that is $\mathfrak{d}_\mathsf{Z} = \sup\{d_\mathsf{Z}(z,z') : z,z' \in \mathsf{Z}\}$. We also write $\mathscr{P}(\mathsf{Z})$ to denote the subspace of Borel probability measures. For $\pi \in \mathscr{P}(\mathsf{X})$, we define the support of π as

$$supp(\pi) = \{ A \in \mathcal{B}(\mathsf{X}) : A \text{ is closed, } \pi(A^{\mathsf{c}}) = 0 \}.$$

For two metric spaces (X, d_X) , (Y, d_Y) , $\mathscr{P}(X \times Y)$ is always defined w.r.t. the product σ -algebra. For $\mathbb{P} \in \mathscr{P}(X \times Y)$, we will write \mathbb{P}_0 , \mathbb{P}_1 to denote the first and second marginals respectively. For $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$, we let

$$\mathbf{T}(\mu,\nu) = \{ \mathbb{P} \in \mathscr{P}(\mathsf{X} \times \mathsf{Y}) : \mathbb{P}_0 = \mu, \, \mathbb{P}_1 = \nu \}.$$

For a function $f: X \to \mathbb{R}^d$, we write $||f||_{\infty} = \sup_{x \in X} ||f(x)||$, where $||\cdot||$ denotes the usual Euclidean norm. For a function $f: X \to Y$, we define its Lipschitz constant Lip(f) by

$$Lip(f) = \inf\{C \ge 0 : d_{\mathsf{Y}}(f(x_0), f(x_1)) \le Cd_{\mathsf{X}}(x_0, x_1), x_0, x_1 \in \mathsf{X}\}.$$

We also define

$$\operatorname{Lip}(\mathsf{X},\mathsf{Y}) = \{f : \mathsf{X} \to \mathsf{Y} : \operatorname{Lip}(f) < \infty\}, \qquad \operatorname{Lip}_1(\mathsf{X},\mathsf{Y}) = \{f : \mathsf{X} \to \mathsf{Y} : \operatorname{Lip}(f) \leq 1\},$$

and write C(X,Y) for the class of continuous functions from X to Y.

3 Main results

Let (X, d_X) , (Y, d_Y) be two compact metric spaces and write \mathcal{X}, \mathcal{Y} for their respective Borel σ -algebras. We will use d to denote the metric for both X, Y when the context allows. Let $\pi_0 \in \mathcal{P}(X), \pi_1 \in \mathcal{P}(Y)$. We begin by recalling the Iterative Proportional Fitting Procedure (IPFP) solving the following Schrödinger bridge problem

$$\mathbb{P}^{\star} \in \arg\min\{\mathrm{KL}(\mathbb{P}|\mathbb{Q}) : \mathbb{P} \in \mathscr{P}(\mathsf{X} \times \mathsf{Y}), \mathbb{P}_{0} = \pi_{0}, \mathbb{P}_{1} = \pi_{1}\},\tag{1}$$

where $\mathbb{Q} \in \mathscr{P}(\mathsf{X} \times \mathsf{Y})$ is a reference measure admitting a density w.r.t. $\rho_0 \otimes \rho_1$, with $\rho_0 \in \mathscr{P}(\mathsf{X})$ equivalent to π_0 , and $\rho_1 \in \mathscr{P}(\mathsf{Y})$, equivalent to π_1 ; that is for any $(x, y) \in \mathsf{X} \times \mathsf{Y}$

$$d\mathbb{Q}/d(\rho_0 \otimes \rho_1)(x, y) = K(x, y) = \exp[-c(x, y)]. \tag{2}$$

In the case where X = Y, we have that Problem (1) with the choice $c(x, y) = ||x - y||^2/\varepsilon$ is equivalent to $OT_{\varepsilon}(\mu, \nu)$, see e.g. (Peyré and Cuturi, 2019, Remark 4.2). First, we give a sufficient condition to ensure that the solution of (1) exists and is unique. The proof of this proposition is a straightforward consequence of (Csiszár, 1975, Corollary 3.2).

Proposition 1. Assume that $KL(\pi_i|\rho_i) < +\infty$ for $i \in \{0,1\}$ and that $c \in C(X \times Y, \mathbb{R})$. Then there exists a unique solution to (1).

The following proposition, see (Peyré and Cuturi, 2019, Proposition 4.2) for instance, ensures that we can assume without loss of generality that $\rho_0 = \pi_0$ and $\rho_1 = \pi_1$.

Proposition 2. Assume that $\mathrm{KL}(\pi_i|\rho_i) < +\infty$ for $i \in \{0,1\}$ and that $c \in \mathrm{C}(\mathsf{X} \times \mathsf{Y}, \mathbb{R})$. Let \mathbb{P}^* solution of (1) with \mathbb{Q} given by (2) and $\hat{\mathbb{P}}^*$ the solution of (1) with \mathbb{Q} such that for any $(x,y) \in \mathsf{X} \times \mathsf{Y}$

$$d\mathbb{Q}/d(\pi_0 \otimes \pi_1)(x,y) = K(x,y).$$

Then $\mathbb{P}^* = \hat{\mathbb{P}}^*$.

As a consequence, for the rest of this paper, we assume that $\rho_0 = \pi_0$ and $\rho_1 = \pi_1$. In order to solve (1) we consider the IPFP sequence which iteratively solves each half-bridge problem, *i.e.* we define $(\mathbb{P}^n)_{n\in\mathbb{N}}$ such that for any $n\in\mathbb{N}$

$$\mathbb{P}^{2n+1} = \arg\min\{\mathrm{KL}\left(\mathbb{P}|\mathbb{P}^{2n}\right) : \mathbb{P} \in \mathscr{P}(\mathsf{X} \times \mathsf{Y}), \mathbb{P}_0 = \pi_0\},$$

$$\mathbb{P}^{2n+2} = \arg\min\{\mathrm{KL}\left(\mathbb{P}|\mathbb{P}^{2n+1}\right) : \mathbb{P} \in \mathscr{P}(\mathsf{X} \times \mathsf{Y}), \mathbb{P}_1 = \pi_1\},$$
(3)

with $\mathbb{P}^0 = \mathbb{Q}$ and where we recall that $\mathbb{P}_0, \mathbb{P}_1$ denote the marginals of the joint distribution \mathbb{P} . Note that $(\mathbb{P}^n)_{n\in\mathbb{N}}$ is uniquely defined if $c\in C(X\times Y,\mathbb{R})$, see (Csiszár, 1975, Theorem 3.1). For discrete or compact spaces it is known that IPFP converges at an exponential rate on compact or discrete spaces; see e.g. Chen et al. (2016); Altschuler et al. (2017); Franklin and Lorenz (1989). For the non-compact case, convergence, but without any rates, has been established under various regularity conditions in Rüschendorf (1995).

We are now ready to state our main result which is a quantitative uniform stability estimate for the IPFP.

Theorem 3. Assume that $c \in \text{Lip}(X \times Y, \mathbb{R})$. For any $\pi_0, \hat{\pi}_0 \in \mathscr{P}(X)$, $\pi_1, \hat{\pi}_1 \in \mathscr{P}(Y)$ let $(\mathbb{P}^n)_{n \in \mathbb{N}}$ and $(\hat{\mathbb{P}}^n)_{n \in \mathbb{N}}$ the IPFP sequence with marginals (π_0, π_1) respectively $(\hat{\pi}_0, \hat{\pi}_1)$. Then for any $n \in \mathbb{N}$ we have

$$\mathbf{W}_{1}(\mathbb{P}^{n}, \hat{\mathbb{P}}^{n}) \leq C \left\{ \mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}) \right\},$$

with

$$C = e^{17||c||_{\infty}} \{1 + 15\operatorname{Lip}(c)(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}})\}.$$

Corollary 4, establishing the quantitative stability of the Schrödinger bridge, can be obtained by making minor modifications to the proof of Theorem 3.

Corollary 4. Assume that $c \in \text{Lip}(X \times Y, \mathbb{R})$. For any $\pi_0, \hat{\pi}_0 \in \mathscr{P}(X), \pi_1, \hat{\pi}_1 \in \mathscr{P}(Y)$ let \mathbb{P}^* , respectively $\hat{\mathbb{P}}^*$, be the Schrödinger bridge with marginals (π_0, π_1) , respectively $(\hat{\pi}_0, \hat{\pi}_1)$. Then, we have

$$\mathbf{W}_{1}(\mathbb{P}^{\star}, \hat{\mathbb{P}}^{\star}) \leq C \left\{ \mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}) \right\},$$

with C as in Theorem 3.

Remark 5. Although the constants are far from sharp, Lipschitz continuity in the marginals is the best one can hope. Indeed, for any $\mathbb{P} \in \mathbf{T}(\pi_0, \pi_1)$, $\hat{\mathbb{P}} \in \mathbf{T}(\hat{\pi}_0, \hat{\pi}_1)$ we have that

$$\mathbf{W}_{1}(\mathbb{P}, \hat{\mathbb{P}}) = \sup \left\{ \int_{\mathsf{X} \times \mathsf{Y}} f(x, y) d\mathbb{P}(x, y) - \int_{\mathsf{X} \times \mathsf{Y}} f(x, y) d\hat{\mathbb{P}}(x, y) : f \in \operatorname{Lip}_{1}(\mathsf{X} \times \mathsf{Y}) \right\}$$

$$\geq \sup \left\{ \int_{\mathsf{X} \times \mathsf{Y}} f(x) d\mathbb{P}(x, y) - \int_{\mathsf{X} \times \mathsf{Y}} f(x) d\hat{\mathbb{P}}(x, y) : f \in \operatorname{Lip}_{1}(\mathsf{X}) \right\} \geq \mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}),$$

and a similar calculation shows that $\mathbf{W}_1(\mathbb{P},\hat{\mathbb{P}}) \geq \min\{\mathbf{W}_1(\pi_0,\hat{\pi}_0),\mathbf{W}_1(\pi_1,\hat{\pi}_1)\}$. In the case where $\hat{\pi}_0,\hat{\pi}_1$, are empirical versions of π_0,π_1 respectively with n samples, the Lipschitz continuity in the marginals and the results by Fournier and Guillin (2015) also imply a sample complexity of n^d for learning the Schrödinger bridge when $X = Y = \mathbb{R}^d$.

4 Proof

The proof is divided into four parts. First, we recall that the IPFP sequence is associated with a sequence of potentials. In Section 4.1 we show quantitative regularity and boundedness properties for these potentials. The boundedness is due to a reparameterization by Carlier and Laborde (2020). Then, in Section 4.2 we recall a contraction property and show useful Lipschitz properties of the potentials w.r.t. the Hilbert–Birkhoff metric. We then turn to the proof of the uniform quantitative stability of the potentials w.r.t. this metric in Section 4.3. Finally, in Section 4.4 we show how uniform quantitative bounds on the potentials translate into bounds onto probability measures which concludes the proof.

4.1 Regularity properties of the potentials

In this section, we fix $\pi_0 \in \mathscr{P}(\mathsf{X})$ and $\pi_1 \in \mathscr{P}(\mathsf{Y})$ and let $(\mathbb{P}^n)_{n \in \mathbb{N}}$ the IPFP sequence given by (3). The IPFP sequence can be described by a corresponding sequence of (measurable) potentials $(\tilde{f}_n, \tilde{g}_n)_{n \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$, $\tilde{f}_n : \mathsf{X} \to (0, +\infty)$, $\tilde{g}_n : \mathsf{Y} \to (0, +\infty)$ and $\tilde{f}_0 = \tilde{g}_0 = 1$, see (Csiszár, 1975, Theorem 3.1).

Proposition 6. For any $n \in \mathbb{N}$ and $(x,y) \in X \times Y$ we have

$$(\mathrm{d}\mathbb{P}^{2n}/\mathrm{d}\pi_0\otimes\pi_1)(x,y)=\tilde{f}_n(x)K(x,y)\tilde{g}_n(y),$$

$$(\mathrm{d}\mathbb{P}^{2n+1}/\mathrm{d}\pi_0 \otimes \pi_1)(x,y) = \tilde{f}_{n+1}(x)K(x,y)\tilde{g}_n(y),$$

$$\tilde{f}_{n+1}(x) = \left(\int_{\mathsf{Y}} K(x,y)\tilde{g}_n(y)\mathrm{d}\pi_1(y)\right)^{-1},$$

$$\tilde{g}_{n+1}(y) = \left(\int_{\mathsf{X}} K(x,y)\tilde{f}_{n+1}(x)\mathrm{d}\pi_0(x)\right)^{-1}.$$

For any $n \in \mathbb{N}$, $a_n > 0$ and $(x, y) \in \mathsf{X} \times \mathsf{Y}$ we have also

$$(\mathrm{d}\mathbb{P}^{2n}/\mathrm{d}\pi_0\otimes\pi_1)(x,y)=(a_n\tilde{f}_n(x))K(x,y)(\tilde{g}_n(y)/a_n).$$

In other words, the measure \mathbb{P}^{2n} is invariant w.r.t. rescaling of the potentials \tilde{f}_n and \tilde{g}_n . This observation is at the core of the work of Carlier and Laborde (2020) which proves the geometric convergence of the IPFP w.r.t. the L^p metric for bounded costs. For any $n \in \mathbb{N}$, let $\tilde{\varphi}_n = \log(\tilde{f}_n)$ and $\tilde{\Psi}_n = \log(\tilde{g}_n)$ and let $a_n = \exp[-\int_{\mathbb{X}} \tilde{\varphi}_n(x) d\pi_0(x)]$. Finally, for any $n \in \mathbb{N}$, let $\varphi_n = \tilde{\varphi}_n + \log(a_n)$ and $\Psi_n = \tilde{\Psi}_n - \log(a_n)$. Similarly, for any $n \in \mathbb{N}$ we define

$$f_n = \exp[\varphi_n]$$
, $g_n = \exp[\Psi_n]$.

The log-potentials $(\varphi_n, \Psi_n)_{n \in \mathbb{N}}$ can be computed recursively using the following proposition.

Proposition 7. For any $n \in \mathbb{N}$ and $(x,y) \in X \times Y$ we have

$$\varphi_{n+1}(x) = -\log \left\{ \int_{\mathsf{Y}} K(x,y) \exp[\Psi_n(y)] d\pi_1(y) \right\}$$

$$+ \int_{\mathsf{X}} \log \left\{ \int_{\mathsf{Y}} K(x,y) \exp[\Psi_n(y)] d\pi_1(y) \right\} d\pi_0(x),$$

$$\Psi_{n+1}(y) = -\log \left\{ \int_{\mathsf{X}} K(x,y) \exp[\varphi_{n+1}(y)] d\pi_0(x) \right\},$$

$$(d\mathbb{P}^{2n}/d(\pi_0 \otimes \pi_1))(x,y) = \exp[\varphi_n(x) + \Psi_n(y)] K(x,y).$$

Recall that for any $x, y \in X \times Y$ we have $K(x, y) = \exp[-c(x, y)]$. Using (Carlier and Laborde, 2020, Lemma 3.1) we have the following result.

Proposition 8. For any $n \in \mathbb{N}$ we have $\max(\|\varphi_n\|_{\infty}, \|\Psi_n\|_{\infty}) \leq 3\|c\|_{\infty}$.

We now establish the Lipschitz property of these potentials under the assumption that the cost function c is Lipschitz; this is automatically satisfied in the case where $c(x,y) = \|x-y\|^2/\varepsilon$ and X, Y are compact, or when c is a metric by the triangle inequality.

Proposition 9. Assume that $c \in \text{Lip}(X \times Y, \mathbb{R})$. Then, for any $n \in \mathbb{N}$,

$$\max\{\operatorname{Lip}(\varphi_n),\operatorname{Lip}(\Psi_n)\} < \operatorname{Lip}(c).$$

Proof. Using Proposition 7 and the fact that $c \in \text{Lip}(X \times Y, \mathbb{R})$, we have for any $x, x' \in X$

$$\varphi_{n+1}(x) - \varphi_{n+1}(x')$$

$$= \log \left\{ \int_{\mathbf{Y}} K(x', y) \exp[\Psi_n(y)] d\pi_1(y) / \int_{\mathbf{Y}} K(x, y) \exp[\Psi_n(y)] d\pi_1(y) \right\}$$

$$= \log \left\{ \int_{\mathbf{Y}} \exp[-c(x, y) + c(x, y) - c(x', y) + \Psi_n(y)] d\pi_1(y) \right\}$$

$$- \log \left\{ \int_{\mathbf{Y}} \exp[-c(x, y + \Psi_n(y)] d\pi_1(y) \right\}$$

$$\leq \log \left\{ \int_{\mathbf{Y}} \exp[-c(x, y) + \operatorname{Lip}(c) d_{\mathbf{X}}(x, x') + \Psi_n(y)] d\pi_1(y) \right\}$$

$$- \log \left\{ \int_{\mathbf{Y}} \exp[-c(x, y) + \Psi_n(y)] d\pi_1(y) \right\}$$

$$\leq \operatorname{Lip}(c) d_{\mathbf{X}}(x, x').$$

Similarly we obtain that for any $x, x' \in X$, $\varphi_{n+1}(x') - \varphi_{n+1}(x) \leq \text{Lip}(c)d_X(x, x')$, whence it follows that $\text{Lip}(\varphi_{n+1}) \leq \text{Lip}(c)$. Similarly we have that for any $y, y' \in Y$

$$\Psi_{n+1}(y') - \Psi_{n+1}(y)$$

$$= \log \left\{ \int_{\mathsf{X}} K(x,y) \exp[\varphi_{n+1}(y)] d\pi_{0}(x) / \int_{\mathsf{X}} K(x,y') \exp[\varphi_{n+1}(y)] d\pi_{0}(x) \right\}$$

$$= \log \left\{ \int_{\mathsf{X}} \exp[-c(x,y') + c(x,y') - c(x,y) + \varphi_{n+1}(y)] d\pi_{0}(x) \right\}$$

$$- \log \left\{ \int_{\mathsf{X}} \exp[-c(x,y') + \varphi_{n+1}(y)] d\pi_{0}(x) \right\}$$

$$\leq \log \left\{ \int_{\mathsf{X}} \exp[-c(x,y') + \operatorname{Lip}(c) d_{\mathsf{Y}}(y,y') + \varphi_{n+1}(y)] d\pi_{0}(x) \right\}$$

$$- \log \left\{ \int_{\mathsf{X}} \exp[-c(x,y') + \varphi_{n+1}(y)] d\pi_{0}(x) \right\}$$

$$\leq \operatorname{Lip}(c) d_{\mathsf{Y}}(y,y').$$

Remark 10. Notice that for any $n \in \mathbb{N}$, $\operatorname{Lip}(\varphi_n)$, $\operatorname{Lip}(\Psi_n)$ are independent of the functions φ_n, Ψ_n and only depend on the properties of the kernel $K(\cdot, \cdot)$.

Upon combining the two previous propositions and the fact that for any $s, t \in [0, M]$, $|e^t - e^s| \le e^M |t - s|$, we obtain the following controls on the sequence of rescaled potentials $(f_n, g_n)_{n \in \mathbb{N}}$.

Proposition 11. Assume that $c \in \text{Lip}(X \times Y, \mathbb{R})$. Then, we have that for any $n \in \mathbb{N}$

$$\max(\|f_n\|_{\infty}, \|g_n\|_{\infty}) \le e^{3\|c\|_{\infty}}, \qquad \max(\operatorname{Lip}(f_n), \operatorname{Lip}(g_n)) \le \operatorname{Lip}(c)e^{3\|c\|_{\infty}}.$$

4.2 The Hilbert-Birkhoff metric, contraction and Lipschitz properties

We now recall basic properties of the Hilbert–Birkhoff metric; see Lemmens and Nussbaum (2014); Kohlberg and Pratt (1982); Bushell (1973) for a review. Let E be a real vector space and K a cone in this vector space, *i.e.* K is convex, $\mathsf{K} \cap (-\mathsf{K}) = \{0\}$ and $\lambda \mathsf{K} \subset \mathsf{K}$ for any $\lambda \geq 0$. In what follows, we let C be a part of the cone *i.e.* for any $x,y \in \mathsf{C}$ there exist $\alpha,\beta \geq 0$ such that $\alpha x - y \in \mathsf{K}$ and $\beta y - x \in \mathsf{K}$. In addition, assume that C is convex and that for any $\lambda > 0$ $\lambda \mathsf{K} \subset \mathsf{K}$. In this case we have for any $x,y \in \mathsf{C}$ that

$$M(x,y) = \inf\{\beta \ge 0 : \beta y - x \in K\} > 0.$$

Similarly we define for any $x, y \in \mathsf{C}$

$$m(x, y) = \sup \{ \alpha \ge 0 : x - \alpha y \in \mathsf{K} \}.$$

Note that $m(x,y) = M(y,x)^{-1} > 0$. Finally, the Hilbert-Birkhoff metric is defined for any $x,y \in C$ by

$$d_H(x, y) = \log(M(x, y)/m(x, y)).$$

By (Lemmens and Nussbaum, 2014, Lemma 2.1), d_H is a metric on C/\sim the space C quotiented by the equivalence relation: $x \sim y$ if there exists $\lambda > 0$ such that $y = \lambda x$. In particular, if $\|\cdot\|$ is a norm on V then letting $\tilde{\mathsf{C}} = \{x \in \mathsf{C} : \|x\| = 1\}$, we have that $(\tilde{\mathsf{C}}, d_H)$ is a metric space.

Let $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ be two normed real vector space and $K \subset V$, $K' \subset V'$ two cones. In addition, let C and C' be convex parts of K and K' respectively, such that for any $\lambda > 0$, $\lambda C \subset C$ and $\lambda C' \subset C'$. Let $u : V \to V'$ be a linear mapping such that $u(C) \subset C'$. The projective diameter of u is given by

$$\Delta(u) = \sup\{d_H(u(x), u(y)) : x, y \in \tilde{\mathsf{C}}\}.$$

Similarly, we also define the Birkhoff contraction ratio of u

$$\kappa(u) = \sup\{\kappa : d_H(u(x), u(y)) \le \kappa d_H(x, y), x, y \in \tilde{\mathsf{C}}\}.$$

Using the Birkhoff contraction theorem (Birkhoff, 1957; Bauer, 1965; Hopf, 1963) we have that

$$\kappa(u) \le \tanh(\Delta(u)/4).$$
(4)

In order to use the Birkhoff contraction theorem, we collect the following basic facts on cones in function spaces.

Proposition 12. Let Z be a compact space. $\mathsf{F} = [0, +\infty)^\mathsf{Z}$ is a cone and $\tilde{\mathsf{F}} = \mathsf{C}(\mathsf{Z}, (0, +\infty))$ is a convex part of F such that for any $\lambda > 0$, $\lambda \tilde{\mathsf{F}} \subset \tilde{\mathsf{F}}$. In addition, we have that for any $f, g \in \tilde{\mathsf{F}}$

$$d_H(f,g) = \log(\|f/g\|_{\infty}) + \log(\|g/f\|_{\infty}).$$

In this case, we have that for any $f, g \in \tilde{\mathsf{F}}$, $d_H(f,g)$ is the oscillation of $\log(f/g)$. In what follows, we introduce key mappings which allow us to compute the IPFP potential $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$. Recall that for any $n \in \mathbb{N}$ we have

$$f_{n+1}(x) = a_n \left(\int_{\mathsf{Y}} K(x, y) g_n(y) d\pi_1(y) \right)^{-1},$$

$$a_n = \exp[\int_{\mathsf{X}} \log \left(\int_{\mathsf{Y}} K(x, y) g_n(y) d\pi_1(y) \right) d\pi_0(x)],$$

$$g_{n+1}(y) = \left(\int_{\mathsf{Y}} K(x, y) f_{n+1}(x) d\pi_0(x) \right)^{-1}.$$
(5)

Let $\pi_0 \in \mathscr{P}(\mathsf{X})$ and $\pi_1 \in \mathscr{P}(\mathsf{Y})$. We define $\mathcal{E}^x_{\pi_0}$ and $\mathcal{E}^y_{\pi_1}$ such that for any $f: \mathsf{X} \to [0, +\infty)$ and $g: \mathsf{Y} \to [0, +\infty)$ we have

$$\mathcal{E}_{\pi_0}^x(f)(y) = \int_{\mathbf{Y}} K(x,y) f(x) d\pi_0(x), \quad \mathcal{E}_{\pi_1}^y(g)(x) = \int_{\mathbf{Y}} K(x,y) g(y) d\pi_1(y).$$

The following proposition is a consequence of the Birkhoff contraction theorem, see also Chen et al. (2016).

Proposition 13. For any $\pi_0 \in \mathscr{P}(\mathsf{X})$ and $\pi_1 \in \mathscr{P}(\mathsf{Y})$, $\mathcal{E}^x_{\pi_0}(\mathrm{C}(\mathsf{X},(0,+\infty))) \subset \mathrm{Lip}(\mathsf{Y},(0,+\infty))$ and $\mathcal{E}^y_{\pi_1}(\mathrm{C}(\mathsf{Y},(0,+\infty))) \subset \mathrm{Lip}(\mathsf{X},(0,+\infty))$. In addition, we have

$$\max(\kappa(\mathcal{E}_{\pi_0}^x), \kappa(\mathcal{E}_{\pi_1}^y)) \le \tanh(\|c\|_{\infty}).$$

Proof. Let $\pi_0 \in \mathscr{P}(\mathsf{X})$. Since $K : \mathsf{X} \times \mathsf{Y} \to (0, +\infty)$ is continuous and $\mathsf{X} \times \mathsf{Y}$ is compact we get that for any $f \in \mathrm{C}(\mathsf{X}, (0, +\infty))$, $\mathcal{E}^x_{\pi_0}(f) \in \mathrm{C}(\mathsf{Y}, (0, +\infty))$. In addition, let $u \in \mathrm{C}(\mathsf{Y}, (0, +\infty))$ such that for any $y \in \mathsf{Y}$, u(y) = 1. Then, we have that

$$\Delta(\mathcal{E}_{\pi_0}^x) \le 2 \sup\{ d_H(\mathcal{E}_{\pi_0}^x(f), u) : f \in C(X, (0, +\infty)) \},
\Delta(\mathcal{E}_{\pi_1}^y) \le 2 \sup\{ d_H(\mathcal{E}_{\pi_1}^x(g), u) : g \in C(Y, (0, +\infty)) \}.$$
(6)

In addition, using Proposition 12, we have for any $f \in C(X, (0, +\infty))$

$$d_H(\mathcal{E}_{\pi_0}^x(f), u) = \log(\sup\{\mathcal{E}_{\pi_0}^x(f)(y) : y \in Y\}) - \log(\inf\{\mathcal{E}_{\pi_0}^x(f)(y) : y \in Y\}). \tag{7}$$

For any $f \in \mathcal{C}(\mathsf{X},(0,+\infty))$ and $y \in \mathsf{Y}$ we have

$$\mathcal{E}_{\pi_0}^x(f)(y) \ge \exp[-\|c\|_{\infty}] \int_{\mathsf{X}} f(x) d\pi_0(x), \quad \mathcal{E}_{\pi_0}^x(f)(y) \le \exp[\|c\|_{\infty}] \int_{\mathsf{X}} f(x) d\pi_0(x).$$

Combining this result, (4), (6) and (7) we get that $\Delta(\mathcal{E}_{\pi_0}^x) \leq \tanh(\|c\|_{\infty})$. The proof that $\Delta(\mathcal{E}_{\pi_1}^y) \leq \tanh(\|c\|_{\infty})$ is similar. Lipschitz continuity follows from the definitions of $\mathcal{E}_{\pi_0}^x, \mathcal{E}_{\pi_1}^y$ and the Lipschitz continuity of K. In fact, for any function $f: X \to \mathbb{R}$, resp. $g: Y \to \mathbb{R}$, that does not vanish π_0 a.e., resp. π_1 -a.e., $y \mapsto \mathcal{E}_{\pi_0}^x(f)(y)$, resp. $x \mapsto \mathcal{E}_{\pi_0}^x(g)(x)$, is Lipschitz continuous.

Proposition 14. Let $\pi_0, \hat{\pi}_0 \in \mathscr{P}(X)$ and $\pi_1, \hat{\pi}_1 \in \mathscr{P}(Y)$. Then for any $f \in \operatorname{Lip}(X, (0, +\infty))$ and $g \in \operatorname{Lip}(Y, (0, +\infty))$ we have

$$d_H(\mathcal{E}_{\pi_0}^x(f), \mathcal{E}_{\hat{\pi}_0}^x(f)) \le 2 \|1/f\|_{\infty} \left[\text{Lip}(f) + \text{Lip}(c) \|f\|_{\infty} \right] \exp[2 \|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0),$$

$$d_H(\mathcal{E}_{\pi_1}^y(g), \mathcal{E}_{\hat{\pi}_1}^y(g)) \le 2 \|1/g\|_{\infty} \left[\text{Lip}(g) + \text{Lip}(c) \|g\|_{\infty} \right] \exp[2 \|c\|_{\infty} \mathbf{W}_1(\pi_1, \hat{\pi}_1).$$

Proof. Let $f \in \text{Lip}(X, (0, +\infty))$. We have

$$\mathcal{E}_{\pi_0}^x(f)(y)/\mathcal{E}_{\hat{\pi}_0}^x(f)(y) = 1 + \int_{\mathsf{X}} K(x,y)f(x)\mathrm{d}(\pi_0 - \hat{\pi}_0)(x)/\int_{\mathsf{X}} K(x,y)f(x)\mathrm{d}\hat{\pi}_0(x). \tag{8}$$

In addition, we have for any $x, x' \in X$ and $y \in Y$

$$|K(x,y)f(x) - K(x',y)f(x')|$$

$$\leq |K(x,y)f(x) - K(x',y)f(x)| + |K(x',y)f(x) - K(x',y)f(x')|$$

$$\leq ||f||_{\infty} \operatorname{Lip}(K(\cdot,y))d_{X}(x,x') + ||K(\cdot,\cdot)||_{\infty} \operatorname{Lip}(f)d_{X}(x,x').$$

Since $K(x,y) = \exp[-c(x,y)]$, using the fact that for |s|, |t| < M we have $|\exp[s] - \exp[t]| \le \exp[M]|t-s|$, we have that for any $x, x' \in \mathsf{X}$ and $y \in \mathsf{Y}$

$$|K(x',y) - K(x,y)| \le \exp[\|c\|_{\infty}]|c(x',y) - c(x,y)| \le \exp[\|c\|_{\infty}] \operatorname{Lip}(c) d_{\mathsf{X}}(x,x').$$

Therefore we have that for all $y \in \mathcal{Y}$

$$\operatorname{Lip}(K(\cdot,y)f(\cdot)) \le ||f||_{\infty} \exp[||c||_{\infty}] \operatorname{Lip}(c) + \exp[||c||_{\infty}] \operatorname{Lip}(f).$$

Using this result we get that for any $y \in \mathcal{Y}$

$$\left| \int_{\mathbf{X}} K(x, y) f(x) d(\pi_0 - \hat{\pi}_0)(x) \right| \le \left[\text{Lip}(f) + \text{Lip}(c) \|f\|_{\infty} \right] \exp[\|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0). \tag{9}$$

In addition, we have that for any $y \in \mathsf{Y}$

$$\int_{X} K(x, y) f(x) dx \ge \exp[-\|c\|_{\infty}] / \|1/f\|_{\infty}. \tag{10}$$

Combining (8), (9) and (10) we get that for any $y \in Y$

$$\mathcal{E}_{\pi_0}^x(f)(y)/\mathcal{E}_{\hat{\pi}_0}^x(f)(y) \le 1 + \|1/f\|_{\infty} \left[\text{Lip}(f) + \text{Lip}(c) \|f\|_{\infty} \right] \exp[2 \|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0). \tag{11}$$

Similarly, we have that for any $y \in Y$

$$\mathcal{E}_{\hat{\pi_0}}^x(f)(y)/\mathcal{E}_{\pi_0}^x(f)(y) \le 1 + \|1/f\|_{\infty} \left[\text{Lip}(f) + \text{Lip}(c) \|f\|_{\infty} \right] \exp[2\|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0). \tag{12}$$

Combining Proposition 12, (11), (12) and the fact that for any $t \ge 0$, $\log(1+t) \le t$ we get that

$$d_H(\mathcal{E}_{\pi_0}^x(f), \mathcal{E}_{\hat{\pi}_0}^x(f)) \le 2 \|1/f\|_{\infty} [\operatorname{Lip}(f) + \operatorname{Lip}(c) \|f\|_{\infty}] \exp[2 \|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0).$$

The proof for $d_H(\mathcal{E}^y_{\pi_1}(g), \mathcal{E}^y_{\hat{\pi}_1}(g))$ is similar.

4.3 Quantitative uniform bounds on the potentials

In this section, we derive quantitative uniform bounds on the potentials w.r.t. the Hilbert–Birkhoff metric. More precisely, we show the following theorem.

Theorem 15. For any $\pi_0, \hat{\pi}_0 \in \mathscr{P}(\mathsf{X}), \ \pi_1, \hat{\pi}_1 \in \mathscr{P}(\mathsf{Y})$ let $(f_n, g_n)_{n \in \mathbb{N}}$ and $(\hat{f}_n, \hat{g}_n)_{n \in \mathbb{N}}$ the rescaled IPFP potential associated with (π_0, π_1) , respectively $(\hat{\pi}_0, \hat{\pi}_1)$ and given by (5). Then, for any $n \in \mathbb{N}$ we have

$$d_H(f_n q_n, \hat{f}_n \hat{q}_n) \le 8 \operatorname{Lip}(c) e^{10\|c\|_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

Proof. Let $n \in \mathbb{N}$, $D^x : C(X, (0, +\infty)) \to C(X, (0, +\infty))$ and $D^y : C(Y, (0, +\infty)) \to C(Y, (0, +\infty))$ such that for any $f \in C(X, (0, +\infty))$ and $g \in C(Y, (0, +\infty))$ we have $D^x(f) = 1/f$ and $D^y(g) = 1/g$. We define $\mathcal{E}^y_{\pi_1, \pi_0} : C(Y, (0, +\infty)) \to C(X, (0, +\infty))$ such that for any $g \in C(Y, (0, +\infty))$ we have

$$\mathcal{E}_{\pi_1,\pi_0}^y(g) = \mathcal{E}_{\pi_1}^y(g) \exp[\int_{\mathsf{X}} \log(1/\mathcal{E}_{\pi_1}^y(g)(x)) d\pi_0(x)].$$

Using (5), we have for any $n \in \mathbb{N}$

$$f_{n+1} = D^{x} \circ \mathcal{E}_{\pi_{1},\pi_{0}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}),$$

$$\hat{f}_{n+1} = D^{x} \circ \mathcal{E}_{\hat{\pi}_{1},\hat{\pi}_{0}}^{y} \circ D^{y} \circ \mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}),$$

$$g_{n+1} = D^{y} \circ \mathcal{E}_{\pi_{0}}^{x} \circ D^{x} \circ \mathcal{E}_{\pi_{1},\pi_{0}}^{y}(g_{n}),$$

$$\hat{g}_{n+1} = D^{y} \circ \mathcal{E}_{\hat{\pi}_{0}}^{x} \circ D^{x} \circ \mathcal{E}_{\hat{\pi}_{1},\hat{\pi}_{0}}^{y}(\hat{g}_{n}).$$

$$(13)$$

Using the triangle inequality and Proposition 12 we have for any $n \in \mathbb{N}$

$$d_H(f_n g_n, \hat{f}_n \hat{g}_n) \le d_H(f_n g_n, f_n \hat{g}_n) + d_H(f_n \hat{g}_n, \hat{f}_n \hat{g}_n) \le d_H(g_n, \hat{g}_n) + d_H(f_n, \hat{f}_n). \tag{14}$$

Recall that $f_0 = \hat{f}_0 = 1$ and therefore $d_H(f_0, \hat{f}_0) = 0$. Using Proposition 12, Proposition 13, (13) and the fact that D^x, D^y are isometries, we have for any $n \in \mathbb{N}$

$$d_H(f_{n+1}, \hat{f}_{n+1}) = d_H(\mathbf{D}^x \circ \mathcal{E}^y_{\pi_1, \pi_0} \circ \mathbf{D}^y \circ \mathcal{E}^x_{\pi_0}(f_n), \mathbf{D}^x \circ \mathcal{E}^y_{\hat{\pi}_1, \hat{\pi}_0} \circ \mathbf{D}^y \circ \mathcal{E}^x_{\hat{\pi}_0}(\hat{f}_n))$$

$$= d_{H}(\mathcal{E}_{\pi_{1},\pi_{0}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1},\hat{\pi}_{0}}^{y} \circ D^{y} \circ \mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}))$$

$$= d_{H}(\mathcal{E}_{\pi_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}))$$

$$\leq d_{H}(\mathcal{E}_{\pi_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}))$$

$$+ d_{H}(\mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}))$$

$$\leq d_{H}(\mathcal{E}_{\pi_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n})) + \kappa d_{H}(\mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}))$$

$$\leq d_{H}(\mathcal{E}_{\pi_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y} \circ D^{y} \circ \mathcal{E}_{\pi_{0}}^{x}(f_{n}))$$

$$+ \kappa d_{H}(\mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}), \mathcal{E}_{\pi_{0}}^{x}(\hat{f}_{n})) + \kappa^{2} d_{H}(f_{n}, \hat{f}_{n})$$

$$= d_{H}(\mathcal{E}_{\pi_{1}}^{y}(g_{n}), \mathcal{E}_{\hat{\pi}_{1}}^{y}(g_{n})) + \kappa d_{H}(\mathcal{E}_{\hat{\pi}_{0}}^{x}(\hat{f}_{n}), \mathcal{E}_{\pi_{0}}^{x}(\hat{f}_{n})) + \kappa^{2} d_{H}(f_{n}, \hat{f}_{n}),$$

$$(15)$$

with $\kappa = \tanh(\|c\|_{\infty}) \ge \max\{\kappa(\mathcal{E}_{\hat{\pi}_1}^y), \kappa(\mathcal{E}_{\hat{\pi}_0}^y)\}$. Using Proposition 14 we have for any $n \in \mathbb{N}$

$$d_H(\mathcal{E}_{\hat{\pi}_0}^x(\hat{f}_n), \mathcal{E}_{\pi_0}^x(\hat{f}_n)) \leq 2\|1/\hat{f}_n\|_{\infty}(\operatorname{Lip}(\hat{f}_n) + \operatorname{Lip}(c)\|\hat{f}_n\|_{\infty}) \exp[2\|c\|_{\infty}] \mathbf{W}_1(\pi_0, \hat{\pi}_0),$$

$$d_H(\mathcal{E}_{\pi_1}^y(g_n), \mathcal{E}_{\hat{\pi}_1}^y(g_n)) \leq 2\|1/g_n\|_{\infty}(\operatorname{Lip}(g_n) + \operatorname{Lip}(c)\|g_n\|_{\infty}) \exp[2\|c\|_{\infty}] \mathbf{W}_1(\pi_1, \hat{\pi}_1).$$

Combining this result and Proposition 11, we have for any $n \in \mathbb{N}$

$$d_H(\mathcal{E}_{\hat{\pi}_0}^x(\hat{f}_n), \mathcal{E}_{\pi_0}^x(\hat{f}_n)) \le 4 \operatorname{Lip}(c) e^{8||c||_{\infty}} \mathbf{W}_1(\pi_0, \hat{\pi}_0),$$

$$d_H(\mathcal{E}_{\pi_1}^y(g_n), \mathcal{E}_{\hat{\pi}_1}^y(g_n)) \le 4 \operatorname{Lip}(c) e^{8||c||_{\infty}} \mathbf{W}_1(\pi_1, \hat{\pi}_1).$$

Combining this result and (15) we get that for any $n \in \mathbb{N}$

$$d_H(f_{n+1}, \hat{f}_{n+1}) \le \tanh(\|c\|_{\infty}) d_H(f_n, \hat{f}_n) + 4 \operatorname{Lip}(c) e^{8\|c\|_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

Since $d_H(f_0, \hat{f}_0) = 0$ we have that for any $n \in \mathbb{N}$

$$d_{H}(f_{n+1}, \hat{f}_{n+1}) \leq 4 \operatorname{Lip}(c) e^{8\|c\|_{\infty}} \sum_{k=0}^{n} \tanh(\|c\|_{\infty})^{k} (\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}))$$

$$\leq 4 \operatorname{Lip}(c) e^{8\|c\|_{\infty}} (1 - \tanh(\|c\|_{\infty}))^{-1} (\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}))$$

$$\leq 2 \operatorname{Lip}(c) e^{8\|c\|_{\infty}} (1 + e^{2\|c\|_{\infty}}) (\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}))$$

$$\leq 4 \operatorname{Lip}(c) e^{10\|c\|_{\infty}} (\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1})). \tag{16}$$

Similarly, we get that for any $n \in \mathbb{N}$

$$d_H(g_n, \hat{g}_n) \le 4 \operatorname{Lip}(c) e^{10||c||_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)). \tag{17}$$

Combining (14), (16) and (17) concludes the proof.

Unfortunately controlling d_H is not enough to control the distance between \mathbb{P}^n and $\hat{\mathbb{P}}^n$ for any $n \in \mathbb{N}$. Indeed, using the Hilbert–Birkhoff metric we control the oscillations of $f_n g_n/(\hat{f}_n \hat{g}_n)$ for any $n \in \mathbb{N}$ but in order to control probability distances between \mathbb{P}^n and $\hat{\mathbb{P}}^n$ for any $n \in \mathbb{N}$ we need to derive an upper-bound for $||f_n g_n - \hat{f}_n \hat{g}_n||_{\infty}$ for any $n \in \mathbb{N}$. The next lemma is key in order to obtain such bounds.

Lemma 16. For any $\pi_0, \hat{\pi}_0 \in \mathscr{P}(\mathsf{X}), \ \pi_1, \hat{\pi}_1 \in \mathscr{P}(\mathsf{Y})$ let $(f_n, g_n)_{n \in \mathbb{N}}$ and $(\hat{f}_n, \hat{g}_n)_{n \in \mathbb{N}}$ the rescaled IPFP potential associated with (π_0, π_1) , respectively $(\hat{\pi}_0, \hat{\pi}_1)$ and given by (5). Then, for any $n \in \mathbb{N}$ there exist $x_n^{\dagger} \in \mathsf{X}$ and $y_n^{\dagger} \in \mathsf{Y}$ such that

$$|f_n(x_n^{\dagger})g_n(y_n^{\dagger})/(\hat{f}_n(x_n^{\dagger})\hat{g}_n(y_n^{\dagger})) - 1| \le 4\operatorname{Lip}(c)e^{10||c||_{\infty}}(\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

Proof. Let $n \in \mathbb{N}$ and

$$\Delta := 4 \operatorname{Lip}(c) e^{10||c||_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

Using that $\mathbb{P}^{2n}(X \times Y) = \hat{\mathbb{P}}^{2n}(X \times Y) = 1$, we have

$$\int_{\mathsf{X}\times\mathsf{Y}} \{f_n(x)g_n(y)/(\hat{f}_n(x)\hat{g}_n(y)) - 1\} \hat{f}_n(x)\hat{g}_n(y) d\hat{\pi}_0(x) d\hat{\pi}_1(y)
= \int_{\mathsf{X}\times\mathsf{Y}} f_n(x)g_n(y) d\hat{\pi}_0(x) d\hat{\pi}_1(y) - 1
= \int_{\mathsf{X}\times\mathsf{Y}} f_n(x)g_n(y) d\hat{\pi}_0(x) d\hat{\pi}_1(y) - \int_{\mathsf{X}\times\mathsf{Y}} f_n(x)g_n(y) d\pi_0(x) d\pi_1(y).$$
(18)

In addition, using Proposition 11 we have

$$\operatorname{Lip}(f_n g_n) \le \|f_n\|_{\infty} \operatorname{Lip}(g_n) + \|g_n\|_{\infty} \operatorname{Lip}(f_n) \le 2 \operatorname{Lip}(c) e^{6\|c\|_{\infty}}.$$

Combining this result, (18) and the fact that $\mathbf{W}_1(\pi_0 \otimes \pi_1, \hat{\pi}_0 \otimes \hat{\pi}_1) \leq \mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)$, we get that

$$\left| \int_{\mathbf{X} \times \mathbf{Y}} \{ f_n(x) g_n(y) / (\hat{f}_n(x) \hat{g}_n(y)) - 1 \} d\hat{\mathbb{P}}^{2n}(x, y) \right|$$

$$\leq 2 \operatorname{Lip}(c) e^{6||c||_{\infty}} \mathbf{W}_1(\pi_0 \otimes \pi_1, \hat{\pi}_0 \otimes \hat{\pi}_1)$$

$$\leq 2 \operatorname{Lip}(c) e^{6||c||_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)) < \Delta/2.$$

Therefore

$$1 - \Delta/2 \le \int_{\mathbf{Y} \setminus \mathbf{Y}} f_n(x) g_n(y) / (\hat{f}_n(x) \hat{g}_n(y)) d\hat{\mathbb{P}}^{2n}(x, y) \le 1 + \Delta/2.$$
 (19)

Assume that for any $(x, y) \in X \times Y$ we have that

$$|f_n(x)g_n(y)/(\hat{f}_n(x)\hat{g}_n(y)) - 1| > \Delta.$$

Combining this with (19) there exist $(x_n^+, y_n^+) \in X \times Y$ and $(x_n^-, y_n^-) \in X \times Y$ such that

$$f_n(x_n^+)g_n(y_n^+)/(\hat{f}_n(x_n^+)\hat{g}_n(y_n^+)) > 1 + \Delta, \quad f_n(x_n^-)g_n(y_n^-)/(\hat{f}_n(x_n^-)\hat{g}_n(y_n^-)) < 1 - \Delta,$$

whence it follows that $\Delta < 1$. Therefore, by Theorem 15 we get that

$$d_{H}(f_{n}g_{n}, \hat{f}_{n}\hat{g}_{n})$$

$$\geq \log(f_{n}(x_{n}^{+})g_{n}(y_{n}^{+})/(\hat{f}_{n}(x_{n}^{+})\hat{g}_{n}(y_{n}^{+}))) - \log(f_{n}(x_{n}^{-})g_{n}(y_{n}^{-})/(\hat{f}_{n}(x_{n}^{-})\hat{g}_{n}(y_{n}^{-})))$$

$$> \log((1 + \Delta)/(1 - \Delta)) \geq 2\Delta \geq d_{H}(f_{n}g_{n}, \hat{f}_{n}\hat{g}_{n}),$$

which is absurd.

Finally, we conclude this section by deriving bounds on $||f_n g_n - \hat{f}_n \hat{g}_n||_{\infty}$ combining Theorem 15 with Lemma 16.

Theorem 17. For any $\pi_0, \hat{\pi}_0 \in \mathscr{P}(\mathsf{X}), \ \pi_1, \hat{\pi}_1 \in \mathscr{P}(\mathsf{Y}) \ let \ (f_n, g_n)_{n \in \mathbb{N}} \ and \ (\hat{f}_n, \hat{g}_n)_{n \in \mathbb{N}} \ the \ rescaled \ IPFP$ potential associated with (π_0, π_1) , respectively $(\hat{\pi}_0, \hat{\pi}_1)$ and given by (5). Then, for any $n \in \mathbb{N}$ we have

$$||f_n g_n - \hat{f}_n \hat{g}_n||_{\infty} \le 12 \operatorname{Lip}(c) e^{16||c||_{\infty}} (\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

Proof. Let $n \in \mathbb{N}$, $x \in X$ and $y \in Y$. Using Proposition 8 and the fact that for any $s, t \in \bar{\mathrm{B}}(0, M)$ with $M \geq 0$ we have $|e^s - e^t| \leq e^M |s - t|$ we get

$$|f_n(x)g_n(y) - \hat{f}_n(x)\hat{g}_n(y)| \le e^{6\|c\|_{\infty}} |\log(f_n(x)g_n(y)/(\hat{f}_n(x)\hat{g}_n(y)))|$$
(20)

Assume that $f_n(x)g_n(y)/(\hat{f}_n(x)\hat{g}_n(y)) \ge 1$. Then using that for any t > 0, $\log(t) \le t - 1$, Theorem 15 and Lemma 16 we have, with $(x_n^{\dagger}, y_n^{\dagger})$ from Lemma 16,

$$\begin{aligned} |\log(f_{n}(x)g_{n}(y)/(\hat{f}_{n}(x)\hat{g}_{n}(y)))| &= \log(f_{n}(x)g_{n}(y)/(\hat{f}_{n}(x)\hat{g}_{n}(y))) \\ &+ \log(\hat{f}_{n}(x_{n}^{\dagger})\hat{g}_{n}(y_{n}^{\dagger})/(f_{n}(x_{n}^{\dagger})g_{n}(y_{n}^{\dagger}))) + \log(f_{n}(x_{n}^{\dagger})g_{n}(y_{n}^{\dagger})/(\hat{f}_{n}(x_{n}^{\dagger})\hat{g}_{n}(y_{n}^{\dagger}))) \\ &\leq d_{H}(f_{n}g_{n}, \hat{f}_{n}\hat{g}_{n}) + \log(f_{n}(x_{n}^{\dagger})g_{n}(y_{n}^{\dagger})/(\hat{f}_{n}(x_{n}^{\dagger})\hat{g}_{n}(y_{n}^{\dagger}))) \\ &\leq d_{H}(f_{n}g_{n}, \hat{f}_{n}\hat{g}_{n}) + f_{n}(x_{n}^{\dagger})g_{n}(y_{n}^{\dagger})/(\hat{f}_{n}(x_{n}^{\dagger})\hat{g}_{n}(y_{n}^{\dagger})) - 1 \\ &\leq 12 \operatorname{Lip}(c) e^{10||c||_{\infty}}(\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1})). \end{aligned}$$

Combining this result and (20) we get that

$$|f_n(x)g_n(y) - \hat{f}_n(x)\hat{g}_n(y)| \le 12 \operatorname{Lip}(c)e^{16\|c\|_{\infty}}(\mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)).$$

The proof in the case where $f_n(x)g_n(y)/(\hat{f}_n(x)\hat{g}_n(y)) \leq 1$ is similar.

4.4 From potentials to probability metrics

Using Theorem 17 we are now ready to prove Theorem 3.

Proof of Theorem 3. Let $n \in \mathbb{N}$ and $F \in \text{Lip}(X \times Y, \mathbb{R})$, that is $F : X \times Y \to \mathbb{R}$ such that for any $x_0, x_1 \in X$ and $y_0, y_1 \in Y$ we have

$$|F(x_0, y_0) - F(x_1, y_1)| \le d_{\mathsf{X}}(x_0, x_1) + d_{\mathsf{Y}}(y_0, y_1).$$

We will be considering quantities of the form $\int_{\mathsf{X}\times\mathsf{Y}} F(x,y)[\mathrm{d}\mu - \mathrm{d}\mu'](x,y)$ where $\mu,\mu' \in \mathscr{P}(\mathsf{X}\times\mathsf{Y})$; therefore possibly replacing F with F-a for some constant a, we may assume that there exist $\bar{x}\in\mathsf{X}$ and $\bar{y}\in\mathsf{Y}$ such that $F(\bar{x},\bar{y})=0$. Therefore, we have that

$$||F||_{\infty} = \sup\{|F(x,y) - F(\bar{x},\bar{y})| : x \in X, y \in Y\} < \mathfrak{d}_X + \mathfrak{d}_Y.$$

We write $\operatorname{Lip}^* = \{ F \in \operatorname{Lip}_1 : F(\bar{x}, \bar{y}) = 0 \}$. Using this result and Proposition 11, we get that

$$\operatorname{Lip}(FKf_{n}g_{n}) \leq \operatorname{Lip}(F) \|K\|_{\infty} \|f_{n}\|_{\infty} \|g_{n}\|_{\infty} + \operatorname{Lip}(K) \|F\|_{\infty} \|f_{n}\|_{\infty} \|g_{n}\|_{\infty} + \operatorname{Lip}(f_{n}) \|F\|_{\infty} \|K\|_{\infty} \|g_{n}\|_{\infty} + \operatorname{Lip}(g_{n}) \|F\|_{\infty} \|K\|_{\infty} \|f_{n}\|_{\infty} \leq e^{7\|c\|_{\infty}} + 3 \operatorname{Lip}(c) e^{7\|c\|_{\infty}} (\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}}).$$

Combining this result with Theorem 17 and the fact that $\mathbf{W}_1(\pi_0 \otimes \pi_1, \hat{\pi}_0 \otimes \hat{\pi}_1) \leq \mathbf{W}_1(\pi_0, \hat{\pi}_0) + \mathbf{W}_1(\pi_1, \hat{\pi}_1)$, we get that

$$\int_{\mathsf{X}\times\mathsf{Y}} F(x,y)K(x,y)f_{n}(x)g_{n}(y)\mathrm{d}\pi_{0}(x)\mathrm{d}\pi_{1}(y) - \int_{\mathsf{X}\times\mathsf{Y}} F(x,y)K(x,y)\hat{f}_{n}(x)\hat{g}_{n}(y)\mathrm{d}\hat{\pi}_{0}(x)\mathrm{d}\hat{\pi}_{1}(y) \\
\leq \int_{\mathsf{X}\times\mathsf{Y}} F(x,y)K(x,y)f_{n}(x)g_{n}(y)\mathrm{d}\pi_{0}(x)\mathrm{d}\pi_{1}(y) - \int_{\mathsf{X}\times\mathsf{Y}} F(x,y)K(x,y)f_{n}(x)g_{n}(y)\mathrm{d}\hat{\pi}_{0}(x)\mathrm{d}\hat{\pi}_{1}(y) \\
+ \int_{\mathsf{X}\times\mathsf{Y}} F(x,y)K(x,y)\sup\|f_{n}g_{n} - \hat{f}_{n}\hat{g}_{n}\|_{\infty}\mathrm{d}\hat{\pi}_{0}(x)\mathrm{d}\hat{\pi}_{1}(y) \\
\leq \left[\operatorname{Lip}(FKf_{n}g_{n}) + 12(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}})\operatorname{Lip}(c)\mathrm{e}^{17\|c\|_{\infty}}\right]\left[\mathbf{W}_{1}(\pi_{0},\hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1},\hat{\pi}_{1})\right] \\
\leq \left(\mathrm{e}^{7\|c\|_{\infty}} + 3\operatorname{Lip}(c)(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}})\mathrm{e}^{7\|c\|_{\infty}} + 12(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}})\operatorname{Lip}(c)\mathrm{e}^{17\|c\|_{\infty}}\right)\left[\mathbf{W}_{1}(\pi_{0},\hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1},\hat{\pi}_{1})\right] \\
\leq \mathrm{e}^{17\|c\|_{\infty}} \{1 + 15\operatorname{Lip}(c)(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}})\}\left(\mathbf{W}_{1}(\pi_{0},\hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1},\hat{\pi}_{1})\right).$$

Therefore, we have that

$$\begin{aligned} \mathbf{W}_{1}(\mathbb{P}^{2n}, \hat{\mathbb{P}}^{2n}) &= \sup\{ \int_{\mathsf{X} \times \mathsf{Y}} F(x, y) d\mathbb{P}^{2n}(x, y) - \int_{\mathsf{X} \times \mathsf{Y}} F(x, y) d\hat{\mathbb{P}}^{2n}(x, y) : F \in \operatorname{Lip} \} \\ &= \sup\{ \int_{\mathsf{X} \times \mathsf{Y}} F(x, y) d\mathbb{P}^{2n}(x, y) - \int_{\mathsf{X} \times \mathsf{Y}} F(x, y) d\hat{\mathbb{P}}^{2n}(x, y) : F \in \operatorname{Lip}^{\star} \} \\ &\leq e^{17\|c\|_{\infty}} \{ 1 + 15 \operatorname{Lip}(c) (\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}}) \} (\mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1})). \end{aligned}$$

Proof of Corollary 4. Let $n \in \mathbb{N}$. Using (Chen et al., 2016, Lemma 4) we have that $d_H(f_{n+1}, f_n) \le \kappa^n d_H(f_1, f_0)$ and $d_H(g_{n+1}, g_n) \le \kappa^n d_H(g_1, g_0)$. Thus

$$d_H(f_{n+1}g_{n+1}, f_ng_n) = d_H(f_{n+1}, f_n) + d_H(g_{n+1}, g_n) \le \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right].$$

As explained earlier this is not enough on its own to control $||f_{n+1}g_{n+1} - f_ng_n||_{\infty}$. However, we can use a similar technique as Lemma 16. We have that

$$\int_{\mathbf{X} \times \mathbf{Y}} [f_{n+1}(x)g_{n+1}(y)/(f_n(x)g_n(y))]K(x,y)f_n(x)g_n(y)d\pi_0(x)d\pi_1(y) = 1.$$
(21)

In what follows, we assume that for all $(x,y) \in \mathsf{X} \times \mathsf{Y}$, $|f_{n+1}(x)g_{n+1}(y)/(f_n(x)g_n(y)) - 1| \ge \Delta$ with $\Delta = \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right]/2$. Combining this with (21), we get that there exist $(x_n^+, y_n^+), (x_n^-, y_n^-) \in \mathsf{X} \times \mathsf{Y}$ such that

$$f_{n+1}(x_n^+)g_{n+1}(y_n^+)/(f_n(x_n^+)g_n(y_n^+)) \ge 1 + \Delta, \quad f_{n+1}(x_n^-)g_{n+1}(y_n^-)/(f_n(x_n^-)g_n(y_n^-)) \le 1 - \Delta.$$

This implies that $\Delta < 1$. Hence, we have that

$$d_H(f_{n+1}g_{n+1}, f_ng_n) \ge \log(f_{n+1}(x_n^+)g_{n+1}(y_n^+)/(f_n(x_n^+)g_n(y_n^+))) - \log(f_{n+1}(x_n^-)g_{n+1}(y_n^-)/(f_n(x_n^-)g_n(y_n^-)))$$

$$\geq \log((1+\Delta)/(1-\Delta)) > 2\Delta \geq d_H(f_{n+1}g_{n+1}, f_ng_n).$$

This is absurd, hence there exists $(x^*, y^*) \in X \times Y$ such that

$$|f_{n+1}(x^*)g_{n+1}(y^*)/(f_n(x^*)g_n(y^*)) - 1| \le \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right].$$

Therefore, we have that for any $x \in X$ and $y \in Y$

$$\begin{aligned} |\log (f_{n+1}(x)g_{n+1}(y)/(f_n(x)g_n(x)))| \\ &\leq |\log (f_{n+1}(x^*)g_{n+1}(y^*)/(f_n(x^*)g_n(y^*)))| + d_H(f_{n+1}g_{n+1}, f_ng_n) \\ &\leq 2\kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right]. \end{aligned}$$

Combining this result, Proposition 11 and the fact that for any $s, t \in [0, M], |e^t - e^s| \le e^M |t - s|$ we get that for any $x \in X$ and $y \in Y$

$$|f_{n+1}(x)g_{n+1}(y) - f_n(x)g_n(y)| \le e^{6||c||_{\infty}} |\log (f_{n+1}(x)g_{n+1}(y)/f_n(x)g_n(x))|$$

$$\le 2e^{6||c||_{\infty}} \kappa^n [d_H(f_1, f_0) + d_H(g_1, g_0)].$$

Therefore, we have

$$||f_{n+1}g_{n+1} - f_ng_n||_{\infty} \le 2e^{6||c||_{\infty}} \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0)\right].$$

Let $F \in \text{Lip}_1(\mathsf{X} \times \mathsf{Y}, \mathbb{R})$, and without loss of generality we may assume that $F(\bar{x}, \bar{y}) = 0$ for a fixed pair $(\bar{x}, \bar{y}) \in \mathsf{X} \times \mathsf{Y}$. We then have

$$\int_{\mathsf{X}\times\mathsf{Y}} F(x,y) f_{n+1}(x) g_{n+1}(y) K(x,y) d\pi_0(x) d\pi_1(y)
- \int_{\mathsf{X}\times\mathsf{Y}} F(x,y) f_n(x) g_n(y) K(x,y) d\pi_0(x) d\pi_1(y)
\leq \int_{\mathsf{X}\times\mathsf{Y}} \|F(x,y)\|_{\infty} \|f_{n+1} g_{n+1} - f_n g_n\|_{\infty} \|K\|_{\infty} d\pi_0(x) d\pi_1(y)
\leq 2(\mathfrak{d}_{\mathsf{X}} + \mathfrak{d}_{\mathsf{Y}}) e^{7\|c\|_{\infty}} \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right].$$

Taking the supremum over $\{F \in \text{Lip}_1(X \times Y, \mathbb{R}) : F(\bar{x}, \bar{y}) = 0\}$, we have that

$$\mathbf{W}_1(\mathbb{P}^{n+1}, \mathbb{P}^n) \le 2(\mathfrak{d}_X + \mathfrak{d}_Y)e^{7\|c\|_{\infty}} \kappa^n \left[d_H(f_1, f_0) + d_H(g_1, g_0) \right].$$

By completeness of $(\mathscr{P}(X \times Y), \mathbf{W}_1)$ we have that \mathbb{P}^n converges in $(\mathscr{P}(X \times Y), \mathbf{W}_1)$ to $\mathbb{P}^* \in \mathscr{P}(X \times Y)$. Similarly $\hat{\mathbb{P}}^n \to \hat{\mathbb{P}}^* \in \mathscr{P}_1(X \times Y)$. Combining these results and Theorem 3 we have

$$\mathbf{W}_{1}(\mathbb{P}^{*}, \hat{\mathbb{P}}^{*}) \leq \mathbf{W}_{1}(\mathbb{P}^{*}, \mathbb{P}^{n}) + \mathbf{W}_{1}(\mathbb{P}^{n}, \hat{\mathbb{P}}^{n}) + \mathbf{W}_{1}(\hat{\mathbb{P}}^{n}, \hat{\mathbb{P}}^{*})$$

$$\leq C \left\{ \mathbf{W}_{1}(\pi_{0}, \hat{\pi}_{0}) + \mathbf{W}_{1}(\pi_{1}, \hat{\pi}_{1}) \right\} + \mathbf{W}_{1}(\mathbb{P}^{*}, \mathbb{P}^{n}) + \mathbf{W}_{1}(\hat{\mathbb{P}}^{n}, \hat{\mathbb{P}}^{*}).$$

We conclude upon letting $n \to +\infty$.

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