

Topological G_2 and $Spin(7)$ strings at 1-loop from double complexes

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ABSTRACT: We study the topological G_2 and $Spin(7)$ strings at 1-loop. We define new double complexes for supersymmetric NSNS backgrounds of string theory using generalised geometry. The 1-loop partition function then has a target-space interpretation as a particular alternating product of determinants of Laplacians, which we have dubbed the analytic torsion. In the case without flux where these backgrounds have special holonomy, we reproduce the worldsheet calculation of the G_2 string and give a new prediction for the $Spin(7)$ string. We also comment on connections with topological strings on Calabi–Yau and K3 backgrounds.

Contents

1	Introduction	1
2	Review of topological strings	4
2.1	The A- and B-models	6
2.2	The G_2 string	13
2.3	The $Spin(7)$ string	16
3	G-structure complexes for special holonomy manifolds	18
3.1	A G_2 complex and Hodge theory	19
3.2	A $Spin(7)$ complex and Hodge theory	22
4	The $G_2 \times G_2$ complex	24
4.1	Generalised $G_2 \times G_2$ structures	24
4.2	Torsion-free generalised $G_2 \times G_2$ structures	25
4.3	The double complex	29
4.4	Hodge theory	33
5	Relation to the topological G_2 string	37
5.1	1-loop partition function	37
5.2	A quadratic target-space action	39
6	The $Spin(7) \times Spin(7)$ complex	46
6.1	Generalised $Spin(7) \times Spin(7)$ structures	47
6.2	The double complex	47
6.3	Hodge theory	50
7	Relation to the topological $Spin(7)$ string	52
8	Some other examples	55
8.1	A- and B-models with background H -flux	56
8.2	Topological strings on K3	59
9	Conclusions and future directions	60

A	Conventions and useful identities	63
A.1	Conventional geometry	63
A.2	G_2	64
A.3	$Spin(7)$	65
B	Determinants and partition functions	69
B.1	ζ -regularised determinants	69
B.2	Direct calculation of partition function	72
C	Review of $O(d, d) \times \mathbb{R}^+$ generalised geometry	77
C.1	$O(d) \times O(d)$ structures	80
C.2	Generalised Calabi–Yau	82
C.3	The generalised Hitchin functional for integrable $G_2 \times G_2$ structures	84

1 Introduction

Topological string models with Calabi–Yau target spaces provide us with subsectors of string theory in which certain quantities can be computed exactly. The key to this is that $(2, 2)$ σ -models with Calabi–Yau targets admit certain topological twists. To be precise, there are two distinct twists which give the A- and B-models [1]. In both of these models, the metric is not a fundamental degree of freedom – the A- and B-models are theories of Kähler and complex structures respectively – which suggests that the resulting theories may be topological. At the quantum level, the A-model can be defined on any Kähler manifold, while the B-model can be defined consistently only on Calabi–Yau targets. With these assumptions, one can show that observables indeed do not depend on the metric and so deserve the name topological.

The connection between topological strings and geometries captured by invariant functionals was first discussed in [2–4], where the partition function of the topological B-model and its conjugate [1, 5–7] on a six-dimensional target space M was argued to be encoded in the Hitchin functional for an $SL(3, \mathbb{C})$ structure on the same target space [8]. Pestun and Witten later observed that there is a discrepancy between the two at 1-loop, and showed that the 1-loop partition function of the B-model is actually given by the partition function of an extended Hitchin functional [9]. Generalising the real three-form that characterises an $SL(3, \mathbb{C})$ structure, this extended functional is written in terms of a polyform which defines a generalised Calabi–Yau structure. A key

insight here was that although the critical points of the two functionals agree, at the quantum level the fluctuating degrees of freedom of the two structures are different. Thus, it was essential to view the target space as a background in generalised geometry in order to match the topological B-model calculation.

One can view the above calculation by first starting with a conventional σ -model with a Calabi–Yau target space. In the large-volume limit, the worldsheet theory is captured by an effective theory on the target space. The topological twist that leads to the B-model corresponds to a subsector of the target-space theory described by a generalised Calabi–Yau structure. There is a similar construction for σ -models with a G_2 holonomy target space. One starts with the G_2 worldsheet algebra, which contains an extended $(1, 1)$ supersymmetry algebra [10]. Importantly, this algebra has a $c = 7/10$ subalgebra, known as a tri-critical Ising model, which can be used to define a topological twist of the σ -model [11]. One might expect that there is a subsector of the target-space theory that captures this twisted sector. An attempt at constructing this theory was made in [11], where a target-space action was proposed by starting from a Hitchin functional for a generalised $G_2 \times G_2$ structure. In this case, the 1-loop partition function of the topological G_2 string disagreed with the target-space calculation, differing by a factor of the Ray–Singer torsion of the background G_2 manifold.

The goal of this paper is to resolve this discrepancy and give a target-space interpretation of the topological G_2 string calculation. Rather than starting from a functional on the target space, we follow a different path and suggest that these results can be obtained by considering a certain double complex that arises naturally in generalised geometry.

To be precise, we will give a double complex for $G_2 \times G_2$ that realises the BRST complex of the topological G_2 string. The degrees of freedom of the worldsheet theory break into right- and left-moving sectors, each with their own BRST operators, with the string states given by tensor products of these sectors. Physical states are then cohomology classes of the total BRST operator. We will give a target-space interpretation for each of these ingredients, culminating in an expression for the 1-loop partition function that agrees with the worldsheet calculation of de Boer et al. [11] and a target-space action whose BV quantisation reproduces this answer. Given the conjectured existence of theories in seven and eight dimensions that unify the A- and B-models [2–4, 12, 13], it seems sensible to consider how topological $Spin(7)$ strings might also be captured by generalised geometry. Following the same logic as for the G_2 string, we make a conjecture for its 1-loop partition function.

Our construction does not require that the target space has special holonomy, but instead requires only the weaker condition of being a purely NSNS Minkowski background

(with metric, dilaton and B field) preserving at least $N = 1$ supersymmetry. In outline, starting with an $O(d, d) \times \mathbb{R}^+$ generalised geometry description of the target space, we show that supersymmetry implies the existence of a torsion-free $G \times G$ structures. From these, one can construct a compatible, torsion-free generalised connection which, together with certain projectors onto representations of $G \times G$, can be used to define a pair of differentials (d_+, d_-) . These differentials give the maps in a double complex for $G \times G \subset O(d) \times O(d) \subset O(d, d)$. After working out the Hodge theory and the analogue of Kähler identities for these differentials, we conjecture that a certain alternating sum of determinants of the Laplacian defined by $\hat{d} = d_+ + d_-$ determines the 1-loop partition function of the corresponding worldsheet theory. Upon restricting to honest special holonomy backgrounds with vanishing H flux, our expression reduces to the known result for the G_2 string and the A- and B-models, and gives a prediction for the $Spin(7)$ string.

Though our work gives a target-space description of the 1-loop partition function for these topological strings, we have not been able to find target-space actions that reproduce these calculations upon quantisation in all cases. The central result of [9] was the construction of a target-space theory based on an extended Hitchin functional for $SL(3, \mathbb{C})$ whose BV quantisation gives precisely the 1-loop partition function of the B-model on a Calabi–Yau target. For the G_2 string, a similar calculation was attempted in [14] with less success – the quantisation of neither the conventional nor the extended G_2 Hitchin functionals reproduced the 1-loop partition function of the G_2 string. To the authors’ knowledge, there has been no attempt to repeat this for the $Spin(7)$ string. In this paper, we give a target-space action whose quantisation does agree with the G_2 string, but it is not based on an invariant functional that we are aware of. For the $Spin(7)$ string, we have not been able to write down a target-space action.

A summary of our results follows:

- In Section 4, we introduce a new double complex for $G_2 \times G_2$ structures on seven-dimensional manifolds within generalised geometry. The differential operators that appear in this double complex are defined using a generalised connection that is compatible with the $G_2 \times G_2$ structure. We show that the operators are nilpotent and commute in the correct manner (so that they define a complex) if the generalised connection is torsion-free, which implies that the underlying string background is an NSNS Minkowski solution preserving at least $N = 1$ supersymmetry. We define Laplacians for these operators and their Hodge theory.
- We conjecture that the 1-loop partition function of the corresponding topological string is given by a certain alternating product of determinants of the Laplace

operators acting on the double complex. In Section 5 we restrict to the case of a G_2 holonomy background, and show that our general expression for the 1-loop partition function agrees with the worldsheet calculation of de Boer et al. [11].

- We give a target-space action whose BV quantisation reproduces our expression for the 1-loop partition function in Section 5.2. It does not seem that this action comes from considering variations of an invariant functional, unlike the case of the B-model [9].
- In Section 6 we repeat the above analysis for $Spin(7) \times Spin(7)$ structures on eight-dimensional manifolds. We again show that one can define a double complex provided there exists a torsion-free connection compatible with the generalised structure, equivalent to the corresponding NSNS Minkowski background preserving some supersymmetry. In Section 7 we compute the alternating product of determinants, and in the case of a $Spin(7)$ holonomy manifold conjecture that this gives the 1-loop partition function of the $Spin(7)$ string. We find it to be

$$Z_1 = (\det' \Delta_{\mathbf{1}})^{-1} (\det' \Delta_{\mathbf{7}}) (\det' \Delta_{\mathbf{21}})^{-1/2} (\det' \Delta_{\mathbf{27}})^{-1/2}, \quad (1.1)$$

where $\Delta_{\mathbf{r}}$ is the Laplacian acting on the \mathbf{r} representation of $Spin(7)$ and \det' is the ζ -regularised determinant.

- We outline in Section 8 how our formalism applies to the A- and B-models with flux and show that the relevant Laplacian is associated to the Lie algebroid defined by the corresponding generalised complex structure, agreeing with [15]. We also comment on topological strings on K3 surfaces where one finds that the 1-loop contribution is trivial.

We begin with an overview of the worldsheet theories for the A- and B-models, G_2 and $Spin(7)$ in Section 2. We then review the complexes that one can define on manifolds with G -structures and outline their Hodge theory in Section 3, before moving onto the results outlined above. The appendices contain our conventions and useful identities, a discussion of determinants and partition functions, and a quick review of $O(d, d) \times \mathbb{R}^+$ generalised geometry.

2 Review of topological strings

Since we will be proposing a target-space interpretation of various topological theories, we will first spend some time reviewing topological strings from the worldsheet, starting

with the well-known A/B-models [1, 5, 7, 16] and then moving on to the topological G_2 [10, 11, 14, 17] and $Spin(7)$ strings [10].

In each of these cases, special holonomy of the target space implies the existence of an extended worldsheet symmetry which allows a twisting procedure that renders the theory topological. In brief, one looks for an operator ρ , often related to the extended symmetry, with which to ‘twist’ the energy-momentum tensor

$$T \longrightarrow T_{\text{twist}} \sim T + \partial^2 \rho, \quad (2.1)$$

such that the central charge c of the twisted theory vanishes. This twisted energy-momentum tensor endows operators of the theory with new charges under Lorentz transformations. Interestingly, the twisting operator ρ is intimately related to the spectral flow operator, or analogues thereof, which is used to generate target-space supersymmetry.

Next, one looks for a nilpotent scalar¹ operator Q :

$$Q^2 = 0. \quad (2.2)$$

Typically, Q is built from the supersymmetry generators and then identified as the relevant BRST operator. One then requires that T_{twist} is a Q -trivial operator.² This is usually done by requiring the action to be written as a Q -exact piece, plus terms independent of the target-space metric. If this is the case, one can use localisation techniques to obtain exact results for correlators by evaluating them on fixed points of the BRST symmetry [1]. We further require that physical operators fall into Q -cohomology classes. These physical fields form a closed ring under the OPE, called the chiral ring, which is often related to certain cohomological data of the target space. The chiral ring generates certain highest-weight states in the NS sector, which can be related to the R sector ground states through spectral flow.

For *closed* topological strings, one has independent left- and right-moving sectors. States of the theory are built from tensor products of left- and right-moving states, and the BRST operator can be split into a left- and a right-moving operator

$$Q = Q_L + Q_R, \quad Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0. \quad (2.3)$$

Grading the states by left- and right-moving fermion number (p, q) , we find that

¹This is a scalar with respect to the twisted Lorentz symmetry.

²A necessary condition for this is that the central charge vanishes, hence the need to look for a T_{twist} with vanishing central charge.

observables fit into a double complex

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \vdots \\
& & \uparrow & & \uparrow & & \uparrow \\
\dots & \longrightarrow & \mathcal{O}^{p-1,q+1} & \longrightarrow & \mathcal{O}^{p,q+1} & \longrightarrow & \mathcal{O}^{p+1,q+1} & \longrightarrow & \dots \\
& & \uparrow & & \uparrow Q_R & & \uparrow & & \\
\dots & \longrightarrow & \mathcal{O}^{p-1,q} & \xrightarrow{Q_L} & \mathcal{O}^{p,q} & \xrightarrow{Q_L} & \mathcal{O}^{p,q+1} & \longrightarrow & \dots \\
& & \uparrow & & \uparrow Q_R & & \uparrow & & \\
\dots & \longrightarrow & \mathcal{O}^{p-1,q-1} & \longrightarrow & \mathcal{O}^{p,q-1} & \longrightarrow & \mathcal{O}^{p+1,q-1} & \longrightarrow & \dots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array} \tag{2.4}$$

This double complex is understood for the A/B-model [15]. It is the main goal of this paper to show that there is a nice target-space interpretation of (2.4) for *any* topological string with a special holonomy target space, at least in the infinite-volume limit, and that the 1-loop partition function of the worldsheet theory calculates a particular quantity of the double complex that we have dubbed the *analytic torsion*.

We will now go into more detail about the twisting procedure and the cohomological structure of the topological string for A/B, G_2 and $Spin(7)$ strings.

2.1 The A- and B-models

When the target space M is Kähler and the H -flux vanishes, the worldsheet supersymmetry is enhanced to $N = (2, 2)$. This symmetry is built from a left-moving and a right-moving sector which each have an energy-momentum tensor T , two supercurrents G^\pm , and a $U(1)$ current J . The \pm on the supercharges correspond to their charge under the $U(1)$. We will denote fields in the right-moving sector with a bar.

After the usual mode expansion, the relevant commutators for the left-moving sector are

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{2.5}$$

$$[L_m, G_{n\pm a}^\pm] = \left(\frac{1}{2}m - n \mp a\right)G_{m+n\pm a}^\pm, \tag{2.6}$$

$$[L_m, J_n] = -nJ_{m+n}, \tag{2.7}$$

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}, \tag{2.8}$$

$$[J_m, G_{n\pm a}^\pm] = \pm G_{m+n\pm a}^\pm, \quad (2.9)$$

and similarly for the right-movers. Strikingly, if one defines a twisted energy-momentum operator via

$$T_{\text{twist}} = T + \frac{1}{2}\partial J, \quad (2.10)$$

then the new modes $\tilde{L}_m = L_m - \frac{1}{2}(m+1)J_m$ satisfy

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n}, \quad [\tilde{L}_0, G_{-1/2}^+] = 0. \quad (2.11)$$

Hence we see that the central charge of the twisted algebra *vanishes*, and the supercharge $G_{-1/2}^+$ is a scalar with respect to the new Lorentz symmetry. We can therefore use it as the left-moving BRST operator.

Given this twist, there are two inequivalent twists of the right-moving sector given by [18]

$$\bar{T}_{\text{twist}} = \bar{T} \pm \frac{1}{2}\partial \bar{J}, \quad (2.12)$$

where upper sign corresponds to the B-model and the lower leads to the A-model. Both twists result in algebras with vanishing central charge, but with different nilpotent scalar operators. One then finds that the relevant right-moving BRST operators are

$$[\tilde{\bar{L}}_m, \bar{G}_{-1/2}^\pm] = 0. \quad (2.13)$$

The total BRST operators for each model are then

$$Q_A = G_{-1/2}^+ + \bar{G}_{-1/2}^-, \quad Q_B = G_{-1/2}^+ + \bar{G}_{-1/2}^+. \quad (2.14)$$

One can then examine the cohomology of local observables in each case.

Let us also briefly note how this twist is related to spectral flow. This is a symmetry of the algebras given by

$$L_n^\eta = L_n + \eta J_n + \frac{c}{6}\eta^2 \delta_{n,0}, \quad (2.15)$$

$$G_{n\pm a}^{\eta\pm} = G_{n\pm(a+\eta)}^\pm, \quad (2.16)$$

$$J_n^\eta = J_n + \frac{c}{3}\eta \delta_{n,0}. \quad (2.17)$$

In particular, we see from (2.16) that for $\eta = \frac{1}{2}$, we have a map from the NS to the R sector. This is generated by an operator which one can bosonise to $e^{i\rho/2}$. One then finds

that the $U(1)$ generator can be written in terms of ρ as

$$J = \partial\rho. \tag{2.18}$$

Inserting this into (2.10), one finds a formula for the twisted energy-momentum tensor more like that of (2.1).

The A-model

The A-model action can be written as

$$S = \left\{ Q_A, \int_{\Sigma} V \right\} + \int_{x(\Sigma)} \omega, \tag{2.19}$$

for some V , where ω is the Kähler form on M , Σ is the worldsheet and $x: \Sigma \rightarrow M$ is a map from the worldsheet to the target space. The action is thus Q_A -exact, up to a term that depends only on the homology class of $x(\Sigma) \subset M$ and so is independent of the worldsheet metric. This is sufficient to show that the energy-momentum tensor is Q_A -exact.³ Note further that the second term is also independent of the target-space complex structure and hence all correlators will depend only on the Kähler moduli. This topological string is thus quasi-topological, depending on some but not all the target-space moduli.

Localisation techniques allow us to evaluate correlators exactly by restricting the calculation to solutions of the equations of motion for V . It turns out that, for the bosonic sector, these are

$$\bar{\partial}x = \partial\bar{x} = 0. \tag{2.20}$$

The theory therefore localises on holomorphic maps $x: \Sigma \rightarrow M$. Coupling this theory to gravity, one must then integrate correlators over the moduli space of complex structures on Σ . This ensures the result is indeed independent of the target-space complex structure.

To study the Q_A cohomology ring, and hence the physical local operators, it is useful to go to the infinite-volume limit in which contributions from non-trivial homology classes $x(\Sigma)$ drop out of all correlation functions. In this limit, it is possible to show that the chiral operators take the form

$$\mathcal{O}_{\alpha} = \alpha(x)_{\mu_1 \dots \mu_p \bar{\nu}_1 \dots \bar{\nu}_q} \chi^{\mu_1} \dots \chi^{\mu_p} \bar{\chi}^{\bar{\nu}_1} \dots \bar{\chi}^{\bar{\nu}_q}, \tag{2.21}$$

³In fact, to write S as in (2.19), one has to use the equations of motion. It is possible to show that there exists an operator \tilde{Q}_A , which is related to Q_A by the equations of motion, such that (2.19) also holds off-shell.

where the χ^μ and $\bar{\chi}^{\bar{\nu}}$ are the left- and right-moving worldsheet fermions with $U(1)$ charge $+1$ and -1 respectively. Note that under the twisted Lorentz symmetry, these are scalars and thus dimension-zero operators, as is required for a topological observable. Moreover, one can identify $\chi^\mu \in x^*(T^{1,0})$ and $\bar{\chi}^{\bar{\nu}} \in x^*(T^{0,1})$.⁴ Hence, the space of operators is identified with standard (p, q) -forms on M . Under this identification, one finds that

$$G_{-1/2}^+ \sim \partial, \quad \bar{G}_{-1/2}^- \sim \bar{\partial}, \quad Q_A \sim \partial + \bar{\partial} = d. \quad (2.22)$$

Therefore, the chiral ring is isomorphic to the de Rham cohomology ring of M in the infinite-volume limit. Furthermore, the double complex given in (2.4) maps onto the Dolbeault complex of M .

At finite volume, the chiral ring is deformed by worldsheet instantons coming from the second term in (2.19). While the operators can still be identified with (p, q) -forms, Q_A no longer matches the de Rham operator, and the ring structure does not match the de Rham cohomology. Instead, one finds what is called the quantum-deformed cohomology of M which takes into account holomorphic multiwrappings of the worldsheet on Riemann surfaces in M [19, 20].

The correlators are interesting in their own right as they compute Gromov–Witten invariants [21]. Unfortunately, contributions from worldsheet instantons make them difficult to calculate directly. In practice, one uses either the holomorphic anomaly equations to relate correlators at genus g to lower-genus correlators, or mirror symmetry to relate correlators in the A-model to those in the B-model.

The B-model

Unlike the A-model, which exists for any Kähler target space, the axial R-symmetry generated by the pair $(J, -\bar{J})$ is anomalous unless $c_1(M) = 0$. Hence, for the B-model twist to be well defined at the quantum level, one must restrict to Calabi–Yau target spaces.

With this restriction, it is possible to write the action as a Q_B -exact piece, plus a term that is independent of both the worldsheet metric and the target-space Kähler form.⁵ Thus one finds that correlators depend only on the complex structure moduli, and again one has a quasi-topological theory. The theory localises on solutions to

$$dx = d\bar{x} = 0, \quad (2.23)$$

⁴Here, and throughout the paper, we will use the shorthand $T = TM$, $T^{1,0} = T^{1,0}M$, and so on.

⁵We do not give the action explicitly as we did for the A-model as it is not enlightening in this case. It can be found in e.g. [22]

and hence we can calculate exact results by restricting to constant maps $x: \Sigma \rightarrow M$. This observation often makes B-model correlators easier to calculate (though perhaps less interesting from a mathematical point of view).

To study the Q_B cohomology ring, it is once again useful to go to the infinite-volume limit. There one finds the dimension-zero operators take the form

$$\mathcal{O}_\beta = \beta(x)^{\mu_1 \dots \mu_p} \bar{\nu}_1 \dots \bar{\nu}_q \theta_{\mu_1} \dots \theta_{\mu_p} \bar{\eta}^{\bar{\nu}_1} \dots \bar{\eta}^{\bar{\nu}_q}, \quad (2.24)$$

where the θ_μ and $\bar{\eta}^{\bar{\nu}}$ are left- and right-moving fermions (though scalars under the twisted Lorentz symmetry), both with $U(1)$ charge $+1$. We can identify $\theta_\mu \in x^*(T^{*1,0})$ and $\bar{\eta}^{\bar{\nu}} \in x^*(T^{0,1})$, and so the space of operators corresponds to sections of $\Lambda^q T^{*0,1} \otimes \Lambda^p T^{1,0}$. Using this identification, one finds

$$G_{-1/2}^+ \sim \frac{1}{2}(\bar{\partial} + \partial^\dagger), \quad \bar{G}_{-1/2}^+ \sim \frac{1}{2}(\bar{\partial} - \partial^\dagger), \quad Q_B \sim \bar{\partial}. \quad (2.25)$$

The chiral ring of physical operators is therefore isomorphic to the bundle-valued Dolbeault cohomology groups

$$H_{\bar{\partial}}^\bullet(M, \Lambda^\bullet T^{1,0}) = \bigoplus_{p,q} H_{\bar{\partial}}^q(M, \Lambda^p T^{1,0}). \quad (2.26)$$

The holomorphic $(n, 0)$ -form of the Calabi–Yau target space then gives an isomorphism between this and the usual Dolbeault complex on $(n - p, q)$ -forms.

Looking at (2.25), we see that the left and right BRST operators do not correspond to the Dolbeault operators, but instead raise the antiholomorphic degree while lowering the holomorphic degree of forms. This means that the left and right fermion numbers cannot be matched to the holomorphic and antiholomorphic degree of the form respectively, and the BRST double complex (2.4) cannot be identified with a Dolbeault complex on the target space. However, the *total* cohomology can still be identified with the Dolbeault cohomology as above [7]. Moreover, given that the B-model is independent of the Kähler moduli, this chiral ring is exact at finite volume even though it was derived at infinite volume.

The 1-loop partition function

Finally, we will briefly review the 1-loop partition functions of the A- and B-models. One can calculate the 1-loop partition function from the free energy which is given by [7, 23, 24]

$$\mathcal{F}_1 = \frac{1}{2} \int \frac{d\tau d\bar{\tau}}{\tau_2} (-1)^F F_L F_R e^{2\pi i \tau H_L} e^{-2\pi i \bar{\tau} H_R}, \quad (2.27)$$

where F_L and F_R are the left- and right-moving fermion number operators respectively, $F = F_L + F_R$ is the total fermion number operator, $H_L = \{Q_L, Q_L^\dagger\}$ is the left-moving Hamiltonian, and similarly for H_R . The BRST operators are given by $Q_L = G_{-1/2}^+$ and $Q_R = \bar{G}_{-1/2}^\pm$ depending on whether we are in the A- or B-model. Integrating over the upper half-plane, this can be shown to be equal to [7, 14]

$$\mathcal{F}_1 = \delta(H_L - H_R) \frac{1}{2} \log \left[\prod_{F_L, F_R} (\det'(H_L + H_R))^{(-1)^{F_L F_R}} \right], \quad (2.28)$$

with the partition function then given by $e^{-\mathcal{F}_1}$.

From a worldsheet perspective, this is simply an alternating product of determinants of Hamiltonians acting on the double BRST complex (2.4). To understand what this calculates on the target space, one needs to use the target-space identification of (2.4). For the A-model at infinite volume ($\omega \rightarrow \infty$), this identification is clear and (2.28) becomes an alternating product of Laplacians acting on the Dolbeault complex:

$$Z_1^A \stackrel{\omega \rightarrow \infty}{=} \left[\prod_{p,q} (\det' \Delta^{p,q})^{(-1)^{p+q} pq} \right]^{-1/2}, \quad (2.29)$$

which can be written in terms of holomorphic Ray–Singer torsions:

$$Z_1^A \stackrel{\omega \rightarrow \infty}{=} \frac{I_1}{I_0^3}. \quad (2.30)$$

For finite volume, the answer will receive contributions from strings wrapping cycles in M [24].

For the B-model, understanding the 1-loop calculation in terms of the double BRST complex is more opaque as the target-space BRST complex is more complicated. Despite this, one can show that, up to moduli-independent terms (i.e. a multiplicative constant), one obtains the same answer as for the A-model [7]:

$$Z_1^B = \frac{I_1}{I_0^3}. \quad (2.31)$$

This holds for arbitrary Calabi–Yau target spaces, even for finite volume.

We emphasise the form of the 1-loop partition function given by the right-hand side of (2.29), as it will appear again when we look at the 1-loop partition function of the G_2 and $Spin(7)$ strings. The 1-loop partition function calculates a quantity related to the target-space BRST double complex, given by a particular product of determinants

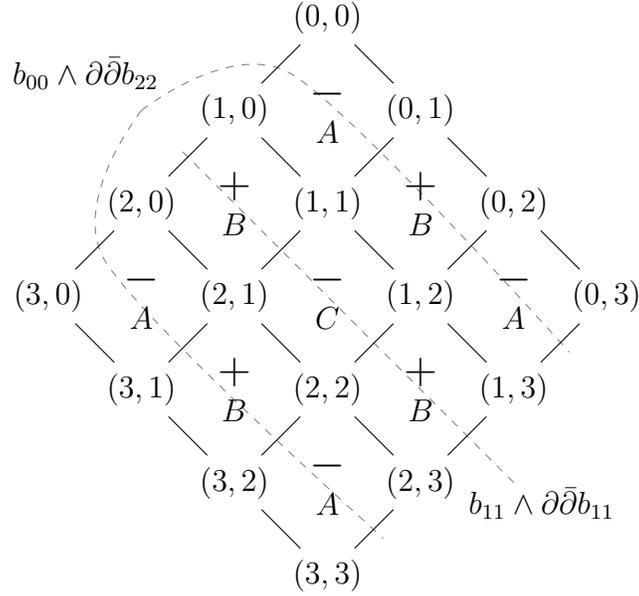


Figure 1. Figure adapted from [9]. Complex conjugation, Hodge duality and contraction with the holomorphic 3-form Ω leave only three independent determinants which all $\det' \Delta^{p,q}$ can be expressed in terms of. For example, $\det' \Delta^{0,0} = A$ and $\det' \Delta^{1,1} = AB^2C$. The analytic torsion (the 1-loop partition function) is then given by $(A^{-4}B^4C^{-1})^{1/2}$, in agreement with (2.29). Note that upon BV quantising (2.32), the first and second terms contribute $A^{-4}B^2$ and B^2C^{-1} respectively, corresponding to the products of the determinants along the dotted lines in the figure.

of Laplacians on that complex as shown. Given the similarity to the analytic torsion of one-dimensional complexes [25, 26], we shall refer to the quantity (2.29), when applied to arbitrary complexes, as the *analytic torsion* of the double complex.

In [9], Pestun and Witten showed that this result for Z_1^{B} could be obtained by BV quantising the target-space theory defined by

$$S = \int_M b_{00} \wedge \partial \bar{\partial} b_{22} + b_{11} \wedge \partial \bar{\partial} b_{11}, \quad (2.32)$$

where the subscripts denote the (p, q) -form degree. Furthermore, they showed that this action has a natural interpretation as the quadratic variation of the Hitchin functional for a generalised Calabi–Yau structure, where the variation is taken within a fixed cohomology class [27]. We review the generalised Hitchin functional in appendix C.2. This provides a link between topological strings at 1-loop and geometric structures in the $O(d, d)$ geometry of Hitchin that we will explore further in this paper.

Note that the 1-loop partition function has a nice pictorial interpretation in terms of the Dolbeault complex, as we illustrate in Figure 1 [9]. Briefly, the determinant of $\Delta^{p,q}$ can be decomposed into a product of Laplacians acting on the subspaces appearing in the Hodge decomposition of $\Omega^{p,q}(M)$. These four spaces are represented by the four squares surrounding each vertex in the complex. By Hodge duality and complex conjugation, there are only three independent values these can take, represented by A , B and C in the diamond. It turns out that the 1-loop partition function can be read off from the Hodge diamond by multiplying these factors together with alternating powers of $\pm\frac{1}{2}$, in a “checkerboard pattern”, as shown in the figure. We give a brief review of ζ -regularised determinants of Laplacians in Appendix B.1.

2.2 The G_2 string

The existence of a topological string with G_2 target space was conjectured in [10] and further studied in [11, 14, 17], yet its properties are still not fully understood. Evidence for the twisting procedure comes from the extended worldsheet symmetry implied by G_2 holonomy of the target space. Indeed, given a G_2 structure $\varphi \in \Omega^3(M)$, one can define the operators

$$\Phi = \frac{1}{3!} \varphi_{\mu\nu\rho} \psi^\mu \psi^\nu \psi^\rho, \quad (2.33)$$

$$K = \frac{1}{2} \varphi_{\mu\nu\rho} \psi^\mu \psi^\nu \partial x^\rho, \quad (2.34)$$

$$X = -\frac{1}{4!} (*\varphi)_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \psi^\rho \psi^\sigma - \frac{1}{2} g_{\mu\nu} \psi^\mu \partial \psi^\nu, \quad (2.35)$$

$$M = -\frac{1}{3!} (*\varphi)_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \psi^\rho \partial x^\sigma - \frac{1}{2} g_{\mu\nu} \partial x^\mu \partial \psi^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \partial^2 x^\nu. \quad (2.36)$$

These operators along with the $N = 1$ superconformal operators (T, G) define a closed algebra denoted by $\mathcal{SW}_{[0, \frac{21}{2}]}(\frac{3}{2}, \frac{3}{2}, 2)$, a particular supersymmetric \mathcal{W} -algebra.⁶

As in the A/B-models, one can identify a spectral flow-like operator which implements the twisting. It turns out that, in this case, it is easier to understand the theory via the states, rather than the chiral ring. In the topological theory, the chiral ring is in one-to-one correspondence with the R ground states of the untwisted theory. These

⁶This is true in the free-field or infinite-volume limit. More generally, properties like the Jacobi identity hold only modulo the ideal generated by the null field N defined by [28]

$$N = 4GX - 2\Phi K - 4\partial M - \partial^2 G. \quad (2.37)$$

become the physical states in the twisted theory and hence one obtains an equivalent description of the theory.

To study the states of the theory, we introduce the operators $T_I = -\frac{1}{5}X$ and $G_I = \frac{i}{\sqrt{15}}\Phi$. One finds they define an $N = 1$ superconformal algebra of central charge $c = \frac{7}{10}$. This is a minimal model known as the tri-critical Ising model. One can write the original energy-momentum tensor T as

$$T = T_I + T_r, \quad T_I(z)T_r(w) = \text{regular}, \quad (2.38)$$

where T_r defines a Virasoro algebra commuting with T_I with central charge $c = \frac{98}{10}$. States are then labeled by two quantum numbers, $|\Delta_I, \Delta_r\rangle$, specifying their weights under T_I and T_r . Since T_I defines a minimal model, we know the weights of the conformal primaries of the theory. They split into an NS and an R sector:

$$\text{NS: } 0, \quad \frac{1}{10}, \quad \frac{6}{10}, \quad \frac{3}{2}, \quad \text{R: } \frac{7}{16}, \quad \frac{3}{80}. \quad (2.39)$$

We also know that the R ground states of the full theory must have total weights $\Delta = \Delta_I + \Delta_r = \frac{d}{16} = \frac{7}{16}$. Therefore, we find that the R ground states are

$$|\frac{7}{16}, 0\rangle, \quad |\frac{3}{80}, \frac{2}{5}\rangle. \quad (2.40)$$

One can then use the state with only non-zero tri-critical Ising weight to define a map between the R-sector ground states and certain special NS states. This is the analogue of the spectral flow operator of $N = (2, 2)$ theories. Indeed, using the fusion rules of the tri-critical Ising model, we have the following NS states:

$$|0, 0\rangle, \quad |\frac{1}{10}, \frac{2}{5}\rangle, \quad |\frac{6}{10}, \frac{2}{5}\rangle, \quad |\frac{3}{2}, 0\rangle. \quad (2.41)$$

By examining the total weight of the states, we notice that these states are respectively generated by the operators⁷

$$f(x), \quad A_\mu(x)\psi^\mu, \quad B_{\mu\nu}(x)\psi^\mu\psi^\nu, \quad C_{\mu\nu\rho}(x)\psi^\mu\psi^\nu\psi^\rho, \quad (2.42)$$

and so they define target-space 0-, 1-, 2-, and 3-forms. To ensure they have the correct weight under T_I , the coefficients must be restricted to lie in particular G_2 representations. In particular, they must fall into the irreducible representations that appear in the G_2

⁷One could ask why we do not include fields of the form ∂x or $\partial\psi$. The reason is that the derivative ensures that these have conformal weight ≥ 1 and hence cannot be scalars under a twisted Lorentz symmetry. They should therefore not be included in a set of local physical topological operators.

instanton complex of [29], which will be explored in more detail in the next section.

One can also use the R ground state to twist the model to produce an energy-momentum tensor T_{twist} whose algebra has vanishing central charge. Bosonising the theory, one can write

$$\Phi = \exp\left(\frac{3i}{\sqrt{5}}\rho\right), \quad (2.43)$$

$$X = (\partial\rho)^2 + \frac{1}{4\sqrt{5}}\partial^2\rho, \quad (2.44)$$

$$|\frac{7}{16}, 0\rangle = \exp\left(\frac{-5i}{4\sqrt{5}}\rho\right). \quad (2.45)$$

Given that one can relate twisted and untwisted correlators of the A/B-models by $2g - 2$ insertions of the spectral flow operator at genus g , one may guess that a twisted G_2 string is obtained by inserting $2g - 2$ copies of (2.45). This has the effect of shifting the energy-momentum tensor induced by X to

$$X_{\text{twist}} = (\partial\rho)^2 + \frac{3}{2\sqrt{5}}\partial^2\rho. \quad (2.46)$$

Taking the total twist $T_{\text{twist}} = \frac{1}{5}X_{\text{twist}} + T_r$, one finds an algebra with vanishing central charge. The NS states of (2.41) then have a shifted Δ_I weight and become

$$|0, 0\rangle, \quad |-\frac{2}{5}, \frac{2}{5}\rangle, \quad |-\frac{2}{5}, \frac{2}{5}\rangle, \quad |0, 0\rangle. \quad (2.47)$$

In particular, they have total weight zero – a necessary condition for a physical state in a topological theory.

It remains to be seen whether there exists a nilpotent operator Q that is a scalar with respect to the twisted Lorentz algebra such that T_{twist} is Q -exact. In [11], it was argued that the correct operator is a particular conformal block of the supersymmetry generator $G_{-1/2}$, which was denoted by $G_{-1/2}^\downarrow$.⁸ While it was not shown that T_{twist} is exact with respect to this operator, it was argued that $G_{-1/2}^\downarrow$ is indeed nilpotent and maps the special NS states within themselves:

$$|0, 0\rangle \xrightarrow{G_{-1/2}^\downarrow} |\frac{1}{10}, \frac{2}{5}\rangle \xrightarrow{G_{-1/2}^\downarrow} |\frac{6}{10}, \frac{2}{5}\rangle \xrightarrow{G_{-1/2}^\downarrow} |\frac{3}{2}, 0\rangle \quad (2.48)$$

⁸Note that Fiset and Gaberdiel [30] show that the cohomology of $G_{-1/2}^\downarrow$ is not restricted to the chiral ring and so it cannot be the exact BRST operator which captures the geometry of the target space (though the honest BRST operator is likely related to $G_{-1/2}^\downarrow$). For our purposes, we need only the identification of the complex of special NS states as later we will identify the analogue of the correct BRST operator in the target space.

This complex has a target-space interpretation as the G_2 Dolbeault complex (3.5) which we will describe in the following section. The physical states should therefore be in the cohomology of this complex. In addition, a heuristic argument was given that the path integral localises on constant maps $x: \Sigma \rightarrow M$ and so one would not expect instanton corrections at finite volume.

One finds completely analogous results for the right-moving sector and so the total BRST operator should be

$$Q = G_{-1/2}^\downarrow + \bar{G}_{-1/2}^\downarrow, \quad (2.49)$$

with the physical states given by tensor products of left- and right-moving states, each in (2.47), that are annihilated by Q . This poses the question: what is the target-space interpretation of the BRST double complex (2.4)? One of the results of this paper is to show that there exists a double complex on any G_2 manifold which naturally represents this worldsheet complex. Moreover, we will examine its relation to the 1-loop partition function and compare our results to those found in [14].

2.3 The *Spin*(7) string

The topological *Spin*(7) string was also conjectured to exist in [10] but there has been little further study since then.⁹ We will now outline some of the evidence for its existence.

As before, a target space with *Spin*(7) holonomy implies an extended worldsheet symmetry which is required for the twisting procedure. If $\Theta \in \Omega^4(M)$ is the self-dual 4-form defining the *Spin*(7) structure, we can define the operators

$$\tilde{X} = \frac{1}{4!} \Theta_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \psi^\rho \psi^\sigma + \frac{1}{2} g_{\mu\nu} \psi^\mu \partial \psi^\nu, \quad (2.50)$$

$$\tilde{M} = \frac{1}{3!} \Theta_{\mu\nu\rho\sigma} \psi^\mu \psi^\nu \psi^\rho \partial x^\sigma - \frac{1}{2} g_{\mu\nu} \partial x^\mu \partial \psi^\nu + \frac{1}{2} g_{\mu\nu} \psi^\mu \partial^2 x^\nu. \quad (2.51)$$

These, together with the $N = 1$ superconformal generators (T, G) , form a closed algebra. The rescaled operator $T_I = \frac{1}{8} \tilde{X}$ generates a Virasoro algebra with central charge $c = \frac{1}{2}$, known as the bosonic Ising model. This plays the same role as the tri-critical Ising model in the G_2 string and will be important for the putative twisting procedure.

Once again, it is easier to understand the theory via the states. We can write the total energy-momentum tensor as

$$T = T_I + T_r, \quad T_I(z)T_r(w) = \text{regular}, \quad (2.52)$$

⁹Though see, for example, [31–34]

where T_r defines a Virasoro algebra commuting with T_I of central charge $c = \frac{23}{2}$. We can therefore label states as $|\Delta_I, \Delta_r\rangle$ with respect to their weights under T_I and T_r . Since the bosonic Ising model is a minimal model, we know the possible weights are given by

$$\Delta_I: \quad 0, \quad \frac{1}{16}, \quad \frac{1}{2}. \quad (2.53)$$

Since the total weight of the R ground states must be equal to $\frac{d}{16} = \frac{8}{16} = \frac{1}{2}$, one finds that they must be

$$|0, \frac{1}{2}\rangle, \quad |\frac{1}{16}, \frac{7}{16}\rangle, \quad |\frac{1}{2}, 0\rangle. \quad (2.54)$$

Once again, we find a state with only non-vanishing bosonic Ising weight which we can use as a spectral flow-like operator to define a map between the R ground states and certain NS highest-weight states. Indeed, using the fusion rules, the states in the NS sector are

$$|0, 0\rangle, \quad |\frac{1}{16}, \frac{7}{16}\rangle, \quad |\frac{1}{2}, \frac{1}{2}\rangle. \quad (2.55)$$

By examining the total weight, we see that these states are generated by the operators¹⁰

$$f(x), \quad A_\mu(x)\psi^\mu, \quad B_{\mu\nu}(x)\psi^\mu\psi^\nu. \quad (2.56)$$

Hence, the states are related to target-space 0-, 1-, and 2-forms. To ensure the states have the correct T_I weights, we find that $B_{\mu\nu}$ must be restricted to lie in the $\mathbf{7}$ of $Spin(7)$. Intriguingly, these representations are precisely those that appear in the $Spin(7)$ instanton complex [29].

We can also use the R ground state to form a twisted energy-momentum tensor with vanishing central charge. Indeed, bosonising the theory, one can write

$$\tilde{X} = (\partial\rho)^2 + \frac{1}{4\sqrt{3}}\partial^2\rho, \quad (2.57)$$

$$|\frac{1}{2}, 0\rangle = \exp\left(\frac{3i}{2\sqrt{3}}\rho\right). \quad (2.58)$$

The insertion of $2g - 2$ copies of (2.58) into correlators is equivalent to twisting the energy-momentum tensor to

$$\tilde{X}_{\text{twist}} = (\partial\rho)^2 + \frac{5}{4\sqrt{3}}\partial^2\rho. \quad (2.59)$$

Taking the twist of the full theory to be $T_{\text{twist}} = \frac{1}{8}\tilde{X}_{\text{twist}} + T_r$, one finds a Virasoro

¹⁰Once again, we do not include terms with $\partial\phi$ or $\partial\psi$ as they cannot be scalars with respect to a twisted Lorentz symmetry.

algebra with vanishing central charge. Furthermore, the weights of the NS states under this twisted algebra become

$$|0, 0\rangle, \quad \left|-\frac{7}{16}, \frac{7}{16}\right\rangle, \quad \left|-\frac{1}{2}, \frac{1}{2}\right\rangle. \quad (2.60)$$

These have total weight zero under the twisted Lorentz symmetry, a necessary condition for the physical states of a topological theory.

It is still unknown whether there is an appropriate nilpotent operator Q such that T_{twist} is Q -exact. However, we find it highly suggestive that states of weight zero in the NS sector define the vector spaces of the $Spin(7)$ instanton complex of [29], much like we saw for the G_2 string.¹¹ We therefore expect that the correct operator is some sub-operator of G , suitably projected so that one gets the correct target-space complex. We will provide some evidence for this in Section 7. The full theory contains states that are tensor products of the left- and right-moving sectors, and the physical operators in the chiral ring again correspond to cohomology classes of $Q = Q_L + Q_R$.

Despite not knowing the precise worldsheet theory, we will show that there exists a natural double complex on any $Spin(7)$ target space that seems to encode the left- and right-moving states and gives candidates for the left- and right-moving BRST operators. We will use this to make a conjecture for the partition function at 1-loop.

3 G -structure complexes for special holonomy manifolds

It is very striking that the left- and right-moving states selected by the topological twist precisely form the vector spaces in the instanton complexes of [29]. These are particular complexes that arise on manifolds with G -structure $G \subset O(d)$ as a subcomplex of de Rham.¹² Given the appearance of these complexes in topological strings, we will briefly review them for G_2 and $Spin(7)$ holonomy manifolds and analyse their Hodge theory. We will find a doubled version of these complexes in later sections by lifting to $O(d, d) \times \mathbb{R}^+$ geometry, and match them to the BRST double complex. We will mirror the techniques used in this section when analysing the properties of these double complexes.

¹¹In fact, it is also possible to formulate the left- and right-moving sectors of the A- and B-models in terms of the instanton complex for $SU(n)$ structures.

¹²The cohomology of these complexes is also related to the moduli space of G -instantons on these manifolds.

3.1 A G_2 complex and Hodge theory

Let (M, φ) be a seven-dimensional manifold with a (possibly torsionful) G_2 structure. The G_2 structure defines a unique metric g and hence a Hodge star operator $*$. The intrinsic torsion of the structure is encoded by $d\varphi$ and $d*\varphi$, which both vanish if and only if the intrinsic torsion vanishes [35]. Any such manifold admits a decomposition of differential forms into irreducible G_2 representations as [36]

$$\Lambda^0 T^* = \Lambda_{\mathbf{1}}^0 T^*, \quad (3.1)$$

$$\Lambda^1 T^* = \Lambda_{\mathbf{7}}^1 T^*, \quad (3.2)$$

$$\Lambda^2 T^* = \Lambda_{\mathbf{7}}^2 T^* \oplus \Lambda_{\mathbf{14}}^2 T^*, \quad (3.3)$$

$$\Lambda^3 T^* = \Lambda_{\mathbf{1}}^3 T^* \oplus \Lambda_{\mathbf{7}}^3 T^* \oplus \Lambda_{\mathbf{27}}^3 T^*, \quad (3.4)$$

where the subscript denotes the dimension of the G_2 representation and we are using the shorthand $T^* \equiv T^*M$. Higher-degree differential forms have similar decompositions via Hodge duality. A precise definition of the subspaces in terms of $(\varphi, *\varphi)$ is given in Appendix A.2. We will denote the space of sections of p -forms in the r -dimensional representation as $\Omega_{\mathbf{r}}^p(M)$, and the projection onto those subspaces by $\mathcal{P}_{\mathbf{r}}^p$.

Given such a decomposition, consider the following sequence of maps defined by composing the de Rham differential with certain projections [37, 38]

$$\check{d}: \Omega_{\mathbf{1}}^0(M) \xrightarrow{d} \Omega_{\mathbf{7}}^1(M) \xrightarrow{\mathcal{P}_{\mathbf{7}}^2 d} \Omega_{\mathbf{7}}^2(M) \xrightarrow{\mathcal{P}_{\mathbf{1}}^3 d} \Omega_{\mathbf{1}}^3(M) \quad (3.5)$$

Provided the intrinsic torsion of the G_2 structure has no component in the $\mathbf{14}$, one finds $\check{d}^2 = 0$ and so the above sequence is actually a complex – we will then refer to (3.5) as the “ G_2 complex” [29]. For the remainder of this section we will restrict to torsion-free G_2 structures, i.e. those with $d\varphi = d*\varphi = 0$, and hence G_2 holonomy.

Given the G_2 complex, we can introduce an inner product on each of the vector spaces and consider the Laplacian defined by $\check{\Delta} = \check{d}\check{d}^\dagger + \check{d}^\dagger\check{d}$. We do this in a way that will allow for comparison to the usual de Rham Laplacian. First, we define isomorphisms between spaces with the same G_2 representation as

$$\begin{aligned} \theta_{\mathbf{1}}: \Lambda_{\mathbf{1}}^0 T^* &\longrightarrow \Lambda_{\mathbf{1}}^3 T^*, & \theta_{\mathbf{7}}: \Lambda_{\mathbf{7}}^1 T^* &\longrightarrow \Lambda_{\mathbf{7}}^2 T^*, \\ f &\longmapsto k_1 f \varphi, & \lambda &\longmapsto k_2 \lambda^a \varphi_{abc}, \end{aligned} \quad (3.6)$$

where k_1 and k_2 are constants we will determine later and indices are raised and lowered using the G_2 metric defined by φ . Next we fix an inner product (\cdot, \cdot) to be the standard

inner product on 0- and 1-forms:

$$(f, g)_0 = \int_M \text{vol } fg, \quad (\lambda, \nu)_1 = \int_M \text{vol } \lambda \lrcorner \nu. \quad (3.7)$$

We extend this to an inner product on the whole complex by demanding that it depends only on the representation and not the degree of the p -form:

$$(\theta_1(f), \theta_1(f'))_3 = (f, f')_0, \quad (\theta_7(\lambda), \theta_7(\lambda'))_2 = (\lambda, \lambda')_1, \quad (3.8)$$

where $(\cdot, \cdot)_p$ denotes restriction to p -forms. Note that this forces (\cdot, \cdot) to be the usual inner product on differential forms, up to possible multiplicative constants that are determined by k_1 and k_2 .

We can fix the constants by demanding that the following diagram commutes

$$\begin{array}{ccccccc} \Omega_1^0 & \xrightarrow{\check{d}} & \Omega_7^1 & \xrightarrow{\check{d}} & \Omega_7^2 & \xrightarrow{\check{d}} & \Omega_1^3 \\ \downarrow \theta_1 & & \downarrow \theta_7 & & \downarrow \theta_7^{-1} & & \downarrow \theta_1^{-1} \\ \Omega_1^3 & \xrightarrow{\check{d}^\dagger} & \Omega_7^2 & \xrightarrow{\check{d}^\dagger} & \Omega_7^1 & \xrightarrow{\check{d}^\dagger} & \Omega_1^0 \end{array} \quad (3.9)$$

If this holds the Laplacians acting on isomorphic G_2 representations are equivalent in the sense that

$$\theta_1 \check{\Delta}^0 = \check{\Delta}^3 \theta_1, \quad \theta_7 \check{\Delta}^1 = \check{\Delta}^2 \theta_7, \quad (3.10)$$

where $\check{\Delta}^p$ is the restriction of $\check{\Delta}$ to p -forms. We can therefore unambiguously write $\check{\Delta}_{\mathbf{r}}$ for the Laplacian acting on differential forms in the G_2 representation \mathbf{r} . This will be important later when we consider determinants of these Laplacians. A quick calculation shows that the diagram commutes and the Laplacians are isomorphic for

$$\frac{k_1}{k_2} = -\frac{3}{7}. \quad (3.11)$$

Finally, we can fix the coefficient k_2 (and hence k_1) up to an irrelevant overall sign by demanding that

$$\check{\Delta}_7 = \Delta_7, \quad (3.12)$$

where $\Delta = dd^\dagger + d^\dagger d$ is the de Rham Laplacian.¹³ It is possible to show that Δ also

¹³We emphasise that the de Rham adjoint d^\dagger is defined by the usual inner product on forms and not the rescaled inner product we have defined for the \check{d} complex.

commutes with the projection operators and only depends on the G_2 representation of the form, not the degree, and hence (3.12) is well defined. Note that this is a non-trivial constraint since Δ_7 contains terms coming from $\mathcal{P}_{14}^2 d$, while $\check{\Delta}_7$ does not. Fortunately, for a torsion-free G_2 structure and any $\lambda \in \Omega_7^1$, we have [39]

$$d^\dagger \mathcal{P}_{14}^2 d\lambda = 2d^\dagger \mathcal{P}_7^2 d\lambda. \quad (3.13)$$

With this it is easy to check that (3.11) and (3.12) impose

$$k_2 = -\frac{1}{3}, \quad k_1 = \frac{1}{7}. \quad (3.14)$$

The inner product is then given by

$$(\alpha, \beta)_p = \kappa_p \int_M \alpha \wedge * \beta, \quad \kappa_p = \begin{cases} 1 & p = 0, 1, \\ 3 & p = 2, \\ 7 & p = 3. \end{cases} \quad (3.15)$$

Having fixed the coefficients k_1 and k_2 we can now define explicitly the operators $(\check{d}, \check{d}^\dagger, \check{\Delta})$. Since we have assumed that the G_2 structure is torsion-free, we can replace the de Rham differential in the definition of \check{d} in (3.5) with the Levi-Civita connection ∇ compatible with the G_2 structure. This simplifies calculations as derivatives will then commute with the projection operators since the \mathcal{P}_r^p are defined in terms of φ and $*\varphi$ which are covariantly constant (see Appendix A.2 for more details).

In terms of the Levi-Civita connection, the \check{d} operator acting on p -forms becomes

$$(\check{d}\omega)_{a_1 \dots a_{p+1}} = (p+1)(\mathcal{P}_r^{p+1})_{a_1 \dots a_{p+1}}{}^{b_1 \dots b_{p+1}} \nabla_{[b_1} \omega_{b_2 \dots b_{p+1}]}. \quad (3.16)$$

The adjoint operator \check{d}^\dagger becomes

$$(\check{d}^\dagger \omega)_{a_1 \dots a_{p-1}} = -C_p \nabla^b \omega_{b a_1 \dots a_{p-1}}, \quad C_p = \begin{cases} 1 & p = 1, \\ 3 & p = 2, \\ \frac{7}{3} & p = 3. \end{cases} \quad (3.17)$$

Note that we do not need to include projectors in the definition of the adjoint operator as we are assuming that ω lives in one of the spaces in (3.5). Again, these definitions have the useful properties that the natural differential operators that one can construct depend only on the G_2 representation and not the p -form degree of the object on which

they act. For example, acting on 1-, 2- or 3-forms we have¹⁴

$$\check{d}^\dagger|_1 = \theta_1^{-1}\check{d}\theta_7, \quad \check{d}^\dagger|_2 = \theta_7^{-1}\check{d}\theta_7^{-1}, \quad \check{d}^\dagger|_3 = \theta_7\check{d}\theta_1^{-1}. \quad (3.18)$$

Finally, the Laplacian $\check{\Delta}$ can be written as

$$\check{\Delta}^0 f = -\nabla^a \nabla_a f = \Delta_1 f, \quad (3.19)$$

$$\check{\Delta}^1 \lambda = -6(\mathcal{P}_7^2)_{ba}{}^{cd} \nabla^b \nabla_c \lambda_d - \nabla_a \nabla^b \lambda_b = \Delta_7 \lambda, \quad (3.20)$$

$$\check{\Delta}^2 \mu = -7(\mathcal{P}_1^3)_{ba_1 a_2}{}^{def} \nabla^b \nabla_d \mu_{ef} - 6(\mathcal{P}_7^2)_{a_1 a_2}{}^{bc} \nabla_b \nabla^d \mu_{dc} = \theta_7 \Delta_7 \theta_7^{-1} \mu, \quad (3.21)$$

$$\check{\Delta}^3 \rho = -7(\mathcal{P}_1^3)_{abc}{}^{def} \nabla_d \nabla^g \rho_{gef} = \theta_1 \Delta_1 \theta_1^{-1} \rho. \quad (3.22)$$

Given these operators, it is natural to ask if there is some kind of Hodge decomposition for the spaces $\Omega_{\mathbf{r}}^p$. We defined the inner product (\cdot, \cdot) in terms of the usual inner product on differential forms, differing on 2- and 3-forms by a positive multiplicative factor. The usual inner product is positive definite and is G_2 invariant,¹⁵ hence it reduces to a positive-definite inner product on the irreducible G_2 representations in (3.1)–(3.4). In particular, it is positive definite on the spaces $(\Omega_1^0, \Omega_7^1, \Omega_7^2, \Omega_1^3)$, and hence (\cdot, \cdot) is positive definite. We therefore have a decomposition of the spaces in (3.5) as

$$\Omega_{\mathbf{r}}^p = \check{H}^p \oplus \check{d}\Omega_{\mathbf{r}'}^{p-1} \oplus \check{d}^\dagger \Omega_{\mathbf{r}''}^{p-1}, \quad (3.23)$$

where \check{H}^p is the space of $\check{\Delta}$ -harmonic p -forms. Of course, with our choice of Laplacian, we have

$$\check{H}^0 \simeq \check{H}^3 \simeq H_1 \simeq \mathbb{R}, \quad \check{H}^1 \simeq \check{H}^2 \simeq H_7 = 0, \quad (3.24)$$

where $H_{\mathbf{r}}$ is the de Rham cohomology group restricted to forms in the G_2 representation \mathbf{r} . The fact that the cohomology group depends only on the representation follows from the equivalent statement for Δ . The final equality holds on any manifold of G_2 holonomy [40].

3.2 A *Spin*(7) complex and Hodge theory

An eight-dimensional *Spin*(7) manifold M is defined by a 4-form Θ which, unlike the G_2 case, is not stable but instead lives in a particular $GL(8)$ orbit. Any such admissible 4-form defines a metric g as in [41] with respect to which we have $\Theta = *\Theta$. We then say

¹⁴Cf. these with the relations given by Bryant [39] using the de Rham differentials, which in the notation of that paper read $(d_{\mathbf{q}}^p)^\dagger = d_{\mathbf{p}}^q$.

¹⁵This is inherited from the $SL(7, \mathbb{R})$ invariance of the inner product.

the $Spin(7)$ structure is integrable if there exists a torsion-free compatible connection¹⁶ which is the case if and only if $d\Theta = 0$. As in the G_2 case, we can decompose the exterior algebra into $Spin(7)$ representations:

$$\Lambda^0 T^* = \Lambda_{\mathbf{1}}^0 T^*, \quad (3.25)$$

$$\Lambda^1 T^* = \Lambda_{\mathbf{8}}^1 T^*, \quad (3.26)$$

$$\Lambda^2 T^* = \Lambda_{\mathbf{7}}^2 T^* \oplus \Lambda_{\mathbf{21}}^2 T^*, \quad (3.27)$$

$$\Lambda^3 T^* = \Lambda_{\mathbf{8}}^3 T^* \oplus \Lambda_{\mathbf{48}}^3 T^*, \quad (3.28)$$

$$\Lambda^4 T^* = \Lambda_{\mathbf{1}}^4 T^* \oplus \Lambda_{\mathbf{7}}^4 T^* \oplus \Lambda_{\mathbf{27}}^4 T^* \oplus \Lambda_{\mathbf{35}}^4 T^*. \quad (3.29)$$

The definition of these spaces along with the relevant projectors are given in Appendix A.2. Following [29], one can define a sequence of maps built from the de Rham differential and suitable projectors:

$$\check{d}: \Omega_{\mathbf{1}}^0(M) \xrightarrow{d} \Omega_{\mathbf{8}}^1(M) \xrightarrow{\mathcal{P}_{\mathbf{7}}^2 d} \Omega_{\mathbf{7}}^2(M) \quad (3.30)$$

which, for integrable $Spin(7)$ structures, defines a complex.

Again, we would like to define an inner product on this complex such that the induced Laplacians $\check{\Delta}$ match the conventional Laplacians Δ evaluated on $\Omega_{\mathbf{r}}^p$, possibly up to an overall scaling (which drops out when evaluating determinants of $\check{\Delta}$). To do so, we adapt the arguments made in [39] for G_2 manifolds and find that

$$d^\dagger \mathcal{P}_{\mathbf{21}}^2 d|_{\Omega_{\mathbf{8}}^1} = 3d^\dagger \mathcal{P}_{\mathbf{7}}^2 d|_{\Omega_{\mathbf{8}}^1}, \quad \mathcal{P}_{\mathbf{7}}^2 d^\dagger \mathcal{P}_{\mathbf{48}}^3 d|_{\Omega_{\mathbf{7}}^2} = \frac{12}{7} \mathcal{P}_{\mathbf{7}}^2 d^\dagger \mathcal{P}_{\mathbf{8}}^3 d|_{\Omega_{\mathbf{7}}^2}. \quad (3.31)$$

One can then use these identities to show

$$\Delta_{\mathbf{1}}^0 = d^\dagger d, \quad \Delta_{\mathbf{8}}^1 = dd^\dagger + 4d^\dagger \mathcal{P}_{\mathbf{7}}^2 d, \quad \Delta_{\mathbf{7}}^2 = 4\mathcal{P}_{\mathbf{7}}^2 dd^\dagger. \quad (3.32)$$

Taking the following inner product on (3.30), it is easy to check that $\check{\Delta}_{\mathbf{r}}^p = \Delta_{\mathbf{r}}^p$ as required:

$$(\alpha, \beta)_p = \kappa_p \int_M \alpha \wedge * \beta, \quad \kappa_p = \begin{cases} 1 & p = 0, 1, \\ 4 & p = 2. \end{cases} \quad (3.33)$$

¹⁶Since Θ defines a metric g , this is equivalent to saying that the Levi-Civita connection is compatible and hence has $Spin(7)$ holonomy.

In terms of the Levi-Civita connection, the \check{d} operator acting on p -forms becomes

$$(\check{d}\omega)_{a_1\dots a_{p+1}} = (p+1)(\mathcal{P}_{\mathbf{r}}^{p+1})_{a_1\dots a_{p+1}}{}^{b_1\dots b_{p+1}}\nabla_{[b_1}\omega_{b_2\dots b_{p+1}]}, \quad (3.34)$$

with the adjoint operator \check{d}^\dagger given by

$$(\check{d}^\dagger\omega)_{a_1\dots a_{p-1}} = -C_p\nabla^b\omega_{ba_1\dots a_{p-1}}, \quad C_p = \begin{cases} 1 & p = 1, \\ 4 & p = 2. \end{cases} \quad (3.35)$$

4 The $G_2 \times G_2$ complex

In the previous section we reviewed the G_2 complex and its Hodge theory in the case where the G_2 structure is torsion-free. In this section, we will give an extension of these ideas which is relevant for type II backgrounds with NSNS flux. In particular, we will see that the relevant geometric structure is that of a torsion-free $G_2 \times G_2$ structure, which naturally gives rise to a double complex and Laplace-type operators that will turn out to capture information about the topological G_2 string. This will be described using the formalism of $O(7,7) \times \mathbb{R}^+$ generalised geometry

Generalised geometry has been of great use for understanding supergravity backgrounds that preserve some amount of supersymmetry and thus admit generalised G -structures. These structures are characterised by the presence of additional objects, usually in the form of globally defined non-vanishing tensors, that reduce the structure group of the generalised tangent bundle from $O(d,d) \times \mathbb{R}^+$ to some subgroup G . For example, the generalised complex / Calabi–Yau structures of Hitchin and Gualtieri [27, 42] are respectively $U(\frac{d}{2}, \frac{d}{2})$ and $SU(\frac{d}{2}, \frac{d}{2})$ structures. These have found many applications in string theory including formulating topological strings [9, 15, 43, 44]. We mostly follow the conventions of [45], and provide a brief review of the key concepts we will be using in Appendix C.

4.1 Generalised $G_2 \times G_2$ structures

Let M be a seven-dimensional Riemannian spin manifold and E its $O(7,7) \times \mathbb{R}^+$ generalised tangent bundle. Introducing an $O(7) \times O(7)$ generalised metric G , or equivalently a Riemannian metric g , a two-form gauge field B and a scalar ϕ , corresponds to specifying an orthogonal decomposition $E = C_+ \oplus C_-$, with each $C_\pm \cong T$. Let us now also assume that there exist two globally defined real spinors $\epsilon_+ \in S(C_+)$ and

$\epsilon_- \in S(C_-)$. Each define a G_2 structure on M given by φ_\pm .¹⁷ When the spinors ϵ_\pm are linearly independent, the G_2 structures are orthogonal and intersect on an $SU(3)$ structure. There may, however, be points on the manifold where the ϵ_\pm align and hence the G_2 structures coincide. In this case, the manifold does not admit a conventional global G -structure. However, within generalised geometry they define a single global generalised $G_2 \times G_2$ structure on E .¹⁸

It turns out that certain supersymmetric backgrounds of string theory compactified to three dimensions can be described by such $G_2 \times G_2$ structures. For concreteness, consider a type IIB NSNS background of the form $\mathbb{R}^{2,1} \times M$ where M is seven-dimensional – this is the case first described in [47]. We have two supersymmetry parameters of opposite chirality ε_\pm which decompose under the reduction $Spin(9, 1) \rightarrow Spin(2, 1) \times Spin(7)$ as

$$\varepsilon_\pm = \zeta_\pm \otimes \epsilon_\pm, \quad (4.1)$$

where ζ_\pm and ϵ_\pm are irreducible $Spin(2, 1)$ and $Spin(7)$ spinors respectively.

For the background to preserve supersymmetry, the variations of the gravitinos and dilatinos under ϵ_\pm must vanish. These conditions give the Killing spinor equations for the supersymmetry parameters. Under the decomposition (4.1), for vanishing RR fields these equations impose that ζ_\pm is a constant spinor on $\mathbb{R}^{2,1}$, and on M we need

$$\begin{aligned} (\gamma^\mu \partial_\mu \phi \mp \frac{1}{12} \gamma^{\mu\nu\rho} H_{\mu\nu\rho}) \epsilon_\pm &= 0, \\ (\nabla_\mu \mp \frac{1}{8} \gamma^{\nu\rho} H_{\mu\nu\rho}) \epsilon_\pm &= 0, \end{aligned} \quad (4.2)$$

where the γ_μ are gamma matrices for the $O(7)$ structure defined by g , and ∇ is the associated Levi-Civita connection. As was shown in [47], these equations are satisfied if and only if ϵ_\pm define a generalised torsion-free $G_2 \times G_2$ structure. For generic (g, H, ϕ) , these equations describe a background preserving minimal supersymmetry in three dimensions. However, when H vanishes and ϕ is constant, as must be the case for compact backgrounds [48], these equations imply the preservation of four supercharges or $N = 2$ supersymmetry in three dimensions.

4.2 Torsion-free generalised $G_2 \times G_2$ structures

Recall that one can always find a torsion-free generalised connection that is compatible with the $O(7) \times O(7)$ generalised metric structure on M , giving the analogue of the

¹⁷These are sometimes labelled $G_{2\pm}$. Moving forward we will mostly omit the signs on φ_\pm since they can generally be deduced from the context.

¹⁸Note that this is different from the $SU(7)$ structure defined in [46], which generalises G_2 geometry to M-theory or string backgrounds with RR flux.

Levi-Civita connection in generalised geometry. As we review in Appendix C, this connection is not uniquely defined, but there are certain combinations of it which give a unique generalised Ricci tensor and scalar. The generic form of a generalised Levi-Civita connection D in terms of the background fields is given in (C.26).

We begin by finding the conditions that this generalised Levi-Civita connection must satisfy in order to be compatible with a $G_2 \times G_2$ structure. Since the generalised Levi-Civita connection is torsion-free, the resulting $G_2 \times G_2$ -compatible connection will also be torsion-free. However, unlike generalised metric structures, the existence of such a compatible connection is, in general, obstructed by the intrinsic torsion of the $G_2 \times G_2$ structure. That is, if D is a generalised Levi-Civita connection, the conditions $D\epsilon_+ = D\epsilon_- = 0$ can be solved only if the generalised intrinsic torsion vanishes.

Using similar logic to [49, 50], it can be shown that this constraint is equivalent to the background preserving minimal supersymmetry with vanishing RR fluxes, i.e. that equations (4.2) are satisfied. Using the expression for a generalised Levi-Civita connection given in (C.26), one has that the compatibility conditions which must be imposed are

$$\begin{aligned} D_a \epsilon_+ &= \nabla_a \epsilon_+ - \frac{1}{24} H_{abc} \gamma^{bc} \epsilon_+ - \frac{1}{6} \partial_b \phi \gamma_a^b \epsilon_+ + \frac{1}{4} A_{abc}^+ \gamma^{bc} \epsilon_+ = 0, \\ D_{\bar{a}} \epsilon_+ &= \nabla_{\bar{a}} \epsilon_+ - \frac{1}{8} H_{\bar{a}bc} \gamma^{bc} \epsilon_+ = 0, \end{aligned} \quad (4.3)$$

which ensure that the connection is compatible with the G_2 structure defined by ϵ_+ . There are then similar conditions for compatibility with the G_2 structure defined by ϵ_- .

The second equation should be familiar, as it says that $D_{\bar{a}}$ must act on C_+ as the ϵ_+ -preserving Hull connection ∇^- . This connection exists if and only if the ordinary intrinsic torsion of the G_{2+} structure has no component in the **14** [51]. We can combine the two equations and derive a purely algebraic relation between the components of the generalised connection:

$$X_{abc} \gamma^{bc} \epsilon_+ \equiv \left(\frac{1}{12} H_{abc} \gamma^{bc} - \frac{1}{6} \partial_b \phi \gamma_a^b + \frac{1}{4} A_{abc}^+ \gamma^{bc} \right) \epsilon_+ = 0. \quad (4.4)$$

Note that this equation holds only when X_{abc} acts on the structure-defining spinor, not for a generic spinor. To find the constraints this imposes, we can use G_2 representation theory (for the G_2 factor defined by ϵ_+). In general, X is a 1-form taking values in the 21-dimensional adjoint representation of $Spin(7)$, so under G_2 it decomposes as $X \in \mathbf{7} \times \mathbf{7} + \mathbf{7} \times \mathbf{14}$. The second term gives a 1-form valued in the adjoint of G_2 , i.e. it is the component of X that is compatible with ϵ_+ , and so drops out of (4.4). Therefore, it is the components of X in the $\mathbf{7} \times \mathbf{7}$ that must be set to vanish. Now consider the G_2

decompositions of the fields

$$\partial\phi \in \mathbf{7}, \quad H \in \mathbf{1} + \mathbf{7} + \mathbf{27}, \quad A^+ \in \mathbf{14} + \mathbf{27} + \mathbf{64}. \quad (4.5)$$

One can quickly check that the representations $\mathbf{1}$ and $\mathbf{14}$ occur only in the tensor product $\mathbf{7} \times \mathbf{7}$ while the $\mathbf{64}$ is only in $\mathbf{7} \times \mathbf{14}$, and the remainder may appear in both. As a result, we immediately conclude that (4.4) sets: 1) $A^+|_{\mathbf{14}} = 0$ – recall that the A^+ tensor simply parametrises the freedom one has within the family of generalised Levi-Civita connections, and so this is not a constraint on the background; 2) $H|_{\mathbf{1}} = 0$ – this is an actual constraint on the structure.¹⁹ On the other hand, since the component $A^+|_{\mathbf{64}}$ drops out entirely from (4.4), it is left unconstrained, implying that $G_2 \times G_2$ -compatible torsion-free connections, if they exist, are not unique.

For the $\mathbf{7}$ components, we isolate the relevant terms by writing

$$\partial_a\phi|_{\mathbf{7}} = \partial_a\phi, \quad H_{abc}|_{\mathbf{7}} = (*\varphi)_{abc}{}^d H_d, \quad (4.6)$$

which gives

$$\left(\frac{1}{12}(*\varphi)_{ad}{}^{bc} H^d \gamma_{bc} - \frac{1}{6}\partial_b\phi\gamma_a{}^b\right)\epsilon_+ = 0. \quad (4.7)$$

Next we note that $\zeta^T\gamma_{ab}\epsilon_+ \in \mathbf{7}$ for any spinor ζ , and so we can use the expression (A.13) for the projector onto the $\mathbf{7}$ representation to write

$$(*\varphi)_{ab}{}^{cd}\gamma_{cd}\epsilon_+ = 4\gamma_{ab}\epsilon_+. \quad (4.8)$$

We then have

$$\frac{1}{3}\left(H_b - \frac{1}{2}\partial_b\phi\right)\gamma_a{}^b\epsilon_+ = 0. \quad (4.9)$$

This will vanish for $H_a = \frac{1}{2}\partial_a\phi$, which must thus be the choice which is necessary for a $G_2 \times G_2$ -compatible connection.

Next consider the $\mathbf{27}$ components, which we pick out by writing

$$H_{abc}|_{\mathbf{27}} = H_{e[a}\varphi^e{}_{bc]}, \quad A_{abc}^+|_{\mathbf{27}} = A_{ea}^+\varphi^e{}_{bc} - A_{e[a}^+\varphi^e{}_{bc]}, \quad (4.10)$$

where H_{ab} and A_{ab}^+ are symmetric and traceless. Plugging this into the expression for

¹⁹The vanishing of the singlet component of the H flux matches the physical observation that this component of the torsion can be related to the cosmological constant in a supersymmetric background, and so it must be set to zero for the Minkowski solutions that we are considering.

X , we then have

$$\begin{aligned} & \frac{1}{4} \left(\frac{1}{3} H_{e[a} \varphi^e{}_{bc]} + (A_{ea}^+ \varphi^e{}_{bc} - A_{e[a}^+ \varphi^e{}_{bc]}) \right) \gamma^{bc} \epsilon_+ \\ & = \frac{1}{4} (\alpha_{ae} \varphi^e{}_{bc} + \beta_{be} \varphi^e{}_{ca}) \gamma^{bc} \epsilon_+ = 0, \end{aligned} \quad (4.11)$$

where

$$\alpha_{ae} = \frac{1}{9} H_{ae} + \frac{2}{3} A_{ae}^+, \quad \beta_{be} = \frac{2}{9} H_{be} - \frac{2}{3} A_{be}^+. \quad (4.12)$$

To see how these two terms are related, one can contract (4.8) with $\alpha^a{}_e \varphi^{efb}$ and use (A.9) to show that

$$(\alpha_{ae} \varphi^e{}_{bc} + 6\alpha_{be} \varphi^e{}_{ca}) \gamma^{bc} \epsilon^+ = 0. \quad (4.13)$$

Thus the precise combination of the **27**s which appears in (4.11) is $\beta - 6\alpha$, which vanishes for

$$A_{ab}^+ = -\frac{2}{21} H_{ab}. \quad (4.14)$$

This is the choice which is necessary for a connection compatible with ϵ_+ . Note that since we are simply using the freedom in choosing the A^+ tensor to obtain this cancellation, the background flux $H|_{\mathbf{27}}$ is entirely unconstrained, in agreement with the G -structure analysis of [48, 52].

The calculations for compatibility with ϵ_- are analogous, with the result

$$D\epsilon_- = 0 \quad \Leftrightarrow \quad \begin{aligned} \nabla^+ \varphi_- &= 0, & H_{\bar{a}\bar{b}\bar{c}} \varphi^{\bar{a}\bar{b}\bar{c}} &= 0, \\ H_{\bar{a}} &= -\frac{1}{2} \partial_{\bar{a}} \phi, & A_{\bar{a}\bar{b}}^- &= \frac{2}{21} H_{\bar{a}\bar{b}}. \end{aligned} \quad (4.15)$$

The remaining unfixed components of the connection are the parts of A^+ and A^- in the (64, 1) + (1, 64). These simply parametrise the family of torsion-free connections which are compatible with the same $G_2 \times G_2$ structure.

Putting this all together, a compatible, torsion-free $G_2 \times G_2$ -generalised connection takes the form

$$\begin{aligned} D_a v^b &= \nabla_a v^b - \frac{5}{42} \varphi_{bcd} H^d{}_a v^c - \frac{1}{42} \varphi_{cad} H^d{}_b v^c - \frac{1}{42} \varphi_{abd} H^d{}_c v^c \\ &\quad - \frac{1}{12} (*\varphi)_{abcd} \partial^d \phi v^c - \frac{1}{3} \delta_a^b \partial_c \phi v^c + \frac{1}{3} \partial^b \phi v_a + (A^+|_{\mathbf{64}})_a{}^b{}_c v^c, \\ D_{\bar{a}} v^b &= \nabla_{\bar{a}}^- v^b \equiv \nabla_{\bar{a}} v^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c v^c, \\ D_a v^{\bar{b}} &= \nabla_a^+ v^{\bar{b}} \equiv \nabla_a v^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_c v^{\bar{c}}, \\ D_{\bar{a}} v^{\bar{b}} &= \nabla_{\bar{a}} v^{\bar{b}} + \frac{5}{42} \varphi_{\bar{b}\bar{c}\bar{d}} H^{\bar{d}}{}_{\bar{a}} v^{\bar{c}} + \frac{1}{42} \varphi_{\bar{c}\bar{a}\bar{d}} H^{\bar{d}}{}_{\bar{b}} v^{\bar{c}} + \frac{1}{42} \varphi_{\bar{a}\bar{b}\bar{d}} H^{\bar{d}}{}_{\bar{c}} v^{\bar{c}} \\ &\quad - \frac{1}{12} (*\varphi)_{\bar{a}\bar{b}\bar{c}\bar{d}} \partial^{\bar{d}} \phi v^{\bar{c}} - \frac{1}{3} \delta_{\bar{a}}^{\bar{b}} \partial_{\bar{c}} \phi v^{\bar{c}} + \frac{1}{3} \partial^{\bar{b}} \phi v_{\bar{a}} + (A^-|_{\mathbf{64}})_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}}. \end{aligned} \quad (4.16)$$

4.3 The double complex

We now introduce the analogue of the G_2 complex within $O(7,7) \times \mathbb{R}^+$ generalised geometry. Given a $G_2 \times G_2$ structure, we can consider a decomposition of $\Lambda^n E$ into irreducible representations of $G_2 \times G_2$. In particular, we will be interested in the spaces

$$\mathcal{A}_{\mathbf{m},\mathbf{n}}^{p,q} := \Gamma(\Lambda_{\mathbf{m}}^p C_+ \wedge \Lambda_{\mathbf{n}}^q C_-), \quad (4.17)$$

where \mathbf{m} and \mathbf{n} correspond to irreducible $G_{2\pm}$ representations defined by φ_{\pm} . We write (p,q) -forms $\omega \in \mathcal{A}^{p,q}$ as

$$\omega = \frac{1}{p!q!} \omega_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} E^{+a_1 \dots a_p} \otimes E^{-\bar{b}_1 \dots \bar{b}_q}, \quad (4.18)$$

where $\{E^{+a}\}$ and $\{E^{-\bar{b}}\}$ are a basis for C_+ and C_- respectively.

Moreover, using a generalised connection we can build maps between the spaces to give the following diagram

$$\begin{array}{ccccccc}
 & & & \mathcal{A}_{1,1}^{0,0} & & & \\
 & & & \swarrow & & \searrow & \\
 & & & \mathcal{A}_{7,1}^{1,0} & & \mathcal{A}_{1,7}^{0,1} & \\
 & & & \swarrow & & \searrow & \\
 & & & \mathcal{A}_{7,1}^{2,0} & & \mathcal{A}_{7,7}^{1,1} & & \mathcal{A}_{1,7}^{0,2} \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow \\
 & & & \mathcal{A}_{1,1}^{3,0} & & \mathcal{A}_{7,7}^{2,1} & & \mathcal{A}_{7,7}^{1,2} & & \mathcal{A}_{1,1}^{0,3} \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow \\
 & & & \mathcal{A}_{1,7}^{3,1} & & \mathcal{A}_{7,7}^{2,2} & & \mathcal{A}_{7,7}^{1,3} & & \\
 & & & \swarrow & & \searrow & & \swarrow & & \searrow \\
 & & & \mathcal{A}_{1,7}^{3,2} & & \mathcal{A}_{7,7}^{2,3} & & & & \\
 & & & \swarrow & & \searrow & & & & \\
 & & & \mathcal{A}_{1,1}^{3,3} & & & & & &
 \end{array} \quad (4.19)$$

where we have defined

$$(d_+ \omega)_{a_1 \dots a_{p+1} \bar{a}_1 \dots \bar{a}_q} = (p+1) (\mathcal{P}_{\mathbf{m}}^+)_{a_1 \dots a_{p+1}}{}^{b_1 \dots b_{p+1}} D_{b_1} \omega_{b_2 \dots b_{p+1} \bar{a}_1 \dots \bar{a}_q}, \quad (4.20)$$

$$(d_- \omega)_{a_1 \dots a_p \bar{a}_1 \dots \bar{a}_{q+1}} = (-1)^p (q+1) (\mathcal{P}_{\mathbf{m}}^-)_{\bar{a}_1 \dots \bar{a}_{q+1}}{}^{\bar{b}_1 \dots \bar{b}_{q+1}} D_{\bar{b}_1} \omega_{a_1 \dots a_p \bar{b}_2 \dots \bar{b}_{q+1}}, \quad (4.21)$$

where $\omega \in \mathcal{A}_{\mathbf{m},\mathbf{n}}^{p,q}$, and $\mathcal{P}_{\mathbf{m}}^{\pm}$ are the projectors onto the relevant $G_{2\pm}$ representation as given in Appendix A.2. Here we assume that D is a $G_2 \times G_2$ -compatible connection so that it commutes with the projectors – such a connection always exists (though it may

not be torsion-free).

We now ask when (4.19) is actually a double complex. That is, when do we have

$$d_{\pm}^2 = 0, \quad d_+ d_- + d_- d_+ = 0. \quad (4.22)$$

We will show that a sufficient condition is that the $G_2 \times G_2$ structure is torsion-free, which corresponds physically to a supersymmetric NSNS Minkowski background. Then we can take the generalised connections in (4.20) to be of the form (4.16).

At first sight this statement might worry the reader – since these connections are not uniquely determined, it would seem that we need some extra information (beyond that of the supergravity background) to further constrain the connection, as otherwise the operators might not be uniquely defined. As we mentioned earlier, the generalised Levi-Civita connection is also not unique however one can construct unique operators from it, such as the generalised versions of the Ricci tensor and scalar. Something similar happens here, namely the d_{\pm} operators do not depend on the undetermined $(\mathbf{64}, \mathbf{1}) + (\mathbf{1}, \mathbf{64})$ components of the connection and so they are actually unique, i.e. they depend only on the data of the torsion-free $G_2 \times G_2$ structure itself. To see this, note that the double complex consists solely of maps between the $G_2 \times G_2$ representations $(\mathbf{1}, \mathbf{1})$, $(\mathbf{7}, \mathbf{1})$, $(\mathbf{1}, \mathbf{7})$ and $(\mathbf{7}, \mathbf{7})$. Now, simple representation theory tells us that the $(\mathbf{64}, \mathbf{1})$ or $(\mathbf{1}, \mathbf{64})$ cannot give such maps. In other words, any tensor transforming in those representations must be projected out. Therefore, we can compute the double complex with any choice of $A^{\pm}|_{\mathbf{64}}$ tensor in (4.16) and obtain a unique answer.

Another useful result for the torsion-free case is that one may use the following “simplified” connection to define the d_{\pm} operators:

$$\begin{aligned} \hat{D}_a v^b &= \nabla_a v^b, \\ \hat{D}_{\bar{a}} v^b &= \nabla_{\bar{a}}^- v^b = \nabla_{\bar{a}} v^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c v^c, \\ \hat{D}_a v^{\bar{b}} &= \nabla_a^+ v^{\bar{b}} = \nabla_a v^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}}, \\ \hat{D}_{\bar{a}} v^{\bar{b}} &= \nabla_{\bar{a}} v^{\bar{b}}, \end{aligned} \quad (4.23)$$

where the Hull connections ∇^{\mp} are assumed to preserve the G_2 structures φ_{\pm} . As a generalised connection \hat{D} is neither torsion-free nor is it compatible with the $G_2 \times G_2$ structure, and yet the operators d_{\pm} defined from it coincide with the ones defined using D . Remarkably, this means that in the torsion-free case, the double complex can be described using just the ordinary Levi-Civita and Hull connections.

To verify this, take for example $\alpha \in \mathcal{A}_{7,7}^{1,1}$, and let D be a generalised Levi-Civita

connection of the form (C.26). Then we have

$$\begin{aligned}
\frac{1}{2}(\mathrm{d}_+^D \alpha - \mathrm{d}_+^{\hat{D}} \alpha)_{ab\bar{a}} &= \mathcal{P}_{ab}{}^{cd}(D_c - \hat{D}_c)\alpha_{d\bar{a}} \\
&= \mathcal{P}_{ab}{}^{cd}\left(\frac{1}{6}H_c{}^e{}_d\alpha_{e\bar{a}} - \frac{1}{3}\partial_c\phi\alpha_{d\bar{a}} - (A^+)_c{}^e{}_d\alpha_{e\bar{a}}\right) \\
&= \mathcal{P}_{ab}{}^{cd}\left(\frac{1}{6}H\varphi_c{}^e{}_d\alpha_{e\bar{a}} - \frac{1}{6}H^f(*\varphi)_{cdef}\alpha_{\bar{a}}^e - \frac{1}{6}H^f{}_{[c}\varphi_{de]}f\alpha_{\bar{a}}^e\right. \\
&\quad \left.- \frac{1}{3}\partial_c\phi\alpha_{d\bar{a}} + (A^+)^f{}_{c\varphi def}\alpha_{\bar{a}}^e - (A^+)^f{}_{[c}\varphi_{de]}f\alpha_{\bar{a}}^e\right) \\
&= \frac{1}{6}H\varphi_a{}^e{}_b\alpha_{e\bar{a}} + \frac{1}{3}\mathcal{P}_{ab}{}^{cd}(2H_c - \partial_c\phi)\alpha_{d\bar{a}} \\
&\quad - \frac{1}{18}(H^{cd} - \frac{21}{2}(A^+)^{cd})\varphi_{abc}v_d.
\end{aligned} \tag{4.24}$$

The difference between the operators vanishes precisely when D is a $G_2 \times G_2$ -compatible, torsion-free connection (these are the same conditions we found in the previous section). It should also be clear from this calculation that one could consider the action on any other element of the complex (4.19) and obtain analogous constraints. Thus, the two operators coincide if and only if the generalised structure has vanishing torsion. Assuming this is the case, we see that the operators agree and so we are free to use \hat{D} to define d_\pm .

This simplified connection makes checking the nilpotency conditions (4.22) substantially easier. First consider d_+^2 . We have that the simplified connection satisfies

$$\begin{aligned}
[\hat{D}_{[a_1}, \hat{D}_{a_2}]\omega_{a_3\dots a_{p+2}}\bar{b}_1\dots\bar{b}_q &= -p\mathcal{R}_{[a_1a_2}{}^e{}_{a_3}\omega_{|e|a_4\dots a_k}\bar{b}_1\dots\bar{b}_q - q\mathcal{R}_{[a_1a_2]}^+{}^{\bar{c}}{}_{[\bar{b}_1}\omega_{|a_3\dots a_k]|^{\bar{c}}\bar{b}_1\dots\bar{b}_q} \\
&= -q\mathcal{R}_{[a_1a_2]}^+{}^{\bar{c}}{}_{[\bar{b}_1}\omega_{|a_3\dots a_k]|^{\bar{c}}\bar{b}_1\dots\bar{b}_q},
\end{aligned} \tag{4.25}$$

where we have used (A.2) to write the commutator of connections in terms of curvatures. Now notice that because ∇^+ is compatible with φ_- it follows that $\mathcal{R}^+ \in \Lambda^2 T \otimes \mathfrak{g}_2^-$, and similarly $\mathcal{R}^- \in \Lambda^2 T \otimes \mathfrak{g}_2^+$. But since $\mathrm{d}H = 0$, we have that $\mathcal{R}_{a_1a_2b_1b_2}^+ = \mathcal{R}_{b_1b_2a_1a_2}^-$ and so actually $\mathcal{R}^+ \in \mathfrak{g}_2^+ \otimes \mathfrak{g}_2^-$ (and $\mathcal{R}^- \in \mathfrak{g}_2^- \otimes \mathfrak{g}_2^+$). Therefore, (4.25) vanishes when the projectors in the definition of d_+ are applied to it. So for $\omega \in \mathcal{A}^{0,q}$ one has

$$\begin{aligned}
\frac{1}{2}(\mathrm{d}_+^2 \omega)_{a_1a_2\bar{b}_1\dots\bar{b}_q} &= \mathcal{P}_{a_1a_2}{}^{c_1c_2}D_{c_1}D_{c_2}\omega_{\bar{b}_1\dots\bar{b}_q} \\
&= \mathcal{P}_{a_1a_2}{}^{c_1c_2}\hat{D}_{[c_1}\hat{D}_{c_2]}\omega_{\bar{b}_1\dots\bar{b}_q} = 0,
\end{aligned} \tag{4.26}$$

and if $\omega \in \mathcal{A}^{1,q}$

$$\begin{aligned}
\frac{1}{6}(d_+^2 \omega)_{a_1 a_2 a_3 \bar{b}_1 \dots \bar{b}_q} &= \mathcal{P}_{a_1 a_2 a_3}{}^{d_1 d_2 d_3} D_{d_1} \mathcal{P}_{d_2 d_3}{}^{c_1 c_2} \hat{D}_{c_1} \omega_{c_2 \bar{b}_1 \dots \bar{b}_q} \\
&= \mathcal{P}_{a_1 a_2 a_3}{}^{d_1 d_2 d_3} \mathcal{P}_{d_2 d_3}{}^{c_1 c_2} D_{d_1} \hat{D}_{c_1} \omega_{c_2 \bar{b}_1 \dots \bar{b}_q} \\
&= \mathcal{P}_{a_1 a_2 a_3}{}^{c_1 c_2 c_3} D_{c_1} \hat{D}_{c_2} \omega_{c_3 \bar{b}_1 \dots \bar{b}_q} \\
&= \mathcal{P}_{a_1 a_2 a_3}{}^{c_1 c_2 c_3} \hat{D}_{[c_1} \hat{D}_{c_2} \omega_{c_3] \bar{b}_1 \dots \bar{b}_q} = 0,
\end{aligned} \tag{4.27}$$

where we have used that the projectors commute with the compatible connection D and that $(\mathcal{P}_1)_{a_1 a_2 a_3}{}^{b_1 b_2 b_3} (\mathcal{P}_7)_{b_1 b_2}{}^{c_1 c_2} = (\mathcal{P}_1)_{a_1 a_2 a_3}{}^{c_1 c_2 b_3}$. Obviously, if $\omega \in \mathcal{A}^{p>1,q}$ then $d_+^2 \omega = 0$ trivially, and one can repeat this reasoning to also conclude that $d_-^2 = 0$.

To see that $\{d_+, d_-\} = 0$, consider first $\alpha \in \mathcal{A}_{7,7}^{1,1}$. Then

$$\begin{aligned}
[\hat{D}_a, \hat{D}_{\bar{b}}] \alpha_{c\bar{d}} &= [\nabla_a, \nabla_{\bar{b}}] \alpha_{c\bar{d}} + \frac{1}{2} (\nabla_a H_{\bar{b}}{}^e{}_c) \alpha_{e\bar{d}} + \frac{1}{2} (\nabla_{\bar{b}} H_a{}^{\bar{e}}{}_{\bar{d}}) \alpha_{c\bar{e}} \\
&\quad (1-1) \frac{1}{2} H_{\bar{b}}{}^e{}_c \alpha_{e\bar{d}} + (1-1) \frac{1}{2} H_a{}^{\bar{e}}{}_{\bar{d}} \nabla_{\bar{b}} \alpha_{c\bar{e}} \\
&\quad - \frac{1}{2} (H_a{}^{\bar{e}}{}_{\bar{b}} \nabla_{\bar{e}} \alpha_{c\bar{d}} + H_{\bar{b}}{}^e{}_a \nabla_e \alpha_{c\bar{d}}) \\
&= - \left(\mathcal{R}_{ab}{}^e{}_c - \frac{1}{2} \nabla_a H_{\bar{b}}{}^e{}_c + \frac{1}{4} H_a{}^{\bar{f}}{}_{\bar{b}} H_{\bar{f}}{}^e{}_c \right) \alpha_{e\bar{d}} \\
&\quad - \left(\mathcal{R}_{a\bar{b}}{}^{\bar{e}}{}_{\bar{d}} - \frac{1}{2} \nabla_{\bar{b}} H_a{}^{\bar{e}}{}_{\bar{d}} + \frac{1}{4} H_{\bar{b}}{}^f{}_a H_f{}^{\bar{e}}{}_{\bar{d}} \right) \alpha_{c\bar{e}}.
\end{aligned} \tag{4.28}$$

Antisymmetrising on $[ac]$ and $[\bar{b}\bar{d}]$, this becomes

$$-\frac{1}{2} \mathcal{R}_{ac}{}^e{}_{\bar{b}} \alpha_{e\bar{d}} + \frac{1}{2} \mathcal{R}_{\bar{b}\bar{d}}{}^{\bar{e}}{}_a \alpha_{c\bar{e}}. \tag{4.29}$$

Again, due to the representations that \mathcal{R}^+ and \mathcal{R}^- live in, this is projected out in $\{d_+, d_-\} \alpha$.

To show that on an element $\beta \in \mathcal{A}_{7,1}^{2,0}$ we also have $\{d_+, d_-\} \beta = 0$, it is actually simpler to use a torsion-free compatible connection. One then has

$$\varphi^{abc} (D_a D_{\bar{a}} - D_{\bar{a}} D_a) \beta_{bc} = [D_a, D_{\bar{a}}] \varphi^{abc} \beta_{bc} = R_{a\bar{a}}^0 \varphi^{abc} \beta_{bc} = 0, \tag{4.30}$$

where in the first equality we used compatibility to commute the G_2 3-form through the derivatives, then we used the definition of the generalised Ricci tensor of a torsion-free connection (C.28), and finally we used the fact that generalised torsion-free $G_2 \times G_2$ manifolds are generalised Ricci-flat.

The action of $\{d_+, d_-\}$ on the remaining spaces of the double complex can be computed similarly to these two examples and leads to the same result. One can therefore conclude that if a generalised $G_2 \times G_2$ structure has vanishing intrinsic torsion, then there exists a double complex of the form (4.19).

4.4 Hodge theory

Let us now see how several concepts familiar from the conventional G_2 complex naturally generalise to the $G_2 \times G_2$ double complex.

We start by defining the adjoint operators in direct analogy with Section 3.1. Consider two tensors in $\mathcal{A}^{1,0}$

$$\alpha = \alpha_a E^{+a}, \quad \beta = \beta_a E^{+a}. \quad (4.31)$$

We can define an inner product between them as

$$\begin{aligned} (\alpha, \beta) &= \int_M \Phi G(\alpha, \beta) = \int_M \Phi \alpha_a \beta_b \eta(E^{+a}, E^{+b}) \\ &= \int_M \Phi \alpha_a \beta_b \eta^{ab} = \int_M e^{-2\phi} \text{vol } \alpha \lrcorner \beta, \end{aligned} \quad (4.32)$$

where both Φ and G are $O(d) \times O(d)$ invariant, so that the inner product is also invariant. More generally for $\alpha, \beta \in \mathcal{A}^{p,q}$ one has

$$\int_M \Phi G(\alpha, \beta) = \kappa_p \kappa_q \int_M e^{-2\phi} \text{vol } \alpha \lrcorner \beta, \quad (4.33)$$

where the constants κ_p and κ_q are defined by

$$\kappa_p = \begin{cases} 1 & p = 0, 1, \\ 3 & p = 2, \\ 7 & p = 3. \end{cases} \quad (4.34)$$

This ensures that this inner product agrees with the usual inner product for each G_2 factor. Given this definition, one can check that the adjoint operators, defined by $(\alpha, d_{\pm} \beta) = (d_{\pm}^{\dagger} \alpha, \beta)$, act on (p, q) -forms as

$$(d_{+}^{\dagger} \alpha)_{c_2 \dots c_p \bar{d}_1 \dots \bar{d}_q} = -\gamma_p D^{c_1} \alpha_{c_1 \dots c_p \bar{d}_1 \dots \bar{d}_q}, \quad (4.35)$$

$$(d_{-}^{\dagger} \alpha)_{c_1 \dots c_p \bar{d}_2 \dots \bar{d}_q} = \gamma_q (-1)^{p+1} D^{\bar{d}_1} \alpha_{c_1 \dots c_p \bar{d}_1 \dots \bar{d}_q}, \quad (4.36)$$

where

$$\gamma_p = \begin{cases} 1 & p = 1, \\ 3 & p = 2, \\ \frac{7}{3} & p = 3. \end{cases} \quad (4.37)$$

In this way, the adjoint operators inherit many of the properties of the those for the usual G_2 complex, as described in Section 3.1. For example, using (3.18), acting on a tensor $\alpha \in \mathcal{A}^{3,1}$, we have that $d_+^\dagger \alpha = (\theta_+)_{\mathbf{7}} d_+ (\theta_+)_{\mathbf{1}}^{-1} \alpha$.

4.4.1 Kähler identities

There exist useful anticommutation relations between the differentials and their adjoints, which are the $G_2 \times G_2$ analogues of the Kähler identities of the Dolbeault complex. Taking $\lambda \in \mathcal{A}^{1,0}$, in components we have that

$$(d_+^\dagger d_- \lambda)_{\bar{a}} = D^a D_{\bar{a}} \lambda_a, \quad (d_- d_+^\dagger \lambda)_{\bar{a}} = -D_{\bar{a}} D^a \lambda_a, \quad (4.38)$$

and so

$$(d_+^\dagger d_- \lambda + d_- d_+^\dagger \lambda)_{\bar{b}} = [D_a, D_{\bar{b}}] \lambda^a = R_{\bar{a}b}^0 \lambda^a \equiv 0, \quad (4.39)$$

since the generalised Ricci tensor vanishes for a torsion-free $G_2 \times G_2$ structure. Note that this is essentially the same calculation as (4.30). Indeed, because of the isomorphisms (3.18), the Kähler identities are automatically implied by the anticommutation relations of the d_\pm operators, and vice-versa. For example, acting on $\mu \in \mathcal{A}^{2,0}$ satisfying $\mu = (\theta_+)_{\mathbf{7}} \lambda$ for some $\lambda \in \mathcal{A}^{1,0}$, we have

$$(d_+^\dagger d_- + d_- d_+^\dagger) \mu = (d_+^\dagger d_- + d_- d_+^\dagger) (\theta_+)_{\mathbf{7}} \lambda = (\theta_+)_{\mathbf{7}}^{-1} (d_+ d_- + d_- d_+) \lambda = 0, \quad (4.40)$$

since $\{d_+, d_-\} \lambda = 0$.

We conclude that the differentials and their adjoints over the $G_2 \times G_2$ complex satisfy

$$d_\pm^2 = (d_\pm^\dagger)^2 = \{d_\pm, d_\mp\} = \{d_\pm^\dagger, d_\mp^\dagger\} = 0. \quad (4.41)$$

4.4.2 Laplacians

We define Laplacians for both the “plus” and “minus” differentials as usual:

$$\Delta_\pm = d_\pm^\dagger d_\pm + d_\pm d_\pm^\dagger. \quad (4.42)$$

Much as for the G_2 complex in Section 3.1, it follows from the properties of the adjoints that the Laplacians depend only on the $G_2 \times G_2$ representation of the object on which they act and not on the (p, q) degree of the form. For instance, taking $\alpha \in \mathcal{A}_{\mathbf{7},\mathbf{1}}^{2,3}$, there

exists some $\beta \in \mathcal{A}_{\mathbf{7}, \mathbf{1}}^{1,0}$ such that $\alpha = (\theta_+)_{\mathbf{7}}(\theta_-)_{\mathbf{1}}\beta$. Then

$$\begin{aligned}
\Delta_+ \alpha &= \left(d_+^\dagger d_+ + d_+ d_+^\dagger \right) (\theta_+)_{\mathbf{7}} (\theta_-)_{\mathbf{1}} \beta = (\theta_-)_{\mathbf{1}} \left(d_+^\dagger d_+ + d_+ d_+^\dagger \right) (\theta_+)_{\mathbf{7}} \beta \\
&= (\theta_-)_{\mathbf{1}} \left(d_+^\dagger (\theta_+)_{\mathbf{7}}^{-1} d_+^\dagger + d_+ (\theta_+)_{\mathbf{7}}^{-1} d_+ \right) \beta \\
&= (\theta_-)_{\mathbf{1}} (\theta_+)_{\mathbf{7}} \left(d_+ d_+^\dagger + d_+^\dagger d_+ \right) \beta = (\theta_-)_{\mathbf{1}} (\theta_+)_{\mathbf{7}} \Delta_+ \beta.
\end{aligned} \tag{4.43}$$

Note as well that if we consider the combined differential $\hat{d} = d_+ + d_-$, the Kähler identities imply that its Laplacian coincides with the sum of the Laplacians for d_+ and d_- :

$$\hat{\Delta} = \hat{d}^\dagger \hat{d} + \hat{d} \hat{d}^\dagger = \Delta_+ + \Delta_- . \tag{4.44}$$

We will now show that the Δ_+ and Δ_- Laplacians are in fact equal, as is the case for Δ_∂ and $\Delta_{\bar{\partial}}$ of the Dolbeault complex. Considering first $f \in \mathcal{A}^{0,0}$, we have that

$$\begin{aligned}
\Delta_+ f &= d_+^\dagger d_+ f = -D^a D_a f = -\nabla^2 f + 2\partial^a \phi \nabla_a f, \\
\Delta_- f &= d_-^\dagger d_- f = -D^{\bar{a}} D_{\bar{a}} f = -\nabla^2 f + 2\partial^{\bar{a}} \phi \nabla_{\bar{a}} f,
\end{aligned} \tag{4.45}$$

and so $\Delta_+ f = \Delta_- f$. Now take $\lambda \in \mathcal{A}^{1,0}$. For Δ_- we have

$$\begin{aligned}
(\Delta_- \lambda)_b &= (d_-^\dagger d_- \lambda)_b = -D^{\bar{a}} D_{\bar{a}} \lambda_b \\
&= -\nabla^2 \lambda_b - H_{\bar{a}cb} \nabla^{\bar{a}} \lambda^c + 2\partial^{\bar{a}} \phi \nabla_{\bar{a}} \lambda_b + \frac{1}{4} H^{\bar{a}d}{}_b H_{\bar{a}dc} \lambda^c \\
&\quad - \frac{1}{2} (\nabla^{\bar{a}} H_{\bar{a}bc} - 2\partial^{\bar{a}} \phi H_{\bar{a}bc}) \lambda^c.
\end{aligned} \tag{4.46}$$

For the two terms in Δ_+ , we find

$$(d_+ d_+^\dagger \lambda)_b = -D_b D^a \lambda_a = -\nabla_b \nabla^a \lambda_a + 2\partial^a \phi \nabla_b \lambda_a + 2(\nabla_b \nabla_a \phi) \lambda^a, \tag{4.47}$$

and

$$\begin{aligned}
(d_+^\dagger d_+ \lambda)_b &= -2D^a D_{[a} \lambda_{b]} - \varphi_{abcd} D^a D^c \lambda^d \\
&= -\nabla^2 \lambda_b + \nabla^a \nabla_b \lambda_a + 4\partial^a \phi \nabla_{[a} \lambda_{b]} \\
&\quad - \frac{1}{2} (*\varphi)_{bcde} \partial^c \phi \nabla^d \lambda^e - \varphi_{c[bd} H^c{}_{e]} \nabla^d \lambda^e \\
&= -\nabla^2 \lambda_b + \nabla^a \nabla_b \lambda_a + 4\partial^a \phi \nabla_{[a} \lambda_{b]} - H_{bcd} \nabla^c \lambda^d,
\end{aligned} \tag{4.48}$$

where we used the compatibility condition $\frac{1}{2} \partial_a \phi = H_a$ and the G_2 decomposition of H in reverse $H_{abc} = (*\varphi)_{abcd} H^d + \varphi_{e[ab} H^e{}_{c]}$ for the final step. Putting this together, we

can compare the two Laplacians to find

$$\begin{aligned}
(\Delta_- \lambda - \Delta_+ \lambda)_b &= 2(D^a D_{[a} \lambda_{b]} + \frac{1}{2}(*\varphi)_{abcd} D^a D^c \lambda^d) + D_b D^a \lambda_a - D^{\bar{a}} D_{\bar{a}} \lambda_b \\
&= [\nabla_b, \nabla_a] \lambda^a - 2(\nabla_a \nabla_b \phi) \lambda^a + \frac{1}{4} H^{\bar{a}d} H_{\bar{a}dc} \lambda^c \\
&\quad - \frac{1}{2}(\nabla^{\bar{a}} H_{\bar{a}bc} - 2\partial^{\bar{a}} \phi H_{\bar{a}bc}) \lambda^c + H_{\bar{a}bc} \nabla^{\bar{a}} \lambda^c + H_{bcd} \nabla^c \lambda^d \\
&\quad + 2\partial^{\bar{a}} \phi \nabla_{\bar{a}} \lambda_b - 2\partial^a \phi \nabla_b \lambda_a - 2\partial^a \phi \nabla_{[a} \lambda_{b]} \\
&= 0,
\end{aligned} \tag{4.49}$$

which can be checked in a gauge where the C_{\pm} frames are aligned, and using the fact that the equations of motion are automatically satisfied for a generalised $G_2 \times G_2$ background. It should also be clear that the computation of the actions of Δ_{\pm} on an element $\tilde{\lambda} \in \mathcal{A}^{0,1}$ would be entirely symmetrical, and so $\Delta_+ \tilde{\lambda} = \Delta_- \tilde{\lambda}$ as well.

Now let us consider the action of the Laplacians on an element $\zeta \in \mathcal{A}^{1,1}$. Using $(*\varphi)^{cd}{}_{ef} \mathcal{R}_{cdab}^+ = -2\mathcal{R}_{efab}^+$, which follows from $\mathcal{R}^+ \in \mathfrak{g}_2^+ \otimes \mathfrak{g}_2^-$, we perform a similar calculation to find the “plus” Laplacian

$$\begin{aligned}
(\Delta_+ \zeta)_{a\bar{a}} &= -D_a D^b \zeta_{b\bar{a}} - 2D^a D_{[a} \zeta_{b]\bar{a}} - \varphi_{abcd} D^a D^c \zeta^d{}_{\bar{a}} \\
&= -\nabla^2 \zeta_{a\bar{a}} + H_{acb} H^c{}_{\bar{a}\bar{b}} w^{b\bar{b}} + 2\partial^b \phi \nabla_b \zeta_{a\bar{a}} - H_{abc} \nabla^b w^c{}_{\bar{a}} + H_{b\bar{b}\bar{a}} \nabla^b w_a{}^{\bar{b}} \\
&\quad - [\nabla_a, \nabla_b] \zeta^b{}_{\bar{a}} + 2(\nabla_a \nabla_b \phi) \zeta^b{}_{\bar{a}} - \frac{1}{4} H^b{}_{\bar{c}\bar{a}} H_b{}^{\bar{b}\bar{c}} \zeta_{a\bar{b}} \\
&\quad + \mathcal{R}_{ab\bar{b}\bar{a}}^+ \zeta^{b\bar{b}} + \nabla_{[a} H_{b]\bar{b}\bar{a}} \zeta^{b\bar{b}} + \frac{1}{2} H_{[b}{}^{\bar{c}}{}_{|\bar{a}}|} H_{a]\bar{b}\bar{c}} w^{b\bar{b}} \\
&= -\nabla^2 \zeta_{a\bar{a}} + H_{acb} H^c{}_{\bar{a}\bar{b}} \zeta^{b\bar{b}} + 2\mathcal{R}_{ab\bar{b}\bar{a}}^+ \zeta^{b\bar{b}} \\
&\quad + 2\partial^b \phi \nabla_b \zeta_{a\bar{a}} - H_{abc} \nabla^b \zeta^c{}_{\bar{a}} + H_{b\bar{b}\bar{a}} \nabla^b \zeta_a{}^{\bar{b}}.
\end{aligned} \tag{4.50}$$

We can immediately deduce the “minus” Laplacian by exchanging barred and unbarred indices and taking $H \rightarrow -H$:

$$\begin{aligned}
(\Delta_- \zeta)_{a\bar{a}} &= -\nabla^2 \zeta_{a\bar{a}} + H_{\bar{a}\bar{c}\bar{b}} H^{\bar{c}}{}_{ab} \zeta^{b\bar{b}} + 2\mathcal{R}_{\bar{a}\bar{b}\bar{b}a}^- \zeta^{b\bar{b}} \\
&\quad + 2\partial^{\bar{b}} \phi \nabla_{\bar{b}} \zeta_{a\bar{a}} + H_{\bar{a}\bar{b}\bar{c}} \nabla^{\bar{b}} \zeta_a{}^{\bar{c}} - H_{\bar{b}\bar{b}a} \nabla^{\bar{b}} \zeta^b{}_{\bar{a}},
\end{aligned} \tag{4.51}$$

and so we directly observe that $\Delta_+ \zeta = \Delta_- \zeta$ too.

Finally, since the Laplacians depend only on the $G_2 \times G_2$ representation, the cases we have covered are actually sufficient to conclude that over the entire double complex $\Delta_+ = \Delta_- = \frac{1}{2} \hat{\Delta}$.

5 Relation to the topological G_2 string

In this section we will show that the double complex (4.19) is the target-space realisation of the worldsheet BRST complex of the topological G_2 string. Indeed, if one studies the left- and right-moving sectors separately, one finds that the states in the topological theory are the following [10, 11, 14, 17]

$$\begin{array}{cccc} |0, 0\rangle & \left| \frac{1}{10}, \frac{2}{5} \right\rangle & \left| \frac{6}{10}, \frac{2}{5} \right\rangle & \left| \frac{3}{2}, 0 \right\rangle \\ \Omega_1^0 & \Omega_7^1 & \Omega_7^2 & \Omega_1^3 \end{array} . \quad (5.1)$$

Here, the states are labelled as in Section 2 and the second row shows the interpretation of these states as differential forms on the target space. By studying the OPE of the supercurrent G^+ with states of the form $A_{\mu_1 \dots \mu_k}(X)\psi^{\mu_1} \dots \psi^{\mu_k}$, one can show that the left-moving BRST operator $Q_L = G_{-1/2}^\downarrow \sim \check{d}$ acts as

$$\begin{array}{ccccccc} |0, 0\rangle & \xrightarrow{Q_L} & \left| \frac{1}{10}, \frac{2}{5} \right\rangle & \xrightarrow{Q_L} & \left| \frac{6}{10}, \frac{2}{5} \right\rangle & \xrightarrow{Q_L} & \left| \frac{3}{2}, 0 \right\rangle \\ \Omega_1^0 & \xrightarrow{\check{d}} & \Omega_7^1 & \xrightarrow{\check{d}} & \Omega_7^2 & \xrightarrow{\check{d}} & \Omega_1^3 \end{array} \quad (5.2)$$

Similar results hold for the right-moving sector as well, with $Q_R = \bar{G}_{-1/2}^\downarrow$.

The full string states are tensor products of left- and right-moving states from (5.1). As target-space tensors, we find that the string states correspond to

$$\Omega_{\mathbf{m}}^p \otimes \Omega_{\mathbf{n}}^q = \mathcal{A}_{\mathbf{m}, \mathbf{n}}^{p, q}, \quad (5.3)$$

where the spaces $\mathcal{A}_{\mathbf{m}, \mathbf{n}}^{p, q}$ are as in (4.17). Moreover, given (5.2) and the fact that $Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0$, we see that the natural target-space identification of the BRST operators is

$$Q_L \sim d_+, \quad Q_R \sim d_-. \quad (5.4)$$

Physical states then correspond to cohomology classes of $Q = Q_L + Q_R$ in the Hilbert space, or equivalently, harmonic forms under the Laplacian $\hat{\Delta}$ given in (4.44). By the analysis in Section 4.4, we see that these are precisely the harmonic forms of Δ_\pm .

5.1 1-loop partition function

We can use these observations to understand the 1-loop partition function of the topological G_2 string from the target space. As was shown in [14], one can find the 1-loop partition function using the standard formula for the 1-loop free energy of the

topological string [7]:

$$\mathcal{F}_1 = \frac{1}{2} \int \frac{d\tau d\bar{\tau}}{\tau_2} \text{tr}((-1)^F F_L F_R e^{2\pi i \tau H_L - 2\pi i \bar{\tau} H_R}), \quad (5.5)$$

where F_L and F_R are the left- and right-moving fermion number operators respectively, $F = F_L + F_R$ is the total fermion number operator, $H_L = \{Q_L, Q_L^\dagger\}$ is the left-moving Hamiltonian, and similarly for H_R . Taking the domain of τ to be the upper half plane, evaluating the integral gives

$$\mathcal{F}_1 = \frac{1}{2} \delta(H_L - H_R) \log \left[\prod_{F_L, F_R} \det(2\pi(H_L + H_R))^{(-1)^F F_L F_R} \right]. \quad (5.6)$$

Our target space picture provides a clear interpretation of this object. It is precisely the product²⁰

$$\mathcal{F}_1 = \frac{1}{2} \log \left[\prod_{p,q} (\det' \hat{\Delta}^{p,q})^{(-1)^{p+q} pq} \right]. \quad (5.7)$$

It is instructive to compare this with the analogous result for the topological B-model [7, 9]. Indeed, the free energy in (5.7) is of precisely the same form as that for the B-model on a Calabi–Yau threefold, but with the Dolbeault complex replaced with the $G_2 \times G_2$ complex found in the previous section. This striking fact will become important when we consider the topological $Spin(7)$ string in Section 7, about which far less is known.

Using the usual normalisation of the partition function in terms of the free energy, $Z = e^{-\mathcal{F}}$, the corresponding 1-loop partition function is

$$Z_1 = \left[\prod_{p,q} (\det' \hat{\Delta}^{p,q})^{(-1)^{p+q} pq} \right]^{-1/2} \quad (5.8)$$

$$= (\det' \hat{\Delta}_{1,1})^{-9/2} (\det' \hat{\Delta}_{7,1})^{3/2} (\det' \hat{\Delta}_{1,7})^{3/2} (\det' \hat{\Delta}_{7,7})^{-1/2}, \quad (5.9)$$

where in the second line the subscript denotes the $G_2 \times G_2$ representation that $\hat{\Delta}$ acts on, and we have used the fact that the determinant depends only on the representation on which $\hat{\Delta}$ acts. Comparing (5.8) to (2.29), we see that, much like in the A/B-model, the 1-loop partition function calculates the analytic torsion of the G_2 double complex.

In the case of the topological G_2 string, the target manifold has G_2 holonomy

²⁰We are using the ζ -regularised determinant of the Laplacians with zero modes removed, denoted by \det' .

with vanishing H -flux. This means we can further simplify the partition function by considering the diagonal subgroup $G_2 \subset G_2 \times G_2$, and then using the decomposition $\mathbf{7} \times \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27}$. With this, and the fact that $\hat{\Delta} \simeq \Delta$ on these subspaces, the 1-loop partition function of the topological G_2 string is given by

$$Z_1 = (\det' \Delta_{\mathbf{1}})^{-5} (\det' \Delta_{\mathbf{7}})^{5/2} (\det' \Delta_{\mathbf{14}})^{-1/2} (\det' \Delta_{\mathbf{27}})^{-1/2}, \quad (5.10)$$

which exactly matches the expression given in [14]. Much like in the A/B-models, we can read this result off immediately from the double complex, as shown in Figure 2. For the pure G_2 case, we find three independent Laplacians assigned to the faces of the squares in the diamond. These once again correspond to determinants of Laplacians restricted to the subspaces in the Hodge decomposition of $\mathcal{A}^{p,q}$. The partition function is then given by the product of these values with alternating powers of $\pm \frac{1}{2}$ in a checkerboard pattern, as shown in the figure.

We can use the work of Pestun and Witten [9] on the B-model to identify the target-space theory that reproduces this 1-loop expression. An attempt was made in [14] to describe the topological G_2 string in terms of a target-space theory defined by a Hitchin-like functional (see Equation (C.52)), but this did not reproduce the partition function that one calculates from the worldsheet. Rather than starting from an invariant functional, we will simply write down a target-space action whose BV quantisation matches Z_1 .

5.2 A quadratic target-space action

Using our double complex (4.19), and by direct analogy with the Dolbeault complex of complex geometry and Pestun and Witten [9], we propose the following quadratic target-space action:

$$S_0 = \int_M \Phi (\theta_+)_1^{-1} (\theta_-)_1^{-1} \left(\frac{1}{2} b_{11} \wedge d_+ d_- b_{11} + a_{00} \wedge d_+ d_- c_{22} \right) \quad (5.11)$$

$$\stackrel{\text{s.h.}}{\sim} \int_M e^{-2\phi} \text{vol} \varphi^{mnp} \varphi^{qrs} \left(-\frac{1}{2} (b_{11})_{mq} \nabla_n \nabla_r (b_{11})_{ps} + \frac{1}{4} a_{00} \nabla_m \nabla_q (c_{22})_{nprs} \right), \quad (5.12)$$

where the fields a_{00} , b_{11} and c_{22} are real elements of $\mathcal{A}^{p,q}$. The integrand now sits in $\mathcal{A}_{1,1}^{3,3}$ multiplied by the $G_2 \times G_2$ volume form $\Phi = e^{-2\phi} \sqrt{g}$ (which ensures one can integrate by parts). In the second line, we have written the action for the case of G_2 special holonomy, with ∇ denoting the Levi-Civita connection. The volume form and projections will be omitted from hereon and taken as part of the integration measure.

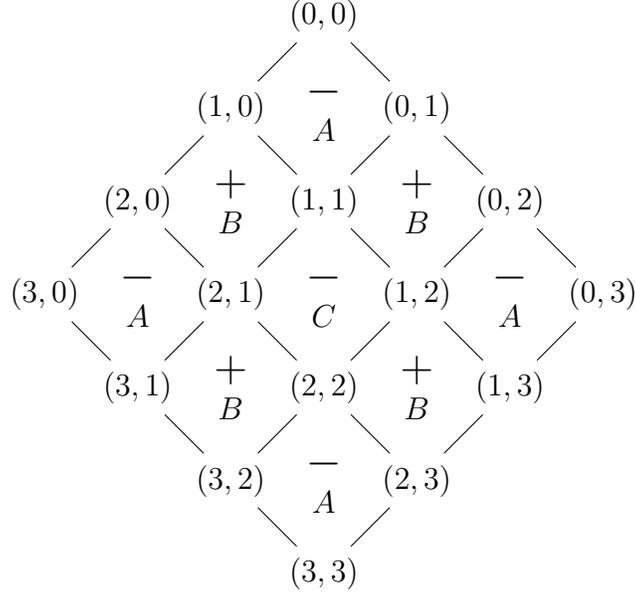


Figure 2. For a global G_2 structure with vanishing flux, equality of Δ_{\pm} and the isomorphisms provided by the 3-form φ mean that $\det' \Delta^{p,q}$ can be expressed in terms of three independent determinants. For example, $\det' \hat{\Delta}^{0,0} = \det' \hat{\Delta}_{1,1} = \det' \Delta_1 \equiv A$, $\det' \hat{\Delta}^{1,0} = \det' \hat{\Delta}_{7,1} = \det' \Delta_7 \equiv AB$, and $\det' \hat{\Delta}^{1,1} = \det' \hat{\Delta}_{7,7} = \det' \Delta_1 \det' \Delta_{21} \det' \Delta_{27} \equiv AB^2C$. The analytic torsion (the 1-loop partition function) is then given by $(A^{-4}B^4C^{-1})^{1/2}$, in agreement with (5.10).

Since in what follows we will be tackling the two terms in the action separately, we will denote the first term involving c_{11} by S_0^a , and the second term by S_0^b .

The idea then is that the partition function of this theory should match the 1-loop partition function calculated in (5.9). We can compute this partition function in two ways. The first is to use the standard BRST-BV quantisation procedure [53], in analogy with [9]. We will follow this path here, assuming all relevant cohomologies of M vanish to simplify the presentation. The second approach is by direct computation, as also demonstrated in [9]. Given a quadratic action, this method is usually robust and perhaps more illustrative if the reader is unfamiliar with the BV approach. We illustrate this computation in Appendix B.2.

$b_{11} \wedge d_+ d_- b_{11}$

We begin by considering the first term S_0^a involving the field b_{11} and constructing the BV action. The gauge symmetries of this term lead to the following BRST transformations:

$$\begin{aligned} Qb_{11} &= d_+ b_{01} + d_- b_{10}, \\ Qb_{10} &= d_+ b_{00}, \\ Qb_{01} &= d_- b_{00}. \end{aligned} \tag{5.13}$$

These are similar to the transformations of [9] but without any reality constraints as the fields involved are real. We have introduced ghosts b_{10} and b_{01} , and a ghost for ghosts b_{00} . The fields b_{pq} have statistics $(-1)^{(p+q)}$.

As in [9] we introduce antifields. The antifield of b_{pq} is a field $b_{(3-p)(3-q)}^*$ of ghost number $p + q - 3$ and statistics $(-1)^{p+q+1}$. The master action then takes the form

$$S = S_0^a + \sum_{p,q} \int_M b_{(3-p)(3-q)}^* \wedge Qb_{pq}. \tag{5.14}$$

This action reduces to S_0^a when the antifields are zero, and satisfies the usual requirements that

$$\{S, S\} = 0, \quad \{S, \Psi_i\} = Q\Psi_i, \tag{5.15}$$

for any field Ψ_i , where the antibracket between two functionals F and G is given by

$$\{F, G\} = \sum_i \left(\frac{\delta F}{\delta \Psi_i} \cdot \frac{\delta G}{\delta \Psi_i^*} - \frac{\delta F}{\delta \Psi_i^*} \cdot \frac{\delta G}{\delta \Psi_i} \right), \tag{5.16}$$

where Ψ_i^* is the antifield for Ψ_i . Using the master action, one can also derive the BRST transformations of the antifields via $Q\Psi_i^* = \{S, \Psi_i^*\}$:

$$\begin{aligned} Qb_{22}^* &= d_+ d_- b_{11}, \\ Qb_{23}^* &= d_- b_{22}^*, \\ Qb_{32}^* &= d_+ b_{32}^*. \end{aligned} \tag{5.17}$$

Next one chooses a Lagrangian submanifold. We choose this so as to remove the kernels of kinetic terms in the master action. This is done by projecting each field onto a subspace orthogonal to its variation under gauge transformations. From the classical

part S_0^a , the Hodge decomposition implies that we should set

$$b_{11} = d_+^\dagger d_-^\dagger d_{22}. \quad (5.18)$$

The term involving fermionic ghosts and antifields reads

$$S_1^a = \int_M b_{22}^* \wedge (d_+ b_{01} + d_- b_{10}). \quad (5.19)$$

Note that d_+ acting on b_{01} has no kernel (assuming vanishing cohomologies), and likewise for d_- on b_{10} . This implies that we can decompose b_{01} and b_{10} as

$$\begin{aligned} b_{01} &= d_+^\dagger d_{02} + d_- d_{00}, \\ b_{10} &= d_+^\dagger d_{20} + d_+ \tilde{d}_{00}. \end{aligned} \quad (5.20)$$

Plugging this into (5.19), we see that terms involving the adjoint operators are orthogonal and so cannot cancel. However, terms involving the differentials cancel when $\tilde{d}_{00} = d_{00}$. To remove the kernel, we should set $\tilde{d}_{00} = -d_{00}$. Finally, the bosonic action involving ghosts of ghosts reads

$$S_2^a = \int_M (b_{23}^* \wedge d_+ b_{00} + b_{32}^* \wedge d_- b_{00}). \quad (5.21)$$

Again assuming vanishing cohomologies, this term puts no constraints on b_{00} .

The antifields are constrained by demanding

$$\sum_i \int_M \Psi_i^* \wedge \Psi_i = 0. \quad (5.22)$$

As b_{00} is unconstrained, we are forced to set $b_{33}^* = 0$. From the constraint on b_{11} , we can derive a constraint on b_{22}^* :

$$b_{22}^* = d_+^\dagger d_{32}^* + d_-^\dagger d_{23}^*. \quad (5.23)$$

The constraints on b_{23}^* and b_{32}^* come from requiring that

$$\int_M (b_{32}^* \wedge b_{01} + b_{23}^* \wedge b_{10}) = 0. \quad (5.24)$$

This holds provided we set

$$b_{32}^* = d_-^\dagger d_{33}, \quad b_{23}^* = d_+^\dagger d_{33}. \quad (5.25)$$

To show this requires an integration by parts, and the fact that the Laplacians Δ_+ and Δ_- are equal.

We now compute contribution to the partition function of each term in the master action of S_0^a . For the classical term, the result is $(\det' d_+ d_-)^{1/2}$ as the term is quadratic in b_{11} , which is bosonic. Note however that with the projection onto the Lagrangian submanifold, the operator $d_+ d_-$ should be thought of as acting on $d_+^\dagger d_-^\dagger$ -exact $(1, 1)$ -forms. The determinant is then

$$(\det'(d_+ d_-))^{-1/2} = \left(\det'(d_+^\dagger d_-^\dagger d_+ d_-) \right)^{-1/4} = \left(\det' \hat{\Delta}_{\bullet}^{1,1} \right)^{-1/2}, \quad (5.26)$$

as again the Laplacians are equal. The dot below denotes the fact that we are acting on $d_+^\dagger d_-^\dagger$ -exact forms, as explained in Appendix B.1. This is also the determinant we have referred to as C in Figure 2.

Next, let us compute the partition function of the bosonic action S_2^a involving ghosts and ghosts of ghosts. We can write the ghost of ghost action as

$$\int_M (b_{23}^* + b_{32}^*) \wedge (d_+ + d_-) b_{00}. \quad (5.27)$$

With the projection onto the Lagrangian submanifold, the result is

$$(\det'(d_+ + d_-))^{-1} = \left(\det' \hat{\Delta}_{\bullet}^{0,0} \right)^{-1/2} = A^{-1/2}. \quad (5.28)$$

Finally, the fermionic ghost term reads

$$S_1^a = \int_M (b_{01} + b_{10}) \wedge (d_+ + d_-) b_{22}^*. \quad (5.29)$$

The Lagrangian submanifold projects b_{22}^* onto $\bullet \mathcal{A}^{2,2} \oplus \hat{\mathcal{A}}_{\bullet}^{2,2} \oplus \mathcal{A}^{\bullet,2,2}$, as defined in Appendix B.1. The contribution to the partition function of this term is therefore

$$\det'(d_+ + d_-) = \left(\det' \hat{\Delta}_{\bullet}^{2,2} \det' \hat{\Delta}_{\bullet}^{2,2} \det' \hat{\Delta}_{\bullet}^{\bullet,2,2} \right)^{1/2} = (AB'B)^{1/2}, \quad (5.30)$$

where in the general $G_2 \times G_2$ case with flux there is a potential asymmetry between B and B' when mirroring the diagram of Figure 2 about the vertical axis. In this case we

denote the Laplacians of the middle upper-left square and the middle lower-right square B , and the middle upper-right and lower-left squares are denoted B' . For a global G_2 structure, one has $B = B'$, which is the case shown in Figure 2. Putting this together, the partition function of the term S_0^a in the classical action is

$$Z^a = \left(\frac{BB'}{C} \right)^{1/2}. \quad (5.31)$$

$a_{00} \wedge \mathbf{d}_+ \mathbf{d}_- b_{22}$

Next we consider the term S_0^b in the classical action. The BRST transformations read

$$\begin{aligned} Qc_{22} &= \mathbf{d}_+ c_{12} + \mathbf{d}_- c_{21}, & Qc_{20} &= \mathbf{d}_+ c_{10}, \\ Qc_{12} &= \mathbf{d}_+ c_{02} + \mathbf{d}_- c_{11}, & Qc_{02} &= \mathbf{d}_- c_{01}, \\ Qc_{21} &= \mathbf{d}_- c_{20} + \mathbf{d}_+ c_{11}, & Qc_{10} &= \mathbf{d}_+ c_{00}, \\ Qc_{11} &= \mathbf{d}_+ c_{01} + \mathbf{d}_- c_{10}, & Qc_{01} &= \mathbf{d}_+ c_{00}, \end{aligned} \quad (5.32)$$

while the field a_{00} is gauge invariant. For each field and ghost c_{pq} we again introduce an antifield $c_{(3-p)(3-q)}^*$ of statistics $(-1)^{p+q+1}$ and ghost number $p + q - 5$. We also introduce a fermionic antifield a_{33}^* for the field a_{00} .

The master action is now given as

$$S = S_0^b + \sum_{p,q} \int_M c_{(3-p)(3-q)}^* \wedge Qc_{pq}, \quad (5.33)$$

which is easily checked to satisfy $\{S, S\} = 0$, and generates the BRST transformations as $Qc_{pq} = \{S, c_{pq}\}$. Similarly the BRST transformations of the antifields are given as $Qc_{pq}^* = \{S, c_{pq}^*\}$ and $Qa_{33}^* = \{S, a_{33}^*\}$. We proceed by introducing a Lagrangian submanifold to project out kernels of kinetic terms in the master action. This is again done by projecting each field onto a subspace orthogonal to its variation under gauge transformations. Assuming vanishing cohomologies, for the classical fields c_{22} and a_{00} we get

$$c_{22} = \mathbf{d}_+^\dagger \mathbf{d}_-^\dagger d_{33}, \quad (5.34)$$

or $c_{22} \in \mathcal{A}^{\bullet, 2, 2}$ and with no conditions on a_{00} . We can immediately compute the partition function of S_0^b :

$$Z_0^b = A^{-1}. \quad (5.35)$$

Next consider the first-level fermionic ghost action

$$S_1^b = \int_M c_{11}^* \wedge (d_+ c_{12} + d_- c_{21}). \quad (5.36)$$

The contribution from this is most easily computed by considering the BRST transformation of the antifield c_{11}^* :

$$Qc_{11}^* = \{S, c_{11}^*\} = d_+ d_- c_{00}. \quad (5.37)$$

The Lagrangian submanifold should hence project c_{11}^* to $\bullet\mathcal{A}^{1,1} \oplus \mathcal{A}^{\bullet,1} \oplus \mathcal{A}^{\bullet,1}$, and the contribution from S_1^b is straightforwardly computed as

$$Z_1^b = (CBB')^{1/2}. \quad (5.38)$$

The second-level bosonic ghost action reads

$$S_2^b = \int_M c_{21}^* \wedge (d_+ c_{02} + d_- c_{11}) + \int_M c_{12}^* \wedge (d_- c_{20} + d_+ c_{11}). \quad (5.39)$$

Consider first the terms involving the field c_{11} . The gauge transformation of c_{11} requires us to project c_{11} to $\mathcal{A}^{\bullet,1}$ giving a contribution $C^{-1/2}$ from these terms. The gauge transformation of c_{02} suggests projecting to $\mathcal{A}^{0,2}$, while c_{20} is projected to $\mathcal{A}^{\bullet,2,0}$. Both of these terms hence contribute a factor $A^{-1/2}$, giving

$$Z_2^b = A^{-1}C^{-1/2}. \quad (5.40)$$

The third-level fermionic action reads

$$S_3^b = \int_M (c_{22}^* \wedge (d_+ c_{01} + d_- c_{10}) + c_{31}^* \wedge d_- c_{01} + c_{13}^* \wedge d_+ c_{10}). \quad (5.41)$$

The contribution from this action is again most easily computed by considering the BRST transformation of the antifields. We have

$$Qc_{22}^* = d_+ c_{12} + d_- c_{21}, \quad (5.42)$$

which tells us that we should project c_{22}^* to $\mathcal{A}^{\bullet,2,2}$. The term involving c_{22}^* then contributes

a factor $A^{1/2}$ to the partition function. Similarly, the BRST transformation of c_{31}^* is

$$Qc_{31}^* = d_+c_{21}^*. \quad (5.43)$$

Neglecting cohomologies, we find that we should set $c_{31}^* = 0$ as part of the Lagrangian projection. Similarly, we also set $c_{13}^* = 0$. The third-level action thus contributes

$$Z_3^b = A^{1/2} \quad (5.44)$$

to the partition function.

The final term to consider is the bosonic action

$$S_4^b = \int_M (c_{23}^* + c_{32}^*) \wedge (d_+ + d_-)c_{00}. \quad (5.45)$$

As the ghost field c_{00} is gauge invariant we have no constraints on this field. This term hence contributes

$$Z_4^b = A^{-1/2}. \quad (5.46)$$

Collecting all contributions, we thus find

$$Z^b = \frac{(BB')^{1/2}}{A^2}. \quad (5.47)$$

Final result

Putting together (5.31) and (5.47), the full partition function for the target-space action (5.11) is

$$Z = Z^a Z^b = \frac{BB'}{C^{1/2}A^2}. \quad (5.48)$$

It is straightforward to check that this expression agrees with the general $G_2 \times G_2$ expression (5.8), and in particular the expression (5.10) for the special case of a global G_2 structure where $B' = B$.

6 The $Spin(7) \times Spin(7)$ complex

So far we have seen that we can use $O(d, d) \times \mathbb{R}^+$ generalised geometry to build a double complex that gives the target-space analogue of the worldsheet BRST complex. We used this double complex to compute the 1-loop partition function of the topological G_2 string and then identify a target-space action which reproduces this result. These arguments can, in fact, be extended to any subgroup $G \subset O(d)$ identified in [29] where

one “doubles” the complexes discussed in that work. These should provide the 1-loop partition function to a suitably twisted σ -model on a target space with the corresponding G -structure. Here, we will focus on the $Spin(7)$ case and use our results to provide a prediction for the conjectured topological $Spin(7)$ string [10]. The calculations here are entirely analogous to those in the $G_2 \times G_2$ case, so we will be light on details and simply sketch out some of the proofs while stating the key results.

6.1 Generalised $Spin(7) \times Spin(7)$ structures

The set up is much like Section 4.1, except now we take M to be eight-dimensional, so that we are working in $O(8,8) \times \mathbb{R}^+$ geometry. In this case, two globally non-vanishing chiral²¹ spinors $\epsilon_{\pm} \in S(C_{\pm})$ each define a $Spin(7)$ structure given by Θ_{\pm} . When the spinors are linearly independent, the $Spin(7)$ structures are orthogonal and intersect on a G_2 or $SU(4)$ structure depending on the relative chirality of ϵ_{\pm} [54, 55]. Much like in the G_2 case, however, there may be places where the spinors align and the structure degenerates to $Spin(7)$. If this is the case, the manifold admits only a local conventional G -structure, however the spinors still define a global $Spin(7) \times Spin(7)$ structure within generalised geometry.

One can describe certain backgrounds of type II strings compactified down to two dimensions in terms of $Spin(7) \times Spin(7)$ structures. In this case, we assume that we have a decomposition of the chiral ten-dimensional spinors as $\epsilon_{\pm} = \zeta_{\pm} \otimes \epsilon_{\pm}$ into irreducible $Spin(1,1)$ and $Spin(8)$ spinors respectively. For the background to preserve supersymmetry, we need the supersymmetry variations of the gravitinos and dilatinos to vanish under ϵ_{\pm} . Under the decomposition above, and the assumption of vanishing RR flux, we find that we need ζ_{\pm} to be constant spinors on $\mathbb{R}^{1,1}$, and that ϵ_{\pm} must satisfy the Killing spinor equations (4.2) on M . It was shown in [54] that these equations hold if and only if ϵ_{\pm} define a torsion-free $Spin(7) \times Spin(7)$ structure.

6.2 The double complex

A generalised $Spin(7) \times Spin(7)$ structure defines a generalised metric and hence a decomposition $E = C_+ \oplus C_-$ as discussed in Appendix C.1. As we did for $G_2 \times G_2$ structures, we can use the G -structure to define a refinement of the exterior algebra of this space and take

$$\mathcal{A}_{\mathbf{m},\mathbf{n}}^{p,q} = \Gamma(\Lambda_{\mathbf{m}}^p C_+ \oplus \Lambda_{\mathbf{n}}^q C_-), \quad (6.1)$$

²¹Note that the subscript \pm identifies which spinor bundle the spinors are sections of (as in Section 4.1), and *not* the chirality of the spinors.

where \mathbf{m} and \mathbf{n} now denote $Spin(7)$ representations. Then, given some compatible connection D , we can define a doubling of the complex (3.30) through the following diagram:

$$\begin{array}{ccccc}
& & \mathcal{A}_{1,1}^{0,0} & & \\
& \swarrow d_+ & & \searrow d_- & \\
& & \mathcal{A}_{8,1}^{1,0} & & \mathcal{A}_{1,8}^{0,1} \\
& \swarrow & & \searrow & \\
\mathcal{A}_{7,1}^{2,0} & & & & \mathcal{A}_{1,7}^{0,2} \\
& \swarrow & & \searrow & \\
& & \mathcal{A}_{8,8}^{1,1} & & \\
& \swarrow & & \searrow & \\
& & \mathcal{A}_{7,8}^{2,1} & & \mathcal{A}_{8,7}^{1,2} \\
& \swarrow & & \searrow & \\
& & \mathcal{A}_{7,7}^{2,2} & &
\end{array} \tag{6.2}$$

where for $\omega \in \mathcal{A}_{\mathbf{m},\mathbf{n}}^{p,q}$ we have defined

$$(d_+\omega)_{a_1\dots a_{p+1}\bar{a}_1\dots\bar{a}_q} = (p+1)(\mathcal{P}_{\mathbf{m}'}^+)_{a_1\dots a_{p+1}}{}^{b_1\dots b_{p+1}}D_{b_1}\omega_{b_2\dots b_{p+1}\bar{a}_1\dots\bar{a}_q}, \tag{6.3}$$

$$(d_-\omega)_{a_1\dots a_p\bar{a}_1\dots\bar{a}_{q+1}} = (-1)^p(q+1)(\mathcal{P}_{\mathbf{n}'}^-)_{\bar{a}_1\dots\bar{a}_{q+1}}{}^{\bar{b}_1\dots\bar{b}_{p+1}}D_{\bar{b}_1}\omega_{a_1\dots a_p\bar{b}_2\dots\bar{b}_{q+1}}. \tag{6.4}$$

We will see that, when D is torsion-free, (6.2) defines a double complex and the restriction to $(\mathcal{A}^{\bullet,0}, d_+)$ is isomorphic to (3.30).

First, we find the condition on the components of a generalised Levi-Civita connection for it to be $Spin(7) \times Spin(7)$ compatible. As before, this comes from taking a type IIB NSNS background of the form $\mathbb{R}^{1,1} \times M$ where M is eight-dimensional, and then considering the Killing spinor equations on M . The conditions on (H, ϕ, A^\pm) that we need to impose are

$$D_a\epsilon_+ = \nabla_a\epsilon_+ - \frac{1}{24}H_{abc}\gamma^{bc}\epsilon_+ - \frac{1}{7}\partial^b\phi\gamma_a{}^b\epsilon_+ + \frac{1}{4}A_{abc}^+\gamma^{bc}\epsilon_+ = 0, \tag{6.5}$$

$$D_{\bar{a}}\epsilon_+ = \nabla_{\bar{a}}\epsilon_+ - \frac{1}{8}H_{\bar{a}bc}\gamma^{bc}\epsilon_+ = 0, \tag{6.6}$$

where ϵ_\pm are the internal spinors that appear in the Killing spinor equations and hence define the $Spin(7) \times Spin(7)$ structure. The conditions above imply compatibility with the first $Spin(7)$ factor, while the analogous conditions for ϵ_- imply compatibility with the second $Spin(7)$ factor.

Decomposing the fields under the first $Spin(7)$, one finds

$$\partial\phi \in \mathbf{8}, \quad H \in \mathbf{8} + \mathbf{48}, \quad A^+ \in \mathbf{48} + \mathbf{112}. \tag{6.7}$$

We can therefore write

$$H_{abc} = H^d \Theta_{dabc} + \tilde{H}_{abc}, \quad (6.8)$$

$$A_{abc}^+ = \left(\tilde{A}_{ade} \Theta_{bc}{}^{de} - \tilde{A}_{[a|de} \Theta_{|bc]}{}^{de} \right) + \hat{A}_{abc}, \quad (6.9)$$

where $H_d \in \Omega_{\mathfrak{8}}^1$, $\tilde{H}, \tilde{A} \in \Omega_{\mathfrak{48}}^3$, and \hat{A} transforms in the **112** representation. Using the fact that

$$(\mathcal{P}_{\mathfrak{21}}^2)_{ab}{}^{cd} \gamma_{cd} \epsilon_+ = 0, \quad \tilde{A}_{[a|de} \Theta_{|bc]}{}^{de} = \frac{2}{3} \tilde{A}_{abc}, \quad (6.10)$$

we find that the conditions for D to be a compatible connection are

$$H_d = \frac{2}{7} \partial_d \phi, \quad \tilde{H} = 20 \tilde{A}, \quad (6.11)$$

while \hat{A} is unfixed. One finds analogous relations for the second $Spin(7)$ factor.

As in the $G_2 \times G_2$ case, we find that the operators in the double complex can be defined in terms of a “simplified” connection \hat{D} which is neither torsion-free, nor compatible. Nonetheless, it can be used to check the nilpotency and anticommutivity of d_{\pm} . The simplified connection is

$$\begin{aligned} \hat{D}_a v^b &= \nabla_a v^b, \\ \hat{D}_{\bar{a}} v^b &= \nabla_{\bar{a}}^- v^b = \nabla_{\bar{a}} v^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c v^c, \\ \hat{D}_a v^{\bar{b}} &= \nabla_a^+ v^{\bar{b}} = \nabla_a v^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}}, \\ \hat{D}_{\bar{a}} v^{\bar{b}} &= \nabla_{\bar{a}} v^{\bar{b}}. \end{aligned} \quad (6.12)$$

Note that the first line of (6.12) immediately implies that $(\mathcal{A}^{*,0}, d_+)$ is isomorphic to (3.30) as required.

We now check the conditions for (3.30) to be a double complex. Firstly note that, as in (4.25), we find that d_+^2 acting on $\omega \in \mathcal{A}^{0,p}$ is given by $(\mathcal{P}_7^2)_+$ acting on the following:

$$[\hat{D}_{a_1}, \hat{D}_{a_2}] \omega_{\bar{b}_1 \bar{b}_2} = -q \mathcal{R}_{a_1 a_2}^+{}_{\bar{b}_1}{}^{\bar{c}} \omega_{\bar{c} \bar{b}_2}. \quad (6.13)$$

However, after taking the projection, this vanishes because $\mathcal{R}^+ \in \mathfrak{spin}_7^+ \otimes \mathfrak{spin}_7^-$. It is clear therefore, that $d_+^2 = 0$ acting on any vertex of (6.2). The $d_-^2 = 0$ condition follows analogously, and so we just need to check $\{d_+, d_-\} = 0$ on $\mathcal{A}^{0,0}$, $\mathcal{A}^{1,0}$, $\mathcal{A}^{0,1}$ and $\mathcal{A}^{1,1}$. The calculation is very similar to those done in Section 4.3 and so we will demonstrate

it only for $\omega \in \mathcal{A}_{\mathbf{8},\mathbf{1}}^{1,0}$. First consider

$$[\hat{D}_a, \hat{D}_{\bar{b}}]\omega_{c\bar{d}} = (\mathcal{R}_{a\bar{b}}{}^e{}_c - \frac{1}{2}\nabla_a H_{\bar{b}c}{}^e - \frac{1}{4}H_{a\bar{b}}{}^{\bar{d}}H_{\bar{d}c}{}^e)\omega_e. \quad (6.14)$$

Antisymmetrising on $[ac]$, this becomes

$$-\frac{1}{2}\mathcal{R}_{ac}{}^+{}^e{}_{\bar{b}}\omega_e, \quad (6.15)$$

which is annihilated by $(\mathcal{P}_7^2)_+$. The others follow similarly and hence we see that if D is a torsion-free $Spin(7) \times Spin(7)$ connection, (3.30) defines a double complex.

6.3 Hodge theory

Next, we define Laplacians and analyse the Hodge theory. To do so, we introduce a metric on the complex as we did in (4.33). That is, for $\alpha, \beta \in \mathcal{A}^{p,q}$, we have

$$(\alpha, \beta) = \int_M \Phi G(\alpha, \beta) = \kappa_p \kappa_q \int_M e^{-2\phi} \text{vol } \alpha \lrcorner \beta, \quad (6.16)$$

where now

$$\kappa_p = \begin{cases} 1 & p = 0, 1, \\ 4 & p = 2. \end{cases} \quad (6.17)$$

Defining adjoint operators d_{\pm}^{\dagger} with this metric, we find their action on (p, q) -forms to be

$$d_{+}^{\dagger}\alpha = -\gamma_p D^{a_1} \alpha_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q}, \quad (6.18)$$

$$d_{-}^{\dagger}\alpha = (-1)^{p+1} \gamma_q D^{\bar{b}_1} \alpha_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q}, \quad (6.19)$$

where

$$\gamma_p = \begin{cases} 1 & p = 1, \\ 4 & p = 2. \end{cases} \quad (6.20)$$

6.3.1 Kähler identities

As in the $G_2 \times G_2$ case, the operators d_{\pm} satisfy the analogue of the Kähler identities:

$$d_{\pm}^2 = (d_{\pm}^{\dagger})^2 = \{d_{\pm}^{\dagger}, d_{\mp}\} = 0. \quad (6.21)$$

Unlike in the $G_2 \times G_2$ case, however, we do not have isomorphisms connecting different vertices in the complex. We must therefore check the relations on all vertices. For the condition $\{d_{+}^{\dagger}, d_{-}\} = 0$, the only non-trivial checks are on $\mathcal{A}^{1,0}$, $\mathcal{A}^{1,1}$, $\mathcal{A}^{2,0}$ and $\mathcal{A}^{2,1}$. For

$\lambda_{1,0} \in \mathcal{A}^{1,0}$, the condition follows simply from (C.28) and the fact that $R_{ab}^0 = 0$ for torsion-free $Spin(7) \times Spin(7)$ manifolds:

$$(\{d_+^\dagger, d_-\}\lambda_{1,0})_{\bar{a}} = [D_a, D_{\bar{a}}]\lambda^a = R_{a\bar{a}}^0\lambda^a = 0. \quad (6.22)$$

For $\lambda_{1,1} \in \mathcal{A}^{1,1}$ we find

$$\begin{aligned} (\{d_+^\dagger, d_-\}\lambda_{1,1})_{\bar{a}\bar{b}} &= 2(\mathcal{P}_7^2)_{\bar{a}\bar{b}}^{\bar{c}\bar{d}} \left(\nabla_{\bar{c}}\nabla_a\lambda^a_{\bar{d}} - \nabla_a\nabla_{\bar{c}}\lambda^a_{\bar{d}} \right. \\ &\quad \left. + (-2\nabla_{\bar{c}}\nabla_a\phi + \frac{1}{2}\nabla_b H_{\bar{c}}^b{}_a + \frac{1}{4}H_{b\bar{c}}^{\bar{e}}H_{\bar{e}}^b{}_a + \partial_b\phi H_{\bar{c}}^b{}_a)\lambda^a_{\bar{d}} \right. \\ &\quad \left. + (\frac{1}{2}\nabla_{\bar{c}}H_{a\bar{d}}^{\bar{e}} + \frac{1}{4}H_{b\bar{d}}^{\bar{e}}H_{\bar{c}}^b{}_a)\lambda^a_{\bar{e}} \right) \\ &= 2(\mathcal{P}_7^2)_{\bar{a}\bar{b}}^{\bar{c}\bar{d}} \left(-R_{\bar{c}\bar{a}}^0\lambda^a_{\bar{d}} + \frac{1}{2}\mathcal{R}_{\bar{c}\bar{d}}^{+\bar{e}}{}_a\lambda^a_{\bar{e}} \right) = 0. \end{aligned} \quad (6.23)$$

The first term vanishes as before, and the second term vanishes because $\mathcal{R}^+ \in \mathfrak{spin}_7^+ \otimes \mathfrak{spin}_7^-$ and hence is annihilated by \mathcal{P}_7^2 . For the remaining vertices, $\mathcal{A}^{2,0}$ and $\mathcal{A}^{2,1}$, the calculation is similar:

$$\begin{aligned} (\{d_+^\dagger, d_-\}\lambda_{2,0})_{a\bar{a}} &= -4R_{\bar{a}b}^0\lambda^b{}_a - 2\mathcal{R}_{bc\bar{a}a}^-\lambda^{bc} \\ &= 0, \end{aligned} \quad (6.24)$$

$$\begin{aligned} (\{d_+^\dagger, d_-\}\lambda_{2,1})_{a\bar{a}\bar{b}} &= 8(\mathcal{P}_7^2)_{\bar{a}\bar{b}}^{\bar{c}\bar{d}} \left(-R_{\bar{c}b}^0\lambda^b{}_{a\bar{d}} - \frac{1}{2}\mathcal{R}_{bc\bar{a}\bar{c}}^-\lambda^{bc}{}_{\bar{d}} - \frac{1}{2}\mathcal{R}_{\bar{c}\bar{d}}^{+\bar{e}}{}_b\lambda^b{}_{a\bar{e}} \right) \\ &= 0. \end{aligned} \quad (6.25)$$

The terms involving \mathcal{R}^- vanish since contraction with λ gives a contraction between the **21** and **7** which is necessarily zero. The analysis for $\{d_-^\dagger, d_+\}$ then mirrors what we have done above. Putting this all together, one sees

$$\{d_\pm^\dagger, d_\mp\} = 0. \quad (6.26)$$

One can also check that $(d_\pm^\dagger)^2 = 0$. Unlike in the $G_2 \times G_2$ case, this does not immediately follow from $d_\pm^2 = 0$ and certain automorphisms of the complex. Instead, one needs to do the calculation explicitly. For example, for $\lambda \in \mathcal{A}^{2,q}$, one finds

$$((d_+^\dagger)^2\lambda)_{\bar{a}_1\dots\bar{a}_q} = -4R_{ab}^0\lambda^{ab}{}_{\bar{a}_1\dots\bar{a}_q} - 2q\mathcal{R}_{ab}^{\bar{b}}{}_{[\bar{a}_1]}\lambda^{ab}{}_{\bar{b}|\dots\bar{a}_q]} = 0, \quad (6.27)$$

and similarly for $(d_-^\dagger)^2$. This proves the Kähler identities in (6.21).

6.3.2 Laplacians

Finally, we can define the Laplacians of the “plus” and “minus” differentials via

$$\Delta_{\pm} = d_{\pm}^{\dagger}d_{\pm} + d_{\pm}d_{\pm}^{\dagger}. \quad (6.28)$$

Taking the combined differential $\hat{d} = d_{+} + d_{-}$, we find that the Kähler identities imply $\hat{d}^2 = 0$, and that

$$\hat{\Delta} = \hat{d}^{\dagger}\hat{d} + \hat{d}\hat{d}^{\dagger} = \Delta_{+} + \Delta_{-}. \quad (6.29)$$

While we omit the rather lengthy calculations, one can follow the same arguments as in Section 4.4.2 to show that the Laplacians are again equal: $\Delta_{+} = \Delta_{-} = \frac{1}{2}\hat{\Delta}$.

7 Relation to the topological *Spin*(7) string

The topological *Spin*(7) string was postulated in [10] but is far less understood than its G_2 counterpart. Despite this, we can use the double complex we have derived, along with intuition gained from the A/B-model and the G_2 -string, to provide a conjecture for its 1-loop partition function.

The key idea is to take the double complex above as the target space interpretation of the BRST complex and assume (5.8) holds for any topological string. That is, we assume that the 1-loop partition function calculates the analytic torsion of the *Spin*(7) double complex:

$$Z_1 = \left[\prod_{p,q} (\det' \hat{\Delta}^{p,q})^{(-1)^{p+q}pq} \right]^{-1/2}, \quad (7.1)$$

where now the product is taken over the vertices in the *Spin*(7) double complex. Again, using the fact that the determinants depend only on the *Spin*(7) \times *Spin*(7) representation, we find

$$Z_1 = (\det' \hat{\Delta}_{7,7})^{-2} (\det' \hat{\Delta}_{8,7}) (\det' \hat{\Delta}_{7,8}) (\det' \hat{\Delta}_{8,8})^{-1/2}. \quad (7.2)$$

As before, the subscripts denote the *Spin*(7) \times *Spin*(7) representation that $\hat{\Delta}$ is acting on.

The topological *Spin*(7) string should correspond to the case where we have a global *Spin*(7) structure with vanishing flux, so that the metric on M has special holonomy. The above expression can then be rewritten using representations of the diagonal *Spin*(7)

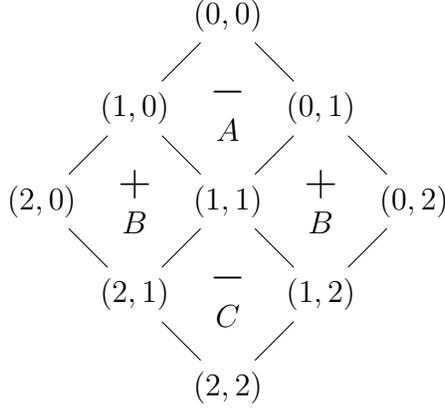


Figure 3. For a global $Spin(7)$ structure with vanishing flux, equality of Δ_{\pm} means that all $\det' \Delta^{p,q}$ can be expressed in terms of three independent determinants. For example, $\det' \hat{\Delta}^{0,0} = \det' \hat{\Delta}_{1,1} = \det' \Delta_1 \equiv A$, $\det' \hat{\Delta}^{2,0} = \det' \hat{\Delta}_{7,1} = \det' \Delta_7 \equiv B$, and $\det' \hat{\Delta}^{2,2} = \det' \hat{\Delta}_{7,7} = \det' \Delta_1 \det' \Delta_{21} \det' \Delta_{27} \equiv C$. The analytic torsion (the 1-loop partition function) is then given by $(A^{-1}B^2C^{-1})^{1/2}$, in agreement with (7.4).

and the fact that, by construction, $\hat{\Delta} \simeq \Delta$ on these subspaces:

$$Z_1 = (\det' \Delta_1)^{-5/2} (\det' \Delta_7)^{-1/2} (\det' \Delta_8)^2 (\det' \Delta_{21})^{-5/2} (\det' \Delta_{27})^{-2} (\det' \Delta_{35})^{-1/2} (\det' \Delta_{48})^2. \quad (7.3)$$

It is possible to further simplify this as, for a $Spin(7)$ manifold, the Laplacians are not independent. Using the relations outlined in Appendix A.3 we find that

$$Z_1 = (\det' \Delta_1)^{-1} (\det' \Delta_7) (\det' \Delta_{21})^{-1/2} (\det' \Delta_{27})^{-1/2}. \quad (7.4)$$

Once more, this result can be read off directly from the double complex by assigning values to the squares in the diamond, using the isomorphisms to see which are equal, and multiplying them together with alternating powers of $\pm \frac{1}{2}$. This is illustrated in Figure 3.

Before providing more motivation for why this is might be the correct 1-loop partition function, we make some brief comments about the result. Firstly, this combination of determinants does not define a topological invariant of the $Spin(7)$ manifold, much like the G_2 case.²² This is not a surprise given that the G -structure defines a unique metric

²²One can calculate the combinations of determinants that are topological, much like in [14]. For $Spin(7)$ there are four independent combinations. We find that Z_1 in (7.4) cannot be written as a combination of these invariants.

and the partition functions clearly depend on the G -structure. Secondly, there does not appear to be a quadratic target-space action whose partition function reproduces (7.4) since the natural “top form” one would write down transforms in the $(7, 7)$ and hence is not a $Spin(7) \times Spin(7)$ invariant. Despite this, our analysis provides a natural geometric interpretation of the 1-loop partition function as the product of ζ -regularised determinants of Laplacians acting on the double complex (6.2).

Let us now motivate the above result by finding worldsheet operators that act as d_{\pm} . As we saw in Section 2.3, the special NS highest-weight states selected out by the topological string are the following:

$$\begin{array}{ccc} |0, 0\rangle & \left| \frac{1}{16}, \frac{7}{16} \right\rangle & \left| \frac{1}{2}, \frac{1}{2} \right\rangle \\ \Omega_1^0 & \Omega_8^1 & \Omega_7^2 \end{array}, \quad (7.5)$$

where we have written their interpretation as target-space differential forms underneath. These states should be generated by the chiral ring of the theory, which in turn should be annihilated by some nilpotent operator Q we are yet to find. Given the discussion in this and the previous section, we expect that this Q should have a left-moving piece that corresponds to the differential \check{d} described in (3.30).

Typically, the operator Q is built from the supersymmetry generators of the theory.²³ We therefore examine the OPE of the supercurrent G with the operators defining the NS states above, and find

$$G(z)f(w) = \dots - \frac{\partial_{\mu} f \psi^{\mu}(w)}{2(z-w)} + \dots \quad (7.6)$$

$$G(z)A_{\mu}\psi^{\mu}(w) = \dots - \frac{\partial_{\mu} A_{\nu} \psi^{\mu} \psi^{\nu}(w)}{2(z-w)} + \frac{\partial x^{\mu} A_{\mu}(w)}{(z-w)} + \dots \quad (7.7)$$

where we have expressed only the order-one poles which give the action of $G_{-1/2}$. We see that, up to some term that involves ∂x^{μ} , we have $G_{-1/2} \sim -\frac{1}{2}d$. In fact, if we decompose the expressions above according to their weight under $T_I = \frac{1}{8}\tilde{X}$, we find

$$\partial_{\mu} f \psi \sim \left[\frac{1}{16} \right], \quad (7.8)$$

$$(\mathcal{P}_7^2)_{\mu\nu}{}^{\rho\sigma} \partial_{\rho} A_{\sigma} \psi^{\mu} \psi^{\nu} \sim \left[\frac{1}{2} \right], \quad (7.9)$$

$$-\frac{1}{2}(\mathcal{P}_{21}^2)_{\mu\nu}{}^{\rho\sigma} \partial_{\rho} A_{\sigma} \psi^{\mu} \psi^{\nu} + \partial x^{\mu} A_{\mu} \sim [0]. \quad (7.10)$$

We would like to pick some sub-operator of $G_{-1/2}$ that selects (7.8) and (7.9), but leaves

²³At least this is true in the A/B-models, and there is evidence that a version of this is true for the G_2 string.

out (7.10).

To see what this could be, we need to understand better the fusion rules of the Ising model. The Ising model is a minimal model at $p = 3$. It therefore has three states α_n for $n = 1, 2, 3$, where the weight of α_n is

$$\alpha_n \sim \frac{(3n-4)^2 - 1}{48} = \begin{cases} 0 & n = 1, \\ \frac{1}{16} & n = 2, \\ \frac{1}{2} & n = 3. \end{cases} \quad (7.11)$$

Plugging in the different values for n reproduces the weights we have written above as expected. By examining the fusion rules, one finds [10]

$$\alpha_1 \alpha_2 \sim \alpha_2, \quad \alpha_2 \alpha_2 \sim \alpha_1 + \alpha_3, \quad \alpha_3 \alpha_2 \sim \alpha_2. \quad (7.12)$$

Using the conformal block decomposition of a state – in which we view a state α_n as a collection of maps between states dictated by the fusion rules – we find that we can write

$$\alpha_2 = \alpha_2^+ + \alpha_2^-, \quad \alpha_2^\pm: \alpha_n \rightarrow \alpha_{n\pm 1}, \quad (7.13)$$

where we take $\alpha_n = 0$ for $n \notin \{1, 2, 3\}$.

Looking at (7.8)–(7.10), we see that, at least in the Ising sector, $G_{-1/2} \sim \alpha_2$. Decomposing the action of $G_{-1/2}$ into conformal blocks as we did for α_2 , the sub-operator that selects only the special NS states in (7.5), and hence the natural candidate for the left-moving BRST operator, is

$$Q_L = G_{-1/2}^+ \sim d_-. \quad (7.14)$$

One finds similar results for the right-moving sector with $Q_R = \bar{G}_{-1/2}^+$, and hence we reproduce the $Spin(7) \times Spin(7)$ double complex of Section 6. Note that this construction is analogous to that of the BRST operator in [11] for the topological G_2 string. We view this as strong evidence that this, or a small variation of this, is the correct worldsheet and target-space interpretation of the BRST operator for the topological $Spin(7)$ string.

8 Some other examples

Though we have focused on the cases of $G_2 \times G_2$ and $Spin(7) \times Spin(7)$, and the diagonal subgroups relevant for topological strings, our construction is in fact far more general. Once one has identified a group $G \subset O(d)$ and the relevant instanton complex

from [29], one can find a doubled version of the complex by lifting to a $G \times G \subset O(d, d)$ structure and then using a torsion-free generalised connection that is compatible with the structure. Moreover, this construction will work for non-vanishing H -flux even if this breaks integrability of the conventional G -structure.

Since the proofs are essentially the same as for the G_2 and $Spin(7)$ case with the relevant groups and representations exchanged, we will not show explicitly that the diagrams we give are double complexes, that the Kähler identities hold, nor that the left and right Laplacians are equal. Instead, we simply write down the complexes in a few cases of interest and comment on connections to 1-loop partition functions.

8.1 A- and B-models with background H -flux

As we have seen, the double complex exists only for certain choices of H -flux. While these choices may break integrability of the conventional G -structure, if the flux preserves integrability of the generalised $G \times G$ structure, and hence the background is still supersymmetric, the double complex is well defined.²⁴ This allows us to describe topological strings on backgrounds with non-vanishing flux where one necessarily needs to use generalised geometry and the doubled complex. In particular, it is interesting to see how our double complex describes the A- or B-models with flux.

First, let us build the double complex that applies to six-dimensional supersymmetric backgrounds. These backgrounds have an integrable $SU(3) \times SU(3)$ structure²⁵ which means we can decompose the bundles C_{\pm} under this group into complex conjugate representations

$$C_{\pm} = C_{\pm}^{1,0} \oplus C_{\pm}^{0,1}. \quad (8.1)$$

In this case, we can build two inequivalent double complexes out of this decomposition that we suggestively call the A- and B-complexes. These are respectively defined by

$$\mathcal{A}_A^{p,q} = \Gamma(\Lambda^p C_+^{0,1} \otimes \Lambda^q C_-^{1,0}), \quad (8.2)$$

$$\mathcal{A}_B^{p,q} = \Gamma(\Lambda^p C_+^{0,1} \otimes \Lambda^q C_-^{0,1}). \quad (8.3)$$

We can build the corresponding differentials from a torsion-free compatible connection as before, this time with the projections mapping onto the vector spaces above. The 1-loop partition function then follows immediately from the analytic torsion formula we

²⁴Recall that integrability of the generalised structure forces some components of H to vanish, while others are related to $\partial\phi$ or left unconstrained.

²⁵In fact, one only needs the target space to be generalised Kähler, which is a slightly weaker structure. However, restricting to $SU(3) \times SU(3)$ ensures there are no anomalies in the twist [15].

used previously, namely

$$Z_1 = \left[\prod_{p,q} (\det' \hat{\Delta}^{p,q})^{(-1)^{p+q}pq} \right]^{-1/2}. \quad (8.4)$$

Let us now see how this relates to the A- and B-models with flux, as studied in [15].

The relation of the A/B-models to generalised geometry has been studied in great detail [44, 56, 57]. In general, a two-dimensional $N = (2, 2)$ σ -model with H -flux has a target space with a twisted generalised Kähler structure [42, 58]. This is defined by two commuting, integrable generalised complex structures \mathcal{J}_1 and \mathcal{J}_2 . Individually, they give a decomposition of the generalised tangent bundle into eigenbundles:

$$E_{\mathbb{C}} = L_1 \oplus \overline{L_1} = L_2 \oplus \overline{L_2}, \quad (8.5)$$

where L_i is the $+i$ eigenbundle of \mathcal{J}_i . Since the two complex structures commute, we can find mutual eigenbundles and write

$$E_{\mathbb{C}} = \underbrace{L_1^+}_{(1,1)} \oplus \underbrace{L_1^-}_{(1,-1)} \oplus \underbrace{\overline{L_1^+}}_{(-1,-1)} \oplus \underbrace{\overline{L_1^-}}_{(-1,1)}, \quad (8.6)$$

where we have indicated the charges under $(\mathcal{J}_1, \mathcal{J}_2)$. We see that L_1 and L_2 can be identified with

$$L_1 = L_1^+ \oplus L_1^-, \quad L_2 = L_1^+ \oplus \overline{L_1^-}. \quad (8.7)$$

As usual, the two generalised complex structures define a generalised metric via $G = -\mathcal{J}_1 \mathcal{J}_2$, which in turn defines the subspaces C_{\pm} . It turns out that the decompositions (8.1) and (8.5) are then related via

$$C_+^{1,0} = L_1^+, \quad C_-^{1,0} = L_1^-. \quad (8.8)$$

Using this, the vector spaces (8.2) and (8.3) that we have dubbed the A- and B-complexes are given by

$$\mathcal{A}_A^{p,q} = \Gamma(\Lambda^p \overline{L_1^+} \otimes \Lambda^q L_1^-) \subseteq \Gamma(\Lambda^{p+q} \overline{L_2}), \quad (8.9)$$

$$\mathcal{A}_B^{p,q} = \Gamma(\Lambda^p \overline{L_1^+} \otimes \Lambda^q \overline{L_1^-}) \subseteq \Gamma(\Lambda^{p+q} \overline{L_1}), \quad (8.10)$$

so that the total space of the A- and B-complexes are $\Lambda^\bullet L_2^*$ and $\Lambda^\bullet L_1^*$ respectively.

Using the results of [42], we know that for any generalised complex structure \mathcal{J} , its

$+i$ eigenbundle L defines a Lie algebroid under the Courant bracket. This means that there is an associated differential d_L that makes $(\Lambda^\bullet L^*, d_L)$ a complex. Furthermore, if \mathcal{J} is part of a generalised Kähler structure, then the differential splits as

$$d_L = \partial_L^+ + \partial_L^-, \quad (8.11)$$

with

$$\partial_L^+ : \Lambda^{p,q} L^* \rightarrow \Lambda^{p+1,q} L^*, \quad \partial_L^- : \Lambda^{p,q} L^* \rightarrow \Lambda^{p,q+1} L^*, \quad (8.12)$$

where we have defined $\Lambda^{p,q} L_1^* = \Lambda^p \overline{L_1^+} \otimes \Lambda^q \overline{L_1^-}$, and similarly for L_2 (see also [59]). One can show that these differentials coincide with the those in the double complex defined via generalised connections, i.e. $\partial_L^\pm = d_\pm$. Hence, the total BRST operator is

$$Q = Q_L + Q_R = d_+ + d_- = \partial_L^+ + \partial_L^- \quad (8.13)$$

$$= d_L, \quad (8.14)$$

where L is L_1 for the B-model and L_2 for the A-model. The chiral ring then corresponds to the cohomology associated to one of the generalised complex structures, with the choice of structure fixed by whether one uses the A or B twist. This exactly reproduces the results of [15]. We can now interpret the 1-loop partition function (8.4) more concretely with the knowledge that the relevant Laplacian $\hat{\Delta}$ is that associated to the differential d_L .²⁶

For completeness, note that one can recover the chiral rings of the A- and B-models by defining a generalised Kähler structure from a conventional Kähler structure (I, ω) via

$$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \quad (8.15)$$

With this choice, the A- and B-complexes in (8.2) and (8.3) reproduce the chiral rings of the A- and B-models respectively, with the 1-loop partition function in (8.4) reducing to (2.29).

²⁶As was shown in [15], depending on whether one uses \mathcal{J}_1 or \mathcal{J}_2 , Z_1 may receive instanton corrections at finite volume, and so we should view (8.4) as the 1-loop partition function at infinite volume.

8.2 Topological strings on K3

A K3 manifold has an $SU(2) \subset Spin(4)$ structure, for which the relevant instanton complex is isomorphic to the Dolbeault complex

$$\Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1} \xrightarrow{\bar{\partial}} \Omega^{0,2}, \quad (8.16)$$

where we can choose any combination of the three commuting complex structures to define the $\bar{\partial}$ operator. Lifting to a generalised $SU(2) \times SU(2) \subset Spin(4,4)$ structure, we find two possible ways to define the doubling of the complex, corresponding to the A- and B-models on K3. In both cases, in the infinite-volume limit where instanton corrections can be neglected, the complex is naturally isomorphic to the Dolbeault complex:²⁷

$$\begin{array}{ccccc}
 & & \Omega^{0,0} & & \\
 & \swarrow \partial & & \searrow \bar{\partial} & \\
 & \Omega^{1,0} & & \Omega^{0,1} & \\
 & \swarrow & & \swarrow & \\
 \Omega^{2,0} & & \Omega^{1,1} & & \Omega^{0,2} \\
 & \swarrow & & \swarrow & \\
 & \Omega^{2,1} & & \Omega^{1,2} & \\
 & \swarrow & & \swarrow & \\
 & \Omega^{2,2} & & &
 \end{array} \quad (8.17)$$

One can then find the 1-loop partition function of the topological string on K3 using the analytic torsion (8.4). Thanks to the Calabi–Yau structure of a K3, one finds that the ζ -regularised determinants of Laplacians have many identifications:

$$\det' \Delta^{p,q} = \det' \Delta^{q,p} = \det' \Delta^{2-p,q}, \quad (8.18)$$

$$\det' \Delta^{1,1} = (\det' \Delta^{1,0})^2 = (\det' \Delta^{0,0})^4. \quad (8.19)$$

Applying these to (8.4), we find that the 1-loop partition function is trivial, $Z_1^{\text{K3}} = 1$. This matches the result that the partition functions for the A- and B-models are trivial on a K3 in the large-volume limit [7].²⁸ Note that we were able to show this directly from the target-space geometry without a detailed description of the worldsheet theory.

²⁷By changing the choice of complex structure within the hyperkähler structure, one can continuously interpolate between the A- and B-model [57].

²⁸One needs to use the worldsheet theory to see that it is not possible to absorb fermion zero-modes in order to show that the partition function for the A-model remains trivial at finite volume.

9 Conclusions and future directions

In this paper, we have given a prescription for calculating the 1-loop partition function of certain topological string models whose target spaces admit torsion-free $G \times G$ structures within $O(d, d) \times \mathbb{R}^+$ generalised geometry. We reviewed how there are natural complexes for both G_2 and $Spin(7)$ structures. We then extended these to double complexes for $G_2 \times G_2$ and $Spin(7) \times Spin(7)$, with the relevant differentials constructed from torsion-free compatible generalised connections. We showed that such connections exist provided the target space satisfies certain differential conditions that correspond to it being a supersymmetric NSNS background for a Minkowski spacetime. In each case, there existed an analogue of Kähler identities and Hodge theory which allowed us to define Laplacians acting on representations of $G \times G$. Starting from the conjecture that the 1-loop partition function is given by a certain alternating product of determinants of these Laplacians, we showed that our formalism reproduced the known worldsheet results for the A- and B-models and the G_2 string. Our result for the $Spin(7)$ string is new. Finally, as further examples, we discussed how our formalism captures topological strings on K3 surfaces and the A- and B-models with flux.

An overarching theme of our work is that $G \times G \subset O(d, d) \times \mathbb{R}^+$ structures within generalised geometry should be thought of as the correct target-space language for describing worldsheet models, with the left- and right-moving sectors captured by the spaces C_+ and C_- respectively (as mentioned previously in [45, 60]). By moving to the twisted theory, one is restricted to special subspaces of C_{\pm} selected by the G -structure.

Unfortunately, we were not able to give a target-space action for the $Spin(7)$ string, nor for backgrounds with general $G \times G$ structure without special holonomy. Following the logic of [9], one might imagine that there are functionals whose quantisation leads to the 1-loop partition functions we have calculated in this paper. It may be that one needs to consider RR degrees of freedom and extend to exceptional generalised geometry [61–65] in order understand these, or if one wants to understand the story in M-theory. We hope to tackle this in the future, for example by building on the work defining invariant functionals in [46, 66–68]. As a first step, one could imagine quantising variations of the “hypermultiplet structure” of [66], which in type IIA would give the analogue of [9] but for the A-model (or Kähler gravity). However, in order to extend the $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ constructions introduced in this paper one would need to identify the corresponding structures in $E_{8(8)}$ and $E_{9(9)}$ generalised geometries respectively, which have not yet been formulated (though certain subsectors of $E_{8(8)}$ generalised geometry have been introduced in [69] which might provide clues on how to build the invariant functionals).

There are a number of research directions opened up by our work. In [14], the quantised G_2 target-space theory was compared with the results of Pestun and Witten [9] by reducing the theory on a circle. One could perform a similar check by reducing the $Spin(7)$ double complex to G_2 in the cases with and without H flux. Staying in eight dimensions, as another direction one might consider embedding a global $SU(4)$ structure in $Spin(7) \times Spin(7)$ where it should give a non-critical version of the B-model on a fourfold.

Another relatively straightforward extension of this work would be to consider the cases where the generalised intrinsic torsion of the $G_2 \times G_2$ or $Spin(7) \times Spin(7)$ structures does not (entirely) vanish. Recall that we used the fact that we were examining supersymmetric Minkowski backgrounds to immediately conclude that the $G \times G$ generalised structures must be torsion-free, and that was sufficient to prove the existence of the corresponding double complexes. However, it is possible that one may be able to weaken this constraint for other backgrounds – in particular, supersymmetric AdS backgrounds are described in generalised geometry by constant singlet torsion [70–72]. For the $G_2 \times G_2$ case, one could then hope to use the concepts developed in this work to make contact with worldsheet computations for NSNS AdS₃ backgrounds [30].

It would also be worthwhile to understand whether there is a physical interpretation of the double complexes when the groups for right- and left-movers on the worldsheet are not matched. For example, one could imagine taking $SU(3) \times G_2 \subset O(7) \times O(7)$. Such a generalised structure would be defined by three global spinors, a pair of orthogonal $\epsilon_+^{1,2}$ and an ϵ_- , so that the seven-dimensional manifold would generically have an $SU(2)$ structure that becomes $SU(3)$ wherever ϵ_- is parallel to either ϵ_+^i . Again, one can write down the conditions for the structure to be torsion-free and construct differentials using the corresponding torsion-free compatible connection. Closely related to this would be considering the generalised geometric description of heterotic supergravity, where the relevant group is $O(d) \times O(d+n)$, with the gauge group G embedding in the second factor [67, 73–79]. If a double complex can be constructed in this case, one imagines it could be related to G -instantons.

Given the importance of the A- and B-models for understanding mirror symmetry on Calabi–Yau threefolds, one might wonder if these double complexes could be used to probe mirror symmetry on G_2 or $Spin(7)$ manifolds [10, 32, 80–83], or if the 1-loop partition functions can be expressed in terms of the “ ρ -characteristic” of [84] which has special properties for self-mirror manifolds. As a consistency check, G_2 mirror symmetry appears on the worldsheet as a certain automorphism of the right-moving extended algebra [85–87], suggesting that Figure 2 should be symmetric when reflected along the diagonal from the top left to the bottom right, which indeed it is.

Our work may also have applications in K-theory and index formulae, which can be seen by reinterpreting the construction of the double complexes in terms of generalised spinors. Indeed, for $Spin(7) \times Spin(7)$, one can check that the total space of the double complex is isomorphic to the space of generalised spinors. The operators $\mathcal{D}_\pm = d_\pm + d_\pm^\dagger$ then define new elliptic operators on this space which are related to, but not exactly, (twisted) Dirac operators on $\Omega^\bullet(M)$. If these operators, or some construction related to them, has a parallel $Cliff(8, 8)$ action, it would descend to the finite-dimensional space $\ker \mathcal{D}_\pm$, giving it the structure of a $Cliff(8, 8)$ module. The residue of this, as defined in e.g. [88], would give a \mathbb{Z} -valued index for the manifold which, under general arguments, should be invariant under continuous deformations of the operator [89, 90]. We would like to see if and how this index is related to other indices on eight-dimensional manifolds. Something even more curious happens in the $G_2 \times G_2$ case. Here, the total space of the double complex is isomorphic to two copies of the generalised spinors, possibly indicating that the correct description should be in terms of pinors. In any case, if one can find a parallel $Cliff(7, 7)$ action with respect to \mathcal{D}_\pm , then the finite-dimensional space $\ker \mathcal{D}_\pm$ becomes a $Cliff(7, 7)$ module. Due to the split signature of the Clifford algebra, the residue of this representation does not trivially vanish as one would expect for seven-dimensional manifolds. This may give a new index that could be used to distinguish G_2 structures.

More speculatively, one might hope that higher-loop contributions to the partition function can also be captured by the generalised geometry of the target space. Similarly, one might wonder whether one can use the double complexes to compute twisted worldsheet indices in the spirit of Cecotti et al. [91]. In another direction, there has also been recent progress in both the physics and mathematics literature in understanding instantons, invariants and enumerative geometry in the exceptional setting, see e.g. [92–96]. As mentioned, for instantons and their counting, the single complexes of [29] play a natural role. Then, in analogy with how the open-closed duality of the A-model and gauge theory can be used to compute Gromov–Witten invariants of Calabi–Yau manifolds [97], one could ask about the relations between our double complexes and the counting of, for example, associative submanifolds and G_2 instantons.

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A Conventions and useful identities

In this appendix, we collect our conventions together with a number of useful identities and projectors for G_2 and $Spin(7)$.

A.1 Conventional geometry

Given a conventional connection ∇ on M , we can express its torsion $T \in \Gamma(TM \otimes \Lambda^2 T^*M)$ as

$$\begin{aligned} \nabla_m v^n &= \partial_m v^n + \Gamma_m{}^n{}_p v^p, \\ T(v, w) &= \nabla_v w - \nabla_w v - [v, w], \\ T^m{}_{np} &= \Gamma_n{}^m{}_p - \Gamma_p{}^m{}_n, \end{aligned} \tag{A.1}$$

where $[,]$ is the Lie bracket. The curvature of ∇ is then given by the Riemann tensor $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes \text{End } TM)$, defined by

$$\begin{aligned} \mathcal{R}(u, v)w &= [\nabla_u, \nabla_v]w - \nabla_{[u, v]}w, \\ \mathcal{R}_{mn}{}^p{}_q w^q &= [\nabla_m, \nabla_n]w^p - T^q{}_{mn} \nabla_q w^p, \end{aligned} \tag{A.2}$$

with the Ricci tensor and Ricci scalar defined by

$$\mathcal{R}_{mn} = \mathcal{R}_{pm}{}^p{}_n, \quad \mathcal{R} = g^{mn} \mathcal{R}_{mn}. \tag{A.3}$$

We define the generalised Kronecker delta as

$$\delta_{n_1 \dots n_p}^{m_1 \dots m_p} = \delta_{n_1}^{[m_1} \dots \delta_{n_p}^{m_p]}, \tag{A.4}$$

so that its components are zero or $\pm \frac{1}{p!}$. In particular, this convention implies

$$\delta_{n_1 \dots n_p}^{m_1 \dots m_p} \alpha^{n_1 \dots n_p} = \alpha^{[m_1 \dots m_p]}. \tag{A.5}$$

A.2 G_2

We use a (conventional) orthonormal frame $g_{mn} = \delta_{mn}$ and take the seven-dimensional gamma matrices to furnish a representation of $\text{Cliff}(7; \mathbb{R})$ with $\gamma^{(8)} = \gamma^1 \dots \gamma^7 = -i\mathbb{1}$.²⁹ We take the G_2 structure to be defined by a Majorana spinor ϵ normalised such that $\bar{\epsilon}\epsilon = 1$. The G_2 -invariant 3-form φ and its Hodge dual $*\varphi$ are defined as

$$\varphi_{mnp} = -i\bar{\epsilon}\gamma_{mnp}\epsilon, \quad (*\varphi)_{m_1\dots m_4} = -\bar{\epsilon}\gamma_{m_1\dots m_4}\epsilon. \quad (\text{A.6})$$

In an orthonormal frame, these can be written as

$$\begin{aligned} \varphi &= e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567}, \\ *\varphi &= e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457}. \end{aligned} \quad (\text{A.7})$$

Identities

Using Fierz identities on products of four ϵ 's, one can show the following identities hold:

$$\begin{aligned} \varphi^{m_1 m_2 p} \varphi_{n_1 n_2 p} &= 2\delta_{n_1 n_2}^{m_1 m_2} + (*\varphi)^{m_1 m_2}_{n_1 n_2}, \\ \varphi^{m p_1 p_2} \varphi_{n p_1 p_2} &= 6\delta^m_n, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \varphi^{m_1 m_2 m_3} \varphi_{m_1 m_2 m_3} &= 42, \\ (*\varphi)^{m_1 m_2 m_3 p} \varphi_{n_1 n_2 p} &= -6\delta^{[m_1}_{[n_1} \varphi_{n_2]}^{m_2 m_3]}, \\ (*\varphi)^{m_1 m_2 p_1 p_2} \varphi_{n p_1 p_2} &= 4\varphi^{m_1 m_2}_n. \end{aligned} \quad (\text{A.9})$$

One also has

$$\begin{aligned} (*\varphi)^{m_1\dots m_4} (*\varphi)_{n_1\dots n_4} &= 24\delta_{n_1\dots n_4}^{m_1\dots m_4} + 72\delta_{[n_1 n_2}^{[m_1 m_2} (*\varphi)^{m_3 m_4]}_{n_3 n_4]} \\ &\quad - 16\delta^{[m_1}_{[n_1} \varphi^{m_2 m_3 m_4]}_{n_2 n_3 n_4]}, \\ (*\varphi)^{m_1 m_2 m_3 p} (*\varphi)_{n_1 n_2 n_3 p} &= 6\delta_{n_1 n_2 n_3}^{m_1 m_2 m_3} + 9\delta^{[m_1}_{[n_1} (*\varphi)^{m_2 m_3]}_{n_2 n_3]} - \varphi^{m_1 m_2 m_3} \varphi_{n_1 n_2 n_3}, \\ (*\varphi)^{m_1 m_2 p_1 p_2} (*\varphi)_{n_1 n_2 p_1 p_2} &= 8\delta_{n_1 n_2}^{m_1 m_2} + 2(*\varphi)^{m_1 m_2}_{n_1 n_2}, \\ (*\varphi)^{m p_1 p_2 p_3} (*\varphi)_{n p_1 p_2 p_3} &= 24\delta^m_n, \\ (*\varphi)^{p_1\dots p_4} (*\varphi)_{p_1\dots p_4} &= 168. \end{aligned} \quad (\text{A.10})$$

²⁹One can take the gamma matrices to be imaginary and antisymmetric, so that a Majorana spinor is real and obeys $\bar{\epsilon} = \epsilon^T$ [98].

Other useful identities include

$$\begin{aligned}\varphi^{m_1 m_2 m_3} \varphi_{n_1 n_2 n_3} &= 3\varphi^{[m_1 m_2}_{[n_1} \varphi^{m_3]}_{n_2 n_3]} + 6\delta^{[m_1}_{[n_1} (*\varphi)^{m_2 m_3]}_{n_2 n_3]}, \\ \varphi_{[m_1 m_2 m_3} \varphi_{m_4 m_5]}^p &= (*\varphi)_{[m_1 \dots m_4} \delta_{m_5]}^p.\end{aligned}\tag{A.11}$$

Projectors on forms

It is useful to have explicit expressions for the various projectors onto representations of G_2 . We define the projectors $\mathcal{P}_{\mathbf{r}}^p$, which project a p -form onto the \mathbf{r} representation, so that

$$\begin{aligned}\lambda_{mm'} &= (\mathcal{P}_{\mathbf{7}}^2 + \mathcal{P}_{\mathbf{14}}^2)_{mm'}{}^{nn'} \lambda_{nn'}, \\ \sigma_{m_1 m_2 m_3} &= (\mathcal{P}_{\mathbf{1}}^3 + \mathcal{P}_{\mathbf{7}}^3 + \mathcal{P}_{\mathbf{27}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} \sigma_{n_1 n_2 n_3}.\end{aligned}\tag{A.12}$$

In indices, these projectors are given by

$$\begin{aligned}(\mathcal{P}_{\mathbf{7}}^2)_{mm'}{}^{nn'} &= \frac{1}{3} \left(\delta_{mm'}^{nn'} + \frac{1}{2} (*\varphi)_{mm'}{}^{nn'} \right), \\ (\mathcal{P}_{\mathbf{14}}^2)_{mm'}{}^{nn'} &= \frac{1}{3} \left(2\delta_{mm'}^{nn'} - \frac{1}{2} (*\varphi)_{mm'}{}^{nn'} \right), \\ (\mathcal{P}_{\mathbf{1}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} &= \frac{1}{42} \varphi_{m_1 m_2 m_3} \varphi^{n_1 n_2 n_3}, \\ (\mathcal{P}_{\mathbf{7}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} &= \frac{1}{4} \frac{1}{3!} (*\varphi)_{m_1 m_2 m_3 q} (*\varphi)^{n_1 n_2 n_3 q}, \\ (\mathcal{P}_{\mathbf{27}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} &= \frac{3}{4} \delta_{m_1 m_2 m_3}^{n_1 n_2 n_3} - \frac{3}{8} \delta_{[m_1}^{[n_1} (*\varphi)_{m_2 m_3]}^{n_2 n_3]} + \frac{1}{56} \varphi_{m_1 m_2 m_3} \varphi^{n_1 n_2 n_3}.\end{aligned}\tag{A.13}$$

From these we can obtain useful relations like

$$*\varphi_{mn}{}^{pq} (\lambda_{\mathbf{7}})_{pq} = 4(\lambda_{\mathbf{7}})_{mn}, \quad *\varphi_{mn}{}^{pq} (\lambda_{\mathbf{14}})_{pq} = -2(\lambda_{\mathbf{14}})_{mn}.\tag{A.14}$$

A.3 *Spin*(7)

We use an orthonormal frame $g_{mn} = \delta_{mn}$ and take the eight-dimensional gamma matrices to furnish a representation of $\text{Cliff}(8; \mathbb{R})$ with $\gamma^{(9)} = \gamma^1 \dots \gamma^8 = \mathbb{1}$.³⁰ We take the *Spin*(7) structure to be defined by a chiral Majorana spinor ϵ , with chirality $\gamma^{(9)}\epsilon = \epsilon$ normalised such that $\bar{\epsilon}\epsilon = 1$. The self-dual *Spin*(7)-invariant 4-form is defined as

$$\Theta_{mnpq} = \bar{\epsilon} \gamma_{mnpq} \epsilon.\tag{A.15}$$

³⁰One can take the gamma matrices to be real and symmetric, so that a Majorana spinor obeys $\bar{\epsilon} = \epsilon^T$ [98].

In an orthonormal frame, this can be written as

$$\begin{aligned} \Theta = & -e^{1234} - e^{1256} - e^{1278} - e^{3456} - e^{3478} - e^{5678} - e^{1357} \\ & + e^{1368} + e^{1458} + e^{1467} + e^{2358} + e^{2367} + e^{2457} - e^{2468}. \end{aligned} \quad (\text{A.16})$$

Identities

Again, using Fierz rearrangement one can show the following identities hold:

$$\Theta^{mnpq}\Theta_{mnpq} = 336, \quad (\text{A.17})$$

$$\Theta^{qmnpr}\Theta_{rmnp} = 42\delta_r^q, \quad (\text{A.18})$$

$$\Theta^{pqmnr}\Theta_{rsmn} = 12\delta_{rs}^{pq} - 4\Theta^{pq}{}_{rs}, \quad (\text{A.19})$$

$$\Theta^{ijkm}\Theta_{pqrm} = 6\delta_{pqr}^{ijk} - 9\Theta^{[ij}{}_{[pq}\delta_r^{k]}. \quad (\text{A.20})$$

Projectors on forms

We define the projectors $\mathcal{P}_{\mathbf{r}}^p$, which project a p -form onto the \mathbf{r} representation of $Spin(7)$, so that

$$\lambda_{mm'} = (\mathcal{P}_{\mathbf{7}}^2 + \mathcal{P}_{\mathbf{21}}^2)_{mm'}{}^{nn'} \lambda_{nn'}, \quad (\text{A.21})$$

$$\sigma_{m_1 m_2 m_3} = (\mathcal{P}_{\mathbf{8}}^3 + \mathcal{P}_{\mathbf{48}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} \sigma_{n_1 n_2 n_3}, \quad (\text{A.22})$$

$$\tau_{m_1 \dots m_4} = (\mathcal{P}_{\mathbf{1}}^4 + \mathcal{P}_{\mathbf{7}}^4 + \mathcal{P}_{\mathbf{27}}^4 + \mathcal{P}_{\mathbf{35}}^4)_{m_1 \dots m_4}{}^{n_1 \dots n_4} \tau_{n_1 \dots n_4}. \quad (\text{A.23})$$

In indices, these projectors are given by

$$(\mathcal{P}_{\mathbf{7}}^2)_{mm'}{}^{nn'} = \frac{1}{4} \left(\delta_{mm'}^{nn'} - \frac{1}{2} \Theta_{mm'}{}^{nn'} \right), \quad (\text{A.24})$$

$$(\mathcal{P}_{\mathbf{21}}^2)_{mm'}{}^{nn'} = \frac{3}{4} \left(\delta_{mm'}^{nn'} + \frac{1}{6} \Theta_{mm'}{}^{nn'} \right), \quad (\text{A.25})$$

$$(\mathcal{P}_{\mathbf{8}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} = \frac{1}{7} \left(\delta_{m_1 m_2 m_3}^{n_1 n_2 n_3} - \frac{3}{2} \Theta_{[m_1 m_2}{}^{[n_1 n_2} \delta_{m_3]}^{n_3]} \right), \quad (\text{A.26})$$

$$(\mathcal{P}_{\mathbf{48}}^3)_{m_1 m_2 m_3}{}^{n_1 n_2 n_3} = \frac{1}{7} \left(6 \delta_{m_1 m_2 m_3}^{n_1 n_2 n_3} + \frac{3}{2} \Theta_{[m_1 m_2}{}^{[n_1 n_2} \delta_{m_3]}^{n_3]} \right), \quad (\text{A.27})$$

$$(\mathcal{P}_{\mathbf{1}}^4)_{m_1 \dots m_4}{}^{n_1 \dots n_4} = \frac{1}{336} \Theta_{m_1 \dots m_4}{}^{n_1 \dots n_4}, \quad (\text{A.28})$$

$$\begin{aligned} (\mathcal{P}_{\mathbf{7}}^4)_{m_1 \dots m_4}{}^{n_1 \dots n_4} = & \frac{1}{8} \left(\delta_{m_1 \dots m_4}^{n_1 \dots n_4} - \frac{3}{2} \Theta_{[m_1 m_2}{}^{[n_1 n_2} \delta_{m_3 m_4]}^{n_3 n_4]} \right. \\ & \left. - \frac{1}{6} \Theta_{[m_1 \dots m_3}{}^{[n_1 \Theta_{m_4]}{}^{n_2 \dots n_4]} \right), \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned} (\mathcal{P}_{\mathbf{27}}^4)_{m_1 \dots m_4}{}^{n_1 \dots n_4} = & \frac{1}{8} \left(3 \delta_{m_1 \dots m_4}^{n_1 \dots n_4} + \frac{15}{2} \Theta_{[m_1 m_2}{}^{[n_1 n_2} \delta_{m_3 m_4]}^{n_3 n_4]} \right. \\ & \left. - \frac{1}{2} \Theta_{[m_1 \dots m_3}{}^{[n_1 \Theta_{m_4]}{}^{n_2 \dots n_4]} + \frac{1}{7} \Theta_{m_1 \dots m_4}{}^{n_1 \dots n_4} \right), \end{aligned} \quad (\text{A.30})$$

$$\begin{aligned}
(\mathcal{P}_{35}^4)_{m_1 \dots m_4}{}^{n_1 \dots n_4} &= \frac{1}{8} \left(4 \delta_{m_1 \dots m_4}^{n_1 \dots n_4} - 6 \Theta_{[m_1 m_2} [n_1 n_2 \delta_{m_3 m_4}]^{n_3 n_4} \right. \\
&\quad \left. + \frac{2}{3} \Theta_{[m_1 \dots m_3} [n_1 \Theta_{m_4}]^{n_2 \dots n_4}] - \frac{1}{6} \Theta_{m_1 \dots m_4} \Theta^{n_1 \dots n_4} \right). \tag{A.31}
\end{aligned}$$

Note in particular the helpful relations

$$\Theta_{mn}{}^{pq}(\lambda_7)_{pq} = -6(\lambda_7)_{mn}, \quad \Theta_{mn}{}^{pq}(\lambda_{21})_{pq} = 2(\lambda_{21})_{mn}. \tag{A.32}$$

Differential operators

Using the decomposition of differential forms into $Spin(7)$ representations and taking $f \in \Omega_1^0$, $\alpha \in \Omega_8^1$, $\beta \in \Omega_7^2$, $\gamma \in \Omega_{21}^2$, $\delta \in \Omega_{48}^3$, $\mu \in \Omega_{27}^4$ and $\nu \in \Omega_{35}^4$, one can write the exterior derivative as combinations of the following operators

$$d_8^1: \Omega_1^0 \rightarrow \Omega_8^1, \quad d_8^1 f = df, \tag{A.33}$$

$$d_7^8: \Omega_8^1 \rightarrow \Omega_7^2, \quad d_7^8 \alpha = \mathcal{P}_7^2 d\alpha, \tag{A.34}$$

$$d_{21}^8: \Omega_8^1 \rightarrow \Omega_{21}^2, \quad d_{21}^8 \alpha = \mathcal{P}_{21}^2 d\alpha, \tag{A.35}$$

$$d_{35}^8: \Omega_8^1 \rightarrow \Omega_{35}^4, \quad d_{35}^8 \alpha = \mathcal{P}_{35}^4 d * (\alpha \wedge \Theta), \tag{A.36}$$

$$d_{48}^7: \Omega_7^2 \rightarrow \Omega_{48}^3, \quad d_{48}^7 \beta = \mathcal{P}_{48}^3 d\beta, \tag{A.37}$$

$$d_{48}^{21}: \Omega_{21}^2 \rightarrow \Omega_{48}^3, \quad d_{48}^{21} \gamma = \mathcal{P}_{48}^3 d\gamma, \tag{A.38}$$

$$d_{27}^{48}: \Omega_{48}^3 \rightarrow \Omega_{27}^4, \quad d_{27}^{48} \delta = \mathcal{P}_{27}^4 d\delta, \tag{A.39}$$

$$d_{35}^{48}: \Omega_{48}^3 \rightarrow \Omega_{35}^4, \quad d_{35}^{48} \delta = \mathcal{P}_{35}^4 d\delta. \tag{A.40}$$

We also impose $(d_{\mathbf{p}}^{\mathbf{q}})^\dagger = d_{\mathbf{p}}^{\mathbf{q}}$, where the adjoint is defined by the standard inner product on differential forms. Adapting the arguments made in [39], one can find the following decomposition of the exterior derivative:

$$df = d_8^1 f, \tag{A.41}$$

$$d(f\Theta) = d_8^1 f \wedge \Theta, \tag{A.42}$$

$$d\alpha = d_7^8 \alpha + d_{21}^8 \alpha, \tag{A.43}$$

$$d * (\alpha \wedge \Theta) = -\frac{1}{2} d_1^8 \alpha \Theta + \frac{1}{2} (d_7^8 \alpha) \cdot \Theta + d_{35}^8 \alpha, \tag{A.44}$$

$$d(\alpha \wedge \Theta) = d_7^8 \alpha \wedge \Theta + d_{21}^8 \alpha \wedge \Theta, \tag{A.45}$$

$$d * \alpha = - * d_1^8 \alpha, \tag{A.46}$$

$$d\beta = -\frac{3}{7} * (d_8^7 \beta \wedge \Theta) + d_{48}^7 \beta, \tag{A.47}$$

$$d(\beta \cdot \Theta) = -\frac{16}{7} d_8^7 \beta \wedge \Theta + 4 * d_{48}^7 \beta, \tag{A.48}$$

$$d * \beta = *d_{\mathbf{8}}^7 \beta, \quad (\text{A.49})$$

$$d\gamma = \frac{1}{7} * (d_{\mathbf{8}}^{21} \gamma \wedge \Theta) + d_{\mathbf{48}}^{21} \gamma, \quad (\text{A.50})$$

$$d * \gamma = *d_{\mathbf{8}}^{21} \gamma, \quad (\text{A.51})$$

$$d\delta = \frac{1}{8} (d_{\mathbf{7}}^{48} \delta) \cdot \Theta + d_{\mathbf{27}}^{48} \delta + d_{\mathbf{35}}^{48} \delta, \quad (\text{A.52})$$

$$d * \delta = - * d_{\mathbf{7}}^{48} \delta - * d_{\mathbf{21}}^{48} \delta, \quad (\text{A.53})$$

$$d\mu = *d_{\mathbf{48}}^{27} \mu, \quad (\text{A.54})$$

$$d\nu = \frac{1}{7} d_{\mathbf{8}}^{35} \nu \wedge \Theta - *d_{\mathbf{48}}^{35} \nu. \quad (\text{A.55})$$

In the above, we have used the notation $\beta \cdot \Theta$ to denote the isomorphism $\Omega_7^2 \rightarrow \Omega_7^4$ given by

$$(\beta \cdot \Theta)_{abcd} = 4\beta_{[a]i} \Theta_{i|bcd]}. \quad (\text{A.56})$$

Laplacians

One can use the relations above to prove various identities for determinants of Laplacians on $Spin(7)$ manifolds. In particular, one has

$$\det' \Delta_{\mathbf{8}} = (\det' \Delta_{\mathbf{7}})(\det' \Delta_{\mathbf{1}}), \quad (\text{A.57})$$

$$\det' \Delta_{\mathbf{35}} = (\det' \Delta_{\mathbf{27}})(\det' \Delta_{\mathbf{8}}), \quad (\text{A.58})$$

$$\det' \Delta_{\mathbf{48}} = (\det' \Delta_{\mathbf{27}})(\det' \Delta_{\mathbf{21}}). \quad (\text{A.59})$$

As an example, we will prove the first of these relations – the others follow similarly. Using the decomposition of the exterior derivative into $d_{\mathbf{q}}^{\mathbf{p}}$, and the fact that $d^2\alpha = d^2\beta = 0$, for $\alpha \in \Omega_{\mathbf{8}}^1$ and $\beta \in \Omega_7^2$ we have

$$\Delta_{\mathbf{1}} f = d_{\mathbf{1}}^{\mathbf{8}} d_{\mathbf{8}}^{\mathbf{1}} f, \quad (\text{A.60})$$

$$\Delta_{\mathbf{7}} \beta = 4d_{\mathbf{7}}^{\mathbf{8}} d_{\mathbf{8}}^{\mathbf{7}} \beta, \quad (\text{A.61})$$

$$\Delta_{\mathbf{8}} \alpha = d_{\mathbf{8}}^{\mathbf{1}} d_{\mathbf{1}}^{\mathbf{8}} \alpha + 4d_{\mathbf{8}}^{\mathbf{7}} d_{\mathbf{7}}^{\mathbf{8}} \alpha. \quad (\text{A.62})$$

Then one finds

$$\begin{aligned}
\det' \Delta_8 &= \det'(d_8^1 d_1^8 + 4d_8^7 d_7^8) \\
&= \det'((d_8^1 + 2d_8^7)(d_1^8 + 2d_7^8)) \\
&= \det'((d_1^8 + 2d_7^8)(d_8^1 + 2d_8^7)) \\
&= \det'(d_1^8 d_8^1 + 2d_1^8 d_8^7 + 2d_7^8 d_8^1 + 4d_7^8 d_8^7) \\
&= \det'(d_1^8 d_8^1 + 4d_7^8 d_8^7) \\
&= \det'(\Delta_1 + \Delta_7) \\
&= (\det' \Delta_1)(\det' \Delta_7).
\end{aligned} \tag{A.63}$$

Note that, in going from the second to the third line, we used the fact that \det' is the (ζ -regularised) product of non-zero eigenvalues. In going from the fourth line to the fifth, we used $d_7^8 d_8^1 = 0$, which is simply the statement that (3.30) is a complex. The final identity follows from noting that $\Delta_1 \Delta_7 = \Delta_7 \Delta_1 = 0$.

B Determinants and partition functions

B.1 ζ -regularised determinants

We give a brief outline of ζ -regularised determinants and their properties [99]. We follow the notation of [9].

Given an increasing sequence of positive real numbers $A = \{a_1, a_2, \dots\}$, we define the ζ -regularised sum of the numbers to be $\zeta_A(-1)$ where, for large $\text{Re } s$, we define

$$\zeta_A(s) = \sum_n a_n^{-s} \quad \text{Re } s \gg 0, \tag{B.1}$$

and then extended to the whole of \mathbb{C} by analytic continuation. The ζ -regularised product of A is then defined to be

$$e^{-\zeta'_A(0)}, \tag{B.2}$$

where the prime denotes differentiation with respect to the complex parameter s .

Given a vector space V and an operator $A: V \rightarrow V$ with only non-negative real eigenvalues, we define the ζ -regularised determinant, denoted $\det' A$, to be the ζ -regularised product of its non-zero eigenvalues. That is

$$\det' A := e^{-\zeta'_A(0)} \tag{B.3}$$

where we have used the same symbol for the operator and its sequence of non-zero eigenvalues. For an operator $B: V \rightarrow W$, we note the useful identity

$$|\det 'B| := (\det 'B^\dagger B)^{1/2}. \quad (\text{B.4})$$

Note that since we neglect the zero eigenvalues, we also have

$$\det 'B^\dagger B = \det 'B B^\dagger. \quad (\text{B.5})$$

Given two operators $A, B: V \rightarrow V$ that obey $AB = BA = 0$, it is simple to show

$$\det '(A + B) = \det 'A \det 'B. \quad (\text{B.6})$$

These determinants are useful when looking at Laplacians of differential operators. We will highlight some useful identities for these determinants in the de Rham and Dolbeault complexes. The results in the latter case all naturally generalise to the $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ complexes we discuss in this paper.

First consider the de Rham Laplacian $\Delta = dd^\dagger + d^\dagger d$. Denoting the space of p -forms on an n -dimensional manifold M by Ω^p , the Hodge decomposition gives

$$\Omega^p = d\Omega^{p-1} \oplus d^\dagger\Omega^{p+1} \oplus H^p. \quad (\text{B.7})$$

Figure 4(a) shows this pictorially: we can associate the exact (resp. co-exact) subspaces with the left (resp. right) region surrounding the node in the de Rham complex. Note that, by definition, H^p is the zero eigenspace of Δ and so can be neglected when calculating $\det '\Delta$. Hence, we can write

$$\det '\Delta^p = \det '(\Delta^p) \det '(\Delta'^p), \quad (\text{B.8})$$

where Δ^p is the restriction of Δ^p to $d\Omega^{p-1}$, and Δ'^p is the restriction to $d^\dagger\Omega^{p+1}$. Observe that one can identify

$$'\Delta = dd^\dagger, \quad \Delta' = d^\dagger d. \quad (\text{B.9})$$

Due to (B.5), we see that

$$\det '(\Delta^p) = \det '(\Delta'^{p-1}). \quad (\text{B.10})$$

Furthermore, since the Hodge star commutes with the Laplacian, we have

$$\det '\Delta^p = \det '\Delta^{n-p}, \quad (\text{B.11})$$

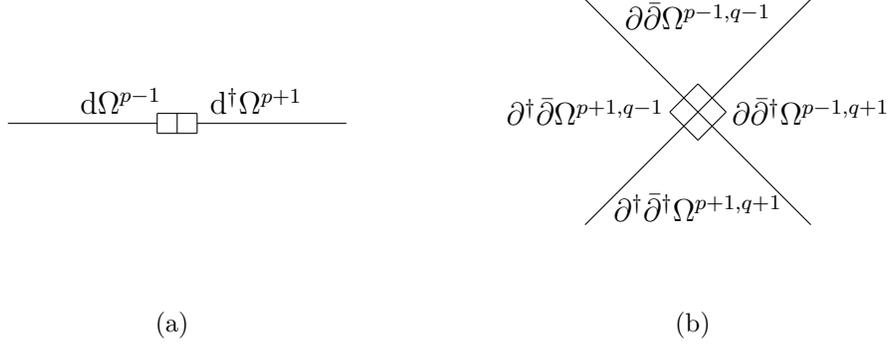


Figure 4. A pictorial representation of the Hodge decomposition of differential forms (neglecting harmonic forms). Figure (a) shows the de Rham decomposition of Ω^p into exact and co-exact pieces which we can view as coming from the left and right of the node respectively. Figure (b) shows the Dolbeault decomposition of $\Omega^{p,q}$. Pictorially, the subspaces can be associated with the squares surrounding the node, corresponding to the direction the double differential maps from.

where $n = \dim M$.

For Kähler manifolds, we can refine this further. We have the Laplacians for ∂ , $\bar{\partial}$ and d which are proportional:

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta. \quad (\text{B.12})$$

We also have the Hodge decompositions of $\Omega^{p,q}$ with respect to ∂ and $\bar{\partial}$:

$$\Omega^{p,q} = \partial\Omega^{p-1,q} \oplus \partial^\dagger\Omega^{p+1,q} \oplus H_\partial^{p,q}, \quad (\text{B.13})$$

$$= \bar{\partial}\Omega^{p,q-1} \oplus \bar{\partial}^\dagger\Omega^{p,q+1} \oplus H_{\bar{\partial}}^{p,q}. \quad (\text{B.14})$$

By (B.12), we have equality of the spaces of harmonic forms, $H_\partial^{p,q} = H_{\bar{\partial}}^{p,q}$, which means that we can combine the Hodge decompositions above and write

$$\Omega^{p,q} = \partial\bar{\partial}\Omega^{p-1,q-1} \oplus \partial\bar{\partial}^\dagger\Omega^{p-1,q+1} \oplus \partial^\dagger\bar{\partial}\Omega^{p+1,q-1} \oplus \partial^\dagger\bar{\partial}^\dagger\Omega^{p+1,q+1} \oplus H^{p,q}. \quad (\text{B.15})$$

Once again, $H^{p,q}$ is the zero eigenspace of Δ and so can be ignored when computing \det' . The Laplacian Δ then decomposes according to its action on the four subspaces in (B.15). One can identify these subspaces with the four squares surrounding a vertex in the Hodge diamond, as shown in Figure 4(b). Given this decomposition, one finds

$$\det' \Delta = (\det' \overset{\bullet}{\Delta})(\det' \Delta \bullet)(\det' \overset{\bullet}{\Delta})(\det' \bullet \Delta), \quad (\text{B.16})$$

where the position of \bullet denotes the restriction of Δ to the relevant subspace as labelled in Figure 4(b). For example

$$\dot{\Delta} = \Delta|_{\partial\bar{\partial}\Omega^{p-1,q-1}}. \quad (\text{B.17})$$

Note that we also use this notation for the spaces $\mathcal{A}^{p,q}$, with the replacement $(\partial, \bar{\partial}) \rightarrow (d_+, d_-)$. Using (B.5) and (B.12) one then finds

$$\det' \dot{\Delta}^{p,q} = \det' (\Delta \bullet)^{p,q-1} = \det' \dot{\Delta}^{p-1,q-1} = \det' (\bullet \Delta)^{p-1,q}. \quad (\text{B.18})$$

Hence, the value of the determinant depends only on the “square” in the Hodge diamond and not the vertex (as is shown in Figure 1).³¹ One can then use the symmetries of the Hodge diamond to relate the value of determinants on different squares. In particular, for a Calabi–Yau n -fold, one can use Hodge duality, complex conjugation, and contraction with the holomorphic n -form to see that

$$\det' \Delta^{p,q} = \det' \Delta^{q,p} = \det' \Delta^{n-p,q} = \det' \Delta^{p,n-q} = \det' \Delta^{n-p,n-q}. \quad (\text{B.19})$$

For a Calabi–Yau threefold, this leaves us with three independent determinants, as shown in Figure 1.

All of this generalises to the $G_2 \times G_2$ and $Spin(7) \times Spin(7)$ complexes, where for G_2 the maps θ_{\pm} play the role of Hodge duality and contraction with the holomorphic n -form. A small distinction is that in general there is no notion of “complex conjugation” and so $\det' \hat{\Delta}^{p,q} \neq \det' \hat{\Delta}^{q,p}$. However, when one has a genuine G_2 or $Spin(7)$ structure, these determinants are in fact equal, leading to Figures 2 and 3.

B.2 Direct calculation of partition function

In Section 5.2 of the main text, we gave a calculation of the partition function of the target-space theory for the $G_2 \times G_2$ double complex (4.19) via BV quantisation. Here, following [9], we will show that this calculation agrees with a direct calculation using formal manipulations of the path integral.

The partition function of the theory is

$$Z = \frac{1}{V(\mathcal{G})} \int \mathcal{D}a \mathcal{D}b \mathcal{D}c e^{-S_0}, \quad (\text{B.20})$$

where the measure is for the fields b_{11} , a_{00} and c_{22} , S_0 is the quadratic target-space action (5.11), and $V(\mathcal{G})$ is the volume of the gauge group.

³¹These were referred to as the determinants of the “face” Laplacians in [9].

$b_{11} \wedge d_+ d_- b_{11}$

Let us start by focusing on S_0^a , the term in the action that depends on b_{11} . Since S_0^a is a quadratic action for a single real bosonic field with a second-order kinetic operator, the path integral over b_{11} is formally given by

$$\begin{aligned} Z^a &= \frac{1}{V(\mathcal{G}^a)} \frac{V(H^{1,1}) V(\bullet \mathcal{A}^{1,1}) V(\dot{\mathcal{A}}^{1,1}) V(\mathcal{A}^{\bullet 1,1})}{\sqrt{\det' d_+ d_- |_{\dot{\mathcal{A}}^{1,1}}}} \\ &= \frac{1}{V(\mathcal{G}^a)} \frac{V(H^{1,1}) V(\bullet \mathcal{A}^{1,1}) V(\dot{\mathcal{A}}^{1,1}) V(\mathcal{A}^{\bullet 1,1})}{\sqrt{\det' \hat{\Delta}_C}}, \end{aligned} \quad (\text{B.21})$$

where we are again denoting the volume of a (formally infinite-dimensional) space Ω by $V(\Omega)$. Here the determinant comes from the integral over the component of b_{11} that is orthogonal to gauge transformations (so that the restricted kinetic operator has no kernel). One then needs to account for b_{11} that do come from gauge transformations, which in this case is simply the product of the volumes of $\bullet \mathcal{A}^{1,1}$, $\dot{\mathcal{A}}^{1,1}$ and $\mathcal{A}^{\bullet 1,1}$. One can think of these as b_{11} that are of the form $d_+ b_{01} + d_- b_{10}$, where b_{10} and b_{01} can be independent of each other. We then need to compute the (formal) volumes of the spaces.

First consider $d_- : \mathcal{A}^{\bullet 1,0} \rightarrow \bullet \mathcal{A}^{1,1}$. Since d_- is an invertible map between real vector spaces, the ratio of the volumes is

$$\frac{V(\bullet \mathcal{A}^{1,1})}{V(\mathcal{A}^{\bullet 1,0})} = \sqrt{\det' d_-^\dagger d_- |_{\mathcal{A}^{\bullet 1,0}}} = \sqrt{\det' \hat{\Delta}_B}, \quad (\text{B.22})$$

where we have observed that the operator $d_-^\dagger d_-$ acting on $\mathcal{A}^{\bullet 1,0}$ is simply $\hat{\Delta}_B$. Using the Hodge decomposition, the volume of $\mathcal{A}^{\bullet 1,0}$ is

$$V(\mathcal{A}^{\bullet 1,0}) = V(\mathcal{A}^{1,0}) V(\mathcal{A}^{\bullet 1,0}) V(H^{1,0}), \quad (\text{B.23})$$

where $V(H^{1,0})$ is the space of $\hat{\Delta}$ -harmonic $(1,0)$ -forms. Finally, using the map $d_+ : \mathcal{A}^{0,0} \rightarrow \mathcal{A}^{\bullet 1,0}$, we have

$$\frac{V(\mathcal{A}^{\bullet 1,0})}{V(\mathcal{A}^{0,0})} = \sqrt{\det' d_+^\dagger d_+ |_{\mathcal{A}^{0,0}}} = \sqrt{\det' \hat{\Delta}_A}. \quad (\text{B.24})$$

Noting that $V(\mathcal{A}^{0,0}) = V(\mathcal{A}^{\bullet,0}) V(H^{0,0})$, we then have

$$V(\bullet\mathcal{A}^{1,1}) = \frac{V(H^{0,0}) \sqrt{\det' \hat{\Delta}_B}}{V(H^{1,0}) \sqrt{\det' \hat{\Delta}_A}} \frac{V(\mathcal{A}^{1,0})}{V(\mathcal{A}^{0,0})}. \quad (\text{B.25})$$

One can find $V(\mathcal{A}^{\bullet,1})$ in a similar fashion:

$$V(\mathcal{A}^{\bullet,1}) = \frac{V(H^{0,0}) \sqrt{\det' \hat{\Delta}_{B'}}}{V(H^{0,1}) \sqrt{\det' \hat{\Delta}_A}} \frac{V(\mathcal{A}^{0,1})}{V(\mathcal{A}^{0,0})}. \quad (\text{B.26})$$

Finally, we need to calculate $V(\dot{\mathcal{A}}^{1,1})$. Given $d_- : \mathcal{A}^{\bullet,1,0} \rightarrow \dot{\mathcal{A}}^{1,1}$, one has

$$\frac{V(\dot{\mathcal{A}}^{1,1})}{V(\mathcal{A}^{\bullet,1,0})} = \sqrt{\det' d_-^\dagger d_- |_{\mathcal{A}^{\bullet,1,0}}} = \sqrt{\det' \hat{\Delta}_A}. \quad (\text{B.27})$$

Using the above expressions for $V(\mathcal{A}^{\bullet,1,0})$, one then finds

$$V(\dot{\mathcal{A}}^{1,1}) = \frac{1}{V(H^{0,0})} \det' \hat{\Delta}_A V(\mathcal{A}^{0,0}). \quad (\text{B.28})$$

Putting this together, one has

$$Z^a = \frac{1}{V(\mathcal{G}^a)} \frac{V(H^{1,1}) V(H^{0,0})}{V(H^{1,0}) V(H^{0,1})} \left(\frac{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}{\det' \hat{\Delta}_C} \right)^{1/2} \frac{V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}{V(\mathcal{A}^{0,0})}. \quad (\text{B.29})$$

Formally, one identifies $V(\mathcal{G}^a)$ with $V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1}) / V(\mathcal{A}^{0,0})$, so that the gauge group \mathcal{G}^a can be thought of as gauge transformations by $(1, 0)$ and $(0, 1)$ fields, modulo $(0, 0)$ -form ghosts. The contribution to the partition function from S_0^a is then

$$Z^a = \frac{V(H^{1,1}) V(H^{0,0})}{V(H^{1,0}) V(H^{0,1})} \left(\frac{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}{\det' \hat{\Delta}_C} \right)^{1/2}. \quad (\text{B.30})$$

$\mathbf{a}_{00} \wedge \mathbf{d}_+ \mathbf{d}_- \mathbf{c}_{22}$

We now compute the contribution of S_0^b , which depends on a_{00} and c_{22} . In this case, since S_0^b is an action for two real bosonic fields with a second-order kinetic operator,

the path integral over a_{00} and c_{22} is formally given by

$$\begin{aligned} Z^b &= \frac{1}{V(\mathcal{G}^b)} \frac{V(H^{0,0}) V(H^{2,2}) V(\bullet\mathcal{A}^{2,2}) V(\dot{\mathcal{A}}^{2,2}) V(\mathcal{A}^{\bullet 2,2})}{\det' d_+ d_-|_{\dot{\mathcal{A}}^{2,2}}} \\ &= \frac{1}{V(\mathcal{G}^b)} \frac{V(H^{0,0}) V(H^{2,2}) V(\bullet\mathcal{A}^{2,2}) V(\dot{\mathcal{A}}^{2,2}) V(\mathcal{A}^{\bullet 2,2})}{\det' \hat{\Delta}_A}, \end{aligned} \quad (\text{B.31})$$

where again the determinant comes from the component of c_{22} that is orthogonal to gauge transformations (so that the restricted kinetic operator has no kernel). Note that a_{00} has no gauge transformations. Again, we need the volumes of the spaces appearing above.

Consider first the map $d_- : \mathcal{A}^{2,1} \rightarrow \bullet\mathcal{A}^{2,2}$ so that

$$\frac{V(\bullet\mathcal{A}^{2,2})}{V(\mathcal{A}^{\bullet 2,1})} = \sqrt{\det' d_-^\dagger d_-|_{\mathcal{A}^{2,1}}} = \sqrt{\det' \hat{\Delta}_{B'}}. \quad (\text{B.32})$$

We also note that

$$V(\mathcal{A}^{2,1}) = V(\dot{\mathcal{A}}^{2,1}) V(\mathcal{A}^{\bullet 2,1}) V(\mathcal{A}^{\bullet 2,1}) V(\bullet\mathcal{A}^{2,1}) V(H^{2,1}). \quad (\text{B.33})$$

We now want to write the volume of the various subspaces of $\mathcal{A}^{2,1}$ in terms of lower-degree $\mathcal{A}^{p,q}$. For example, we have

$$V(\bullet\mathcal{A}^{2,1}) = \frac{V(H^{1,0})}{V(H^{2,0}) V(H^{0,0})} \frac{\det' \hat{\Delta}_A}{\sqrt{\det' \hat{\Delta}_B}} \frac{V(\mathcal{A}^{2,0}) V(\mathcal{A}^{0,0})}{V(\mathcal{A}^{1,0})}. \quad (\text{B.34})$$

Similar calculations for $V(\mathcal{A}^{\bullet 2,1})$ and $V(\dot{\mathcal{A}}^{2,1})$ give

$$V(\mathcal{A}^{\bullet 2,1}) = \frac{V(H^{0,1}) V(H^{1,0})}{V(H^{0,0}) V(H^{1,1})} \frac{\sqrt{\det' \hat{\Delta}_C}}{\det' \hat{\Delta}_B} \frac{V(\mathcal{A}^{1,1}) V(\mathcal{A}^{0,0})}{V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}, \quad (\text{B.35})$$

$$V(\dot{\mathcal{A}}^{2,1}) = \frac{V(H^{0,0})}{V(H^{1,0})} \frac{\det' \hat{\Delta}_B}{\sqrt{\det' \hat{\Delta}_A}} \frac{V(\mathcal{A}^{1,0})}{V(\mathcal{A}^{0,0})}. \quad (\text{B.36})$$

Using these we have

$$V(\bullet\mathcal{A}^{2,2}) = \frac{V(H^{2,0}) V(H^{1,1}) V(H^{0,0})}{V(H^{2,1}) V(H^{1,0}) V(H^{0,1})} \frac{\det' \hat{\Delta}_B}{\sqrt{\det' \hat{\Delta}_A \det' \hat{\Delta}_C}} \frac{V(\mathcal{A}^{2,1}) V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}{V(\mathcal{A}^{2,0}) V(\mathcal{A}^{1,1}) V(\mathcal{A}^{0,0})}, \quad (\text{B.37})$$

from which it is simple to see

$$V(\mathcal{A}\bullet^{2,2}) = \frac{V(H^{0,2}) V(H^{1,1}) V(H^{0,0})}{V(H^{1,2}) V(H^{1,0}) V(H^{0,1})} \frac{\det' \hat{\Delta}_{B'}}{\sqrt{\det' \hat{\Delta}_A \det' \hat{\Delta}_C}} \frac{V(\mathcal{A}^{1,2}) V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}{V(\mathcal{A}^{0,2}) V(\mathcal{A}^{1,1}) V(\mathcal{A}^{0,0})}. \quad (\text{B.38})$$

Finally we need $V(\dot{\mathcal{A}}^{2,2})$ which is given by

$$V(\dot{\mathcal{A}}^{2,2}) = \frac{V(H^{1,0}) V(H^{0,1})}{V(H^{1,1}) V(H^{0,0})} \frac{\det' \hat{\Delta}_C}{\sqrt{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}} \frac{V(\mathcal{A}^{1,1}) V(\mathcal{A}^{0,0})}{V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}. \quad (\text{B.39})$$

Putting this all together, we find that the contribution to the partition function is

$$Z^b = \frac{1}{V(\mathcal{G}^b)} \frac{V(H^{2,2}) V(H^{2,0}) V(H^{0,2}) V(H^{1,1}) V(H^{0,0})^2}{V(H^{2,1}) V(H^{1,2}) V(H^{1,0}) V(H^{0,1})} \frac{\sqrt{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}}{\det'^2 \hat{\Delta}_A} \times \frac{V(\mathcal{A}^{2,1}) V(\mathcal{A}^{1,2}) V(\mathcal{A}^{1,0}) V(\mathcal{A}^{0,1})}{V(\mathcal{A}^{2,0}) V(\mathcal{A}^{0,2}) V(\mathcal{A}^{1,1}) V(\mathcal{A}^{0,0})}. \quad (\text{B.40})$$

Again, taking $V(\mathcal{G}^b)$ to cancel the various volumes of the spaces of forms, this simplifies to

$$Z^b = \frac{V(H^{2,2}) V(H^{2,0}) V(H^{0,2}) V(H^{1,1}) V(H^{0,0})^2}{V(H^{2,1}) V(H^{1,2}) V(H^{1,0}) V(H^{0,1})} \frac{\sqrt{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}}{\det'^2 \hat{\Delta}_A}. \quad (\text{B.41})$$

Final result

Combining the contributions from S_0^a and S_0^b , the partition function is given by

$$Z = \frac{V(H^{2,2}) V(H^{2,0}) V(H^{0,2}) V(H^{1,1})^2 V(H^{0,0})^3}{V(H^{2,1}) V(H^{1,2}) V(H^{1,0})^2 V(H^{0,1})^2} \frac{\det' \hat{\Delta}_B \det' \hat{\Delta}_{B'}}{\det'^2 \hat{\Delta}_A \sqrt{\det' \hat{\Delta}_C}}. \quad (\text{B.42})$$

Upon taking the cohomologies to be trivial and setting $\det' \hat{\Delta}_A = A$, and so on, we have $Z = BB'A^{-2}C^{-1/2}$ in agreement with both the double complex calculation in Section 5.1 and the BV quantisation calculation in Section 5.2.

C Review of $O(d, d) \times \mathbb{R}^+$ generalised geometry

Generalised geometry is a geometric formalism in which one extends the tangent bundle by a sequence of differential forms to create a vector bundle $T \hookrightarrow E$ that has an enlarged structure group $GL(d, \mathbb{R}) \hookrightarrow \mathcal{G}$. The case we will be interested in is the geometry defined by the vector bundle

$$E = T \oplus T^*, \quad (\text{C.1})$$

with sections or *generalised vectors* written as $V = v + \lambda$. This bundle has a natural $O(d, d)$ structure which preserves a symmetric bilinear form

$$\eta(V, V) = v \lrcorner \lambda. \quad (\text{C.2})$$

One can then take tensor products of E and decompose them according to $O(d, d)$ representations, and sections of such bundles are called generalised tensors.

One can also define a bracket $\llbracket \cdot, \cdot \rrbracket$ which gives E the structure of an exact Courant algebroid [100, 101].³² It is called the Courant bracket and is given by

$$\llbracket v + \lambda, w + \mu \rrbracket = \mathcal{L}_v w + \mathcal{L}_v \mu - \mathcal{L}_w \lambda - \frac{1}{2} d(v \lrcorner \mu - w \lrcorner \lambda), \quad (\text{C.3})$$

where $W = w + \mu$ is another section of E .

The Courant bracket is clearly covariant under diffeomorphisms and also under a closed 2-form transformation $b \in \Omega_{\text{cl}}^2(M)$ given by

$$e^b(v + \lambda) = v + \lambda - v \lrcorner b. \quad (\text{C.4})$$

We therefore have an enlarged automorphism group of the Courant algebroid given by $\text{Diff} \hookrightarrow \text{GDiff} = \text{Diff} \ltimes \Omega_{\text{cl}}^2(M)$, whose elements we refer to as generalised diffeomorphisms. These are generated by a local derivative along a generalised vector $V = v + \lambda$ called the Dorfman derivative. The action of the Dorfman derivative is

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \mu - w \lrcorner d\lambda. \quad (\text{C.5})$$

Note that this is not antisymmetric but instead satisfies

$$\frac{1}{2}(L_V W - L_W V) = \llbracket V, W \rrbracket, \quad \frac{1}{2}(L_V W + L_W V) = d\eta(V, W). \quad (\text{C.6})$$

³²One also needs a smooth bundle map $a: E \rightarrow T$ called the anchor in the definition of the Courant algebroid. We will normally take this to just be the projection onto T in (C.1).

We can naturally incorporate the NSNS flux H into the construction by twisting the Dorfman derivative (and hence the Courant bracket) to get the flux twisted derivative

$$L_V^H W = \mathcal{L}_v w + \mathcal{L}_v \mu - w \lrcorner d\lambda + w \lrcorner (v \lrcorner H). \quad (\text{C.7})$$

An alternative but equivalent way to include the flux is to take (C.1) to just be a local definition and allow non-trivial patching by $\Omega_{\text{cl}}^2(M)$. That is, for an open subset $\mathcal{U}_i \in M$ and $V_i = v_i + \lambda_i \in \Gamma(\mathcal{U}_i, E)$, $V_j = v_j + \lambda_j \in \Gamma(\mathcal{U}_j, E)$, there exists a $\Lambda_{ij} \in \Omega^1(\mathcal{U}_i \cap \mathcal{U}_j)$ such that, on $\mathcal{U}_i \cap \mathcal{U}_j$

$$\begin{aligned} v_i &= v_j \\ \lambda_i &= \lambda_j - v_j \lrcorner d\Lambda_{ij} \end{aligned} \quad \Rightarrow \quad V_i = e^{\text{d}\Lambda_{ij}} V_j. \quad (\text{C.8})$$

This patching defines a bundle E^H . The equivalence of (C.7) and (C.8) comes from choosing a global isomorphism $E^H \simeq E$. To do so, one must pick a connection B which is locally a 2-form, and patches as³³

$$\begin{aligned} B_i &= B_j + \text{d}\Lambda_{ij} && \text{on } \mathcal{U}_i \cap \mathcal{U}_j, \\ \Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} &= \text{d}\Lambda_{ijk} && \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k, \end{aligned} \quad (\text{C.9})$$

where $\Lambda_{ijk} \in C^\infty(\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k)$. The flux is determined by this connection via $H = \text{d}B$ locally. It is easy to see from the patching (C.8) and (C.9) that $V \in \Gamma(E)$ if and only if $e^B V \in \Gamma(E^H)$. Moreover, it is easy to check that $L_{e^B V} e^B W = e^B L_V^H W$. Hence a choice of B defines an isomorphism of algebroids

$$(E^H, L) \quad \longleftrightarrow \quad (E, L^H). \quad (\text{C.10})$$

It is possible to show that a twist by H and H' are equivalent as algebroids if and only if $H' = H + \text{d}\alpha$. That is, inequivalent exact Courant algebroids are classified by $[H] \in H^3(M)$ [100].³⁴ This equivalence of twisted bundle versus twisted derivative applies to all generalised tensor bundles and we will often move between the two pictures in (C.10) and will drop the superscript H to avoid cluttering our notation further.

Since it geometrises the H flux, generalised geometry turns out to be naturally well suited to describe the NSNS sector of string backgrounds. In fact, as was shown in [45], one can also account for the dilaton by enlarging the structure group further

³³This non-trivial constraint on triple intersections means B is a connective structure on a gerbe [102].

³⁴In addition, this class must be quantised in string theory.

to $O(d, d) \times \mathbb{R}^+$. All tensors should then be appropriately weighted under the \mathbb{R}^+ by including factors of $\det T^*$ in the bundles. In particular, we can consider weighted generalised vectors $\tilde{V} \in \Gamma(\tilde{E})$ and the induced action of the $O(d, d)$ metric η

$$\tilde{E} = E \otimes \det T^* \quad \Rightarrow \quad \eta(\tilde{V}, \tilde{W}) \in \Gamma((\det T^*)^2). \quad (\text{C.11})$$

The $O(d, d)$ structure defines a Clifford algebra via

$$\{\Gamma^A, \Gamma^B\} = \eta^{AB}, \quad (\text{C.12})$$

where η^{AB} are the components of the $O(d, d)$ inner product η in some orthonormal frame. One can show that this has a natural representation on the exterior algebra, so that weighted p -forms $(\det T^*)^{-1/2} \otimes \Lambda^\bullet T^*$ form a spinor representation of $Spin(d, d) \times \mathbb{R}^+$. We then call any $\rho \in \Gamma((\det T^*)^{-1/2} \otimes \Lambda^\bullet T^*)$ a generalised spinor and denote the vector bundle of generalised spinors by S . Note that S is reducible as an $O(d, d) \times \mathbb{R}^+$ representation. There exists a notion of chirality and we can define even/odd spinors to be even/odd polyforms. That is

$$S = S_+ \oplus S_-, \quad S_\pm = (\det T^*)^{-1/2} \otimes \Lambda^{\text{ev/odd}} T^*. \quad (\text{C.13})$$

More generally we can define the weighted spinor bundles

$$S_\pm^{(p)} = (\det T^*)^p \otimes S_\pm, \quad (\text{C.14})$$

so that $S_\pm^{(1/2)}$ corresponds to unweighted polyforms.

There exists a natural $O(d, d)$ -invariant pairing on S called the Mukai pairing. Taking $\rho, \mu \in \Gamma(S_\pm^{(p)})$, it is given by

$$\langle \rho, \mu \rangle = \sum_i \rho_i \wedge \sigma(\mu_{d-i}) \in \Gamma((\det T^*)^{2p}), \quad (\text{C.15})$$

where ρ_i means the restriction of the polyform ρ to its degree i component, and $\sigma: S \rightarrow S$ is the automorphism defined by³⁵

$$\sigma(\mu_k) = (-1)^{k(k+1)/2} \mu_k. \quad (\text{C.16})$$

Note that for d even, $\langle \cdot, \cdot \rangle$ restricts to a pairing on $S_\pm^{(1/2)}$, and that for $d = 6$ this pairing defines a non-degenerate symplectic structure.

³⁵This convention is different than the one chosen in e.g. [42]

As with conventional geometry, one has a notion of connections, torsion and curvature. A generalised connection is simply a first-order linear differential operator D which acts on a generalised vector in frame indices as

$$D_A V^B = \partial_A V^B + \Omega_A{}^B{}_C V^C, \quad (\text{C.17})$$

where ∂_A denotes the natural embedding of the ordinary partial derivative in E , and the generalised connection one-form Ω takes values in the adjoint representation of $O(d, d) \times \mathbb{R}^+$, so that the action of D has the obvious extension to any generalised tensor with arbitrary conformal weight. The generalised torsion T of such a connection is a generalised tensor defined in terms of the Dorfman derivative (C.5) by

$$T(V) \cdot \alpha = L_V^D \alpha - L_V \alpha, \quad (\text{C.18})$$

where $V \in \Gamma(E)$ and α is a generalised tensor. One might also expect there exists a generalised analogue of the Riemannian curvature, however the naive object one would define turns out to not be tensorial, and we find that there is no useful notion of “generalised curvature” for an arbitrary generalised connection without specifying additional structure.

C.1 $O(d) \times O(d)$ structures

A generalised metric is given by a reduction of $O(d, d) \times \mathbb{R}^+$ to the maximal compact subgroup $O(d) \times O(d)$ [42, 45, 103]. As for many conventional G -structures, it is defined by a set of globally non-vanishing tensors (Φ, G) , where $\Phi \in \Gamma(\det T^*)$ – which specifies the isomorphism between weighted and un-weighted generalised vectors $\tilde{E} \cong E$ – and $G: S^2 E \rightarrow \mathbb{R}$ is a positive-definite inner product on E that is compatible with the $O(d, d)$ metric (C.2) in the following sense. Using η as an isomorphism $E \cong E^*$, we can view $G: E \rightarrow E$ and then require $G^2 = 1$. Given such a G , we get a decomposition of E into eigenbundles of G so that

$$E = C_+ \oplus C_-, \quad (\text{C.19})$$

where C_\pm are η -orthogonal subbundles of E such that $\eta|_{C_\pm}$ is positive (resp. negative) definite. The inner product G can then be written

$$G = \eta|_{C_+} - \eta|_{C_-}. \quad (\text{C.20})$$

Hence a choice of G is equivalent to a choice of decomposition (C.19).

An alternative definition of a generalised metric (Φ, G) is given via a choice of conformal split frame of \hat{E} . This is defined to be a local frame $\{\hat{E}_a^+\} \cup \{\hat{E}_{\bar{a}}^-\}$ such that

$$\eta(\hat{E}_a^+, \hat{E}_b^+) = \Phi^2 \delta_{ab}, \quad (\text{C.21})$$

$$\eta(\hat{E}_{\bar{a}}^-, \hat{E}_{\bar{b}}^-) = -\Phi^2 \delta_{\bar{a}\bar{b}}, \quad (\text{C.22})$$

$$\eta(\hat{E}_a^+, \hat{E}_{\bar{b}}^-) = 0. \quad (\text{C.23})$$

This determines Φ uniquely and defines C_{\pm} to be the span of $\{\hat{E}^{\pm}\}$.

A generalised metric is equivalent to a choice of conventional metric g , B -field, and dilaton ϕ . Indeed, given two independent local orthonormal frames \hat{e}_a^+ , $\hat{e}_{\bar{a}}^-$ of T , the conformal split frame can be written as

$$\begin{aligned} \hat{E}_a^+ &= e^{-2\phi} \sqrt{g} (\hat{e}_a^+ + \iota_{\hat{e}_a^+} g + \iota_{\hat{e}_a^+} B), \\ \hat{E}_{\bar{a}}^- &= e^{-2\phi} \sqrt{g} (\hat{e}_{\bar{a}}^- - \iota_{\hat{e}_{\bar{a}}^-} g + \iota_{\hat{e}_{\bar{a}}^-} B). \end{aligned} \quad (\text{C.24})$$

In several applications, it is useful to evaluate $O(d) \times O(d)$ expressions in which one chooses frames such that $\hat{e}_a^+ = \hat{e}_{\bar{a}}^- = \hat{e}_a$ are aligned.

A generalised G -structure is said to be integrable if there exists a torsion-free generalised connection that is compatible with the structure. An $O(d) \times O(d)$ structure is thus torsion-free if there exists a generalised connection D that satisfies

$$DG = 0, \quad D\Phi = 0, \quad L_V^D = L_V, \quad (\text{C.25})$$

where L_V^D is the Dorfman derivative with all partial derivatives replaced with the connection D . We call a connection that satisfies these constraints a generalised Levi-Civita connection. As was shown in [45], such connections always exist but are not unique. Using a split frame, the torsion-free connection acting on $V = v^a \hat{E}_a^+ + v^{\bar{a}} \hat{E}_{\bar{a}}^-$ takes the form

$$\begin{aligned} D_a v^b &= \nabla_a v^b - \frac{1}{6} H_a{}^b{}_c v^c - \frac{2}{d-1} (\delta_a{}^b \partial_c \phi - \delta_{ac} \partial^b \phi) v^c + A_a^{+b}{}_c v^c, \\ D_{\bar{a}} v^b &= \nabla_{\bar{a}}^- v^b \equiv \nabla_{\bar{a}} v^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c v^c, \\ D_a v^{\bar{b}} &= \nabla_a^+ v^{\bar{b}} \equiv \nabla_a v^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}}, \\ D_{\bar{a}} v^{\bar{b}} &= \nabla_{\bar{a}} v^{\bar{b}} + \frac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v^{\bar{c}} - \frac{2}{d-1} (\delta_{\bar{a}}{}^{\bar{b}} \partial_{\bar{c}} \phi - \delta_{\bar{a}\bar{c}} \partial^{\bar{b}} \phi) v^{\bar{c}} + A_{\bar{a}}^{-\bar{b}}{}_{\bar{c}} v^{\bar{c}}, \end{aligned} \quad (\text{C.26})$$

where ∇ is the Levi-Civita connection for g , $H = dB$ and A^{\pm} are undetermined tensors

satisfying

$$\begin{aligned} A_{abc}^+ &= -A_{acb}^+, & A_{[abc]}^+ &= 0, & A_a^{+a}{}_b &= 0, \\ A_{\bar{a}\bar{b}\bar{c}}^- &= -A_{\bar{a}\bar{c}\bar{b}}^-, & A_{[\bar{a}\bar{b}\bar{c}]}^- &= 0, & A_{\bar{a}}^{-\bar{a}}{}_{\bar{b}} &= 0, \end{aligned} \quad (\text{C.27})$$

so that they do not contribute to the torsion. The A^\pm tensors thus parametrise the failure of the metric-compatibility and vanishing torsion conditions to specify a unique generalised connection.

Thanks to the generalised metric structure, we can use the compatible connection D to define generalised curvatures. The analogue of the Riemann tensor is not unique, i.e., depends on the choice of generalised Levi-Civita, and so it is not a very useful object. However, there exist certain contractions and projections that are uniquely defined. In particular, we can define the generalised Ricci tensor R^0 and generalised Ricci scalar R via the action of D on either generalised vectors [45]

$$R_{ab}^0 w_+^a = [D_a, D_{\bar{b}}] w_+^a, \quad R_{\bar{a}\bar{b}}^0 w_-^{\bar{a}} = [D_{\bar{a}}, D_{\bar{b}}] w_-^{\bar{a}}, \quad (\text{C.28})$$

or spinors

$$\begin{aligned} R_{ab}^0 \gamma^a \epsilon^+ &= [\gamma^a D_a, D_{\bar{b}}] \epsilon^+, & -\frac{1}{4} R \epsilon^+ &= (\gamma^a D_a \gamma^b D_b - D_{\bar{a}} D_{\bar{a}}) \epsilon^+, \\ R_{\bar{a}\bar{b}}^0 \gamma^{\bar{a}} \epsilon^- &= [\gamma^{\bar{a}} D_{\bar{a}}, D_b] \epsilon^-, & -\frac{1}{4} R \epsilon^- &= (\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a) \epsilon^-. \end{aligned} \quad (\text{C.29})$$

Here ϵ^\pm are $S(C_\pm)$ spinors and the γ^a are representations of the Clifford algebra induced by the $O(d)$ structure on C_\pm . Upon explicit evaluation, one finds

$$R_{ab}^0 = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2 \nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab}), \quad (\text{C.30})$$

$$R = \mathcal{R} + 4 \nabla^2 \phi - 4 (\partial \phi)^2 - \frac{1}{12} H^2, \quad (\text{C.31})$$

where we have aligned the frames $\hat{e}_a^+ = \hat{e}_{\bar{a}}^-$, and \mathcal{R}_{ab} and \mathcal{R} are the conventional Ricci tensor and scalar for g . The right-hand side of these are simply the equations of motion in the absence of RR fluxes, and hence both R_{ab}^0 and R vanish on-shell. In particular, since a background which is supersymmetric and solves the Bianchi identity $dH = 0$ automatically solves the equations of motion, both R_{ab}^0 and R must vanish for supersymmetric backgrounds. This crucial result is used many times in the main text.

C.2 Generalised Calabi–Yau

A generalised Calabi–Yau structure is a reduction of the structure group to $SU(n, n)$ where $d = 2n$. It is defined by a nowhere-vanishing complex pure spinor Ψ . Given such

a spinor, one can define the null space L_Ψ

$$L_\Psi = \{V \in \Gamma(E) \mid \not{V}\rho = V^A \Gamma_A \Psi = 0\}. \quad (\text{C.32})$$

A generalised Calabi–Yau structure is then given by a Ψ satisfying

$$\dim_{\mathbb{C}} L_\Psi = d, \quad \langle \Psi, \bar{\Psi} \rangle \neq 0. \quad (\text{C.33})$$

A spinor satisfying the first condition is said to be pure, and the associated null space is said to be maximally isotropic. The generalised Calabi–Yau structure is integrable (i.e. there exists a torsion-free compatible connection) if and only if

$$d\Psi = 0. \quad (\text{C.34})$$

Hitchin showed that for $d = 6$ these structures can be described via a variational problem [27]. Indeed, consider a real chiral spinor ρ which is stable in the sense of [104]. Since $\langle \cdot, \cdot \rangle$ defines an $O(d, d)$ invariant symplectic structure, there is an associated moment map $\mu: S_\pm \rightarrow \mathfrak{g}^*$ given by

$$\mu(\rho)(a) = \frac{1}{2} \langle a \cdot \rho, \rho \rangle \quad \forall a \in O(d, d). \quad (\text{C.35})$$

Then we can consider the following map which is an invariant quartic homogeneous function in ρ

$$q: S_\pm \rightarrow (\det T^*)^2, \quad q(\rho) = \text{tr}(\mu(\rho)^2). \quad (\text{C.36})$$

It turns out that ρ defines an $SU(n, n)$ structure if and only if $q(\rho) < 0$, which is an open condition on ρ . Such a ρ is known as stable. Note that $q(\rho) \in \Gamma((\det T^*)^2)$ which has a canonical orientation and hence a well-defined notion of a negative section. The real spinor ρ then becomes the real part of the complex pure spinor Ψ , with the imaginary part $\hat{\rho}$ given by the first variation of the functional

$$H(\rho) = \int_M \sqrt{-\frac{q(\rho)}{3}} \quad \Rightarrow \quad \delta H = \int_M \langle \delta \rho, \hat{\rho} \rangle. \quad (\text{C.37})$$

Note that H is a homogeneous functional of degree 2 in ρ . Denoting the space of stable spinors of definite chirality by U , one can show that there is an integrable complex structure \mathcal{J} on $\rho \in \Gamma(U)$ and that the second variation of the functional H is given by

$$\delta^2 H(\delta_1 \rho, \delta_2 \rho) = \int_M \langle \delta_1 \rho, \mathcal{J} \delta_2 \rho \rangle. \quad (\text{C.38})$$

Now suppose we fix some $\rho \in \Gamma(U)$ such that $d\rho = 0$ and only allow variations within the cohomology class of ρ . That is, we take $\delta\rho = db$ for some real polyform b . Then by (C.37) we have

$$\delta H(db) = \int_M \langle db, \hat{\rho} \rangle = \int_M \langle b, d\hat{\rho} \rangle = 0 \quad \Rightarrow \quad d\hat{\rho} = 0. \quad (\text{C.39})$$

Therefore, stationary points of H within a fixed cohomology class $[\rho]$ correspond to $SU(3, 3)$ structures with $d\rho = d\hat{\rho} = 0$, that is $d\Psi = d(\rho + i\hat{\rho}) = 0$. Hence, stationary points correspond to integrable $SU(3, 3)$ structures.

C.3 The generalised Hitchin functional for integrable $G_2 \times G_2$ structures

Turning now to the generalised geometry of a seven-dimensional manifold, in the main text we describe $G_2 \times G_2$ structures in terms of a pair of C_\pm spinors. However, following [47, 54], one can also define them through a $Spin(7, 7) \times \mathbb{R}^+$ globally defined nowhere-vanishing real chiral spinor $\rho \in S_\pm$ that is stable in the sense of [104]. By a simple dimension count, one has that the spinor lives in an open orbit of $Spin(7, 7) \times \mathbb{R}^+$.

One can define an operator \square_ρ which maps spinors of one chirality to the other given by

$$\square_\rho: S_\pm \rightarrow S_\mp, \quad \square_\rho(\alpha) = e^B * \sigma(e^{-B}\alpha), \quad (\text{C.40})$$

where α is a generalised spinor, $*$ is the Hodge operator associated to the Riemannian metric g and σ was given in (C.16). If we work in the flux twisted differential picture instead, then we can just write $\square_\rho(\alpha) = *\sigma(\alpha)$. It is possible to show that this is an $O(7) \times O(7)$ covariant map and hence a generalised $G_2 \times G_2$ structure can be equivalently described by either a stable $\rho \in \Gamma(S_\pm)$ or a stable $\square_\rho \rho \in \Gamma(S_\mp)$. The $G_2 \times G_2$ structure is then said to be integrable (there exists a compatible torsion-free generalised connection) if and only if

$$d\rho = d\square_\rho \rho = 0, \quad (\text{C.41})$$

which is the analogue of $d\Psi = d(\rho + i\hat{\rho}) = 0$ for an $SU(3, 3)$ structure. For concreteness, we will take $\rho \in \Gamma(S_-)$, and so $\square_\rho \rho \in \Gamma(S_+)$.

To match the description of generalised $G_2 \times G_2$ structures given in the main text around (4.1) where we consider the spinors $\epsilon_\pm \in S(C_\pm)$, we note that there is also a natural isomorphism between these bundles and the bundle of $O(d, d) \times \mathbb{R}^+$ spinors S as

$$S \simeq S(C_+) \otimes S(C_-), \quad (\text{C.42})$$

and under this isomorphism we associate

$$e^{-2\phi}e^B(\epsilon_+ \otimes \epsilon_-) = \rho + \square_\rho \rho. \quad (\text{C.43})$$

In general, a $G_2 \times G_2$ structure defines a local $SU(3)$ structure on the manifold. We can then write $\rho, \square_\rho \rho$ explicitly in terms of the local $SU(3)$ structure of the manifold. While the results of this paper will hold in general, we will mostly be interested in the case where the generalised structure is induced from a genuine G_2 structure. In that case, we can write

$$\rho = e^{-2\phi}e^B(-\varphi + \text{vol}), \quad (\text{C.44})$$

$$\square_\rho \rho = e^{-2\phi}e^B(1 - *\varphi). \quad (\text{C.45})$$

We can write the generalised $G_2 \times G_2$ structure more explicitly in terms of the local $SU(3)$ structure defined by the ϵ_\pm . This $SU(3)$ structure locally defines a 1-form α , a 2-form ω , and two 3-forms ψ_\pm which can be viewed as the real and imaginary parts of a holomorphic 3-form on some 6-dimensional $\mathcal{D} \subset T$ that is orthogonal to α . There is also a scalar $\cos a$, where a is the angle between ϵ_+ and ϵ_- as 8-dimensional real vectors. Without loss of generality, we can take $\rho \in \Gamma(S_-)$ and can write

$$\rho = e^{-2\phi}e^B(s\alpha - c(\psi_+ + \omega \wedge \alpha) - s\psi_- - s\frac{1}{2}\omega^2 \wedge \alpha + c\text{vol}_g), \quad (\text{C.46})$$

$$\square_\rho \rho = e^{-2\phi}e^B(c + s\omega - c(\psi_- \wedge \alpha + \frac{1}{2}\omega^2) + s\psi_+ \wedge \alpha - s\frac{1}{6}\omega^3), \quad (\text{C.47})$$

where s and c are shorthand for $\sin a$ and $\cos a$, and vol_g is the volume form defined by the metric. While individually the tensors in these expressions are defined only where $s \neq 0$, the precise combinations that appear can be written as bilinears of ϵ_\pm and so are globally defined. When $s = 0$, the spinors ϵ_\pm become parallel and the $SU(3)$ stabiliser degenerates to a G_2 defined by some 3-form φ . At these points one finds

$$\rho = e^{-2\phi}e^B(-\varphi + \text{vol}), \quad (\text{C.48})$$

$$\square_\rho \rho = e^{-2\phi}e^B(1 - *\varphi). \quad (\text{C.49})$$

As for $SU(3, 3)$ structures, one can understand integrable generalised $G_2 \times G_2$ structures via a variational approach [54]. Since $\rho \in \Gamma(S_-)$ must be in an open orbit of $Spin(7, 7) \times \mathbb{R}^+$, we can consider a function

$$q(\rho) = \langle \rho, \square_\rho \rho \rangle \in \Gamma(\det T^*), \quad (\text{C.50})$$

where we think of this as defined on $U \subset S_-$, the space of stable ρ . As shown in [54], this is a homogeneous function of degree 2 in ρ and the first variation is given by

$$\delta q(\delta\rho) = \langle \delta\rho, \square_\rho\rho \rangle. \quad (\text{C.51})$$

Integrating q over M , one obtains the Hitchin functional for $G_2 \times G_2$ structures:

$$H(\rho) = \int_M \langle \rho, \square_\rho\rho \rangle. \quad (\text{C.52})$$

If we assume that $d\rho = 0$ and vary only within a cohomology class $\delta\rho = db$, we find that the extrema of the Hitchin functional are given by

$$\delta H(\delta\rho) = \int_M \langle db, \square_\rho\rho \rangle = \int_M \langle b, d\square_\rho\rho \rangle = 0 \quad \Rightarrow \quad d\square_\rho\rho = 0. \quad (\text{C.53})$$

Hence the functional extremises on integrable $G_2 \times G_2$ structures.

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