

# Robust Kalman Filtering Under Model Uncertainty: the Case of Degenerate Densities

Shenglun Yi and Mattia Zorzi, *Senior Member IEEE*

## Abstract

We consider a robust state space filtering problem in the case that the transition probability density is unknown and possibly degenerate. The resulting robust filter has a Kalman-like structure and solves a minimax game: the nature selects the least favorable model in a prescribed ambiguity set which also contains non-Gaussian probability densities, while the other player designs the optimum filter for the least favorable model. It turns out that the resulting robust filter is characterized by a Riccati-like iteration evolving on the cone of the positive semidefinite matrices. Moreover, we study the convergence of such iteration in the case that the nominal model is with constant parameters on the basis of the contraction analysis in the same spirit of Bougerol. Finally, some numerical examples show that the proposed filter outperforms the standard Kalman filter.

## Index Terms

Robust Kalman filtering, minimax problem, low-rank filtering, least favorable model, contraction analysis.

## I. INTRODUCTION

As a classic estimation algorithm, the standard Kalman filter is extensively used. However, it may perform poorly in the case that the actual model does not coincide with the nominal one. For such a reason, manifold robust versions of the Kalman filter have been considered, see for instance [1]–[5].

In particular, risk sensitive Kalman filters [6]–[10] have been proposed in order to address model uncertainty. The latter consider an exponential quadratic loss function rather than the standard quadratic loss function. This means that large errors are severely penalized according to the so called risk sensitivity parameter. Hereafter, these robust Kalman filters have been proved to be equivalent to solve a minimax problem, [11]–[14]. More precisely, there are two players. One player, say nature, selects the least favorable model in a prescribed ambiguity set which is a ball

S. Yi is with the School of Automation, Beijing Institute of Technology, Beijing 100081, China; M. Zorzi is with the Department of Information Engineering, University of Padova, Via Gradenigo 6/B, 35131 Padova, Italy. Emails: 3120185460@bit.edu.cn, zorzi@dei.unipd.it

This work is partially supported by the SID project “A Multidimensional and Multivariate Moment Problem Theory for Target Parameter Estimation in Automotive Radars” (ZORZ\_SID19\_01) funded by the Department of Information Engineering of the University of Padova, Italy.

about the nominal model and whose radius reflects the amount of uncertainty in respect to the nominal model. The other player designs the optimum filter according to the least favorable model.

Recently, instead of concentrating the entire model uncertainty in a single relative entropy constraint, a new paradigm of risk sensitive filters has been proposed in [15]–[20]. The latter characterizes the uncertainty using separate constraints to each time increment of the model. In other words, the ambiguity set is specified at each time step by forming a ball, in the Kullback-Leibler (KL) topology, about the nominal model, [15], [21]. It is worth noting this ball can be defined by using also the Tau-divergence family, [22], [23]. These filters, however, are well defined only in the case that the nominal and the actual transition probability density functions are non-degenerate. This guarantees that the corresponding distorted Riccati iteration evolves on the cone of the positive definite matrices.

Unfortunately, in many practical applications, like weather forecasting and oceanography, the standard Kalman filter fails to work. More precisely, the Riccati iteration produces numerical covariance matrices which are indefinite because of their large dimension. This issue is typically addressed by resorting to a low-rank Kalman algorithm [24].

The contribution of this paper is to extend the robust paradigm in [15] to the case in which the transition probability density is possibly degenerate. Some preliminary results can be found in [25]. Within our framework, degenerate Gaussian probability densities could be also involved in the dynamic game. Accordingly, the resulting robust Kalman filter corresponds to a low-rank risk sensitive Riccati iteration. Although low-rank and distorted Riccati iterations have been widely studied in the literature, e.g. [26]–[30], our iteration appears to be new. Then, we also derive the least favorable model over a finite simulation horizon. Last but not least, we study the convergence of the distorted Riccati iteration in the case that the nominal model has constant parameters by means of the contraction analysis, [31]. It turns out that the convergence is guaranteed if the nominal model is stabilizable, the reachable subspace is observable and the ambiguity set is “small”.

The outline of the paper is as follows. Section II discusses the low-rank robust static estimation problem where the ambiguity set is characterized by a relative entropy constraint and possibly contains degenerate densities. The robust Kalman filtering problem is presented in Section III. The latter is then reformulated as a static minimax game in Section IV. In Section V, we derive the corresponding least favorable model. Section VI regards the convergence of the proposed low-rank robust Kalman filter in the case of constant parameters. In Section VII some numerical examples are provided. Finally, we draw the conclusions in Section VIII.

*Notation:* The image, the kernel and the trace of matrix  $K$  are denoted by  $\text{Im}(K)$ ,  $\ker(K)$  and  $\text{tr}(K)$ , respectively. Given a symmetric matrix  $K$ :  $K > 0$  ( $K \geq 0$ ) means that  $K$  is positive (semi)definite;  $\sigma_{max}(K)$  is the maximum eigenvalue of  $K$ ;  $K^+$  and  $\det^+(K)$  denote the pseudo inverse and the pseudo determinant of  $K$ , respectively. The Kronecker product between two matrices  $K$  and  $V$  is denoted by  $K \otimes V$ .  $x \sim \mathcal{N}(m, K)$  means that  $x$  is a Gaussian random variable with mean  $m$  and covariance matrix  $K$ .  $\mathcal{Q}^n$  is the vector space of symmetric matrices of dimension  $n$ ;  $\mathcal{Q}_+^n$  denotes the cone of positive definite symmetric matrices of dimension  $n$ , while  $\overline{\mathcal{Q}_+^n}$  denotes its closure. The diagonal matrix whose elements in the main diagonal are  $a_1, a_2, \dots, a_n$  is denoted by  $\text{diag}(a_1, a_2, \dots, a_n)$ ;  $\text{Tp}(A_1, A_2, \dots, A_n)$  denotes the block upper triangular Toeplitz matrix whose first block row is  $[A_1 \ A_2 \ \dots \ A_n]$ .

## II. LOW-RANK ROBUST STATIC ESTIMATION

Consider the problem to estimate a random vector  $x \in \mathbb{R}^n$  from an observation vector  $y \in \mathbb{R}^p$ . We assume that the nominal probability density function of  $z := [x^\top \ y^\top]^\top$  is  $f(z) \sim \mathcal{N}(m_z, K_z)$  where the mean vector  $m_z \in \mathbb{R}^{n+p}$  and the covariance matrix  $K_z \in \overline{\mathcal{Q}}_+^{n+p}$  are such that

$$m_z = \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \quad K_z = \begin{bmatrix} K_x & K_{xy} \\ K_{yx} & K_y \end{bmatrix}.$$

Moreover, we assume that  $K_z$  is possibly a singular matrix and such that  $\text{rank}(K_z) = r + p$  with  $r \leq n$  and  $K_y > 0$ . Therefore,  $f(z)$  is possibly a degenerate density whose support is the  $r + p$ -dimensional affine subspace

$$\mathcal{A} = \{m_z + v, \quad v \in \text{Im}(K_z)\}$$

and

$$f(z) = [(2\pi)^{r+p} \det^+(K_z)]^{-1/2} \times \exp \left[ -\frac{1}{2} (z - m_z)^\top K_z^+ (z - m_z) \right]. \quad (1)$$

Let  $\tilde{f}(z) \sim \mathcal{N}(\tilde{m}_z, \tilde{K}_z)$  denote the actual probability density function of  $z$  and we assume that  $\text{rank}(\tilde{K}_z) = r + p$ . Accordingly,

$$\tilde{f}(z) = [(2\pi)^{r+p} \det^+(\tilde{K}_z)]^{-1/2} \times \exp \left[ -\frac{1}{2} (z - \tilde{m}_z)^\top \tilde{K}_z^+ (z - \tilde{m}_z) \right] \quad (2)$$

with support  $\tilde{\mathcal{A}} = \{\tilde{m}_z + v, \quad v \in \text{Im}(\tilde{K}_z)\}$ . We use the KL-divergence to measure the deviation between the nominal probability density function  $f(z)$  and the actual one  $\tilde{f}(z)$ . Since the KL-divergence is not able to measure the ‘‘deterministic’’ deviations, we have to assume that the two probability density functions have the same support, i.e.  $\mathcal{A} = \tilde{\mathcal{A}}$ . In other words, we have to impose that:

$$\text{Im}(K_z) = \text{Im}(\tilde{K}_z), \quad \Delta m_z \in \text{Im}(K_z) \quad (3)$$

where  $\Delta m_z = \tilde{m}_z - m_z$ . Hence, under assumption (3), the KL-divergence between  $\tilde{f}(z)$  and  $f(z)$  is defined as

$$D(\tilde{f}, f) = \int_{\mathcal{A}} \ln \left( \frac{\tilde{f}(z)}{f(z)} \right) \tilde{f}(z) dz. \quad (4)$$

Then, if we substitute (1) and (2) in (4), we obtain

$$D(\tilde{f}, f) = \frac{1}{2} \left[ \Delta m_z^\top K_z^+ \Delta m_z + \ln \det^+(K_z) - \ln \det^+(\tilde{K}_z) + \text{tr} \left( K_z^+ \tilde{K}_z \right) - (r + p) \right].$$

**Lemma 1.** *Let  $f(z) \sim \mathcal{N}(m_z, K_z)$  and  $\tilde{f}(z) \sim \mathcal{N}(\tilde{m}_z, \tilde{K}_z)$  be Gaussian and possibly degenerate probability density functions with the same  $r + p$ -dimensional support  $\mathcal{A}$ . Let*

$$\mathcal{U} = \{\tilde{m}_z \in \mathbb{R}^{n+p} \text{ s.t. } \tilde{m}_z - m_z \in \text{Im}(K_z)\}$$

$$\mathcal{V} = \{\tilde{K}_z \in \overline{\mathcal{Q}}_+^{n+p} \text{ s.t. } \text{Im}(K_z) = \text{Im}(\tilde{K}_z)\}.$$

Then,  $D(\tilde{f}, f)$  is strictly convex with respect to  $\tilde{m}_z \in \mathcal{U}$  and  $\tilde{K}_z \in \mathcal{V} \cap \overline{\mathcal{Q}_+^{n+p}}$ . Moreover,  $D(\tilde{f}, f) \geq 0$  and equality holds if and only if  $f = \tilde{f}$ .

*Proof:* Let  $U_\tau^\top$  be a matrix whose columns form an orthonormal basis for  $\text{Im}(K_z)$ . Moreover, we define  $K_z^\tau = U_\tau K_z U_\tau^\top$  and  $\tilde{K}_z^\tau = U_\tau \tilde{K}_z U_\tau^\top$ . Since  $f(z)$  and  $\tilde{f}(z)$  have the same support, then  $\tilde{K}_z$  and  $\tilde{m}_z$  are such that  $\text{Im}(\tilde{K}_z) = \text{Im}(K_z)$  and  $\Delta m_z \in \text{Im}(K_z)$ . Thus,  $D(\tilde{f}, f)$  is strictly convex in  $\tilde{K}_z \in \mathcal{V} \cap \overline{\mathcal{Q}_+^{n+p}}$  if and only if it is strictly convex in  $\tilde{K}_z^\tau \in \mathcal{Q}_+^{r+p}$ . Then, it is not difficult to see that

$$D(\tilde{f}, f) = \frac{1}{2} \left[ \Delta m_z^\top U_\tau^\top (K_z^\tau)^{-1} U_\tau \Delta m_z + \text{tr}((K_z^\tau)^{-1} \tilde{K}_z^\tau) - \ln \det((K_z^\tau)^{-1} \tilde{K}_z^\tau) - (r + p) \right]$$

which is strictly convex in  $K_z^\tau \in \mathcal{Q}_+^{n+p}$ , see [32]. Hence, we proved the strict convexity of  $D(\tilde{f}, f)$  in  $\tilde{K}_z \in \mathcal{V} \cap \overline{\mathcal{Q}_+^{n+p}}$ . Using similar reasonings, we can conclude that  $D(\tilde{f}, f)$  is strictly convex in  $\tilde{m}_z \in \mathcal{U}$ . Finally, the unique minimum of  $D(\tilde{f}, f)$  with respect to  $\tilde{f}$  is given by the stationary conditions  $\tilde{m}_z = m_z$  and  $\tilde{K}_z = K_z$ , i.e.  $\tilde{f} = f$ . Since  $D(f, f) = 0$ , we conclude that  $D(\tilde{f}, f) \geq 0$  and equality holds if and only if  $\tilde{f} = f$ .  $\square$

In what follows, we assume that  $f$  is known while  $\tilde{f}$  is not and both have the same support. Then, we assume that  $\tilde{f}$  belongs to the ambiguity set, i.e. a ‘‘ball’’:

$$\mathcal{B} = \{\tilde{f} \sim \mathcal{N}(\tilde{m}_z, \tilde{K}_z) \text{ s.t. } D(\tilde{f}, f) \leq c\}$$

where  $c > 0$ , hereafter called tolerance, represents the radius of this ball. It is worth noting that  $f$  is usually estimated from measured data. More precisely, we fix a parametric and Gaussian model class  $\mathcal{M}$ , then we select  $f \in \mathcal{M}$  according to the maximum likelihood principle. Thus, when the length of the data is sufficiently large, we have

$$f \approx \underset{f \in \mathcal{M}}{\text{argmin}} D(\tilde{f}, f)$$

under standard hypotheses. Therefore, the uncertainty on  $f$  is naturally defined by  $\mathcal{B}$ , i.e. the actual model  $\tilde{f}$  satisfies the constraint  $\tilde{f} \in \mathcal{B}$  with  $c = D(\tilde{f}, f)$ . Finally, an estimate of  $c$  is given by  $\hat{c} = D(\tilde{f}, f)$  where  $\tilde{f}$  is estimated from measured data using a model class  $\tilde{\mathcal{M}}$  sufficiently ‘‘large’’, i.e. it contains many candidate models having diverse features.

In view of Lemma 1,  $\mathcal{B}$  is a convex set. We consider the robust estimator  $\hat{x} = g^0(y)$  solving the following minimax problem

$$(\tilde{f}^0, g^0) = \arg \min_{g \in \mathcal{G}} \max_{\tilde{f} \in \mathcal{B}} J(\tilde{f}, g) \quad (5)$$

where

$$\begin{aligned} J(\tilde{f}, g) &= \frac{1}{2} E_{\tilde{f}} [\|x - g(y)\|^2] \\ &= \frac{1}{2} \int_{\mathcal{A}} \|x - g(y)\|^2 \tilde{f}(z) dz. \end{aligned}$$

$\mathcal{G}$  is the set of estimators for which  $E_{\tilde{f}} [\|x - g(y)\|^2]$  is bounded with respect to all the Gaussian densities in  $\mathcal{B}$ .

**Theorem 1.** Let  $f$  be a Gaussian (possibly degenerate) density defined as in (1) with  $K_y > 0$ . Then, the least favorable Gaussian density  $\tilde{f}^0$  is with mean vector and covariance matrix

$$\tilde{m}_z^0 = m_z = \begin{bmatrix} m_x \\ m_y \end{bmatrix}, \quad \tilde{K}_z^0 = \begin{bmatrix} \tilde{K}_x & K_{xy} \\ K_{yx} & K_y \end{bmatrix} \quad (6)$$

so that, only the covariance of  $x$  is perturbed. Then, the Bayesian estimator

$$g^0(y) = G_0(y - m_y) + m_x,$$

with  $G_0 = K_{xy}K_y^{-1}$ , solves the robust estimation problem. The nominal posterior covariance matrix of  $x$  given  $y$  is given by

$$P := K_x - K_{xy}K_y^{-1}K_{yx}. \quad (7)$$

while the least favorable one is:

$$\tilde{P} = (P^+ - \lambda^{-1}H^\top H)^+$$

where  $H^\top \in \mathbb{R}^{n \times r}$  is a matrix whose columns form an orthonormal basis for  $\text{Im}(P)$ . Moreover, there exists a unique Lagrange multiplier  $\lambda > \sigma_{\max}(P) > 0$  such that  $c = D(\tilde{f}^0, f)$ .

*Proof:* The saddle point of  $J$  must satisfy the conditions

$$J(\tilde{f}, g^0) \leq J(\tilde{f}^0, g^0) \leq J(\tilde{f}^0, g) \quad (8)$$

for all  $\tilde{f} \in \mathcal{B}$  and  $g \in \mathcal{G}$ . The second inequality in (8) is based on the fact that the Bayesian estimator  $g^0$  minimizes  $J(\tilde{f}^0, \cdot)$ . Therefore, it remains to prove the first inequality in (8).

Notice that the minimizer of  $J(\tilde{f}, \cdot)$  is

$$g^*(y) = \tilde{K}_{xy}\tilde{K}_y^{-1}(y - \tilde{m}_y) + \tilde{m}_x$$

where

$$\tilde{m}_z := \begin{bmatrix} \tilde{m}_x \\ \tilde{m}_y \end{bmatrix}, \quad \tilde{K}_z := \begin{bmatrix} \tilde{K}_x & \tilde{K}_{xy} \\ \tilde{K}_{yx} & \tilde{K}_y \end{bmatrix}.$$

Moreover,  $J(\tilde{f}, g^*) = 1/2 \text{tr}(\tilde{P})$  where  $\tilde{P} := \tilde{K}_x - \tilde{K}_{xy}\tilde{K}_y^{-1}\tilde{K}_{yx}$ . Since  $\text{Im}(\tilde{K}_z) = \text{Im}(K_z)$ , then  $\text{Im}(\tilde{P}) = \text{Im}(P) = \text{Im}(H^\top)$ . Let  $\check{H}^\top \in \mathbb{R}^{n \times (n-r)}$  be a matrix whose columns form an orthonormal basis for the orthogonal complement of  $\text{Im}(P)$  in  $\mathbb{R}^n$ . Then,  $\check{H}^\top \check{H} + H^\top H = I_n$ . Therefore,

$$\begin{aligned} J(\tilde{f}, g^*) &= \frac{1}{2} \text{tr}(\tilde{P}(\check{H}^\top \check{H} + H^\top H)) = \frac{1}{2} \text{tr}(\tilde{P}H^\top H) \\ &= \frac{1}{2} \mathbb{E}_{\tilde{f}}[\|H(x - g(y))\|^2]. \end{aligned}$$

This means that the maximization problem can be formulated by reducing the dimension of the random vector  $z$ , i.e. we take

$$z_\tau := \begin{bmatrix} x_\tau \\ y \end{bmatrix} = U_\tau z, \quad U_\tau := \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix},$$

and  $z = U_\tau^\top z_\tau$ . The nominal and actual reduced densities are, respectively,  $f_\tau \sim \mathcal{N}(m_z^\tau, K_z^\tau)$  and  $\tilde{f}_\tau \sim \mathcal{N}(\tilde{m}_z^\tau, \tilde{K}_z^\tau)$  where  $m_z^\tau = U_\tau m_z$ ,  $\tilde{m}_z^\tau = U_\tau \tilde{m}_z$ ,  $K_z^\tau = U_\tau K_z U_\tau^\top$ ,  $\tilde{K}_z^\tau = U_\tau \tilde{K}_z U_\tau^\top$ . Accordingly, the maximization of  $J(\cdot, g^0)$  is equivalent to the maximization  $J(\tilde{f}_\tau, g_\tau^0) = \mathbb{E}_{\tilde{f}_\tau}[\|x_\tau - g_\tau^0(y)\|^2]$  where  $\tilde{f}_\tau \in \mathcal{B}_\tau := \{\tilde{f}_\tau \sim \mathcal{N}(\tilde{m}_z^\tau, \tilde{K}_z^\tau) \text{ s.t. } D(\tilde{f}_\tau, f_\tau) \leq c\}$ ,  $g_\tau^0 := Hg^0$  and

$$D(\tilde{f}_\tau, f_\tau) = \frac{1}{2} \left[ (\Delta m_z^\tau)^\top (K_z^\tau)^{-1} \Delta m_z^\tau + \text{tr}((K_z^\tau)^{-1} \tilde{K}_z^\tau) - \ln \det((K_z^\tau)^{-1} \tilde{K}_z^\tau) - (r + p) \right]$$

where  $\Delta m_z^\tau = \tilde{m}_z^\tau - m_z^\tau$ . Since  $K_z^\tau > 0$  and  $\tilde{K}_z^\tau > 0$ , by Theorem 1 in [15], [21] (see also [25]), we have that the maximizer  $\tilde{f}_\tau^0$  has mean  $m_z^\tau$  and covariance matrix

$$\tilde{K}_z^\tau = \begin{bmatrix} \tilde{K}_x^\tau & K_{xy}^\tau \\ K_{yx}^\tau & K_y^\tau \end{bmatrix}$$

where  $\tilde{K}_x^\tau = \tilde{P}^\tau + \tilde{K}_{xy}^\tau K_y^{-1} \tilde{K}_{yx}^\tau$  with

$$\tilde{P}^\tau = ((P^\tau)^{-1} - \lambda^{-1} I_r)^{-1},$$

$P^\tau := HPH^\top > 0$  and  $\lambda > \sigma_{\max}(P^\tau) > 0$  is the unique solution to

$$\frac{1}{2} \{ \ln \det P^\tau - \ln \det \tilde{P}^\tau + \text{tr}((P^\tau)^{-1} \tilde{P}^\tau - I_r) \} = c. \quad (9)$$

We conclude that the least favorable density  $\tilde{f}^0(z)$  has mean and covariance matrix as in (6). Moreover,

$$\tilde{K}_x = H^\top \tilde{K}_x^\tau H = \underbrace{H^\top \tilde{P}^\tau H}_{=: \tilde{P}} + \underbrace{H^\top K_{xy}^\tau K_y^{-1} K_{yx}^\tau H}_{=: K_{xy}}$$

and

$$\tilde{P} = H^\top ((P^\tau)^{-1} - \lambda^{-1} I_r)^{-1} H = (P^+ - \lambda^{-1} H^\top H)^+.$$

Finally, Equation (9) can be written in terms of  $P$  and  $\tilde{P}$ :

$$\frac{1}{2} \{ \ln \det^+ P - \ln \det^+ \tilde{P} + \text{tr}(P^+ \tilde{P} - I_r) \} = c$$

where  $\lambda > \sigma_{\max}(HPH^\top) = \sigma_{\max}(P)$ .

□

**Remark 1.** If  $P > 0$  then  $f$  is a non-degenerate density. In such a case, Theorem 1 still holds and: the pseudo inverse is replaced by the inverse; moreover,  $H^\top H$  becomes the identity matrix. Therefore, we recover the robust static estimation problem proposed in [21].

**Remark 2.** In Problem (5) we could add the constraint  $\text{rank}(G) \leq q$ , with  $q < r$ , i.e. the estimator  $g(y)$  represents a layered linear feedforward neural network with one low-dimensional hidden layer. Following the reasonings in [33], it is possible to characterize  $g^0$  in terms of  $\tilde{f}$ , however, it is not possible to characterize  $\tilde{f}^0$  in terms of  $g^0$ . In plain words, the introduction of such a constraint does not allow to characterize the saddle point for Problem (5). Most importantly, this constraint destroys the convexity of the set  $\mathcal{G}$ , so that the saddle point approach to the minimax problem is not applicable.

### III. LOW-RANK ROBUST KALMAN FILTER

We consider a nominal Gauss-Markov state space model of the form

$$\begin{cases} x_{t+1} = A_t x_t + B_t v_t \\ y_t = C_t x_t + D_t v_t \end{cases} \quad (10)$$

where  $x_t \in \mathbb{R}^n$  is the state vector,  $y_t \in \mathbb{R}^p$  is the measurement vector,  $A_t \in \mathbb{R}^{n \times n}$ ,  $B_t \in \mathbb{R}^{n \times (m+p)}$ ,  $C_t \in \mathbb{R}^{p \times n}$ ,  $D_t \in \mathbb{R}^{p \times (m+p)}$ ,  $v_t \in \mathbb{R}^{m+p}$  is normalized WGN and  $r_t + p := \text{rank}([B_t^\top \ D_t^\top]^\top) \leq n + p$ . Without loss of generality we can assume that  $[B_t^\top \ D_t^\top]^\top \in \mathbb{R}^{(n+p) \times (r_t+p)}$  and it is a full column rank matrix and  $v_t \in \mathbb{R}^{r_t+p}$ . Indeed, if this is not the case there always exists a lower dimensional equivalent representation of  $v_t$  for which the previous hypothesis holds. Finally, we make the standard assumption that  $D_t D_t^\top$  is positive definite. We define  $z_t := [x_{t+1}^\top \ y_t^\top]^\top$ . Let  $f_0(x_0)$  and  $\tilde{\phi}_t(z_t|x_t)$  denote the probability density of  $x_0$  and  $z_t$  conditioned on  $x_t$ , respectively. An equivalent representation of model (10) in the finite interval  $0 \leq t \leq T$  is given by the nominal joint probability density

$$f(X_{T+1}, Y_T) = f_0(x_0) \prod_{t=0}^T \phi_t(z_t|x_t)$$

where

$$X_{T+1} = \begin{bmatrix} x_0 \\ \vdots \\ x_t \\ \vdots \\ x_{T+1} \end{bmatrix} \quad Y_T = \begin{bmatrix} y_0 \\ \vdots \\ y_t \\ \vdots \\ y_T \end{bmatrix}.$$

It is worth noting that  $\phi_t(z_t|x_t)$  is a degenerate conditional probability density in the case that  $r_t < n$ . Let  $\mathcal{A}_t$  denote the support of  $\phi_t$ . Notice that  $\mathcal{A}_t$  depends on  $x_t$ . We assume that the actual model over the interval  $0 \leq t \leq T$  is described by the joint probability

$$\tilde{f}(X_{T+1}, Y_T) = \tilde{f}_0(x_0) \prod_{t=0}^T \tilde{\phi}_t(z_t|x_t).$$

Moreover, we assume that the support of  $\tilde{\phi}_t(z_t|x_t)$  is  $\mathcal{A}_t$ . In this way  $f(X_{T+1}, Y_T)$  and  $\tilde{f}(X_{T+1}, Y_T)$  have the same support, say  $\mathcal{A}$ . Accordingly, we can measure their mismatch through the KL divergence

$$D(\tilde{f}, f) = \int_{\mathcal{A}} \tilde{f} \ln \left( \frac{\tilde{f}}{f} \right) dX_{T+1} dY_T.$$

Following reasonings similar to the ones in [15], it is not difficult to see that

$$D(\tilde{f}, f) = D(\tilde{f}_0, f_0) + \sum_{t=0}^T D(\tilde{\phi}_t, \phi_t) \quad (11)$$

where

$$\begin{aligned} D(\tilde{\phi}_t, \phi_t) &= \tilde{\mathbb{E}} \left[ \ln \left( \frac{\tilde{\phi}_t}{\phi_t} \right) \right] \\ &= \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{f}_t(x_t) \tilde{\phi}_t(z_t|x_t) \ln \left( \frac{\tilde{\phi}_t}{\phi_t} \right) dz_t dx_t, \end{aligned}$$

and  $\tilde{f}_t(x_t)$  denotes the actual marginal density of  $x_t$  whose support is denoted by  $\tilde{\mathcal{A}}_t$ .

Assume to know the nominal model (10), and thus  $f(X_{T+1}, Y_T)$ , while the actual one does not. In view of the decomposition in (11) and given  $Y_{t-1}$ , we can assume that  $\tilde{\phi}_t$  belongs to the following convex ambiguity set:

$$\mathcal{B}_t := \left\{ \tilde{\phi}_t \text{ s.t. } \tilde{\mathbb{E}} \left[ \ln \left( \frac{\tilde{\phi}_t(z_t|x_t)}{\phi_t(z_t|x_t)} \right) \middle| Y_{t-1} \right] \leq c_t \right\}$$

where

$$\begin{aligned} & \tilde{\mathbb{E}} \left[ \ln \left( \frac{\tilde{\phi}_t(z_t|x_t)}{\phi_t(z_t|x_t)} \right) \middle| Y_{t-1} \right] \\ & := \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) \ln \left( \frac{\tilde{\phi}_t(z_t|x_t)}{\phi_t(z_t|x_t)} \right) dz_t dx_t \end{aligned}$$

and  $\tilde{\mathcal{A}}_t$  denotes the support of  $\tilde{f}_t(x_t|Y_{t-1})$ . In plain words, the ambiguity set is expressed incrementally by forming a ball, in the KL topology, about the nominal model and placing an upper bound  $c_t$  on its radius. Then, we consider the robust estimator of  $x_{t+1}$  given  $Y_t$  solving the following dynamic minimax game:

$$\hat{x}_{t+1} = \operatorname{argmin}_{g_t \in \mathcal{G}_t} \max_{\tilde{\phi}_t \in \mathcal{B}_t} J_t(\tilde{\phi}_t, g_t) \quad (12)$$

where

$$\begin{aligned} J_t(\tilde{\phi}_t, g_t) &= \frac{1}{2} \tilde{\mathbb{E}} \left[ \|x_{t+1} - g_t(y_t)\|^2 \middle| Y_{t-1} \right] \\ &= \frac{1}{2} \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \|x_{t+1} - g_t(y_t)\|^2 \tilde{\phi}_t(z_t|x_t) \\ &\quad \times \tilde{f}_t(x_t|Y_{t-1}) dz_t dx_t; \end{aligned}$$

$\mathcal{G}_t$  denotes the class of estimators with finite second-order moments with respect to all the densities  $\tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1})$  such that  $\tilde{\phi}_t \in \mathcal{B}_t$ . Note that  $\tilde{\phi}_t$  must satisfy the constraint:

$$I_t(\tilde{\phi}_t) \triangleq \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dz_t dx_t = 1. \quad (13)$$

**Lemma 2.** For a fixed estimator  $g_t \in \mathcal{G}_t$ , the density  $\tilde{\phi}_t(z_t|x_t) \in \mathcal{B}_t$  having support  $\mathcal{A}_t$  that maximizes the objective function

$$J_t(\tilde{\phi}_t, g_t) = \tilde{\mathbb{E}} \left[ \|x_{t+1} - g_t(y_t)\|^2 \middle| Y_{t-1} \right]$$

under the constraint  $D(\tilde{\phi}_t, \phi_t) \leq c_t$  is given by

$$\tilde{\phi}_t^0 = \frac{1}{M_t(\lambda_t)} \exp \left( \frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2 \right) \phi_t. \quad (14)$$

Moreover,  $M_t(\lambda_t)$  is the normalizing constant such that (13) holds. Finally, for  $c_t > 0$  sufficiently small, there exists a unique  $\lambda_t > 0$  such that  $D(\tilde{\phi}_t^0, \phi_t) = c_t$ .



*Proof:* For a given  $g_t$ , the Lagrangian for the constrained optimization problem takes the form:

$$\begin{aligned}
L_t(\tilde{\phi}_t, \lambda_t, \mu_t) &= J_t(\tilde{\phi}_t, g_t) + \lambda_t(c_t - D_t(\tilde{\phi}_t, \phi_t)) + \mu_t(1 - I_t(\tilde{\phi}_t)) \\
&= \frac{1}{2} \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \|x_{t+1} - g_t(y_t)\|^2 \tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t \\
&\quad + \lambda_t \left( c_t - \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) \ln \left( \frac{\tilde{\phi}_t}{\phi_t} \right) dz_t dx_t \right) \\
&\quad + \mu_t \left( 1 - \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t \right)
\end{aligned}$$

where  $\lambda_t \geq 0$ . In order to maximize  $L_t(\tilde{\phi}_t, \lambda_t, \mu_t)$  with respect to  $\tilde{\phi}_t$ , we need to prove it is concave in  $\tilde{\phi}_t$ . The first and second variation of  $L_t$  with respect to  $\tilde{\phi}_t$  along the direction  $\delta\tilde{\phi}_t$  are, respectively,

$$\begin{aligned}
\delta L_t(\tilde{\phi}_t, \lambda_t, \mu_t; \delta\tilde{\phi}_t) &= \frac{1}{2} \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \|x_{t+1} - g_t(y_t)\|^2 \delta\tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t \\
&\quad + \lambda_t \left( - \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \delta\tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) \ln \left( \frac{\tilde{\phi}_t}{\phi_t} \right) dz_t dx_t \right. \\
&\quad \left. - \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{f}_t(x_t | Y_{t-1}) \delta\tilde{\phi}_t dz_t dx_t \right) \\
&\quad - \mu_t \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \delta\tilde{\phi}_t \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t \\
\delta L^2(\tilde{\phi}_t, \lambda_t, \mu_t; \delta\tilde{\phi}_t^2) &= -\lambda_t \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \delta\tilde{\phi}_t^2 \tilde{f}_t(x_t | Y_{t-1}) \frac{1}{\tilde{\phi}_t} dz_t dx_t.
\end{aligned}$$

Notice that  $\delta\tilde{\phi}_t(\cdot | x_t)$  is a function whose support is  $\mathcal{A}_t$ . Then, under the assumption that  $\phi_t > 0$  in  $\mathcal{A}_t$  and  $\lambda_t > 0$ , we have that the second variation is negative for any  $\delta\tilde{\phi}_t \neq 0$ . So  $L_t$  is strictly concave in  $\tilde{\phi}_t$ . This implies that the point of maximum is given by imposing the stationarity condition  $\delta L(\tilde{\phi}_t, \lambda_t, \mu_t; \delta\tilde{\phi}_t) = 0$  for all functions  $\delta\tilde{\phi}_t$  (having support  $\mathcal{A}_t$ ) which implies

$$\begin{aligned}
\left[ \frac{1}{2} \|x_{t+1} - g_t(y_t)\|^2 - \lambda_t \ln \left( \frac{\tilde{\phi}_t}{\phi_t} \right) - \lambda_t - \mu_t \right] \\
\times \tilde{f}_t(x_t | Y_{t-1}) = 0.
\end{aligned}$$

Accordingly, for any  $x_t \in \tilde{\mathcal{A}}_t$ , the maximum  $\phi_t^0$  is such that

$$\ln \left( \frac{\tilde{\phi}_t^0}{\phi_t} \right) = \frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2 - \ln M_t(\lambda_t)$$

where

$$\ln M_t(\lambda_t) \triangleq 1 + \frac{\mu_t}{\lambda_t}.$$

Thus,

$$\tilde{\phi}_t^0 = \frac{1}{M_t(\lambda_t)} \exp \left( \frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2 \right) \phi_t$$

and  $M_t(\lambda_t)$  must satisfy condition (13):

$$\int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \frac{1}{M_t(\lambda_t)} \exp\left(\frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2\right) \times \phi_t \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t = 1$$

so that

$$M_t(\lambda_t) = \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \exp\left(\frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2\right) \phi_t \times \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t.$$

In order to find the unique Lagrange multiplier  $\lambda_t > 0$  satisfying the inequality constraint, we consider

$$\begin{aligned} D(\tilde{\phi}_t^0, \phi_t) &= \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \tilde{\phi}_t^0 \tilde{f}_t(x_t | Y_{t-1}) \ln\left(\frac{\tilde{\phi}_t^0}{\phi_t}\right) dz_t dx_t \\ &= \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \left(\frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2 - \ln M_t(\lambda_t)\right) \\ &\quad \times \tilde{\phi}_t^0 \tilde{f}_t(x_t | Y_{t-1}) dz_t dx_t \\ &= \frac{1}{\lambda_t} J_t(\tilde{\phi}_t^0, g_t) - \ln M_t(\lambda_t). \end{aligned}$$

Notice that

$$\begin{aligned} &\frac{d}{d\lambda_t} \ln M_t(\lambda_t) \\ &= \frac{-1}{2\lambda_t^2} \frac{1}{M_t(\lambda_t)} \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \exp\left(\frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2\right) \\ &\quad \times \phi_t \tilde{f}_t(x_t | Y_{t-1}) \|x_{t+1} - g_t(y_t)\|^2 dz_t dx_t \\ &= \frac{-1}{2\lambda_t^2} \int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t} \|x_{t+1} - g_t(y_t)\|^2 \tilde{f}_t(x_t | Y_{t-1}) \tilde{\phi}_t dz_t dx_t \\ &= -\frac{1}{\lambda_t^2} J_t(\tilde{\phi}_t^0, g_t(y_t)) \end{aligned}$$

so that

$$\kappa_t(\lambda_t) \triangleq D(\tilde{\phi}_t^0, \phi_t) = -\lambda_t \frac{d}{d\lambda_t} \ln M_t(\lambda_t) - \ln(M_t(\lambda_t)).$$

Then, the derivative of  $\kappa_t(\lambda_t)$  is given by:

$$\begin{aligned} \frac{d\kappa_t}{d\lambda_t} &= -\lambda_t \frac{d^2}{d\lambda_t^2} \ln M_t - 2 \frac{d}{d\lambda_t} \ln M_t \\ &= -\frac{1}{\lambda_t} \left[ \frac{d}{d\lambda_t} \left( \lambda_t^2 \frac{d}{d\lambda_t} \ln M_t \right) \right] = \frac{1}{\lambda_t} \frac{d}{d\lambda_t} J_t(\tilde{\phi}_t^0, g_t) \\ &= -\frac{1}{4\lambda_t^3} \tilde{\mathbb{E}} \left[ (\|x_{t+1} - g_t(y_t)\|^2 \right. \\ &\quad \left. - \tilde{\mathbb{E}} \left[ \|x_{t+1} - g_t(y_t)\|^2 | Y_{t-1} \right])^2 | Y_{t-1} \right] < 0. \end{aligned}$$

Therefore,  $\kappa_t(\lambda_t)$  is a monotone decreasing function. Notice that when  $\lambda_t \rightarrow \infty$ , we have  $\tilde{\phi}_t^0 \rightarrow \phi_t$ , so that  $\kappa_t(\infty) = 0$ . Accordingly, if  $c_t$  is sufficiently small, then  $c_t$  is in the range of  $\kappa_t$ . Therefore, there exists a unique  $\lambda_t > 0$  such that  $\kappa_t(\lambda_t) = c_t$ .  $\square$

Once we get the function  $\tilde{\phi}_t^0$ , the estimator  $g_t \in \mathcal{G}_t$  minimizing the objective function  $J_t(\tilde{\phi}_t^0, g_t)$  is given by

$$\begin{aligned}\hat{x}_{t+1} &= g_t^0(y_t) = \tilde{\mathbb{E}}[x_{t+1}|Y_t] \\ &= \int_{\mathcal{A}_t^*} x_{t+1} \tilde{f}_{t+1}(x_{t+1}|Y_t) dx_{t+1}\end{aligned}\tag{15}$$

where

$$\tilde{f}_{t+1}(x_{t+1}|Y_t) = \frac{\int_{\tilde{\mathcal{A}}_t} \tilde{\phi}_t^0(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_t}{\int_{\tilde{\mathcal{A}}_t} \int_{\mathcal{A}_t^{**}} \tilde{\phi}_t^0(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_{t+1} dx_t},$$

$\mathcal{A}_t^*$  is the support of  $\tilde{f}_t(x_{t+1}|Y_t)$  and  $\mathcal{A}_t^{**}$  is defined as follows:  $x_{t+1} \in \mathcal{A}_t^{**}$  if and only if there exists at least one  $y_t$  for which  $[x_{t+1}^\top y_t^\top]^\top \in \mathcal{A}_t$ .

#### IV. LOW-RANK ROBUST KALMAN FILTER AS A STATIC GAME

In view of (15), the optimal estimator  $\hat{x}_{t+1}$  depends on the least favorable density  $\tilde{\phi}_t^0(z_t|x_t)$  while from (14), the least favorable density  $\tilde{\phi}_t^0(z_t|x_t)$  depends on the optimal estimator  $\hat{x}_{t+1}$ . In order to break this endless loop, we assume at time  $t$  the *a priori* conditional density of  $x_t$  given  $Y_{t-1}$  is

$$\tilde{f}_t(x_t|Y_{t-1}) \sim \mathcal{N}(\hat{x}_t, \tilde{P}_t)\tag{16}$$

and let  $\text{rank}(\tilde{P}_t) = r_t$ . In addition, we know the nominal conditional transition probability density function of model (10) is  $\phi_t(z_t|x_t) \sim \mathcal{N}(m_{z_t|x_t}, K_{z_t|x_t})$  with

$$m_{z_t|x_t} = \begin{bmatrix} A_t \\ C_t \end{bmatrix} x_t, \quad K_{z_t|x_t} = \begin{bmatrix} B_t B_t^\top & B_t D_t^\top \\ D_t B_t^\top & D_t D_t^\top \end{bmatrix}.$$

As we already noticed, the latter could be degenerate and thus  $K_{z_t|x_t}$  could be singular. We define the Gaussian pseudo nominal density

$$f_t(z_t|Y_{t-1}) = \int_{\tilde{\mathcal{A}}_t} \phi_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_t.$$

Since both  $\phi_t(z_t|x_t)$  and  $\tilde{f}_t(x_t|Y_{t-1})$  are Gaussian, we obtain  $f_t(z_t|Y_{t-1}) \sim \mathcal{N}(m_{z_t}, K_{z_t})$  with

$$m_{z_t} = \begin{bmatrix} A_t \\ C_t \end{bmatrix} \hat{x}_t, \quad K_{z_t} = \begin{bmatrix} K_{x_{t+1}} & K_{x_{t+1}y_t} \\ K_{y_t x_{t+1}} & K_{y_t} \end{bmatrix}.$$

where the parametric form of  $K_{z_t}$  is given by

$$K_{z_t} = \begin{bmatrix} A_t \\ C_t \end{bmatrix} \tilde{P}_t \begin{bmatrix} A_t^\top & C_t^\top \end{bmatrix} + \begin{bmatrix} B_t \\ D_t \end{bmatrix} \begin{bmatrix} B_t^\top & D_t^\top \end{bmatrix}.\tag{17}$$

Then, we have

$$K_{y_t} = C_t \tilde{P}_t C_t^\top + D_t D_t^\top \geq D_t D_t^\top > 0.$$

Next, we define the actual density

$$\tilde{f}_t(z_t|Y_{t-1}) = \int_{\tilde{\mathcal{A}}_t} \tilde{\phi}_t(z_t|x_t) \tilde{f}_t(x_t|Y_{t-1}) dx_t.$$

which has the same support of  $f_t(z_t|Y_{t-1})$ , which is denoted by  $\tilde{\mathcal{A}}_t$ . Moreover, on the basis of Equation (14), it is easy to show that if  $\phi_t(z_t|x_t)$  is Gaussian, the least favorable density

$$\begin{aligned} & \tilde{f}_t^0(z_t|Y_{t-1}) \\ &= \frac{1}{M(\lambda_t)} \exp\left(\frac{1}{2\lambda_t} \|x_{t+1} - g_t(y_t)\|^2\right) f_t(z_t|Y_{t-1}) \end{aligned}$$

is also Gaussian. Furthermore, it is not difficult to see that  $D(\tilde{\phi}_t^0, \phi_t) = D(\tilde{f}_t^0, f_t)$ . Hence, the least favorable density of  $z_t$  conditioned on  $Y_{t-1}$  belongs to the following ambiguity set

$$\tilde{\mathcal{B}}_t = \{\tilde{f}_t \sim \mathcal{N}(\tilde{m}_{z_t}, \tilde{K}_{z_t}) \text{ s.t. } D(\tilde{f}_t, f_t) \leq c_t\}.$$

Accordingly, the dynamic minimax game in (12) is equivalent to the following static minimax game

$$\hat{x}_{t+1} = \operatorname{argmin}_{g_t \in \mathcal{G}_t} \max_{\tilde{f}_t \in \tilde{\mathcal{B}}_t} \bar{J}_t(\tilde{f}_t, g_t) \quad (18)$$

where the least favorable model is now characterized by the conditional density  $\tilde{f}_t(z_t|Y_{t-1})$  and

$$\bar{J}_t(\tilde{f}_t, g_t) = \int_{\tilde{\mathcal{A}}_t} \|x_{t+1} - g_t(y_t)\|^2 \tilde{f}_t(z_t|Y_{t-1}) \, dz_t.$$

It is worth noting that the minimizer of  $\bar{J}_t(f_t, \cdot)$  is the Bayesian estimator and the state prediction error  $x_{t+1} - g_t(y_t)$  is Gaussian distributed with zero mean and covariance matrix

$$P_{t+1} := K_{x_{t+1}} - K_{x_{t+1}, y_t} K_{y_t}^{-1} K_{y_t, x_{t+1}} \quad (19)$$

which could be singular. In such a case,

$$H_t^\top H_t(x_{t+1} - g_t(y_t)) = x_{t+1} - g_t(y_t) \quad (20)$$

where  $H_t^\top$  is a matrix whose columns form an orthonormal basis for  $\operatorname{Im}(P_{t+1})$ . On the other hand, the prediction error under the least favorable model is Gaussian with zero mean and covariance matrix  $\tilde{P}_{t+1}$  such that  $\operatorname{Im}(\tilde{P}_{t+1}) = \operatorname{Im}(P_{t+1})$  because  $\operatorname{Im}(\tilde{K}_{z_t}) = \operatorname{Im}(K_{z_t})$ . Therefore, condition (20) still holds and the least favorable probability density of  $z_t$  given  $x_t$  can be written as

$$\tilde{\phi}_t^0 = \frac{1}{M_t(\lambda_t)} \exp\left(\frac{1}{2\lambda_t} \|H_t(x_{t+1} - g_t(y_t))\|^2\right) \phi_t. \quad (21)$$

**Remark 3.** Condition (16) means that, using the terminology coined by Hansen and Sargent [34], the maximizer in (18) is operating under commitment, i.e. the maximizer is required to commit all the least favorable model components at early stages with the estimating player.

At this point the solution to (18) is given by Theorem 1: substituting  $f, \tilde{f}, g, H$  with  $f_t, \tilde{f}_t, g_t, H_t$ , respectively, it is not difficult to see that all the assumptions are satisfied. Then, the least favorable density is  $\tilde{f}_t^0(z_t|Y_{t-1}) \sim \mathcal{N}(m_{z_t}, \tilde{K}_{z_t})$  where

$$\tilde{K}_{z_t} = \begin{bmatrix} \tilde{K}_{x_{t+1}} & K_{x_{t+1}, y_t} \\ K_{y_t, x_{t+1}} & K_{y_t} \end{bmatrix}.$$

The nominal posterior covariance of  $x_{t+1}$  given  $Y_t$  has been defined in (19). Accordingly, the least favorable posterior covariance of  $x_{t+1}$  given  $Y_t$  is

$$\tilde{P}_{t+1} = \tilde{K}_{x_{t+1}} - K_{x_{t+1}, y_t} K_{y_t}^{-1} K_{y_t, x_{t+1}}.$$

Moreover, in view of Equation (9), we have

$$\tilde{P}_{t+1} = (P_{t+1}^+ - \lambda_t^{-1} H_t^\top H_t)^+ \quad (22)$$

where  $\lambda_t > \sigma_{max}(P_{t+1})$  and the Lagrange multiplier  $\lambda_t$  is selected by the following equation

$$\begin{aligned} \gamma(P_{t+1}, \lambda_t) := & \frac{1}{2} \left\{ \text{Indet}^+(P_{t+1}) - \text{Indet}^+(\tilde{P}_{t+1}) \right. \\ & \left. + \text{tr} \left[ P_{t+1}^+ \tilde{P}_{t+1} - I_{r_{t+1}} \right] \right\} = c_t. \end{aligned} \quad (23)$$

Furthermore, the robust estimator is

$$\hat{x}_{t+1} = A_t \hat{x}_t + G_t (y_t - C_t \hat{x}_t).$$

where  $G_t = K_{x_{t+1}, y_t} K_{y_t}^{-1}$ . Then, in view of Equation (17), we know  $G_t$  and  $P_{t+1}$  take the parametric form

$$\begin{aligned} G_t &= (A_t \tilde{P}_t C_t^\top + B_t D_t^\top) (C_t \tilde{P}_t C_t^\top + D_t D_t^\top)^{-1} \\ P_{t+1} &= A_t \tilde{P}_t A_t^\top - G_t (C_t \tilde{P}_t C_t^\top + D_t D_t^\top) G_t^\top + B_t B_t^\top. \end{aligned}$$

---

**Algorithm 1** Low-rank robust Kalman filter at time  $t$

---

**Require:**  $y_t, \hat{x}_t, \tilde{P}_t, c_t$

- 1:  $G_t = (A_t \tilde{P}_t C_t^\top + B_t D_t^\top) (C_t \tilde{P}_t C_t^\top + D_t D_t^\top)^{-1}$
  - 2:  $\hat{x}_{t+1} = A_t \hat{x}_t + G_t (y_t - C_t \hat{x}_t)$
  - 3:  $P_{t+1} = A_t \tilde{P}_t A_t^\top - G_t (C_t \tilde{P}_t C_t^\top + D_t D_t^\top) G_t^\top + B_t B_t^\top$
  - 4: Select  $H_t^\top H_t$  as projection matrix with image  $\text{Im}(P_{t+1})$
  - 5: Find  $\lambda_t^{-1}$  s.t.  $\gamma(P_{t+1}, \lambda_t) = c_t$
  - 6:  $\tilde{P}_{t+1} = (P_{t+1}^+ - \lambda_t^{-1} H_t^\top H_t)^+$
- 

Algorithm 1 is the summary of the resulting low-rank robust Kalman filter. Note that steps 3-6 correspond to a distorted Riccati iteration involving covariance matrices which are possibly singular. In addition,  $\theta_t := \lambda_t^{-1}$  is the time-varying risk sensitivity parameter. It is worth noticing that when  $c_t = 0$  the actual model coincides with the nominal one, so that  $\theta_t = 0$ , and thus  $P_t = \tilde{P}_t$ . Then, step 3 in Algorithm 1 becomes the usual Riccati equation which means we obtain the standard Kalman filter.

**Remark 4.** *The computational complexity of Algorithm 1 is now discussed. Steps 1-3 have the same complexity of the standard Kalman filter, that is*

$$O(n^3) + O(n(r+p)p) + O(p^2(r+p)) + O(n^2(r+p)).$$

*Step 4 and Step 6 have complexity  $O(n^3)$ . In regard to Step 5, the computation of  $\lambda_t^{-1} \in (0, \sigma_{max}(P_{t+1})^{-1})$  is accomplished by a bisection method, see Algorithm 2 in [20]. The complexity to evaluate  $\gamma(P_{t+1}, \lambda_t)$  is  $O(n^3)$ ,*

thus the complexity of Step 5 is  $O(\log_2(\frac{\sigma_{max}(P_{t+1})^{-1}}{\epsilon})n^3)$  where  $\epsilon > 0$  is the selected accuracy, i.e. the solution found satisfies the condition  $|\gamma(P_{t+1}, \lambda_t) - c_t| \leq \epsilon$ . We conclude that the computational complexity of Algorithm 1 is

$$O(n^3) + O(n(r+p)p) + O(p^2(r+p)) + O(n^2(r+p)) \\ + O(\log_2(\frac{\sigma_{max}(P_{t+1})^{-1}}{\epsilon})n^3).$$

*Low-rank risk-sensitive filtering:* The minimax problem in (18) can be relaxed by imposing the constraint  $\tilde{f}_t \in \bar{\mathcal{B}}_t$  through a penalty term:

$$\hat{x}_{t+1} = \operatorname{argmin}_{g_t \in \mathcal{G}_t} \bar{J}_t(\tilde{f}_t, g_t) - \theta D(\tilde{f}_t, f_t) \quad (24)$$

where  $\theta > 0$  represents the risk sensitivity parameter and it can be understood as a regularization parameter. It is not difficult to see that the solution to (24) is the one in Algorithm 1 where Steps 5-6 are replaced by

$$\tilde{P}_{t+1} = (P_{t+1}^+ - \theta H_t^\top H_t)^+.$$

Such an estimator represents the extension to the singular case of the well known risk sensitive filtering problem, [6], [11].

## V. THE LEAST FAVORABLE MODEL

In this section, we characterize the least favorable model corresponding to (12). Let  $e_{t+1} = x_{t+1} - \hat{x}_{t+1}$  denote the state prediction error. By (10), we have

$$e_{t+1} = (A_t - G_t C_t) e_t + (B_t - G_t D_t) v_t. \quad (25)$$

It is worth noticing that the driving noise  $v_t$  is independent from  $e_t$  under the nominal model. In view of (21), we obtain the least favorable density

$$\tilde{\phi}_t^0 = \frac{1}{M_t(\lambda_t)} \exp\left(\frac{1}{2\lambda_t} \|H_t e_t\|^2\right) \phi_t. \quad (26)$$

The latter is not a normalized density, which means that the hostile player could have the opportunity to change retroactively the least favorable density of  $x_t$ .

Accordingly, it is possible to find the least favorable density with respect to the driving noise  $v_t$ . From model (10), it is not difficult to see that

$$\begin{bmatrix} B_t \\ D_t \end{bmatrix} v_t = z_t - \begin{bmatrix} A_t \\ C_t \end{bmatrix} x_t. \quad (27)$$

where  $z_t$  belongs to the affine space

$$\mathbb{S} = \{[A_t^\top \ C_t^\top]^\top x_t + \bar{w}_t, \bar{w}_t \in \operatorname{Im}(\Gamma_t)\}$$

and  $\Gamma_t := [B_t^\top \ D_t^\top]^\top$  is full column rank. Accordingly, there is a one-to-one correspondence between  $v_t \in \mathbb{R}^{r_t+p}$  and  $z_t \in \mathbb{S}$  through the isomorphism

$$\mathbb{R}^{r_t+p} \rightarrow \mathbb{S}$$

$$v_t \mapsto (\Gamma_t^\top \Gamma_t)^{-1} \Gamma_t^\top \left( z_t - \begin{bmatrix} A_t \\ C_t \end{bmatrix} x_t \right).$$

Therefore, instead of characterizing the least favorable density  $\tilde{\phi}_t^0(z_t|x_t)$  directly, we characterize it in terms of  $v_t$ . The nominal density is denoted by  $\psi_t(v_t)$  and

$$\psi_t(v_t) \sim \exp\left(-\|v_t\|^2/2\right)$$

where  $\sim$  means that the two terms are the same up to constant scale factors. Then, we make the guess that least favorable density of  $v_t$  depends on the state prediction error. Hence, the least favorable density is denoted by  $\tilde{\psi}_t(v_t|e_t)$ . We define it over the interval  $[t+1, T]$ . Then,

$$\prod_{s=t+1}^T \exp\left(\frac{\|H_t e_{s+1}\|^2}{2\lambda_s}\right) \psi_s(v_s)$$

$$\sim \exp\left(\frac{\|e_{t+1}\|_{\Omega_{t+1}^+}^2}{2}\right) \prod_{s=t+1}^T \tilde{\psi}_s(v_s|e_s)$$

where the term  $\exp(\|e_{t+1}\|_{\Omega_{t+1}^+}^2)$  indicates the cumulative error at time  $t+1$ . Note that  $\Omega_{t+1}^+$  could be singular.

Decreasing the time index  $t$  by 1 we obtain:

$$\exp\left(\frac{\|H_t e_{t+1}\|^2}{2\lambda_t}\right) \psi_t(v_t)$$

$$\sim \exp\left(\frac{\|e_t\|_{\Omega_t^+}^2 - \|e_{t+1}\|_{\Omega_{t+1}^+}^2}{2}\right) \tilde{\psi}_t(v_t|e_t).$$

Hence, we have

$$\tilde{\psi}_t(v_t|e_t) \sim \exp\left(\frac{\|e_{t+1}\|_{W_{t+1}}^2 - \|e_t\|_{\Omega_t^+}^2 - \|v_t\|^2}{2}\right)$$

where  $W_{t+1} := \Omega_{t+1}^+ + \lambda_t^{-1} H_t^\top H_t$ . Then, substituting (25) we obtain

$$\tilde{\psi}_t(v_t|e_t) \sim \exp\left(\frac{1}{2} \|(A_t - G_t C_t)e_t + (B_t - G_t D_t)v_t\|_{W_{t+1}}^2 - \frac{1}{2}(\|e_t\|_{\Omega_t^+}^2 - \|v_t\|^2)\right). \quad (28)$$

Accordingly, the least favorable density is

$$\tilde{\psi}_t(v_t|e_t) \sim \mathcal{N}(F_t e_t, K_{v_t}) \quad (29)$$

where

$$K_{v_t} = (I_{r_t} - (B_t - G_t D_t)^\top W_{t+1} (B_t - G_t D_t))^{-1}$$

$$F_t = K_{v_t} (B_t - G_t D_t)^\top W_{t+1} (A_t - G_t C_t).$$

Then, in view of (28) and (29), it is not difficult to see that  $\Omega_t^+$  satisfies the backward recursion:

$$\Omega_t^+ = (A_t - G_t C_t)^\top W_{t+1} (A_t - G_t C_t) + F_t^\top K_{v_t}^{-1} F_t.$$

where  $\Omega_{T+1}^+ = 0$ .

In view of (29), the least favorable noise admits the following decomposition:

$$v_t = F_t e_t + L_t \epsilon_t$$

where  $L_t$  is a square root matrix of  $K_{v_t}$  and  $\epsilon_t$  is normalized WGN. Therefore, taking the state space vector

$$\xi_t \triangleq \begin{bmatrix} x_t \\ e_t \end{bmatrix},$$

we obtain the corresponding least favorable model as:

$$\begin{aligned} \xi_{t+1} &= \tilde{A}_t \xi_t + \tilde{B}_t \epsilon_t \\ y_t &= \tilde{C}_t \xi_t + \tilde{D}_t \epsilon_t \end{aligned}$$

where

$$\begin{aligned} \tilde{A}_t &:= \begin{bmatrix} A_t & B_t F_t \\ 0 & A_t - G_t C_t + (B_t - G_t D_t) F_t \end{bmatrix} \\ \tilde{B}_t &:= \begin{bmatrix} B_t \\ B_t - G_t D_t \end{bmatrix} L_t \\ \tilde{C}_t &:= \begin{bmatrix} C_t & D_t F_t \end{bmatrix}, \quad \tilde{D}_t := D_t L_t. \end{aligned}$$

It is worth noting that the least favorable model is generated by a backward recursion implementation.

Finally, it remains to evaluate the performance of the proposed robust Kalman filter and the standard Kalman filter under the least favorable model. Let  $G'_t$  be the gain of an arbitrary filter of the form

$$\hat{x}'_{t+1} = A_t \hat{x}'_t + G'_t (y_t - C_t \hat{x}'_t).$$

Then, we define the corresponding prediction error  $e'_t = x_t - \hat{x}'_t$ . It is not difficult to see that

$$\begin{aligned} e'_{t+1} &= (A_t - G'_t C_t) e'_t \\ &\quad + (B_t - G'_t D_t) F_t e_t + (B_t - G'_t D) L_t \epsilon_t. \end{aligned}$$

Taking into account (25), we obtain

$$\begin{aligned} \begin{bmatrix} e'_{t+1} \\ e_{t+1} \end{bmatrix} &= \left( \tilde{A}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{C}_t \right) \begin{bmatrix} e'_t \\ e_t \end{bmatrix} \\ &\quad + \left( \tilde{B}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{D}_t \right) \epsilon_t. \end{aligned}$$



Then, it is not difficult to see that the covariance matrix of the augmented error is given by the Lyapunov equation

$$\begin{aligned} \Pi_{t+1} = & \left( \tilde{A}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{C}_t \right) \Pi_t \left( \tilde{A}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{C}_t \right)^\top \\ & + \left( \tilde{B}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{D}_t \right) \left( \tilde{B}_t - \begin{bmatrix} G'_t \\ 0 \end{bmatrix} \tilde{D}_t \right)^\top \end{aligned}$$

where its initial value is given by  $\Pi_0 = \mathbb{1}_2 \otimes \tilde{P}_0$  and  $\mathbb{1}_2$  is the  $2 \times 2$  matrix whose entries are 1. Accordingly, the least favorable covariance matrix of the estimation error is the submatrix of  $\Pi_t$  in position  $(1, 1)$ .

## VI. CONVERGENCE ANALYSIS

In this section we study the convergence of the robust Kalman filter outlined in Algorithm 1 under the assumption that the nominal model (10) has constant parameters, that is  $A_t = A$ ,  $B_t = B$ ,  $C_t = C$  and  $D_t = D$ , and the tolerance is constant, i.e.  $c_t = c$ . Without loss of generality we assume that  $BD^\top = 0$ . Otherwise, we can rewrite the filter in Algorithm 1 with  $\tilde{A} = A - BD^\top(DD^\top)^{-1}C$ ,  $\tilde{B}$  such that  $\tilde{B}\tilde{B}^\top = B(I - D^\top(DD^\top)^{-1}D)B^\top$ ,  $\tilde{C} = C$  and  $\tilde{D} = D$ . In this way  $\tilde{B}\tilde{D}^\top = 0$ . The robust filter converges if and only if the least favorable *a posteriori* covariance matrix  $\tilde{P}_t$  of  $x_t$  given  $Y_{t-1}$  converges as  $t \rightarrow \infty$ . In view of (22),  $\tilde{P}_t$  converges if and only if  $P_t$  converges. Accordingly, the robust filter converges if and only if the iteration

$$P_{t+1} = r_c(P_t), \quad P_0 \in \overline{\mathcal{Q}}_+^n \quad (30)$$

converges. The mapping  $r_c$  is defined as follows

$$r_c(P_t) = A\tilde{P}_tA^\top - A\tilde{P}_tC^\top(C\tilde{P}_tC^\top + R)^{-1}C\tilde{P}_tA^\top + Q,$$

where  $R := DD^\top$ ,  $Q := BB^\top$ ,  $\tilde{P}_{t+1}$  has been defined in (22) and  $\theta_t = \lambda_t^{-1}$  is such that  $\gamma(P_{t+1}, \theta_t^{-1}) = c$  holds. Note that,  $r_c(\cdot)$  is a mapping of  $\overline{\mathcal{Q}}_+^n$ .

**Proposition 1.** *Let  $\bar{P}_t$ ,  $t \geq 0$ , be the sequence generated by the Riccati equation*

$$\bar{P}_{t+1} = A\bar{P}_tA^\top - A\bar{P}_tC^\top(C\bar{P}_tC^\top + R)^{-1}C\bar{P}_tA^\top + Q \quad (31)$$

with  $\bar{P}_0 = P_0$ . Then, the sequence  $P_t$ ,  $t \geq 0$ , generated by (30) is such that

$$\text{Im}(P_t) = \text{Im}(\bar{P}_t). \quad (32)$$

*Proof:* We prove the first claim by induction. Clearly, condition (32) holds for  $t = 0$ . We assume that  $\text{Im}(P_t) = \text{Im}(\bar{P}_t)$ . Let  $P_t = U_t D_t U_t^\top$  and  $\bar{P}_t = U_t \bar{D}_t U_t^\top$  be the corresponding reduced singular value decompositions with  $D_t, \bar{D}_t > 0$  and the columns of  $U_t$  form an orthonormal basis of  $\text{Im}(P_t)$ . Substituting such decomposition of  $\bar{P}_t$  on the right hand side of (31), we obtain

$$\begin{aligned} \bar{P}_{t+1} &= \tilde{A}_t \bar{D}_t \tilde{A}_t^\top - \tilde{A}_t \bar{D}_t \tilde{C}_t^\top (\tilde{C}_t \bar{D}_t \tilde{C}_t^\top + R)^{-1} \tilde{C}_t \bar{D}_t \tilde{A}_t^\top + Q \\ &= \tilde{A}_t (\bar{D}_t^{-1} + \tilde{C}_t^\top R^{-1} \tilde{C}_t)^{-1} \tilde{A}_t^\top + Q \end{aligned}$$

where  $\tilde{A}_t = AU_t$  and  $\tilde{C}_t = CU_t$ . Since  $\bar{D}_t$  is positive definite,  $v \in \ker(\bar{P}_{t+1})$  if and only if

$$v \in \ker(\tilde{A}_t) \text{ and } v \in \ker(Q). \quad (33)$$

In a similar way, substituting the decomposition of  $P_t$  in (30), we obtain

$$P_{t+1} = \tilde{A}_t(D_t^{-1} - \theta_t I + \tilde{C}_t^\top R^{-1} \tilde{C}_t)^{-1} \tilde{A}_t^\top + Q$$

where we exploited the fact that  $H_t^\top H_t = U_t U_t^\top$ . Since  $D_t^{-1} - \theta_t I$  is positive definite,  $v \in \ker(\bar{P}_{t+1})$  if and only if condition (33) holds. We conclude that  $\text{Im}(P_{t+1}) = \text{Im}(\bar{P}_{t+1})$ .  $\square$

If the pair  $(A, B)$  is stabilizable and the pair  $(A, C)$  is detectable, we have that  $\bar{P}_t$  converges to a unique solution, say  $\bar{P}_\infty$ , for any arbitrary initial condition  $\bar{P}_0$ , see [35]. Let  $U$  and  $V$  be two matrices whose columns form an orthonormal basis for  $\text{Im}(P_\infty)$  and its orthogonal complement in  $\mathbb{R}^n$ , respectively; from now on we assume both  $U$  and  $V$  are fixed. By Proposition 1 it follows that there exists  $\bar{t} \in \mathbb{N}$  such that

$$P_t^R := U^\top P_t U > 0, \quad \forall t \geq \bar{t}. \quad (34)$$

Moreover, if  $P_t^R$  converges and  $V^\top P_t V \rightarrow 0$  then  $P_t$  converges. Accordingly, in what follows we prove that the sequence  $P_t^R$  converges and  $V^\top P_t V \rightarrow 0$  for any arbitrary initial condition.

**Lemma 3.** *Let  $P_t$ ,  $t \geq 0$ , be the sequence generated by (30). We assume that  $(A, B)$  and  $(A, C)$  are stabilizable and detectable, respectively. If  $P_0$  is such that  $\text{Im}(P_0) \subseteq \text{Im}(U)$ , then  $\text{Im}(P_t) \subseteq \text{Im}(U)$  for any  $t > 0$ . Moreover, the subsystem  $(U^\top A U, U^\top B, C U, D)$  is reachable.*

*Proof:* In view of Proposition 1, we can prove the claim using  $\bar{P}_t$ . Let  $(\check{A}, \check{B}, \check{C}, \check{D})$  be the reachability standard form of  $(A, B, C, D)$  with  $\check{A} = TAT^{-1}$ ,  $\check{B} = TB$ ,  $\check{C} = CT^{-1}$  and  $T$  is the transformation matrix. Hence, we have

$$\begin{aligned} \check{A} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \check{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ \check{C} &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad \check{D} = D \end{aligned} \quad (35)$$

and the pair  $(A_{11}, B_1)$  is reachable. Let  $\check{P}_t := T\bar{P}_t T^\top$ , then it is not difficult to see that  $\check{P}_t$  obeys the following recursion

$$\check{P}_{t+1} = \check{A}\check{P}_t\check{A}^\top - \check{A}\check{P}_t\check{C}^\top (\check{C}\check{P}_t\check{C}^\top + \check{R})^{-1} \check{C}\check{P}_t\check{A}^\top + \check{Q} \quad (36)$$

with  $\check{R} = R$ ,  $\check{Q} = TQT^\top$ . In plain words, (36) is (31) in the new coordinates. Accordingly, we can prove the claim with respect to the new coordinates. Since the system is stabilizable and detectable, the iteration in (36) converges to a unique solution  $\check{P}_\infty$  which solves the corresponding algebraic equation. By direct computation it is not difficult to see that

$$\check{P}_\infty = \begin{bmatrix} \check{P}_\infty^R & 0 \\ 0 & 0 \end{bmatrix}$$

and  $\check{P}_\infty^R$  is the solution to the algebraic Riccati equation corresponding to the system  $(A_{11}, B_1, C_1, D) = (U^\top AU, U^\top B, CU, D)$  with  $U = [I_l \ 0]^\top$  and  $l$  is the dimension of  $\check{P}_\infty^R$ . Since this subsystem is reachable, we have that  $\check{P}_\infty^R$  is positive definite and thus

$$\text{Im}(U) = \text{Im}(\check{P}_\infty) = \text{Im} \left( \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Finally, by direct computation it is not difficult to see that

$$\check{P}_t = \begin{bmatrix} \star & 0 \\ 0 & 0 \end{bmatrix} \subseteq \text{Im}(U) \implies \check{P}_{t+1} = \begin{bmatrix} \star & 0 \\ 0 & 0 \end{bmatrix} \subseteq \text{Im}(U)$$

which concludes the proof.  $\square$

**Proposition 2.** *Consider the sequence generated by (30) with  $\text{Im}(P_0) \subseteq \text{Im}(U)$ . Assume that  $(A, B)$  is stabilizable and the reachable subsystem is also observable. Then, there always exists  $c_{MAX} > 0$  such that if  $0 < c < c_{MAX}$  then the sequence in (34) converges to a unique point  $P_\infty^R$ .*

*Proof:* By Lemma 3 we have that

$$P_t = UP_t^R U^\top, \quad t \geq 0 \quad (37)$$

and by (34) we have that  $\text{Im}(P_t) = \text{Im}(U)$  for  $t \geq \bar{t}$ . In view of (30) and (37), for  $t \geq \bar{t}$  we have

$$P_{t+1}^R = U^\top r_c(P_t)U = U^\top r_c(UP_t^R U^\top)U = r_c^R(P_t^R) \quad (38)$$

where

$$\begin{aligned} r_c^R(P_t^R) &:= A_R((P_t^R)^{-1} - \theta_t I + C_R^\top R^{-1} C_R)^{-1} A_R^\top + Q_R \\ A_R &:= U^\top AU \\ Q_R &:= U^\top BB^\top U \\ C_R &:= CU \end{aligned}$$

and we exploited the fact that  $\text{Im}(H_t^\top) = \text{Im}(P_t) = \text{Im}(U)$ , for  $t \geq \bar{t}$ , and thus

$$\begin{aligned} U^\top \check{P}_t U &= U^\top (P_t^+ - \theta_t H_t^\top H_t)^+ U \\ &= ((P_t^R)^{-1} - \theta_t I)^{-1}. \end{aligned}$$

Next, we consider the subsequence  $P_k^{Rd} := P_{kN}^R$  with  $k \geq \lceil \bar{t}/N \rceil$  and  $N \geq n$ . Then, from (38) we have that

$$P_{k+1}^{Rd} = r_c^{Rd}(P_k^{Rd}) \quad (39)$$

where  $r_c^{Rd}(\cdot)$  is the  $N$ -fold composition of  $r_c^R(\cdot)$  and it is defined as follows

$$r_c^{Rd}(P^{Rd}) = \mathbf{A}_N[(P^{Rd})^{-1} + \mathbf{R}_N]^{-1} \mathbf{A}_N + \mathbf{Q}_N \quad (40)$$

where  $\mathbf{A}_N \in \mathbb{R}^{l \times l}$  and whose precise definition (not useful here) can be found in [36], [37];  $\mathbf{Q}_N \in \overline{\mathcal{Q}}_+^l$  and  $\mathbf{R}_N \in \overline{\mathcal{Q}}_+^l$  are defined as follows:

$$\begin{aligned}
\mathbf{Q}_N &:= \mathcal{R}_N \mathcal{P}_N \mathcal{R}_N^\top \\
\mathbf{R}_N &:= \mathcal{O}_N^\top (\mathcal{D}_N \mathcal{D}_N^\top + \mathcal{H}_N \mathcal{H}_N^\top)^{-1} \mathcal{O}_N + \mathcal{J}_N^\top S_N^{-1} \mathcal{J}_N \\
S_N &:= -\Theta_N^{-1} + \mathcal{L}_N (I + \mathcal{H}_N^\top (\mathcal{D}_N \mathcal{D}_N^\top)^{-1} \mathcal{H}_N)^{-1} \mathcal{L}_N^\top \\
\mathcal{P}_N &:= [I + \mathcal{H}_N^\top (\mathcal{D}_N \mathcal{D}_N^\top)^{-1} \mathcal{H}_N - \mathcal{L}_N^\top \Theta_N \mathcal{L}_N]^{-1} \\
\mathcal{J}_N &:= \mathcal{O}_N^R - \mathcal{L}_N \mathcal{H}_N^\top [\mathcal{D}_N \mathcal{D}_N^\top + \mathcal{H}_N \mathcal{H}_N^\top]^{-1} \mathcal{O}_N \\
\Theta_N &:= \text{diag}(\tilde{\lambda}_N^{-1}, \tilde{\lambda}_{N-1}^{-1}, \dots, \tilde{\lambda}_1^{-1}) \otimes I \\
\mathcal{R}_N &:= \begin{bmatrix} Q_R^{1/2} & A_R Q_R^{1/2} & \dots & A_R^{N-1} Q_R^{1/2} \end{bmatrix} \\
\mathcal{O}_N &:= \begin{bmatrix} (C_R A_R^{N-1})^\top & \dots & (C_R A_R)^\top & C_R^\top \end{bmatrix}^\top \\
\mathcal{O}_N^R &:= \begin{bmatrix} (A_R^{N-1})^\top & \dots & A_R^\top & I \end{bmatrix}^\top \\
\mathcal{D}_N &:= I_N \otimes D \\
\mathcal{H}_N &:= \text{Tp}(0, H_1, H_2, \dots, H_{N-2}, H_{N-1}) \\
\mathcal{L}_N &:= \text{Tp}(0, L_1, L_2, \dots, L_{N-2}, L_{N-1}) \\
H_t &:= \begin{cases} C_R A_R^{t-1} Q_R^{1/2} & t \geq 1 \\ 0 & \text{otherwise} \end{cases} \\
L_t &:= \begin{cases} A_R^{t-1} Q_R^{1/2} & t \geq 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where  $Q_R^{1/2}$  is a square root matrix of  $Q_R$ . Then, the parameters  $\tilde{\lambda}_l$ ,  $l = 1 \dots N$ , are defined as follows. If  $P^{Rd} = P_k^{Rd}$ , then these parameters are given by

$$\gamma(P_{kN+l-1}^R, \tilde{\lambda}_l) = c, \quad l = 1 \dots N. \quad (41)$$

Since the system  $(A_R, Q_R^{1/2}, C_R, D)$  is reachable, by Lemma 3, and observable, by assumption, then [36, Proposition 5.3] guarantees that there exists  $c_{MAX} > 0$  such that the mapping  $r_c^{Rd}(\cdot)$  is a strict contraction for  $0 < c < c_{MAX}$  in the metric space  $(\overline{\mathcal{Q}}_+^l, \delta)$  where  $\delta(X, Y)$  is the Thompson part metric, [38]–[40]:

$$\delta(X, Y) := \ln \max\{\sigma_{max}(Y^{-1}X), \sigma_{max}(X^{-1}Y)\}. \quad (42)$$

Thus, there exists  $0 < \eta_N < 1$  such that

$$\delta(r_c^{Rd}(X), r_c^{Rd}(Y)) \leq \eta_N \delta(X, Y), \quad \forall X, Y \in \overline{\mathcal{Q}}_+^l. \quad (43)$$

It is worth noticing that  $\mathbf{Q}_N$  and  $\mathbf{R}_N$  in (40) depend on  $P^{Rd}$ ; however,  $\eta_N$  does not. Since this metric space is complete, we can exploit the Banach fixed point theorem. Therefore, the iteration in (39) converges to a unique point  $P_\infty^R \in \overline{\mathcal{Q}}_+^l$  for any  $P_0 \geq 0$  such that  $\text{Im}(P_0) \subseteq \text{Im}(U)$ . Moreover, since  $r_c^R(\cdot)$  is a mapping in  $\overline{\mathcal{Q}}_+^l$ , it follows that (38) converges to  $P_\infty^R$ .  $\square$

The next corollary is a consequence of Lemma 3 and Proposition 2.

**Corollary 1.** *Consider the sequence generated by (30) with  $\text{Im}(P_0) \subseteq \text{Im}(U)$ . Assume that  $(A, B)$  is stabilizable and the reachable subsystem is also observable. Then, there always exists  $c_{MAX} > 0$  such that if  $0 < c < c_{MAX}$  then the sequence in (30) converges to  $P_\infty := UP_\infty^R U^\top$ .*

**Remark 5.** *The upper bound  $c_{MAX}$  for the tolerance can be computed explicitly. Following the same reasonings in [36], [37] we have  $c_{MAX} = \gamma(\bar{P}_q^R, \phi_N^{-1})$ , if  $\phi_N^{-1} > \sigma_{\max}(\bar{P}_q^R)$ ; otherwise  $c_{MAX} = \infty$ . Here:  $\bar{P}_q^R$  is the  $q$ -th element of the sequence generated by the standard Riccati iteration for the reachable subsystem  $(A_R, Q_R^{1/2}, C_R, D)$  with initial condition  $P_0^R = 0$ ;  $\phi_N \in (0, 1/\sigma_{\max}(\mathcal{L}_N(I + \mathcal{H}_N^\top(\mathcal{D}_N \mathcal{D}_N^\top)^{-1} \mathcal{H}_N)^{-1} \mathcal{L}_N))$  is the maximum value for which the matrix*

$$\begin{aligned} & \mathcal{O}_N^\top (\mathcal{D}_N \mathcal{D}_N^\top + \mathcal{H}_N \mathcal{H}_N^\top)^{-1} \mathcal{O}_N \\ & + \mathcal{J}_N^\top \{-\phi_N^{-1} I + \mathcal{L}_N(I + \mathcal{H}_N^\top(\mathcal{D}_N \mathcal{D}_N^\top)^{-1} \mathcal{H}_N)^{-1} \mathcal{L}_N^\top\}^{-1} \mathcal{J}_N \end{aligned}$$

is positive definite. Clearly, the user has two degrees of freedom for computing  $c_{MAX}$  that is  $N$  and  $q$ . The larger  $q$  is the larger  $c_{MAX}$  is, while no specific properties in terms of  $N$  have been found.

Proposition 2 hinges on the fact that the image of  $P_t$  is fixed for  $t \geq \bar{t}$  and thus the convergence is proved in the corresponding subspace. However, in the general case, i.e. when  $\text{Im}(P_0) \not\subseteq \text{Im}(U)$ , the image of  $P_t$  is not fixed because  $V^\top P_t V$  could be different from the null matrix; thus, such a case needs to be addressed carefully.

**Lemma 4.** *Consider the sequence generated by (30) with  $P_0 \in \overline{\mathcal{Q}}_+^n$ . Assume that  $(A, B)$  is stabilizable and  $(A, C)$  is detectable. Then, there always exists  $c_{MAX} > 0$  such that if  $0 < c < c_{MAX}$  then*

$$V^\top P_t V \rightarrow 0.$$

*Proof:* We prove the claim by performing the same change of basis used in (35). Let  $\check{P}_t = TP_t T^\top$ , then we have

$$\begin{aligned} \check{P}_{t+1} &= \check{A} \check{P}_t \check{A}^\top - \check{A} \check{P}_t \check{C}^\top (\check{C} \check{P}_t \check{C}^\top + \check{R})^{-1} \check{C} \check{P}_t \check{A}^\top + \check{Q} \\ \check{P}_{t+1} &= (\check{P}_{t+1}^+ - \theta_t \check{H}_t^\top \check{H}_t)^+ \end{aligned} \quad (44)$$

where  $\check{H}_t = H_t T^\top$ . We partition  $\check{P}_t$  and  $\check{P}_t$  conformably with (35):

$$\check{P}_t = \begin{bmatrix} \check{P}_{11,t} & \check{P}_{12,t} \\ \check{P}_{12,t}^\top & \check{P}_{22,t} \end{bmatrix}, \quad \check{P}_t = \begin{bmatrix} \check{P}_{11,t} & \check{P}_{12,t} \\ \check{P}_{12,t}^\top & \check{P}_{22,t} \end{bmatrix}.$$

Taking into account (44), it is not difficult to see that

$$\check{P}_{22,t+1} \leq A_{22} \check{P}_{22,t} A_{22}^\top. \quad (45)$$

Notice that  $\check{P}_t \geq \check{P}_t$  and they have the same eigenvectors corresponding to the nonnull eigenvalues, thus by (23) we have

$$\frac{1}{2} \sum_i -\ln \left( \frac{\sigma_i(\check{P}_t)}{\sigma_i(\check{P}_t)} \right) + \frac{\sigma_i(\check{P}_t)}{\sigma_i(\check{P}_t)} - 1 = c \quad (46)$$

where  $\sigma_i(\check{P}_t)$  and  $\sigma_i(\tilde{\check{P}}_t)$  denote the  $i$ -th largest nonnull eigenvalue of  $\check{P}_t$  and  $\tilde{\check{P}}_t$ , respectively. Since the terms on the left hand side of (46) are nonnegative, it follows that

$$-\ln\left(\frac{\sigma_i(\tilde{\check{P}}_t)}{\sigma_i(\check{P}_t)}\right) + \frac{\sigma_i(\check{P}_t)}{\sigma_i(\tilde{\check{P}}_t)} - 1 \leq 2c. \quad (47)$$

By (47) it follows that

$$\tilde{\check{P}}_t \leq \rho \check{P}_t \quad (48)$$

where  $\rho > 1$  is the unique solution to the equation

$$-\ln \rho + \rho - 1 = 2c. \quad (49)$$

Substituting (48) in (45), we obtain

$$\check{P}_{22,t} \leq \rho A_{22} \tilde{\check{P}}_{22,t} A_{22}^\top.$$

Accordingly, if  $\sqrt{\rho}|\sigma_1(A_{22})| < 1$  then  $\check{P}_{22,t} \rightarrow 0$  which also implies  $\check{P}_{12,t} \rightarrow 0$  because  $\check{P}_t \geq 0$ . Since in these coordinates  $V = [0 \ I_{n-l}]^\top$ , it follows that  $V^\top \check{P}_t V \rightarrow 0$ .

The existence of  $c_{MAX} > 0$  such that  $\sqrt{\rho}|\sigma_1(A_{22})| < 1$  follows from the fact that the left hand side of (49) is monotone increasing for  $\rho > 1$  and it approaches zero as  $\rho \rightarrow 1$ .  $\square$

**Theorem 2.** *Consider the sequence generated by (30) with  $P_0 \in \overline{\mathcal{Q}_+^n}$ . Assume that  $(A, B)$  is stabilizable and the reachable subsystem is also observable. Then, there always exists  $c_{MAX} > 0$  such that if  $0 < c < c_{MAX}$  then the sequence in (34) converges to  $P_\infty^R$ .*

*Proof:* Consider the fixed point  $P_\infty^R$  of Proposition 2 and the subsequences  $P_k^d := P_{kN}$  and  $P_k^{Rd} := P_{kN}^R$  with  $k \geq \lceil \bar{t}/N \rceil$ ,  $N \geq n$  and  $\bar{t}$  is such that (34) holds. Accordingly,  $P_\infty^R$  and  $P_k^{Rd}$ ,  $k \geq \lceil \bar{t}/N \rceil$ , are two elements of the metric space  $(\mathcal{Q}_+^l, \delta)$ . Moreover, we have

$$P_{k+1}^d = r_c^d(P_k^d), \quad P_{k+1}^{Rd} = r_c^{Rd}(P_k^d)$$

where  $r_c^d(\cdot), r_c^{Rd}(\cdot)$  are the  $N$ -fold composition of  $r_c(\cdot), r_c^R(\cdot)$ , respectively. Since  $r_c(\cdot)$  is a continuous mapping of  $\overline{\mathcal{Q}_+^n}$ , then also  $r_c^d(\cdot)$  is a continuous mapping of  $\overline{\mathcal{Q}_+^n}$ . By Lemma 4 we have that  $V^\top P_t V \rightarrow 0$  which implies  $V^\top P_k^d V \rightarrow 0$  and hence

$$r_c^d(P_k^d) = r_c^d(UU^\top P_k^d UU^\top) + M_k$$

with  $M_k \rightarrow 0$ . Accordingly,

$$\begin{aligned} P_{k+1}^{Rd} &= U^\top r_c^d(P_k^d) U = U^\top r_c^d(UU^\top P_k^d UU^\top) U + N_k \\ &= r_c^{Rd}(P_k^{Rd}) + N_k \end{aligned}$$

where  $N_k = U^\top M_k U \in \mathcal{Q}^l$  and  $\|N_k\|$  tends to zero as  $k \rightarrow \infty$ . Since the Thompson part metric is continuous with respect to the arguments, and in view of (43), we have that

$$\begin{aligned} \delta(P_{k+1}^{Rd}, P_\infty^R) &= \delta(r_c^{Rd}(P_k^{Rd}) + N_k, P_\infty^R) \\ &= \delta(r_c^{Rd}(P_k^{Rd}), P_\infty^R) + \varepsilon_k \\ &= \delta(r_c^{Rd}(P_k^{Rd}), r_c^{Rd}(P_\infty^R)) + \varepsilon_k \\ &\leq \eta_N \delta(P_k^{Rd}, P_\infty^R) + \varepsilon_k \end{aligned}$$

where  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . Accordingly,

$$\begin{aligned} \delta(P_{k+1}^{Rd}, P_\infty^R) &\leq \eta_N \delta(P_k^{Rd}, P_\infty^R) + \varepsilon_k \\ &\leq \eta_N^2 \delta(P_{k-1}^{Rd}, P_\infty^R) + \eta_N \varepsilon_{k-1} + \varepsilon_k \leq \dots \\ &\leq \eta_N^{k+1} \delta(P_0^{Rd}, P_\infty^R) + \sum_{l=0}^k \eta_N^{k-l} \varepsilon_l \\ &= \eta_N^{k+1} \delta(P_0^{Rd}, P_\infty^R) + h * \varepsilon(k) \end{aligned}$$

where  $h * \varepsilon(k)$  denotes the convolution in  $k$  between the sequences  $h_k = \eta_N^k$  and  $\varepsilon_k$  with  $k \geq 0$ . Accordingly,  $h * \varepsilon(k)$  can be understood as the output at time  $k$  of a stable first order filter fed by a signal which tends to zero in the steady state. As a consequence  $h * \varepsilon(k) \rightarrow 0$  as  $k \rightarrow \infty$ . Accordingly, we conclude that

$$\delta(P_{k+1}^{Rd}, P_\infty^R) \rightarrow 0$$

as  $k \rightarrow \infty$  and thus  $P_k^{Rd} \rightarrow P_\infty^R$ . It is not difficult to see that the same conclusion holds in the case that  $P_k^{Rd} := P_{kN+l}^R$  with  $l = 1 \dots N - 1$ . Accordingly, we have that  $P_t^R \rightarrow P_\infty^R$  as  $t \rightarrow \infty$ .  $\square$

Finally, in view of the fact that  $V^\top P_t V \rightarrow 0$  we have the following corollary.

**Corollary 2.** *Assume that  $(A, B)$  is stabilizable and the reachable subsystem is also observable. Then, there always exists  $c_{MAX} > 0$  such that if  $0 < c < c_{MAX}$  then the sequence in (30) converges to  $P_\infty = UP_\infty^R U^\top$  for any initial condition  $P_0 \in \overline{\mathcal{Q}_+^n}$ .*

## VII. SIMULATION RESULTS

To evaluate the effectiveness of the robust Kalman filter proposed in Algorithm 1, we consider the nominal model:

$$\begin{aligned} x_{t+1} &= Ax_t + Bv_t \\ y_t &= Cx_t + Dv_t \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0.1 & 0.1 \\ 0 & 0.8407 & -0.3482 \\ 0 & 0.3482 & 0.8407 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}, \end{aligned}$$

and  $x_0 \sim \mathcal{N}(0, \tilde{P}_0)$  with

$$\tilde{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The pairs  $(A, B)$  and  $(A, C)$  are stabilizable and observable, respectively. However, it is worth noticing that the model is not reachable.



Fig. 1. Trace of  $P_t$  for KF, RKF1 with  $c = 10^{-1}$ , and RKF2 with  $c = 2 \cdot 10^{-1}$ .

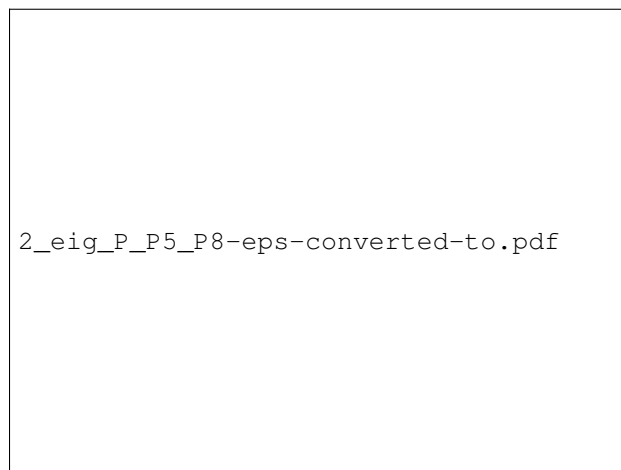


Fig. 2. Minimum nonnull eigenvalue of  $P_t$  as a function of  $t$  for KF, RKF1 with  $c = 10^{-1}$ , and RKF2 with  $c = 2 \cdot 10^{-1}$ .

In what follows, we consider the following three filters over a time interval of length  $T = 100$ : the standard Kalman filter (KF), the robust Kalman filter (RKF) proposed in Algorithm 1 with tolerance  $c_1 = 10^{-1}$  (RKF1) and with tolerance  $c_2 = 2 \cdot 10^{-1}$  (RKF2). In Fig. 1 we show the trace of  $P_t$  generated by these three filters: it converges to a constant value in all the three cases. It is worth noting that  $\text{rank}(P_t) = 2$  for all the algorithms.





Fig. 3. Trace of  $\tilde{P}_t$  for KF, RKF1 with  $c = 10^{-1}$ , and RKF2 with  $c = 2 \cdot 10^{-1}$ . Recall that  $\tilde{P}_t = P_t$  for KF.

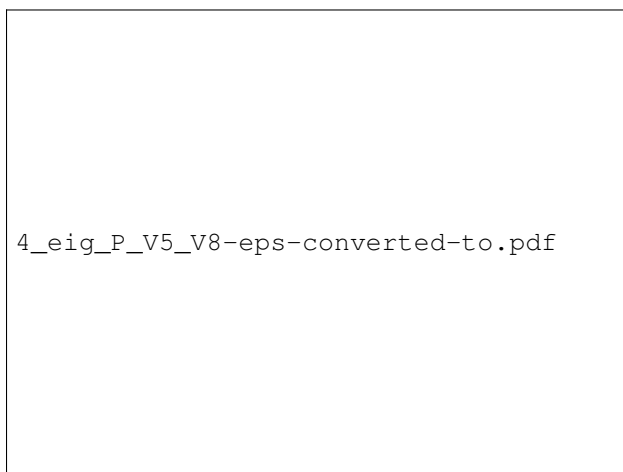


Fig. 4. Minimum nonnull eigenvalue of  $\tilde{P}_t$  as a function of  $t$  for KF, RKF1 with  $c = 10^{-1}$ , and RKF2 with  $c = 2 \cdot 10^{-1}$ .

Fig. 2 shows the minimum nonnull eigenvalue of  $P_t$  as a function of the time: as time increases, its eigenvalue decreases exponentially which means that  $P_t$  has numerical rank equal to 1 as time takes large values. Hence, the corresponding algebraic Riccati equation converges to a unique solution with rank equal to 1, which is in line with our expectations. It is interesting to note that the convergence rate of KF is higher than that of RKF. More precisely, the smaller  $c$  is, the higher the convergence rate is. The trace of  $\tilde{P}_t$  for KF, RKF1 and RKF2 is shown in Fig. 3. Clearly  $\tilde{P}_t = P_t$  for KF. Also in this case  $\text{rank}(\tilde{P}_t) = 2$  and its numerical rank is equal to 1 for large values of  $t$  in all the algorithms, see the minimum nonnull eigenvalue of  $\tilde{P}_t$  depicted in Fig. 4. Finally, as shown in Fig. 5, the risk sensitivity parameter  $\theta_t$  converges to a constant value for both RKFs. Moreover, the smaller  $c$  is, the smaller  $\theta_t$  is.

Fig. 6 shows the variance of the prediction error in the case that the actual model corresponds to the nominal model. In this case, KF performs better than RKF's. On the other hand, when the actual model is the least favorable



Fig. 5. Risk sensitivity parameter  $\theta_t$  as a function of  $t$  for RKF1 with  $c = 10^{-1}$ , and RKF2 with  $c = 2 \cdot 10^{-1}$ .

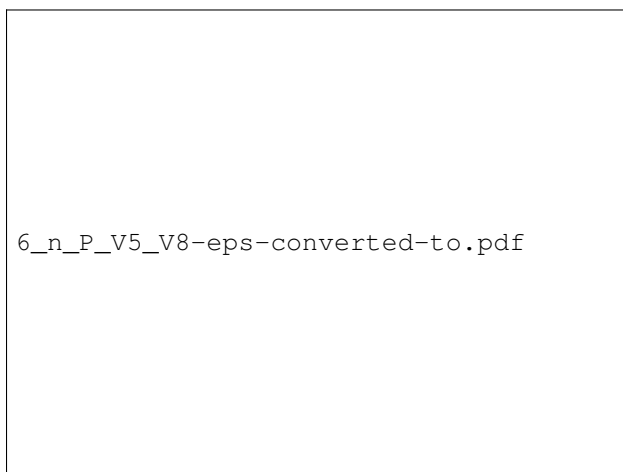


Fig. 6. Scalar variance of the prediction error for KF, RKF1 with  $c = 10^{-1}$  and RKF2 with  $c = 2 \cdot 10^{-1}$  under the assumption that the actual model coincides with the nominal model.

model solution to the corresponding minimax problem, KF is worse than RKF's. Fig. 7 shows the variance of the prediction error for KF and RKF1 in the case that the actual model corresponds to the least favorable model solution to (12) with  $c = 10^{-1}$ . A similar scenario occurs in the case that we compare KF, RKF2 and the actual model is the least favorable one solution to (12) with  $c = 2 \cdot 10^{-1}$ , see Fig. 8. The latter figure shows also the performance of the robust Kalman filters with tolerance  $c = 1$  (RKF3) and  $c = 10^{-2}$  (RKF4). As expected, RKF3 and RKF4 are worse than RKF2, however, the former outperform KF. This means that, even though the ambiguity set is chosen slightly large (case with RKF3) or small (case with RKF4) as compared to the actual model, RKF still outperforms KF. Finally, it is worth noting that the least favorable model depends on tolerance  $c$ , and thus least favorable performance of KF depends  $c$  as well, see Fig. 7 and Fig. 8.

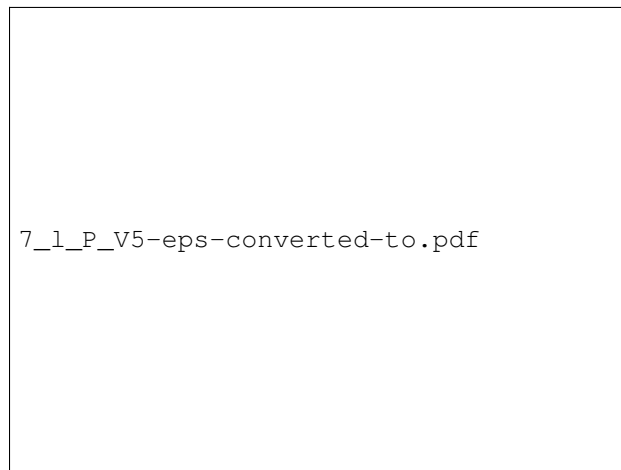


Fig. 7. Scalar variance of the prediction error for KF and RKF1 with  $c = 10^{-1}$  under the assumption that the actual model is the least favorable model solution to the corresponding minimax problem.

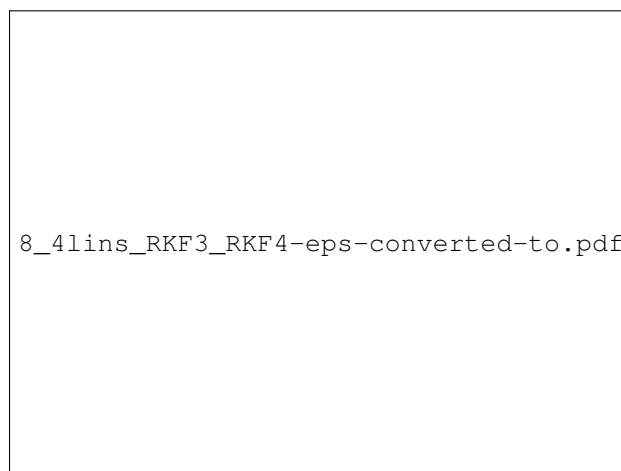


Fig. 8. Scalar variance of the prediction error for KF, RKF2 with  $c = 2 \cdot 10^{-1}$ , RKF3 with  $c = 1$  and RKF4 with  $c = 10^{-2}$  under the assumption that the actual model is the least favorable model solution to the minimax problem with  $c = 2 \cdot 10^{-1}$ .

## VIII. CONCLUSIONS

In this paper, we have considered a robust static estimation problem in the case that the nominal density and the actual density could be possibly degenerate. This approach is characterized by a minimax game: one player selects the least favorable density in a prescribed ambiguity set, while the estimator is designed according to this least favorable density. Then, we have extended this framework to the dynamic case where we have a state space model whose transition probability density is possibly degenerate. The solution is a Kalman-like filter whose gain is updated according to a Riccati-like iteration which evolves on the cone of the positive semidefinite matrices. We have proved that the resulting robust filter converges in the case that the nominal state space model has constant parameters and the ambiguity set has a radius  $c$  sufficiently small. Moreover, we have characterized the least

favorable model over a finite horizon. Finally, some numerical experiments have been presented in order to show the effectiveness of the proposed robust filter.

## IX. ACKNOWLEDGMENTS

Shenglun Yi is partially supported by NSFC under Grant Nos. 61973036 and 61433003.

## REFERENCES

- [1] B. Hassibi, A. Sayed, and T. Kailath, *Indefinite-Quadratic Estimation and Control—A Unified Approach to  $H^2$  and  $H^\infty$  Theories*. Philadelphia: Soc. Indust. Appl. Math., 1999.
- [2] J. Speyer and W. Chung, *Stochastic Processes, Estimation, and Control*. Advances in Design and Control, Philadelphia: Soc. Indust. Appl. Math., 2008.
- [3] L. El Ghaoui and G. Calafiore, “Robust filtering for discrete-time systems with bounded noise and parametric uncertainty,” *IEEE Trans. Autom. Control*, vol. 46, no. 7, pp. 1084–1089, 2001.
- [4] S. Kim, V. Deshpande, and R. Bhattacharya, “Robust Kalman filtering with probabilistic uncertainty in system parameters,” *IEEE Contr. Syst. Lett.*, vol. 5, no. 1, pp. 295–300, 2020.
- [5] B. Li, Y. Tan, L. Zhou, and R. Dong, “Robust-nonsmooth Kalman filtering for stochastic sandwich systems with dead-zone,” *Int. J Control Autom. Syst.*, pp. 1–11, 2020.
- [6] P. Whittle, *Risk-sensitive Optimal Control*. Chichester, England: J. Wiley, 1980.
- [7] R. Banavar and J. Speyer, “Properties of risk-sensitive filters/estimators,” *IEE Proc.—Control Theory Appl.*, vol. 145, no. 1, pp. 106–112, 1998.
- [8] B. Levy and M. Zorzi, “A contraction analysis of the convergence of risk-sensitive filters,” *SIAM J Control Optim.*, vol. 54, no. 4, pp. 2154–2173, 2016.
- [9] J. Huang, D. Shi, and T. Chen, “Distributed robust state estimation for sensor networks: A risk-sensitive approach,” in *Proc. 57th IEEE Conf. Decision and Control (CDC)*, pp. 6378–6383, 2018.
- [10] J. Speyer, C. Fan, and R. Banavar, “Optimal stochastic estimation with exponential cost criteria,” in *Proc. 31st IEEE Conf. Decision Control*, (Tucson, AZ), pp. 2293–2298, Dec. 1992.
- [11] R. Boel, M. James, and I. Petersen, “Robustness and risk-sensitive filtering,” *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 451–461, 2002.
- [12] L. Hansen and T. Sargent, “Robust estimation and control under commitment,” *J. Econom. Theory*, vol. 124, no. 2, pp. 258–301, 2005.
- [13] M. Yoon, V. Ugrinovskii, and I. Petersen, “Robust finite horizon minimax filtering for discrete-time stochastic uncertain systems,” *Syst. Control. Lett.*, vol. 52, no. 2, pp. 99–112, 2004.
- [14] L. Hansen and T. Sargent, “Recursive robust estimation and control without commitment,” *Journal of Economic Theory*, vol. 136, no. 1, pp. 1–27, 2007.
- [15] B. Levy and R. Nikoukhah, “Robust state-space filtering under incremental model perturbations subject to a relative entropy tolerance,” *IEEE Trans. Autom. Control*, vol. 58, pp. 682–695, Mar. 2013.
- [16] M. Zorzi and B. Levy, “Robust Kalman filtering: Asymptotic analysis of the least favorable model,” in *Proc. 57th IEEE Conf. Decision and Control (CDC)*, pp. 7124–7129, Dec 2018.
- [17] S. Abadeh, V. Nguyen, D. Kuhn, and P. Esfahani, “Wasserstein distributionally robust Kalman filtering,” in *Advances in Neural Information Processing Systems*, pp. 8474–8483, 2018.
- [18] M. Zorzi, “Distributed Kalman filtering under model uncertainty,” *IEEE Trans. Control. Netw. Syst.*, vol. 7, no. 2, pp. 990–1001, 2019.
- [19] A. Emanuele, F. Gasparotto, G. Guerra, and M. Zorzi, “Robust distributed Kalman filtering: On the choice of the local tolerance,” *Sensors*, vol. 20, no. 11, p. 3244, 2020.
- [20] A. Zenere and M. Zorzi, “On the coupling of model predictive control and robust Kalman filtering,” *IET Control Theory & Applications*, vol. 12, no. 13, pp. 1873–1881, 2018.
- [21] B. Levy and R. Nikoukhah, “Robust least-squares estimation with a relative entropy constraint,” *IEEE Trans. Informat. Theory*, vol. 50, no. 1, pp. 89–104, 2004.
- [22] M. Zorzi, “Robust Kalman filtering under model perturbations,” *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 2902–2907, 2016.

- [23] M. Zorzi, “On the robustness of the Bayes and Wiener estimators under model uncertainty,” *Automatica*, vol. 83, pp. 133–140, 2017.
- [24] D. P. Dee, “Simplification of the Kalman filter for meteorological data assimilation,” *Quarterly Journal of the Royal Meteorological Society*, vol. 117, no. 498, pp. 365–384, 1991.
- [25] S. Yi and M. Zorzi, “Low-rank Kalman filtering under model uncertainty,” in *Proc. 59th IEEE Conf. Decision and Control (CDC)*, pp. 2930–2935, Dec 2020.
- [26] S. Bonnabel and R. Sepulchre, “The geometry of low-rank Kalman filters,” in *Matrix Information Geometry*, pp. 53–68, Springer, 2013.
- [27] A. Ferrante and L. Ntogramatzidis, “The generalized continuous algebraic Riccati equation and impulse-free continuous-time LQ optimal control,” *Automatica*, vol. 50, no. 4, pp. 1176–1180, 2014.
- [28] A. Ferrante and L. Ntogramatzidis, “The generalised discrete algebraic Riccati equation in linear-quadratic optimal control,” *Automatica*, vol. 49, no. 2, pp. 471–478, 2013.
- [29] G. Baggio and A. Ferrante, “Parametrization of minimal spectral factors of discrete-time rational spectral densities,” *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 396–403, 2019.
- [30] G. Baggio and A. Ferrante, “On the factorization of rational discrete-time spectral densities,” *IEEE Trans. Autom. Control*, vol. 61, no. 4, pp. 969–981, 2016.
- [31] P. Bougerol, “Kalman filtering with random coefficients and contractions,” *SIAM J Control Optim*, vol. 31, pp. 942–959, Jul. 1993.
- [32] T. Cover and J. Thomas, *Information Theory*. New York: Wiley, 1991.
- [33] P. Baldi and K. Hornik, “Neural networks and principal component analysis: Learning from examples without local minima,” *Neural networks*, vol. 2, no. 1, pp. 53–58, 1989.
- [34] L. Hansen and T. Sargent, *Robustness*. Princeton, NJ: Princeton University Press, 2008.
- [35] P. Lancaster and L. Rodman, *Algebraic Riccati equations*. Clarendon press, 1995.
- [36] M. Zorzi and B. C. Levy, “On the convergence of a risk sensitive like filter,” in *Proc. 54th IEEE Conf. Decision and Control (CDC)*, pp. 4990–4995, Dec 2015.
- [37] M. Zorzi, “Convergence analysis of a family of robust Kalman filters based on the contraction principle,” *SIAM J Control Optim*, vol. 55, no. 5, pp. 3116–3131, 2017.
- [38] S. Gaubert and Z. Qu, “The contraction rate in Thompson’s part metric of order-preserving flows on a cone - application to generalized Riccati equations,” *Journal of Differential Equations*, vol. 256, no. 8, pp. 2902–2948, 2014.
- [39] H. Lee and Y. Lim, “Invariant metrics, contractions and nonlinear matrix equations,” *Nonlinearity*, vol. 2, pp. 857–878, 2008.
- [40] J. Lawson and Y. Lim, “The symplectic semigroup and Riccati differential equations,” *J. Dynamical and Control Syst.*, vol. 12, pp. 49–77, Jan. 2006.