

Control Barrier Functions With Unmodeled Dynamics Using Integral Quadratic Constraints

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Abstract—This paper presents a control design method that achieves safety for systems with unmodeled dynamics at the plant input. The proposed method combines control barrier functions (CBFs) and integral quadratic constraints (IQCs). Simplified, low-order models are often used in the design of the controller. Parasitic, unmodeled dynamics (e.g. actuator dynamics, time delays, etc) can lead to safety violations. The proposed method bounds the input-output behavior of these unmodeled dynamics in the time-domain using an α -IQC. The α -IQC is then incorporated into the CBF constraint to ensure safety. The approach is demonstrated with a simple example.

I. INTRODUCTION

This paper focuses on the design of control barrier functions (CBFs) for systems with unmodeled dynamics at the plant input. CBFs are used to design controllers that ensure the system remains within a safe set [1], [2]. Simplified, low-order models are often used for design. However, unmodeled dynamics (e.g. actuator lags, time delays, etc) can lead to safety violations as shown in Section II-B.

This paper presents a method to design CBFs while accounting for unmodeled dynamics. The approach uses the integral quadratic constraint (IQC) framework for analysis of uncertain systems [3]. The main IQC result in [3] provides frequency-domain conditions for stability of uncertain linear time-invariant (LTI) systems. Related results have been formulated using time-domain dissipation inequalities [4], [5]. The specific IQC formulation used in this paper involves a time-domain integral with an exponential weighting (see Section IV-A). This is called an α -IQC¹ and was introduced in [6], [8] for analysis of discrete-time optimization algorithms. A continuous-time formulation was given in [7]. Finally, α -IQCs were used in [9] to bound the effect of unmodeled dynamics in the design of model predictive controllers.

There is a large literature on CBFs with a good overview in [2]. The most closely related work on robust CBFs is briefly summarized. Robust control barrier functions have been developed for guaranteeing safety in the presence of \mathcal{L}_∞ bounded disturbances [10], [11], [12], [13] or stochastic disturbances [14]. The work in [15] and [16] considers robust CBFs to account for variations in the model (changes to the vector fields) and input (sector-bounded) nonlinearities, respectively. A distinguishing feature of our paper is the ability

to handle the effect unmodeled dynamics using α -IQCs. The implication is that the true state of the plant dynamics is only partially observed, i.e. the state of the unmodeled dynamics is not measured. Finally, we note that [17] provides a method to design CBFs for systems with known time delays. Our proposed method can handle unknown (but bounded) delays although with more conservatism than the approach in [17].

II. PRELIMINARIES

A. Control Barrier Functions

This section briefly summarizes the formulation to achieve safety using control barrier functions [1], [2]. Consider the feedback system with plant P , a baseline state feedback controller k , and a safety filter. The plant P is assumed to be given by the following (known) input-affine dynamics:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, and \mathcal{U} defines a set of feasible control inputs. The functions $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$, $g : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_u}$, and baseline state-feedback controller $k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ are all assumed to be locally Lipschitz continuous.

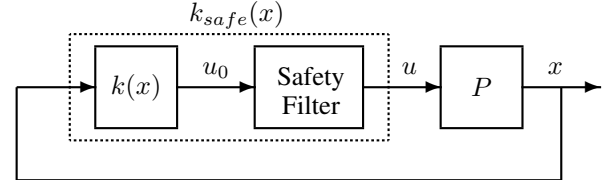


Fig. 1. State-feedback with safety filter

The state-feedback $k(x)$ achieves performance objectives but is not necessarily safe. Specifically, safety is defined by a safe set $\mathcal{C} \subset \mathbb{R}^{n_x}$ and the system is in a safe state at time t if $x(t) \in \mathcal{C}$. We consider a safe set \mathcal{C} defined with a continuously differentiable function $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$:

$$\mathcal{C} := \{x \in \mathbb{R}^{n_x} : h(x) \geq 0\} \quad (2)$$

The boundary and interior of the safe set are denoted $\partial\mathcal{C}$ and $\text{Int}(\mathcal{C})$, respectively. The closed-loop dynamics with the baseline state-feedback controller are:

$$\dot{x}(t) = f(x(t)) + g(x(t))k(x(t)), \quad x(0) = x_0 \quad (3)$$

For simplicity, assume this ordinary differential equation is forward complete, i.e. for every initial condition there exists a unique solution for all $t \geq 0$. The closed-loop is said to be *safe* if $x(0) \in \mathcal{C}$ implies $x(t) \in \mathcal{C}$ for all $t \geq 0$. As noted above, the closed-loop is not necessarily safe when using the baseline state-feedback. Control barrier functions (CBFs) are

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¹The terminology “ ρ -IQC” was first used in [6] for the discrete-time case. Later the term “ α -IQC” was used in [7] for the continuous-time formulation.

one method to design a controller ensuring the closed loop remains in the safe set \mathcal{C} . In particular, the function h is a *control barrier function* if there exists $\alpha > 0$ such that:

$$\sup_{u \in \mathcal{U}} [L_f h(x) + L_g h(x)u] \geq -\alpha h(x) \quad \forall x \in \mathbb{R}^{n_x} \quad (4)$$

where $L_f h := \frac{\partial h}{\partial x} f$ and $L_g h := \frac{\partial h}{\partial x} g$ are the Lie derivatives of h with respect to f and g . If h is a CBF and $x(t) \in \partial \mathcal{C}$ then there exists $u(t) \in \mathcal{U}$ such that $\dot{h}(x(t)) \geq 0$. Thus if the state reaches the boundary of \mathcal{C} then the control can prevent the state from crossing out of the safe set. This is formalized in Theorem 1 below. The CBF constraint (4) ensures that the following set of control inputs is non-empty for all $x \in \mathbb{R}^{n_x}$:

$$\mathcal{U}_{cbf}(x) := \{u \in \mathcal{U} : [L_f h(x) + L_g h(x)u] \geq -\alpha h(x)\}$$

The existence of a control barrier function can be used to design a controller that yields safety for the closed-loop.

Theorem 1. [1], [2] *Consider the nominal plant dynamics in (1). Let $\mathcal{C} \subset \mathbb{R}^{n_x}$ be the superlevel set of a continuously differentiable function $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ as defined in (2). Assume h satisfies (4) for some $\alpha > 0$. Then any controller $k_{safe} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ with $k_{safe}(x) \in \mathcal{U}_{cbf}(x) \quad \forall x \in \mathbb{R}^{n_x}$ renders the set \mathcal{C} forward invariant.*

Proof. This is a special case of Proposition 1 and Corollary 2 in [1]. The trajectories $x(t)$ of the closed-loop with plant (1) and controller $k_{safe}(x)$ satisfy $\dot{h}(x(t)) \geq -\alpha h(x(t))$. It follows from the Grönwall-Bellman lemma [18] that $h(t) \geq h(0)e^{-\alpha t}$ for as long as the solutions exist. Thus $h(0) \geq 0$ implies $h(t) \geq 0$ and the set \mathcal{C} is forward invariant. \square

This summary has simplified some technical details. For example, the CBF condition (4) has the term $-\alpha h(x)$ where $\alpha \in \mathbb{R}$ is a constant. The more general formulation in [1], [2] uses $-\alpha(h(x))$ where α is an extended class- \mathcal{K} function. The simplifying assumptions here allow for a proof using the Grönwall-Bellman lemma. This proof will be adapted later for the case with unmodeled dynamics. The more general results in [1], [2] follow from Nagumo's theorem [19].

Theorem 1 provides flexibility in the choice of the ‘‘safe’’ controller k_{safe} . It is useful to design a safe controller that: (i) ensures the closed loop remains in \mathcal{C} , and (ii) minimally alters the control command from the baseline state-feedback. This is achieved by solving an optimization in real-time:

$$k_{safe}(x) := \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k(x)\|^2 \quad (5)$$

s.t. $L_f h(x) + L_g h(x)u \geq -\alpha h(x)$

If $\mathcal{U} = \mathbb{R}^{n_u}$ then (5) has a quadratic cost with one linear constraint. There is an explicit solution for this special case. If \mathcal{U} is a polytope then (5) is a quadratic program and can be efficiently solved. Finally, note that k_{safe} is not necessarily Lipschitz continuous (and the proof of Theorem 1 using Grönwall-Bellman does not require Lipschitz continuity.)

B. Impact of Unmodeled Dynamics

This section presents a simple example to illustrate the impact of unmodeled dynamics. Consider a two-dimensional

point mass with position $p \in \mathbb{R}^2$ and velocity $\dot{p} \in \mathbb{R}^2$. A double-integrator model for the planar motion is given by:

$$\dot{x}(t) = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \quad (6)$$

where $x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \in \mathbb{R}^4$ is the state and $u(t) \in \mathbb{R}^2$ contains the forces. A baseline state-feedback controller is designed using linear quadratic regulator with cost matrices $Q := \text{diag}(1, 1, 1.75, 1.75)$ and $R := I_2$. This was implemented to track a position reference command $r(t) \in \mathbb{R}^2$:

$$u_0 = K \cdot \left(\begin{bmatrix} r \\ 0 \end{bmatrix} - \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \right) \quad \text{where } K := \begin{bmatrix} 1 & 0 & 1.94 & 0 \\ 0 & 1 & 0 & 1.94 \end{bmatrix}$$

This baseline corresponds to independent proportional-derivative controllers along each dimension. This differs slightly from the feedback diagram in Figure 1 due to the inclusion of the reference command, i.e. the baseline controller is of the form $u_0 = k(x, r)$.

A stationary obstacle of radius $\bar{r} = 1.5$ is assumed to be at the position $\bar{c} = [-0.2]$. The safe set \mathcal{C} is defined by Equation 2 with $h(x) := (p - \bar{c})^T (p - \bar{c}) - \bar{r}^2 \geq 0$. The time derivatives of h along a state trajectory x are given by:

$$\dot{h}(x(t)) = 2(p(t) - \bar{c})^T \dot{p}(t) \quad (7)$$

$$\ddot{h}(x(t), u(t)) = 2(p(t) - \bar{c})^T u(t) + 2\dot{p}(t)^T \dot{p}(t) \quad (8)$$

The function h is not a CBF as defined in the previous section as the control input appears in the second time derivative, i.e. it has relative degree 2.

Exponential CBFs [20], [2] can be used to design safe controllers for barrier functions with relative degree greater than 1. The basic idea can be summarized as follows. Safety is ensured if we can design a controller that achieves $\dot{h}(t) \geq -\alpha h(t)$. Specifically, $\dot{h}(t) \geq -\alpha h(t)$ and $h(0) \geq 0$ implies, under appropriate technical conditions, that $h(t) \geq 0$ for as long as the solution exists. However, the control input u does not appear in $\dot{h}(t)$ in (7). Instead, define a new function $\tilde{h} := \dot{h} + \alpha h$ and note that the desired condition is equivalent to $\tilde{h}(t) \geq 0$. Moreover, $\dot{\tilde{h}} = \ddot{h} + \alpha \dot{h}$. Hence the control input appears in $\dot{\tilde{h}}$ due to (8), i.e. \tilde{h} is relative degree 1. Thus safety is ensured, under appropriate technical conditions, if:

- (i) $h(0) \geq 0$
- (ii) $\tilde{h}(0) \geq 0 \Leftrightarrow \dot{h}(0) \geq -\alpha h(0)$
- (iii) u is chosen so that $\dot{\tilde{h}}(t) \geq -\alpha \tilde{h}(t) \Leftrightarrow u$ is chosen so that $\ddot{h}(t) \geq -\alpha^2 h(t) - 2\alpha \dot{h}(t)$

Roughly, conditions (ii) and (iii) ensure that $\tilde{h}(t) \geq 0$ which, combined with condition (i), ensures $h(t) \geq 0$. The safe controller from the exponential CBF is obtained by solving the following optimization in real-time:

$$k_{safe}(x, r) := \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k(x, r)\|^2 \quad (9)$$

s.t. $\ddot{h}(x, u) \geq -\alpha^2 h(x) - 2\alpha \dot{h}(x)$

Here $\dot{h}(x)$ and $\ddot{h}(x, u)$ denote the expressions in (7) and (8). Additional details on exponential CBFs, including a more rigorous derivation, can be found in [20], [2].

Figure 2 shows a simulation of the two-dimensional point mass with the exponential CBF controller for $\alpha = 5$. The unsafe region due to the obstacle is shaded cyan. The initial conditions are $p(0) = [-10, 0]^T$ and $\dot{p}(0) = [0, 0]^T$. The reference transitions linearly in time from this initial condition to a final desired position of $[+10, 0]^T$ at time $t = 45\text{sec}$. The simulation with the nominal plant model (black line) follows the reference and avoids the obstacle as expected. The figure also shows a simulation (red dashed) with the same controller but on a plant with an input delay $\tau = 0.13\text{sec}$. The zoomed plot on the right of Figure 2 shows that the simulation with delay has small safety violations. Larger delays cause even greater safety violations.

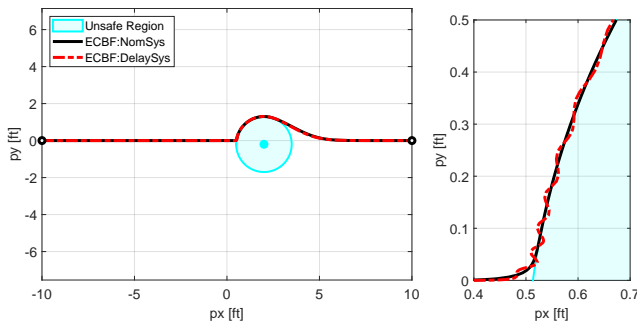


Fig. 2. Position for exponential CBF controller on nominal point mass (black) and with additional delay (red dashed) of $\tau = 0.13\text{sec}$. The left plot shows full trajectory from $(-10, 0)$ to $(10, 0)$. The right plot zooms in on trajectories near boundary of the unsafe region.

Figure 3 shows the the control inputs for the two simulations. The unmodeled delays cause the inputs to oscillate when the exponential CBF is activated (i.e. $k_{safe}(x, r) \neq k(x, r)$) between $t = 25\text{sec}$ to $t = 32\text{sec}$. Similar issues arise due to unmodeled, first-order actuator dynamics.

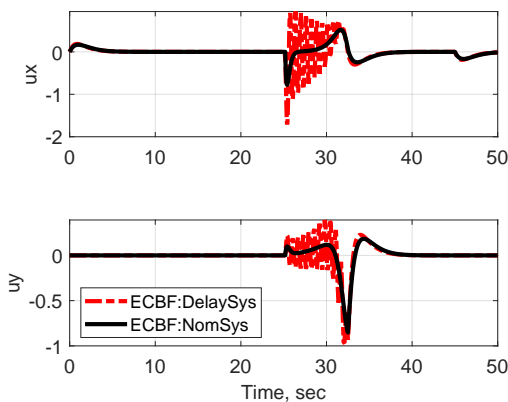


Fig. 3. Control inputs for exponential CBF controller on nominal point mass (black) and with delay (red dashed) of $\tau = 0.13\text{sec}$.

III. PROBLEM FORMULATION: ROBUST CBFs

The safety controllers designed using CBFs or exponential CBFs are often designed using low-order, approximate models. This can cause issues as indicated by the example

in the previous subsection. A method to design CBFs for systems with a known delay is given in [17]. The rest of this paper provides a method to deal with unmodeled (unknown) delays and/or unmodeled dynamics. In particular, we focus on the effect of unmodeled dynamics at the plant input. The plant with uncertainty at the input is:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))(u(t) + w(t)), \quad x(0) = x_0 \\ w(t) &= \Delta(u)(t) \end{aligned} \quad (10)$$

The uncertainty enters due to the additional input $w = \Delta(u)$. If $\Delta = 0$ then this corresponds to the nominal (known) model in Equation 1. However, Δ can have dynamics and account for deviations from the nominal dynamics due to unmodeled effects. This is demonstrated through two examples.

Example 1 (Delay). Assume that the actual plant input is $v = D_\tau(u)$ where D_τ denotes a delay of τ seconds. Thus $v = D_\tau(u)$ corresponds to $v(t) = u(t - \tau)$ for $t \geq \tau$ and $v(t) = 0$ otherwise. The effect of a delay at the plant input is modeled in Equation 10 by setting $w(t) := u(t - \tau) - u(t)$. In this case the perturbation is $\Delta := D_\tau - 1$.

Example 2 (Actuator Dynamics). Assume that the actual plant input is $V(s) = A(s)U(s)$ where $A(s)$ denotes the transfer function for neglected actuator dynamics. The effect of the neglected actuator dynamics at the plant input is modeled in (10) by setting $W(s) := V(s) - U(s) = (A(s) - 1)U(s)$. In this case the perturbation is $\Delta(s) := A(s) - 1$. For example, $A(s) = \frac{p}{s+p}$ corresponds to a simple first-order model for the actuator dynamics yielding $\Delta(s) = \frac{-s}{s+p}$.

In both examples, the signal w represents the deviation from the nominal behavior. Note that w is not simply an exogenous disturbance as it depends on the control signal through the dynamics of Δ . The objective is to design a safe controller that is robust to these unmodeled dynamics. To make this precise, assume the unmodeled dynamics are restricted to be within a known set Δ . This set is described more formally in the next section. For now it is sufficient to state that Δ provides some bounds on the uncertainty.

The objective is to find a condition on the control input u that ensures that the system remains safe for any uncertainty in the uncertainty set Δ . Formally, the goal is to design $u = k_{safe}(x)$ so that the $x(0) \in \mathcal{C}$ implies $x(t) \in \mathcal{C}$ for all $t \geq 0$ and for all $\Delta \in \Delta$. We will use a generalization of CBFs to ensure safety. First note that the nominal CBF constraint in Equation 4 depends only on the state x and the functions (f, g, h) . This is an algebraic condition that can be enforced at each time instant as part of the optimization (5). It is important to emphasize that the uncertainty $w = \Delta(u)$ has dynamics so that $w(t_0)$ depends, in general, on $u(t)$ for $t \leq t_0$. Our notion of robust CBF, defined in Section IV, will account for these dynamic couplings.

IV. ROBUST CBFs WITH UNMODELED DYNAMICS

A. Integral Quadratic Constraints (IQCs)

Our approach relies on IQCs to bound the effect of the unmodeled dynamics. We use a time-domain formulation

with an exponential weighting factor. This is based on a discrete-time formulation introduced in [6] for the analysis of optimization algorithms. A similar formulation has also been used in [8], [7] to analyze convergence rates and in [9] to design robust model-predictive controllers. A special case of a continuous-time α -IQC is defined below.²

Definition 1. Let $F(s)$ be an $n_u \times n_w$ stable, LTI system. A causal operator $\Delta : L_{2e}^{n_u}[0, \infty) \rightarrow L_{2e}^{n_w}[0, \infty)$ satisfies the time-domain α -IQC defined by $F(s)$ if the following inequality holds for all $u \in L_{2e}^{n_u}[0, \infty)$, $w = \Delta(u)$ and $T \geq 0$

$$\int_0^T e^{\alpha t} (z(t)^T z(t) - w(t)^T w(t)) dt \geq 0 \quad (11)$$

where z is the output of $F(s)$ started from zero initial conditions and driven by input u .

Definition 1 is a special case of a more general class of α -IQCs. This special case is used for exposition and more general α -IQCs can be incorporated with CBFs using the method in Section IV-B. The notation $\Delta \in IQC(F, \alpha)$ indicates that Δ satisfies the α -IQC defined by $F(s)$. The α -IQC is a constraint on the input/output pairs of Δ and $IQC(F, \alpha)$ is the set of uncertainties bounded by the α -IQC. As a special case, if Δ is SISO, $\alpha = 0$, and $F(s) = 1$ then (11) simplifies to $\int_0^T w^2(t) dt \leq \int_0^T u^2(t) dt$. This represents a constraint that the output of Δ has less energy (in the L_2 norm) than the input. The dynamics in $F(s)$ can be used to bound the effect of the uncertainty as a function of frequency. This is demonstrated in the next example.

Example 3. The uncertainty due to a delay τ is given by $w = \Delta(u)$ with $\Delta := D_\tau - 1$ as shown in Example 1. The α -IQC is derived using frequency-domain relations. Let $U(s)$, $W(s)$, and $Z(s)$ denote the Laplace Transforms of $u(t)$, $w(t)$, and $z(t)$, respectively. Thus $W(s) = \Delta(s)U(s)$ and $Z(s) = F(s)U(s)$ where $\Delta(s) = (e^{-s\tau} - 1)$. If $\alpha = 0$, we can rewrite the time-domain constraint (11) in the frequency domain using Parseval's theorem [21]:

$$\int_{-\infty}^{\infty} (|F(j\omega)|^2 - |\Delta(j\omega)|^2) \cdot |U(j\omega)|^2 d\omega \geq 0 \quad (12)$$

This condition must hold for all inputs and hence we must select $F(s)$ to satisfy $|F(j\omega)| \geq |\Delta(j\omega)| \forall \omega$. This is done by: (i) generating the frequency response of $\Delta(j\omega)$ for the given τ , and (ii) computing a stable, minimum-phase $F(s)$ with $|F(j\omega)| \geq |\Delta(j\omega)| \forall \omega$. Step (ii) can be performed via convex optimization, e.g. as done in `fitmagfrd` in Matlab. A similar process can be used if the delay is unknown but restricted to $[0, \bar{\tau}]$ for some given $\bar{\tau}$. In this case, $F(s)$ is constructed to bound the frequency responses of $\Delta(j\omega)$ generated for many delay values $\tau \in [0, \bar{\tau}]$. This can again be solved by convex optimization.

The more general case $\alpha > 0$ is handled as follows. Define

²Definition 1 uses an exponential factor $e^{\alpha t}$. Continuous-time α -IQCs have been previously defined using the factor $e^{2\alpha t}$ [7]. Either form can be converted to the other by accounting for the additional factor of 2. The version used here with $e^{\alpha t}$ aligns closely with their use later for CBFs.

$\tilde{w}(t) := e^{\frac{\alpha}{2}t} w(t)$ and similarly for \tilde{u} and \tilde{z} . Multiplication by $e^{\frac{\alpha}{2}t}$ in the time domain causes a shift in the frequency domain: $\tilde{W}(s) = W(s - \frac{\alpha}{2})$. In addition, define $\tilde{\Delta}(s) = \Delta(s - \frac{\alpha}{2})$ and $\tilde{F}(s) = F(s - \frac{\alpha}{2})$. Thus the shifted signals satisfy $\tilde{W}(s) = \tilde{\Delta}(s)\tilde{U}(s)$ and $\tilde{Z}(s) = \tilde{F}(s)\tilde{U}(s)$. The shifted filter $\tilde{F}(s)$ can be constructed to bound the frequency response of $\tilde{\Delta}(s)$ as described above. The filter for the α -IQC is obtained by shifting back: $F(s) = \tilde{F}(s + \frac{\alpha}{2})$. These steps ensure that $F(s)$ defines a valid α -IQC for the delay.

B. CBFs with IQCs

The effect of the uncertainty Δ can be incorporated into the CBF condition using the α -IQC and a Lagrange multiplier. To elaborate on this point, assume the filter $F(s)$ has the following state-space representation:

$$\begin{aligned} \dot{x}_F(t) &= A_F x_F(t) + B_F u(t), \quad x_F(0) = 0 \\ z(t) &= C_F x_F(t) + D_F u(t) \end{aligned} \quad (13)$$

where $x_F(t) \in \mathbb{R}^{n_F}$ is the state of $F(s)$. The integrand in (11) is $e^{\alpha t} I(x_F(t), u(t), w(t))$ where:

$$I(x_F, u, w) := (C_F x_F + D_F u)^T (C_F x_F + D_F u) - w^T w$$

The function h is a *robust CBF* for $\Delta \in IQC(F, \alpha)$ if there exists a Lagrange multiplier $\lambda > 0$ such that:

$$\begin{aligned} \sup_{u \in \mathcal{U}} [L_f h(x) + L_g h(x)(u + w) - \lambda I(x_F, u, w)] &\geq -\alpha h(x) \\ \forall x \in \mathbb{R}^{n_x}, \forall x_F \in \mathbb{R}^{n_F}, \forall w \in \mathbb{R}^{n_w} \end{aligned} \quad (14)$$

If h is a robust CBF then there exists $u(t) \in \mathcal{U}$ such that $\dot{h} - \lambda I \geq -\alpha h$. The following technical lemma verifies that this is sufficient to ensure safety, i.e. $h(0) \geq 0$ implies $h(t) \geq 0$ for all time. The lemma is stated for functions of time and is a variation of the Grönwall-Bellman lemma [18].

Lemma 1. Assume $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is continuously differentiable and $I : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is Lebesgue integrable. In addition, assume the following two conditions hold for some $\alpha, \lambda > 0$:

- (a) $\dot{h}(t) - \lambda I(t) \geq -\alpha h(t)$ for all $t \geq 0$
- (b) $\int_0^T e^{\alpha t} I(t) dt \geq 0$ for all $T \geq 0$

Then $h(t) \geq h(0)e^{-\alpha t}$ for all $t \geq 0$.

Proof. First, use assumption (a) to show the following:

$$\frac{d}{dt} (h(t)e^{\alpha t}) = \left(\dot{h}(t) + \alpha h(t) \right) e^{\alpha t} \stackrel{(a)}{\geq} \lambda e^{\alpha t} I(t) \quad (15)$$

Integrate this inequality from $t = 0$ to $t = T$ and apply (b):

$$h(T)e^{\alpha T} - h(0) \geq \lambda \int_0^T e^{\alpha t} I(t) dt \stackrel{(b)}{\geq} 0 \quad (16)$$

This yields $h(T) \geq h(0)e^{-\alpha T}$ for all $T \geq 0$. \square

The robust CBF constraint (14) ensures that the following set is non-empty for all $x \in \mathbb{R}^{n_x}$ and $x_F \in \mathbb{R}^{n_F}$:

$$\begin{aligned} \mathcal{U}_{rcbf}(x, x_F) := \{ &u \in \mathcal{U} : L_f h(x) + L_g h(x)(u + w) \\ &- \lambda I(x_F, u, w) \geq -\alpha h(x) \forall w \in \mathbb{R}^{n_w} \} \end{aligned}$$

It is emphasized that there is no a-priori bound on $w(t)$ at any point in time. Instead, the α -IQC provides a bound on

the energy (L_2 -norm) of w . Thus the robust CBF condition in the definition of \mathcal{U}_{rcbf} holds for all possible values of w . The next theorem states that the existence of a robust control barrier function can be used to design a controller that yields safety for all possible uncertainties in $IQC(F, \alpha)$.

Theorem 2. *Consider the uncertain plant dynamics in (10) with $\Delta \in IQC(F, \alpha)$ for some stable, LTI system F . Let $\mathcal{C} \subset \mathbb{R}^{n_x}$ be the superlevel set of a continuously differentiable function $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ as defined in (2). Assume h satisfies (14) for some $\alpha, \lambda > 0$. Then any Lipschitz continuous controller $k_{safe} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_F} \rightarrow \mathbb{R}^{n_u}$ with $k_{safe}(x, x_F) \in \mathcal{U}_{rcbf}(x, x_F) \forall (x, x_F) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_F}$ renders the set \mathcal{C} forward invariant for all $\Delta \in IQC(F, \alpha)$.*

Proof. The closed-loop with plant (10), controller $k_{safe}(x, x_F)$, and any $\Delta \in IQC(F, \alpha)$ has trajectories that satisfy the conditions (a) and (b) of Lemma 1. It follows from this Lemma that $h(0) \geq 0$ implies $h(t) \geq 0$ for as long as the solutions exist. \square

The next optimization attempts to match a baseline controller $k(x)$ while satisfying the condition in Theorem 2:

$$k_{safe}(x, x_F) := \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k(x)\|^2 \text{ subject to:}$$

$$L_f h(x) + L_g h(x)(u + w) - \lambda I(x_F, u, w) \geq -\alpha h(x) \forall w \in \mathbb{R}^{n_w}$$

The constraint is quadratic in w . The worst-case value of w is obtained by minimizing the left side to obtain:

$$w^* := -\frac{1}{2\lambda} (L_g h(x))^T \quad (17)$$

The optimization can be equivalently re-written using w^* :

$$k_{safe}(x, x_F) := \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k(x)\|^2 \text{ subject to:} \quad (18)$$

$$L_f h(x) + L_g h(x)(u + w^*) - \lambda I(x_F, u, w^*) \geq -\alpha h(x)$$

The real-time implementation requires a measurement of the state x . This can be used to form w^* . In addition, the filter $F(s)$ must be simulated with input u from initial condition $x_F(0) = 0$ to obtain $x_F(t)$. Given (x, x_F) , the optimization (18) has a convex quadratic constraint on u and a quadratic cost. This is a convex optimization (assuming $u \in \mathcal{U}$ is a convex constraint) and can be efficiently solved in real-time.

Consider the special case with the following assumptions:

(i) the filter is constant with no states, i.e. $F(s) = D_F$,
(ii) $\lambda \rightarrow \infty$, and (iii) $\lambda D_F^T D_F \rightarrow 0$. It follows from (17) and (ii) that w^* and $-\lambda(w^*)^T w^*$ tend to zero. Moreover, (iii) implies that $-\lambda(D_F u)^T (D_F u)$ tends to zero. Thus the robust CBF condition in (18) converges, under these assumptions, to the nominal CBF condition in (5). In other words, we approximately recover the nominal CBF condition by choosing a small (constant) uncertainty level for F and a large value for the Lagrange multiplier $\lambda > 0$. This provides one pragmatic approach to handle unmodeled dynamics with CBFs: Simply use a large Lagrange multiplier and a small constant F to (heuristically) provide some robustness to unmodeled dynamics. A more formal approach is to bound the unmodeled dynamics using $F(s)$ as done in Example 3.

Note that the optimization (18) is not necessarily feasible even if $\mathcal{U} = \mathbb{R}^{n_u}$ due to the quadratic term $-\lambda(C_F x_F + D_F u)^T (C_F x_F + D_F u)$. This is difficult to analyze precisely as past values of u impact the state x_F of the filter F . Smaller values of λ tend to improve feasibility but lead to more conservative paths around the unsafe set. Conversely, larger values of λ tend to degrade feasibility but more closely approximate the performance of the nominal CBF controller.

V. EXAMPLE

We will again consider the two-dimensional point mass dynamics introduced in Section II-B. Recall that we designed an exponential CBF with $\alpha = 5$ and explored the effect of an unmodeled delay of $\tau = 0.13$ sec. In this section we will use a adapt the results in Section IV to derive a robust exponential CBF for the two-dimensional point mass.

The first step is to derive a frequency domain bound on the perturbation due to the unmodeled delay. We assume the true delay τ is unknown but restricted to $[0, \bar{\tau}]$ with $\bar{\tau} = 0.13$. The corresponding perturbation $\Delta(s) = (e^{-s\tau} - 1)$ is bounded using the process described in Example 3 in Section IV-A. Figure 4 shows frequency responses (red-dashed) for $\tilde{\Delta}(s) = \Delta(s - \frac{\alpha}{2})$ with ten values of delay evenly spaced between $[0.1\bar{\tau}, \bar{\tau}]$. The first-order system $\tilde{F}(s) := \frac{2.84s + 5.81}{s + 14.48}$ satisfies $|\tilde{F}(j\omega)| \geq |\tilde{\Delta}(j\omega)| \forall \omega$ and for each delay sample. This choice of $\tilde{F}(s)$ was computed using `fitmagfrd` in Matlab. Next, the α -IQC filter is obtained by shifting the frequency: $F(s) = \tilde{F}(s + \frac{\alpha}{2})$. The state-space data for the resulting filter is $(A_F, B_F, C_F, D_F) = (-16.98, 6.20, -5.70, 2.84)$.

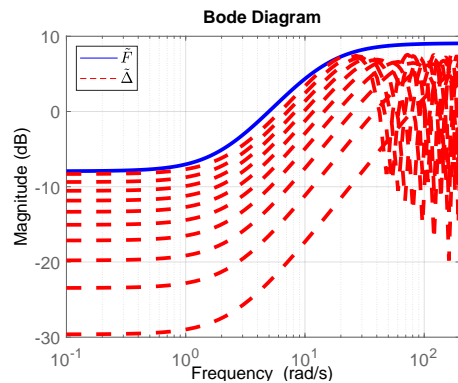


Fig. 4. The delay perturbation is $\Delta(s) = (e^{-s\tau} - 1)$. The figure shows frequency responses of (shifted) perturbation $\tilde{\Delta}(s) = \Delta(s - \frac{\alpha}{2})$ with ten samples of delay (red dashed) and a bound \tilde{F} (blue).

Equation 9 gives the optimization for safe control of the two-dimensional point mass using an exponential CBF. This can be adapted to include the α -IQC using the approach in Section IV. This leads to the following optimization that merges the exponential CBF with the α -IQC:

$$k_{safe}(x, x_F, r) := \arg \min_{u \in \mathcal{U}} \frac{1}{2} \|u - k(x, r)\|^2 \quad (19)$$

$$\text{s.t. } \ddot{h}(x, u) - \lambda I(x_F, u, w^*) \geq -\alpha^2 h(x) - 2\alpha \dot{h}(x)$$

Here $\dot{h}(x)$ and $\ddot{h}(x, u)$ denote the expressions in (7) and (8). Define $\tilde{h} = \dot{h} + \alpha h$ so that the constraint in (19) is $\tilde{h} - \lambda I \geq -\alpha \tilde{h}$. It follows from Theorem 2 that $\tilde{h}(0) \geq 0$ implies $\tilde{h}(t) \geq 0$. Moreover, $\tilde{h}(t) \geq 0$ and $h(0) \geq 0$ imply $h(t) \geq 0$ based on the discussion in Section II-B. Thus this optimization, if feasible at each time, will yield safety.

Figure 5 shows the results of the nominal exponential CBF controller (red dashed) and robust exponential CBF (blue) on the point mass dynamics with delay of $\tau = 0.13\text{sec}$. The plant has two inputs (u_x, u_y) each of which has a delay. An α -IQC for each direction was included for each delay with a Lagrange multiplier $\lambda_x = \lambda_y = 0.1$. Figure 5 shows that the robust exponential CBF controller takes a more cautious (conservative and safe) path around the obstacle. This accounts for the effect of the unmodeled dynamics.

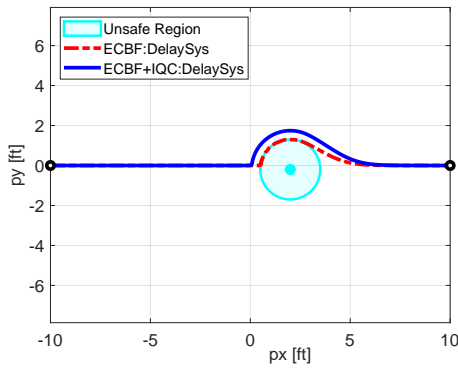


Fig. 5. Position for exponential CBF (red dashed) and robust exponential CBF (blue) controllers on point mass with delay $\tau = 0.13\text{sec}$.

Figure 6 shows the control inputs with the nominal exponential CBF controller (red dashed) and the robust version (blue). The robust version reduces the oscillations in the control signals. Smaller values for the Lagrange multipliers (λ_x, λ_y) further reduce the oscillations but also yield an even more conservative path around the obstacle.

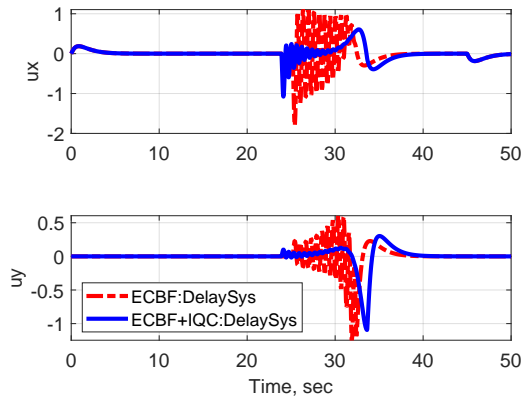


Fig. 6. Control inputs for exponential CBF (red dashed) and robust exponential CBF (blue) controllers on point mass with delay $\tau = 0.13\text{sec}$.

VI. CONCLUSIONS

This paper presented a method to design control barrier functions (CBFs) that are robust to unmodeled dynamics at the plant input, e.g. unmodeled actuator dynamics or time delays. The approach uses α -IQCs to bound the input/output behavior of the uncertainty. A robust CBF condition is derived using a version of the Grönwall-Bellman lemma.

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