

Electromagnetic wave propagation in general Kasner-like metrics

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Abstract

The curved spacetime Maxwell equations are applied to the anisotropically expanding Kasner metrics. Using the application of vector identities we derive 2nd-order differential wave equations for the electromagnetic field components; through this explicit derivation, we find that the 2nd-order wave equations are not uncoupled for the various components (as previously assumed), but that gravitationally-induced coupling between the electric and magnetic field components is generated directly by the anisotropy of the expansion. The lack of such coupling terms in the wave equations from several prior studies may indicate a generally incomplete understanding of the evolution of electromagnetic energy in anisotropic cosmologies. Uncoupling the field components requires the derivation of a 4th-order wave equation, which we obtain for Kasner-like metrics with generalized expansion/contraction rate indices. For the axisymmetric Kasner case, $(p_1, p_2, p_3) = (1, 0, 0)$, we obtain exact field solutions (for general propagation wavevectors), half of which appear not to have been found before in previous studies. For the other axisymmetric Kasner case, $\{p_1, p_2, p_3\} = \{(-1/3), (2/3), (2/3)\}$, we use numerical methods to demonstrate the explicit violation of the geometric optics approximation at early times, showing the physical phase velocity of the wave to be inhibited towards the initial singularity, with $v \rightarrow 0$ as $t \rightarrow 0$.

Keywords: Geometric optics violation; Kasner metric; Cosmology; Curved spacetime Maxwell equations

I. INTRODUCTION

In this paper we study the propagation of electromagnetic waves in the homogeneous but anisotropic Kasner metrics. These metrics are interesting physically, as exact solutions to the Einstein field equations possessing a singularity (for most Kasner metric parameters) at $t = 0$; and they are interesting cosmologically, as building blocks which can be generalized to help construct the Mixmaster universe model [1].

With a convenient alignment of our spatial axes, and noting that we set the speed of light to unity ($c \equiv 1$) throughout this paper, the Kasner metric is written as [2]:

$$ds^2 = -dt^2 + t^{2p_x} dx^2 + t^{2p_y} dy^2 + t^{2p_z} dz^2 , \quad (1)$$

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where the indices are typically assumed to satisfy the conditions:

$$(p_x + p_y + p_z) = (p_x^2 + p_y^2 + p_z^2) = 1 . \quad (2)$$

These Kasner conditions, if applied, ensure the vacuum nature (stress-energy tensor $T^{\mu\nu} = 0$) of the metric.

Modeling the propagation of light as null rays here, using the geometric optics approximation [2], would be comparatively straightforward; but to study much of the interesting physics, a wave treatment using Maxwell's equations is needed. For example, it is known that plane wave solutions will in general not follow null geodesics in anisotropic spacetimes, except in the high frequency (geometric optics) limit; hence effects such as birefringence (and other important phenomena) will be missed in a null ray treatment [3].

Investigations of Maxwell's equations in the Kasner metric, however, have proven to generate wave equations that only in rare or special cases have analytical solutions. Specifically, the problem of finding exact solutions for general wavevectors (all propagation directions), which are good for all values of time t , only appears achievable (among those cases obeying Eq. 2) for the $(p_x, p_y, p_z) = (1, 0, 0)$ case. For that case specifically, analytical wave solutions were derived in Sagnotti and Zwiebach [4] as linear combinations of Bessel functions (i.e., Hankel functions) of purely imaginary order; and derivations by other authors [e.g., 5] have led to equivalent results.

For other sets of Kasner indices – even the other axisymmetric vacuum case, $\{p_x, p_y, p_z\} = \{-1/3, (2/3), (2/3)\}$ – analytical solutions appear to be unavailable, without adopting various simplifications. Using a superpotential formalism developed by Kegeles and Cohen [6] for calculating electromagnetic (as well as neutrino and gravitational) metric perturbations, this Kasner case was studied in a series of papers by different authors [7–9], in which analytical solutions were obtained only within early-time ($t \ll 1$) and late-time ($t \gg 1$) approximations. Alternatively, some authors [4, 5, 10] have found solutions to this case (or for more general choices of Kasner parameters) by restricting the wave propagation direction to be along a single coordinate axis (i.e., along one of the principal expansion axes); and some [e.g., 11] have employed combinations of both approximations.

In some recent papers [5, 12, 13], wave propagation has been studied in the Kasner metric using the *scalar* wave equation:

$$\square_g \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0 , \quad (3)$$

with the d'Alembertian transformed appropriately for the relevant metric being studied. In a lengthier exposition by Petersen [14], it is made clear (for example in their expression, “Scalar wave equation for light”) that this equation is actually being used to study the propagation of *electromagnetic* fields in the Kasner metric. While this may or may not cause discrepancies in some particular study (e.g., deriving cosmological redshifts [5, 14]), it is not a rigorous procedure in general to apply a scalar wave equation individually to the various components of a vector field, implicitly assuming that they are mutually independent.

On the contrary, one of the principal findings of our paper – to be shown via explicit step-by-step derivations using Maxwell’s equations in curved spacetime – is that one cannot specify *homogeneous* 2nd-order differential wave equations for all of the electromagnetic fields in the anisotropic Kasner metric. (Note that for brevity, we will often use the terms “field components” and “fields” interchangeably in this paper.) Assuming a general wave propagation direction, nonhomogeneous terms appear on the right-hand side (RHS) of the equations analogous to Eq. 3 for some of the fields – even in this charge-free spacetime – generated directly by the anisotropy of the axis expansion rates. The RHS term(s) for one given field component are proportional to other electromagnetic field components, thus linking the fields together, making them mutually dependent (an effect which would be absent in any analysis based upon Eq. 3). For any two axes expanding/contracting according to different Kasner parameters ($p_i \neq p_j$), and for a wavevector possessing nonzero components along both of those axes, the electric (E –) or magnetic (B –)field in the direction normal to that “ ij -plane” will experience a driving term from its complementary fields within the plane. (For example, if $p_y \neq p_z$, and wavevector components $(k_y, k_z) \neq 0$, then E_x would be driven by nonzero B_y and/or B_z ; B_x would be driven by nonzero E_y and/or E_z ; and so on.)

The main effect of these driving terms, is that unlike the situation for flat spacetime, the individual electromagnetic field components (for general wavevector directions) are *not* uncoupled in the 2nd-order wave equations; instead, one must go on to derive 4th-order differential equations in order to obtain truly independent wave equations for the six electromagnetic fields. The fact that these fields must satisfy 4th-order equations means that there are actually *four* solutions for each of them, in general, instead of two; and previous results in the literature for such fields that have specified only two field solutions are actually missing half of the solution functions.

Below, we will derive the 4th-order wave equation (which is identical for all six of the

electromagnetic fields), obtaining a general expression that is valid for any values of the Kasner parameters (p_x, p_y, p_z) , including those “Kasner-like” metrics not restricted to obey the vacuum Kasner conditions (Eq. 2). Furthermore, re-examining the $(p_x, p_y, p_z) = (1, 0, 0)$ case, we will find all four solutions (including the two already known), group them into the two natural polarization states for that axisymmetric metric, and obtain the general solution (including all field amplitudes) that completely solves the Maxwell equations in the case of a general propagation direction, with nonzero wavevector components along all distinctly evolving axes.

These additional field solutions – and the field-coupling driving terms on the RHS of the 2nd-order wave equations – have not been seen by us to have been derived in any previous work. Why these additional terms and field solutions have been missed previously (if indeed they have been), is not immediately clear. One likely contributing factor, is that given the difficulty of finding solutions for general propagation directions, the simplification of restricting the wave propagation to lie along one of the principal axes is often made almost immediately [e.g., 3, 10, 11, 15]; and as will be evident in our formulas below, the RHS driving terms in the 2nd-order wave equations can be made to vanish for propagation along a single axis. Alternatively, for the superpotential formalism of Kegeles and Cohen [6] (upon which several other papers are based), the derivation is considerably involved; but it would appear that the homogeneous scalar wave equation (Eq. 3) has been assumed by fiat, as has also been done – appropriately in places, perhaps inappropriately in others – in [5, 12–14, 16]. (Interestingly, the discussion involving Equations 2.1 – 2.3 of Alho *et al.* [13] claims to show the scalar wave equation to be a sufficient condition for energy conservation, $\nabla^a T_{ab} = 0$; but no indication is given there to show the scalar wave equation to be a *necessary* condition for it.) Additionally, in Sagnotti and Zwiebach [4], a subtle argument is made involving the introduction of time-dependent tetrad basis vectors in order to produce their Equation (1.18), a 2nd-order wave equation derived for a Bianchi type-I background metric (of which the Kasner models are a subclass); this formula, from which all of the wave equations for their subsequent Kasner metric analyses are derived, is somehow missing the RHS driving term(s), and thus those terms are absent throughout the paper.

Given that studies such as these are analyzing many important aspects of electromagnetic wave propagation – including field amplitude and energy bounds (i.e., whether or not field solutions “blow up”) close to the Kasner singularity [e.g., 5, 13] – it seems particularly

important to avoid missing any terms which represent the exchange of energy between the different electric and magnetic field components (or, for that matter, between the electromagnetic and gravitational fields). And since this effect is due directly to the anisotropy of the expansion rates, it is reasonable to assume that such gravitationally-induced coupling of energy between the fields may very well occur in *any* anisotropic cosmological metric. It is therefore a real possibility that the evolution of electromagnetic energy in anisotropic cosmologies has been incompletely understood in all previous studies which have relied upon the homogeneous scalar wave equation for understanding vector fields such as light.

In this paper, the derivation of the wave equations for Kasner metrics with general indices will be carried out in Section II, with some of the detailed steps shown in Appendix A. The complete solution of the fields for the $(p_x, p_y, p_z) = (1, 0, 0)$ case, for waves with general propagation directions, will be given in Section III, with some technical discussion of the link between this Kasner case and Minkowski spacetime given in Appendix B.

Additionally, we note that while the Kasner wave equations are certainly apt for numerical calculations, the literature on this topic appears to be quite sparse. (Referring here specifically to numerical studies of the propagating electromagnetic fields on the predefined Kasner background metrics; not to numerical gravitational studies of the evolution of these metrics themselves, or of related spacetimes.) As a step towards providing results in this area, in Section IV we present the findings from a numerical simulation program that we have developed and applied to the study of the $\{p_x, p_y, p_z\} = \{(-1/3), (2/3), (2/3)\}$ case – specifically to the propagating wave behavior of the electromagnetic field component along the axis of expansion rate symmetry. Notably, we demonstrate how the physical phase velocity of the light wave deviates from the null ray speed ($v = c \equiv 1$), detailing how light propagation is slowed and ultimately stopped going back towards the singularity ($t \rightarrow 0$) for this particular Kasner metric. Furthermore, we will recall an interesting class of cosmology-like metrics that reduce to this Kasner case as a vacuum limit, and will note how the formalism and numerical tools developed for this paper can be extended to study that class of metrics in future studies.

Lastly, in Section V, our final discussion and summary of these results will be presented.

II. FORMULATING THE WAVE EQUATIONS

A. Electromagnetic conventions

For a metric $g_{\mu\nu}$ representing a spacetime containing only radiation (or vacuum), and no source charges or currents ($J^\mu = 0$), the curved spacetime Maxwell equations in terms of the electromagnetic field tensor are [2, 17]:

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 , \quad (4a)$$

$$F^{\alpha\beta}{}_{;\alpha} \propto (\sqrt{-|g_{\mu\nu}|} F^{\alpha\beta})_{,\alpha} = 0 , \quad (4b)$$

where $|g_{\mu\nu}|$ is the determinant of the metric tensor, semicolons refer to covariant derivatives, and commas to partial derivatives.

We wish to define the electric and magnetic fields in a way that will usefully represent the observable physical fields as seen by a “fiducial” observer. In a metric with coordinates (t, x, y, z) , and a convenient comoving reference frame, this would represent a stationary observer with $dx = dy = dz = 0$ for all time.

For metrics with some symmetry-breaking physical behavior (such a distinct “flow” of radiation in a particular direction), it might also be interesting to define a different class of fiducial observers (i.e., those stationary with respect to that flow); but for now, we will define our fields by considering a network of observers who are at least *instantaneously* stationary in (x, y, z) position.

Such observers will have timelike worldlines as long as t is a timelike coordinate, with $g_{tt} < 0$. For the formalism presented here, the only assumptions we make are that t is indeed timelike, and that (for considerable simplification) all metrics under consideration will be *diagonal*. We do not assume them to be synchronous or comoving, although all of the metrics used for calculations in this particular paper will be.

With those considerations, the normalized ($g_{\alpha\beta}U^\alpha U^\beta = -1$), future-directed timelike four-velocity of a fiducial observer at a given spacetime location will be:

$$U^\alpha = (1/\sqrt{-g_{tt}}(t, x, y, z), 0, 0, 0) . \quad (5)$$

The electric and magnetic fields, respectively, measured by such an observer will be [18]:

$$E_\alpha = F_{\alpha\beta}U^\beta , \quad (6a)$$

$$B_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}U^\beta, \quad (6b)$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the covariant Levi-Civita totally antisymmetric tensor.

For ease of calculations, we express the Levi-Civita tensor in terms of the Levi-Civita *symbol*, $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$, a tensor density of weight $w = 1$, filled solely with values $(+1, -1, 0)$ [19]. Thus, for a metric that is diagonal:

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-|g_{\mu\nu}|} \tilde{\epsilon}_{\alpha\beta\gamma\delta} = \sqrt{-g_{tt}g_{xx}g_{yy}g_{zz}} \tilde{\epsilon}_{\alpha\beta\gamma\delta}. \quad (7)$$

Using the sign convention from [18] of $\tilde{\epsilon}_{0123} \equiv \tilde{\epsilon}_{txyz} = +1$, each of Equations 6b can individually be inverted to yield the components of $F^{\gamma\delta}$ in terms of the B -fields, and then we can obtain the covariant version of the electromagnetic field tensor via $F_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\delta}F^{\gamma\delta}$. Similarly (and more simply), we can invert Equations 6a to yield: $F_{i0} = -F_{0i} = \sqrt{-g_{tt}}E_i$, for $i \neq 0$. So to sum up, for the sake of clarity – given variations among different authors on the distribution of signs and metric factors – we explicitly write out this tensor as:

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -\sqrt{-g_{tt}}E_x & -\sqrt{-g_{tt}}E_y & -\sqrt{-g_{tt}}E_z \\ \sqrt{-g_{tt}}E_x & 0 & \sqrt{\frac{g_{xx}g_{yy}}{g_{zz}}}B_z & -\sqrt{\frac{g_{xx}g_{zz}}{g_{yy}}}B_y \\ \sqrt{-g_{tt}}E_y & -\sqrt{\frac{g_{xx}g_{yy}}{g_{zz}}}B_z & 0 & \sqrt{\frac{g_{yy}g_{zz}}{g_{xx}}}B_x \\ \sqrt{-g_{tt}}E_z & \sqrt{\frac{g_{xx}g_{zz}}{g_{yy}}}B_y & -\sqrt{\frac{g_{yy}g_{zz}}{g_{xx}}}B_x & 0 \end{bmatrix}. \quad (8)$$

Employing these conventions for the electromagnetic tensor in Equations 4 will produce observationally sensible definitions for the E - and B -fields.

B. Deriving wave equations for the evolving metric

We must now choose a particular class of metrics for the derivation of the curved spacetime Maxwell equations via Eq's. 4,8. Here we are interested in the well known Kasner metrics, the homogeneous but anisotropic spacetimes discussed above in the Introduction. They are as defined in Eq. 1, where vacuum spacetimes are obtained by imposing the usual Kasner index conditions, specified by Eq. 2.

There is no necessity to assume these conditions, however – for example, the isotropic Friedmann universes filled with matter or radiation can be recovered via the choices (respectively) of $p_x = p_y = p_z = 2/3$ or $1/2$. We will place no a priori restrictions (other than realness) on the Kasner indices; we simply note that there must be some physical motivation

for the nature of the cosmic mass-energy that results from a particular choice of (p_x, p_y, p_z) , especially if such a choice violates reasonable energy conditions.

For our work here, we will be considering the application of these wave equations under the commonly used “test field” approximation, in which we assume that the amplitudes and energy densities of the electromagnetic fields being studied are always small enough so that their gravitational perturbations onto the background metric can be neglected. (In cases where gravitational energy from the background metric can pump energy into the fields to increase their amplitudes significantly, it is therefore imperative to make sure that the test fields actually *stay* small enough during the propagation to continue neglecting their gravitational effects.)

Applying the formulas of Section II A to this metric, we see that the electromagnetic field tensor becomes:

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & t^{(p_x+p_y-p_z)} B_z & -t^{(p_x-p_y+p_z)} B_y \\ E_y & -t^{(p_x+p_y-p_z)} B_z & 0 & t^{(-p_x+p_y+p_z)} B_x \\ E_z & t^{(p_x-p_y+p_z)} B_y & -t^{(-p_x+p_y+p_z)} B_x & 0 \end{bmatrix}. \quad (9)$$

Equations 4 then give us eight distinct field equations. Two of these equations (one each from Eq. 4a and Eq. 4b) are modified versions of the usual divergence equations; dividing out common powers of t , we can write them as:

$$t^{-2p_x} E_{x,x} + t^{-2p_y} E_{y,y} + t^{-2p_z} E_{z,z} = 0, \quad (10a)$$

$$t^{-2p_x} B_{x,x} + t^{-2p_y} B_{y,y} + t^{-2p_z} B_{z,z} = 0. \quad (10b)$$

The remaining six equations are the “curl”-like equations, which are:

$$(B_{z,y} - B_{y,z}) = [t^{(-p_x+p_y+p_z)} E_x]_{,t}, \quad (11a)$$

$$(E_{z,y} - E_{y,z}) = -[t^{(-p_x+p_y+p_z)} B_x]_{,t}, \quad (11b)$$

plus four more analogous equations obtained through cyclic permutations over (x, y, z) and corresponding factors indexed to (x, y, z) – such as (p_x, p_y, p_z) , and spatial partial derivatives.

These strongly resemble the usual flat spacetime Maxwell equations, just with the appearance of additional powers of t ; and at first glance, it might seem possible to absorb these

factors through alternative definitions of the fields comprising $F_{\alpha\beta}$. For example, with the redefinitions:

$$E'_x \equiv t^{(-p_x+p_y+p_z)} E_x, \quad B'_x \equiv t^{(-p_x+p_y+p_z)} B_x, \quad (12)$$

(and cyclic permutations thereof), the divergence equations (Eq's. 10) can be simplified to $\nabla \cdot \mathbf{E}' = \nabla \cdot \mathbf{B}' = 0$ (using the flat spacetime version of the dot product here), and the right hand sides of Eq's. 11 reduce simply to the relevant spatial components of $\mathbf{E}'_{,t}$, $\mathbf{B}'_{,t}$. However, additional factors appear on the left hand sides of the equations as a consequence; for example, Eq's. 11 become:

$$[t^{(-p_x-p_y+p_z)} B'_{z,y}] - [t^{(-p_x+p_y-p_z)} B'_{y,z}] = E'_{x,t}, \quad (13a)$$

$$[t^{(-p_x-p_y+p_z)} E'_{z,y}] - [t^{(-p_x+p_y-p_z)} E'_{y,z}] = -B'_{x,t}, \quad (13b)$$

(and cyclic permutations), which does not succeed in removing the (physically meaningful) powers of t in these equations, but just shifts them around. However, these redefinitions do succeed in greatly simplifying the derivations and forms of the wave equations (as well as their eventual solutions), so we will work with these redefined fields from this point on.

Normally, the next step is to convert the coupled 1st-order differential equations for the fields into uncoupled 2nd-order wave equations. In charge-free Minkowski space, where the fields obey $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = -\mathbf{B}_{,t}$, and $\nabla \times \mathbf{B} = \mathbf{E}_{,t}$, the usual trick is to apply vector identities to write $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$, and then use $\nabla \times \nabla \times \mathbf{E} = -(\nabla \times \mathbf{B})_{,t} = -\mathbf{E}_{,t,t}$ to complete the wave equation. But in the Kasner case, there are unavoidable powers of t in Eq's. 13 which do not commute with the time derivatives $\partial/\partial t$ during the aforementioned steps. Therefore, the resulting 2nd-order differential equations are *not* entirely uncoupled here, in general.

Applying the same ‘‘curl of a curl’’ trick with these equations anyway, modified as necessary by these extra powers of t , after some work we derive:

$$E'_{x,t,t} + \left(\frac{p_x}{t}\right) E'_{x,t} - [(t^{-2p_x} E'_{x,x,x}) + (t^{-2p_y} E'_{x,y,y}) + (t^{-2p_z} E'_{x,z,z})] = \frac{(p_z - p_y)}{t} \{ [t^{(-p_x-p_y+p_z)} B'_{z,y}] + [t^{(-p_x+p_y-p_z)} B'_{y,z}] \}, \quad (14)$$

where similar equations for E'_y and E'_z can be obtained from this via cyclic permutations over (x, y, z) and corresponding factors; and the analogous equations for the magnetic fields are obtained via the substitutions $E'_i \rightarrow B'_i$, $B'_i \rightarrow -E'_i$. The full details of this derivation are given in Appendix A.

Note first that the *plus* sign between the two terms on the right hand side (RHS) of Equation 14, instead of a minus sign, means that the RHS cannot be eliminated via any manipulations of the curl-like equations, Eq's. 13. (These are, in fact, the field-coupling nonhomogeneous terms discussed significantly in the Introduction.) Using Eq's. 13 in conjunction with Eq. 14, however, does allow us to present this wave equation in multiple alternative forms. Using the shorthand notation of “ $\{t\nabla^2\}$ ” $\equiv [(t^{-2p_x})\partial^2/\partial x^2 + (t^{-2p_y})\partial^2/\partial y^2 + (t^{-2p_z})\partial^2/\partial z^2]$, we can present the 2nd-order wave equation for E'_x (and analogously for the other fields) in three useful variants for each field:

$$E'_{x,t,t} + \left(\frac{p_x}{t}\right) E'_{x,t} - \{t\nabla^2\} E'_x = \frac{(p_z - p_y)}{t} \{ [t^{(-p_x - p_y + p_z)} B'_{z,y}] + [t^{(-p_x + p_y - p_z)} B'_{y,z}] \}, \text{ or, } \quad (15a)$$

$$E'_{x,t,t} + \left[\frac{p_x + (p_z - p_y)}{t} \right] E'_{x,t} - \{t\nabla^2\} E'_x = 2 \frac{(p_z - p_y)}{t} \{ [t^{(-p_x - p_y + p_z)} B'_{z,y}] \}, \text{ or, } \quad (15b)$$

$$E'_{x,t,t} + \left[\frac{p_x - (p_z - p_y)}{t} \right] E'_{x,t} - \{t\nabla^2\} E'_x = 2 \frac{(p_z - p_y)}{t} \{ [t^{(-p_x + p_y - p_z)} B'_{y,z}] \}. \quad (15c)$$

Rather than creating ambiguity, however, having these different variants available will end up aiding us in deriving further wave equations that are ultimately uncoupled.

These equations take the form of damped (or pumped), driven oscillators. The nonhomogeneous driving terms on the RHS of these 2nd-order wave equations can be interpreted as the anisotropically evolving spacetime behaving somewhat like an active medium, causing the magnetic fields to drive the electric fields, and vice versa, in a more involved manner than happens in Minkowski space [e.g., 20].

In a specific set of cases, the RHS terms do properly equal zero – particularly in the case of axisymmetric Kasner metrics (e.g., $p_y = p_z$), for the fields along the distinctly-evolving direction (e.g., E_x and B_x). But for other fields and cases, dropping the RHS terms causes one to lose about half of the solutions, as we will see below.

Examining these 2nd-order partial differential equations, a way to simplify them immediately is through separation of variables. For each electric or magnetic field, F'_i assume the following solution:

$$\begin{aligned} F'_i(t, x, y, z) &= F''_i(t) S_i(x, y, z) = \\ &= F''_i(t) [c_1 \sin(k_x x) + c_2 \cos(k_x x)] [c_3 \sin(k_y y) + c_4 \cos(k_y y)] [c_5 \sin(k_z z) + c_6 \cos(k_z z)] \end{aligned} \quad (16)$$

where $\{c_1, \dots, c_6\}$ are arbitrary constants dependent upon initial conditions. (To simplify

our numerical calculations, we have chosen to work with real functions throughout.) The constants $(k_x, k_y, k_z) \equiv \mathbf{k}$ are real, positive numbers with $k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$. This form of solution means that the light waves “breathe” with the expansion or contraction along different axis directions, as a particular set of coordinate distances, say $(\Delta x, \Delta y, \Delta z)$, represents different physical distances at different times.

With substitutions like Eq. 16, Equation 14 becomes:

$$E''_{x,t,t} + \left(\frac{p_x}{t}\right) E''_{x,t} + \left(\frac{k_x^2}{t^{2p_x}} + \frac{k_y^2}{t^{2p_y}} + \frac{k_z^2}{t^{2p_z}}\right) E''_x = \frac{1}{S_x(x,y,z)} \times \frac{(p_z - p_y)}{t} \{ [t^{(-p_x - p_y + p_z)} B'_{z,y}] + [t^{(-p_x + p_y - p_z)} B'_{y,z}] \}, \quad (17)$$

where equations for all of the other fields are obtained from this via cyclic permutations as before, and where the alternative versions for each field, as given in Eqn's. 15, still apply.

Now, it should be understood that the Equations 14-17 were written assuming that (k_x, k_y, k_z) are nonzero. Letting one or two of these wavenumbers equal zero (e.g., propagation along a Kasner axis) could cause confusion. The form of the spatial functions $S_i(x, y, z)$, plus $\nabla \cdot \mathbf{E}' = \nabla \cdot \mathbf{B}' = 0$, clearly shows that the wave fields are transverse to the instantaneous propagation direction. So for example, suppose $\mathbf{k} = (k_x, 0, 0)$. Clearly all of the fields then satisfy $F'_{i,y} = F'_{i,z} = 0$, and Equations 15 seem to imply that “adding zero” to the RHS in different ways gives us three meaningfully different versions of the left hand side. But in this case, the transverse nature of the fields tells us that $E'_x = 0$ (as well as $B'_x = 0$), hence all three versions are moot. Alternatively, if we suppose $\mathbf{k} = (0, k_y, 0)$, then all three versions make sense; with the simplest one being Eq. 15c, where the RHS now does equal zero, and the coefficient of the $(E'_{x,t}/t)$ term is $[p_x - (p_z - p_y)]$. If one assumes the usual Kasner condition, $(p_x + p_y + p_z) = 1$, then our resulting equation is equivalent to Equation (3.18) of Sagnotti and Zwiebach [4] for this special case.

To ultimately disentangle the wave equations, we can use the alternative formulations of Eq's. 15 to tailor the couplings between the electric and magnetic fields to group them as three disconnected pairs of fields – for example, $(E_x \leftrightarrow B_y)$, $(E_y \leftrightarrow B_z)$, and $(E_z \leftrightarrow B_x)$. The equations for the first pair would look like this:

$$E'_{x,t,t} + \left[\frac{p_x - (p_z - p_y)}{t}\right] E'_{x,t} - \{t\nabla^2\} E'_x = 2\frac{(p_z - p_y)}{t} \{ [t^{(-p_x + p_y - p_z)} B'_{y,z}] \}, \quad (18a)$$

$$B'_{y,t,t} + \left[\frac{p_y + (p_x - p_z)}{t}\right] B'_{y,t} - \{t\nabla^2\} B'_y = -2\frac{(p_x - p_z)}{t} \{ [t^{(p_x - p_y - p_z)} E'_{x,z}] \}, \quad (18b)$$

We then take $\partial/\partial z$ of Eq. 18a, apply our spatial functions from Eq. 16 as necessary, and invert to get:

$$B'_y = - \left(\frac{1}{k_z^2} \right) \left[\frac{t}{2(p_z - p_y)} \right] [t^{(p_x - p_y + p_z)}] \times \left\{ E'_{x, t, t, z} + \left[\frac{p_x - (p_z - p_y)}{t} \right] E'_{x, t, z} + \left(\frac{k_x^2}{t^2 p_x} + \frac{k_y^2}{t^2 p_y} + \frac{k_z^2}{t^2 p_z} \right) E'_{x, z} \right\} , \quad (19)$$

This expression for B'_y is then inserted back into Eq. 18b to eliminate it in favor of E'_X . Then we apply $\partial/\partial z$ once more – getting another common factor of $(-k_z^2)$ – and then divide out the spatial functions, and any other factors in front of the highest time derivative term. The result is a *fourth*-order ordinary differential equation for $E''_x(t)$. This equation is the *same* for the magnetic fields also (no sign changes or other factors). So for $F''_x(t) \equiv E''_x(t)$ or $B''_x(t)$, we have the formula:

$$\begin{aligned} & F''_{x, t, t, t, t} + \left[\frac{2(1 + 2p_x)}{t} \right] F''_{x, t, t, t} \\ & + \left\{ 2 \left(\frac{k_x^2}{t^2 p_x} + \frac{k_y^2}{t^2 p_y} + \frac{k_z^2}{t^2 p_z} \right) + \frac{1}{t^2} [2p_x + 5p_x^2 - (p_y - p_z)^2] \right\} F''_{x, t, t} \\ & + \left(\frac{2}{t} \left\{ \frac{k_x^2}{t^2 p_x} + [1 + 2(p_x - p_y)] \frac{k_y^2}{t^2 p_y} + [1 + 2(p_x - p_z)] \frac{k_z^2}{t^2 p_z} \right\} + \left\{ \frac{1}{t^3} (2p_x - 1) [p_x^2 - (p_y - p_z)^2] \right\} \right) F''_{x, t} \\ & + \left[\left(\frac{k_x^2}{t^2 p_x} + \frac{k_y^2}{t^2 p_y} + \frac{k_z^2}{t^2 p_z} \right)^2 + \frac{2}{t^2} \{ [(p_x - p_y)(1 + p_x + p_z - 3p_y)] \frac{k_y^2}{t^2 p_y} + [(p_x - p_z)(1 + p_x + p_y - 3p_z)] \frac{k_z^2}{t^2 p_z} \} \right] \\ & \times F''_x = 0 , \end{aligned} \quad (20)$$

with the analogous equations for $\{E''_y, B''_y\}$, $\{E''_z, B''_z\}$ obtained as usual through cyclic permutations over (x, y, z) , (p_x, p_y, p_z) , and (k_x, k_y, k_z) . (Note that we get the exact same 4th-order equations even if we derive them via $(E_x \leftrightarrow B_z)$ instead of $(E_x \leftrightarrow B_y)$, and so on.)

The above procedure is only valid for fields where the RHS of formulas like Eq. 18a are nonzero. For those where the RHS is zero – say, through $k_y = k_z = 0$ (in which case the field itself is zero), or through $p_y = p_z$ (axisymmetric Kasner case), where the RHS for the F''_x fields (but *not* for the F''_y or F''_z fields) are zero – then the 2nd-order wave equation is all we have for that field, and we only have two (if any) nontrivial solutions for it. But, for the general case in which Equation 20 holds, then we actually have four independent solutions for the temporal wave function. In the next section, we will see that the correct set of solutions to use depends upon the *polarization* of the waves.

While Eq. 20 appears well suited for numerical calculations, one might despair of finding analytical solutions (or any clear understanding of the solutions) for a 4th-order differential equation, especially one as complicated as this one. But that is not necessarily the case.

First, just from the nature of the 2nd-order equations, it is clear that the ubiquitous expression $(t^{-2p_x}k_x^2 + t^{-2p_y}k_y^2 + t^{-2p_z}k_z^2)$ tells us a great deal about the effective frequency of oscillations at all times, especially in early-, mid-, or late-time regimes where one or another of the three factors is dominant. Also, the presence of the term $(F''_{i,t}/t)$ implies that we will have Bessel-function-like behavior as $t \rightarrow 0$, but almost pure sinusoidal behavior (with adiabatically-varying frequency) at late times. Much of the qualitative behavior of the fields is therefore obvious.

Furthermore, we will show that it is also possible to “guess” the exact solution to the 4th-order equation, by inferring it from known solutions to a 2nd-order equation like Eq. 17 when the RHS happens to equal zero. (In fact, it may not be a bad trick to try this in general, by temporarily setting the RHS to zero to search for solutions, even if the RHS does not properly equal zero for that case.) It is an unfortunate drawback that even for 2nd-order equations of this type with a RHS equal to zero, known solutions still seem to be rare; but this is a promising path to follow when possible.

For axisymmetric Kasner models, for which two of the parameters p_i are equal, the RHS of the 2nd-order equation for the fields parallel to the axis of symmetry is automatically equal to zero. If one sticks to the usual parameter constraints of Eq. 2, then the only two possibilities are the models: $(p_1, p_2, p_3) = (1, 0, 0)$, and $\{p_1, p_2, p_3\} = \{(-1/3), (2/3), (2/3)\}$. These are therefore the most heavily studied Kasner cases. Although our formalism so far has placed no such restrictions on the p_i parameters, we will nevertheless focus the rest of this paper upon these two cases, because of these reasons: the first case is (almost uniquely) completely solvable for all of the fields, as we will show; the second case exhibits interesting violations of geometric optics as $t \rightarrow 0$; and, the second case is the vacuum limit of a class of non-vacuum, inhomogeneous metrics of particular interest to us for future cosmological study.

III. KASNER SPECIAL CASE (1, 0, 0)

Choosing the axes such that $(p_x, p_y, p_z) = (1, 0, 0)$, the metric becomes:

$$ds^2 = -dt^2 + t^2 dx^2 + dy^2 + dz^2 . \quad (21)$$

Now, the parameter choices $\mathbf{p} = (1, 0, 0)$ (or equivalently $(0, 1, 0)$ or $(0, 0, 1)$) represents a unique Kasner case because it is actually a flat spacetime – all Riemann tensor components are zero. It can therefore be transformed into the Minkowski metric, with the implication that the field equations must be those of ordinary flat spacetime, with their purely sinusoidal solutions. But the formulas to be presented in this section will not *look* very much like those of flat spacetime.

This is due to the fact that the transformation producing the metric in Equation 21 has the effect of nontrivially mixing together the different coordinates and field components. In particular, note that the “separation of variables” from Eq. 16 – with terms like $\sin(k_x x)$ – is not at all the same kind of separation of variables that one would do in static Minkowski spacetime, since the argument “ $k_x x$ ” actually represents a physical distance that increases in time like t , rather than a static wavelength.

It is therefore important to remember that the solutions presented here for this case are merely an unusual way of combining and transforming the different flat spacetime solutions. (A derivation of how Kasner $(1, 0, 0)$ fields are related to the Minkowski frame fields is given in Appendix B.) Nevertheless, it is instructive to use this metric to demonstrate the method of finding solutions, and their properties; and it is probably not a coincidence that the least physically complicated case is also the most mathematically solvable one.

Referring back to Eq. 17, the wave equation for $F''_x(t) \in \{E''_x(t), B''_x(t)\}$ now becomes:

$$F''_{x,t,t} + \left(\frac{1}{t}\right) F''_{x,t} + \left(\frac{k_x^2}{t^2} + k_y^2 + k_z^2\right) F''_x = 0 , \quad (22)$$

which we recognize as an example of the transformed Bessel equation [21] with solutions $J_{\pm ik_x}[(k_y^2 + k_z^2)^{1/2}t]$. For Bessel functions of purely imaginary order like these, real solutions (for $t > 0$) can be constructed [22] as:

$$\begin{aligned} J_{ik_x}^{2+}[(k_y^2 + k_z^2)^{1/2}t] &\equiv \frac{1}{2} \{ J_{ik_x}[(k_y^2 + k_z^2)^{1/2}t] + J_{-ik_x}[(k_y^2 + k_z^2)^{1/2}t] \} , \\ J_{ik_x}^{2-}[(k_y^2 + k_z^2)^{1/2}t] &\equiv \frac{1}{2i} \{ J_{ik_x}[(k_y^2 + k_z^2)^{1/2}t] - J_{-ik_x}[(k_y^2 + k_z^2)^{1/2}t] \} . \end{aligned} \quad (23)$$

These are equivalent to (i.e., linear combinations of) the previously known solutions, as given in (for example) Sagnotti and Zwiebach [4] and Petersen [5]. They turn out to represent only half of the space of solutions for this Kasner case, however.

For the four remaining fields, the right hand sides of the 2nd-order wave equations cannot generally be made equal to zero; but those driving terms can be manipulated, analogously with Eq's. 15, to make them dependent only upon $E'_x(t)$ or $B'_x(t)$. All of those equations end up having the same form:

$$F'_{a,t,t} + \left(\frac{1}{t}\right) F'_{a,t} + \left(\frac{k_x^2}{t^2} + k_y^2 + k_z^2\right) F'_a = 2F'_b, \quad (24)$$

with (F'_a, F'_b) representing the pairs of fields, respectively: $(E'_y, B'_{x,z})$, $(E'_z, -B'_{x,y})$, $(B'_y, -E'_{x,z})$, and $(B'_z, E'_{x,y})$.

We note that this is the same differential equation (with the same temporal solutions) as Eq. 22 for $E''_x(t)$ and $B''_x(t)$, *if* we can set the right hand sides equal to zero. If not, then each field obeys the relevant 4th-order equation instead. A simplification of the physical situation therefore presents itself, where we break the solutions down into the two distinct, nontrivial possibilities: (i) Polarization “ X_E ”, where $E'_x \neq 0$, $B'_x = 0$, so that (E''_x, E''_y, E''_z) are all linear combinations of the functions (J^{2+}, J^{2-}) from Eq's. 23, and (B''_y, B''_z) are solutions of the 4th-order equations; and, (ii) Polarization “ X_B ”, where $E'_x = 0$, $B'_x \neq 0$, and vice-versa ($E \leftrightarrow B$) for the solutions of the nonzero fields. The general solution can then be given as an appropriate combination of Polarization X_E and Polarization X_B .

Now recalling Eq. 20, as applied to this Kasner (1, 0, 0) metric, we obtain the 4th-order wave equation:

$$F''_{i,t,t,t,t} + \left(\frac{2}{t}\right) F''_{i,t,t,t} + \left[2\left(\frac{k_x^2}{t^2} + k_y^2 + k_z^2\right) - \frac{1}{t^2}\right] F''_{i,t,t} + \left[\frac{2}{t}\left(-\frac{k_x^2}{t^2} + k_y^2 + k_z^2\right) + \frac{1}{t^3}\right] F''_{i,t} + \left[\left(\frac{k_x^2}{t^2} + k_y^2 + k_z^2\right)^2 + \left(\frac{4}{t^4}k_x^2\right)\right] F''_i = 0, \quad (25)$$

which turns out to be the same equation applicable for all four remaining fields, $F''_i \in \{E''_y(t), B''_y(t), E''_z(t), B''_z(t)\}$. One can immediately verify that two good solutions to this 4th-order equation are (J^{2+}, J^{2-}) from Eq's. 23; or in other words, $J_{ik_x}[(k_y^2 + k_z^2)^{1/2}t]$ and $J_{-ik_x}[(k_y^2 + k_z^2)^{1/2}t]$ are still good solutions. We must therefore find the two remaining ones.

One interesting possibility for doubling a single Bessel function solution into two solutions, is by recalling their recurrence relations. These Bessel functions of imaginary order satisfy

the usual relation [22]:

$$J_{\pm ik_x}[(k_y^2 + k_z^2)^{1/2}t] = \left[\frac{(k_y^2 + k_z^2)^{1/2}}{(\pm ik_x)} \right] \left(\frac{t}{2} \right) \{ J_{(\pm ik_x)-1}[(k_y^2 + k_z^2)^{1/2}t] + J_{(\pm ik_x)+1}[(k_y^2 + k_z^2)^{1/2}t] \}. \quad (26)$$

Two interesting trial solutions can therefore be obtained by flipping the sign between the two contributing Bessel functions (i.e., the sign in bold above), and using the other recurrence relation involving the Bessel function time derivative:

$$\begin{aligned} & [(k_y^2 + k_z^2)^{1/2}] \left(\frac{t}{2} \right) \{ J_{(\pm ik_x)-1}[(k_y^2 + k_z^2)^{1/2}t] - J_{(\pm ik_x)+1}[(k_y^2 + k_z^2)^{1/2}t] \} \\ &= t \{ J_{\pm ik_x}[(k_y^2 + k_z^2)^{1/2}t], t \}. \end{aligned} \quad (27)$$

Explicit substitution into Eq. 25 does indeed verify that $(t\{J_{\pm ik_x}[(k_y^2 + k_z^2)^{1/2}t], t\})$ are indeed the two remaining solutions. To this author's knowledge, these expressions have not been identified elsewhere by other researchers as solutions to the wave equations in the Kasner $(1, 0, 0)$ metric.

One way of organizing the basis set of four independent solutions could be by using the four variations, $(t\{J_{(\pm ik_x)\pm 1}[(k_y^2 + k_z^2)^{1/2}t]\})$. In practice, however, our choice is to organize them into purely real solutions, which we do as follows. Let $\omega_{yz} \equiv (k_y^2 + k_z^2)^{1/2}$, and $k_R \equiv (\omega_{yz}/k_x)$. Then, using the recurrence relations in Eq's. 26-27, the previously defined solutions (J^{2+}, J^{2-}) from Eq's. 23 become:

$$\begin{aligned} J_{ik_x}^{2+}(\omega_{yz}t) &= \left(\frac{k_R t}{4i} \right) \{ J_{(ik_x)-1}(\omega_{yz}t) + J_{(ik_x)+1}(\omega_{yz}t) - J_{(-ik_x)-1}(\omega_{yz}t) - J_{(-ik_x)+1}(\omega_{yz}t) \}, \\ J_{ik_x}^{2-}(\omega_{yz}t) &= \left(-\frac{k_R t}{4} \right) \{ J_{(ik_x)-1}(\omega_{yz}t) + J_{(ik_x)+1}(\omega_{yz}t) + J_{(-ik_x)-1}(\omega_{yz}t) + J_{(-ik_x)+1}(\omega_{yz}t) \} \end{aligned} \quad (28)$$

Now consider the Bessel time derivative expressions:

$$\begin{aligned} J_{d+} &\equiv \frac{t}{k_x} [J_{ik_x}(\omega_{yz}t), t] = \left(\frac{k_R t}{2} \right) \{ J_{(ik_x)-1}(\omega_{yz}t) - J_{(ik_x)+1}(\omega_{yz}t) \}, \\ J_{d-} &\equiv \frac{t}{k_x} [J_{-ik_x}(\omega_{yz}t), t] = \left(\frac{k_R t}{2} \right) \{ J_{(-ik_x)-1}(\omega_{yz}t) - J_{(-ik_x)+1}(\omega_{yz}t) \}. \end{aligned} \quad (29)$$

These two expressions are clearly complex conjugates of one another, thus we can take the real combinations: $J_{ik_x}^{4+}(\omega_{yz}t) \equiv [(J_{d+} + J_{d-})/2]$, $J_{ik_x}^{4-}(\omega_{yz}t) \equiv [(J_{d+} - J_{d-})/(2i)]$; or, written out in full:

$$\begin{aligned} J_{ik_x}^{4+}(\omega_{yz}t) &= \left(\frac{k_R t}{4} \right) \{ J_{(ik_x)-1}(\omega_{yz}t) - J_{(ik_x)+1}(\omega_{yz}t) + J_{(-ik_x)-1}(\omega_{yz}t) - J_{(-ik_x)+1}(\omega_{yz}t) \}, \\ J_{ik_x}^{4-}(\omega_{yz}t) &= \left(\frac{k_R t}{4i} \right) \{ J_{(ik_x)-1}(\omega_{yz}t) - J_{(ik_x)+1}(\omega_{yz}t) - J_{(-ik_x)-1}(\omega_{yz}t) + J_{(-ik_x)+1}(\omega_{yz}t) \} \end{aligned} \quad (30)$$

We thus take $(J^{2+}, J^{2-}, J^{4+}, J^{4-})$ as our basis set of real solutions for the Kasner $(1, 0, 0)$ case.

Now, it is not enough to specify the form of these temporal solutions, but we must know the amplitudes of all terms as well (for general propagation wavevector \mathbf{k}), to ensure that we have self-consistent solutions to be used to form propagating waves. We can obtain these amplitudes using the constraints from applying the 1st-order Maxwell equations – i.e., $\nabla \cdot \mathbf{E}' = \nabla \cdot \mathbf{B}' = 0$, and Eq's. 13 (with cyclic permutations), as applied for $(p_x, p_y, p_z) = (1, 0, 0)$.

To form waves, we must use combinations of the temporal solutions with the spatial functions (recall Eq. 16), $S_i(x, y, z)$. By analogy, in flat spacetime one defines forward and backward propagating waves via $\cos[kt \mp (k_x x + k_y y + k_z z)] \equiv \cos[kt \mp (\mathbf{k} \cdot \mathbf{r})] = [\cos(kt) \cos(\mathbf{k} \cdot \mathbf{r}) \pm \sin(kt) \sin(\mathbf{k} \cdot \mathbf{r})]$, with $k = (k_x^2 + k_y^2 + k_z^2)^{1/2}$. We can do the same thing here with combinations like $[J_{ik_x}^{2+}(\omega_{yz}t) \cos(\mathbf{k} \cdot \mathbf{r}) \pm J_{ik_x}^{2-}(\omega_{yz}t) \sin(\mathbf{k} \cdot \mathbf{r})]$, and so on (still using expressions like $(\mathbf{k} \cdot \mathbf{r})$ to refer to the flat spacetime version of the dot product). Numerical calculations performed by this author show that these are not precisely the exact forward/backward-propagating combinations, but have some standing wave admixture – something that can be numerically removed using the late-time, almost purely sinusoidal behavior of the Bessel solutions. But for our purposes here (defining real analytical solutions) these are adequate; we just make sure to note that they produce slightly mixed linear combinations of the pure forward/backward waves when used in this way.

Next, we recover the observable electromagnetic fields by undoing the transformation from Eq. 12, to transform $\mathbf{E}' \rightarrow \mathbf{E}$, and $\mathbf{B}' \rightarrow \mathbf{B}$. We assume the wavenumbers (k_x, k_y, k_z) to be real and nonzero (though not necessarily positive). After substantial work to complete the aforementioned steps – including an explicit verification that all of the Maxwell and wave equations described above (as applied to this Kasner case) are satisfied – we obtain the (real by construction) general solutions as:

Polarization X_E , forward/backward propagation:

$$\begin{aligned}
\{E_x, E_y, E_z\} &= E_0^{\text{f/b}} [J_{ik_x}^{2+}(\omega_{yz}t) \cos(\mathbf{k} \cdot \mathbf{r} + \phi_E^{\text{f/b}}) \pm J_{ik_x}^{2-}(\omega_{yz}t) \sin(\mathbf{k} \cdot \mathbf{r} + \phi_E^{\text{f/b}})] \\
&\quad \times \left\{ (t\omega_{yz}^2), \left(-\frac{k_x k_y}{t}\right), \left(-\frac{k_x k_z}{t}\right) \right\}, \\
\{B_x, B_y, B_z\} &= E_0^{\text{f/b}} [J_{ik_x}^{4+}(\omega_{yz}t) \sin(\mathbf{k} \cdot \mathbf{r} + \phi_E^{\text{f/b}}) \mp J_{ik_x}^{4-}(\omega_{yz}t) \cos(\mathbf{k} \cdot \mathbf{r} + \phi_E^{\text{f/b}})] \\
&\quad \times \left\{ 0, \left(-\frac{k_x k_z}{t}\right), \left(\frac{k_x k_y}{t}\right) \right\}; \tag{31a}
\end{aligned}$$

Polarization X_B , forward/backward propagation:

$$\begin{aligned}
\{E_x, E_y, E_z\} &= B_0^{\text{f/b}} [J_{ik_x}^{4+}(\omega_{yz}t) \sin(\mathbf{k} \cdot \mathbf{r} + \phi_B^{\text{f/b}}) \mp J_{ik_x}^{4-}(\omega_{yz}t) \cos(\mathbf{k} \cdot \mathbf{r} + \phi_B^{\text{f/b}})] \\
&\quad \times \left\{ 0, \left(\frac{k_x k_z}{t}\right), \left(-\frac{k_x k_y}{t}\right) \right\}, \\
\{B_x, B_y, B_z\} &= B_0^{\text{f/b}} [J_{ik_x}^{2+}(\omega_{yz}t) \cos(\mathbf{k} \cdot \mathbf{r} + \phi_B^{\text{f/b}}) \pm J_{ik_x}^{2-}(\omega_{yz}t) \sin(\mathbf{k} \cdot \mathbf{r} + \phi_B^{\text{f/b}})] \\
&\quad \times \left\{ (t\omega_{yz}^2), \left(-\frac{k_x k_y}{t}\right), \left(-\frac{k_x k_z}{t}\right) \right\}, \tag{31b}
\end{aligned}$$

where $\{E_0^{\text{f}}, E_0^{\text{b}}, B_0^{\text{f}}, B_0^{\text{b}}\}$ are arbitrary constant amplitudes, and $\{\phi_E^{\text{f}}, \phi_E^{\text{b}}, \phi_B^{\text{f}}, \phi_B^{\text{b}}\}$ are arbitrary constant phases, as determined by the initial conditions.

(Note that if any of the wavevector components, (k_x, k_y, k_z) , happen to be zero – such as for propagation along an axis, or within a plane – then this simplifies matters, since we could use either Eq. 15b or 15c to make the right-hand side of the 2nd-order wave equation for some of the fields go to zero, making the number of solutions for each drop from 4 to 2. For example, $k_z = 0$ means that $B'_{x,z} = 0$ (and $E'_{x,z} = 0$), so that cyclic permutation of Eq. 15b leads to a wave equation for $E'_{y,z}$ (and $B'_{y,z}$) with no driving term and only 2 solutions, $J^{2\pm}$. And this works similarly with other restricted propagation directions, leading to appropriately simplified versions of the most general solutions given above in Eq's. 31.)

To study the energetics of these solutions, we compute the electromagnetic stress-energy tensor via [e.g., 17]: $T^{\mu\nu} = F^\mu{}_\lambda F^{\nu\lambda} - (g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta})/4$. Following Misner *et al.* [2], we write the four-momentum density per unit volume (as measured in an observer's local Lorentz frame) as: $dP^\mu/dV = -T^{\mu\nu} g_{\nu\gamma} U^\gamma$; and by choosing a stationary, comoving observer in the Kasner metric, we have $U^\gamma = (1, 0, 0, 0)$. Then, following Weinberg [17], we write the total integrated four-momentum with a specified volume as $P^\mu = \int (dP^\mu/dV) \sqrt{-|g_{\mu\nu}|} d^3x = [t(dP^\mu/dV)\Delta x\Delta y\Delta z]$.

For simplicity, we restrict the calculation to a single component: the forward-propagating X_E polarization solution. Even so, the results are too complicated to easily evaluate analytically. Instead we study the behavior numerically, selecting a variety of (order-unity) values

for (k_x, k_y, k_z) , adopting some particular value of (x, y, z) in this (homogeneous) model, and examining the behaviors of the energy and momenta P^μ over time.

At large t , these P^μ all oscillate with a period of $\sim (\pi/\sqrt{k_y^2 + k_z^2})$, which makes sense because equations like Eq. 22 become $F''_{x,t,t} + (k_y^2 + k_z^2)F''_x \approx 0$, with obvious solutions $\sim \cos / \sin[(k_y^2 + k_z^2)^{1/2}t]$.

As small t , however, the oscillation frequency increases at an increasing rate and without bound, leading to a self-similar behavior with an infinite number of oscillations as $t \rightarrow 0$. Analytically this makes sense because the $z \rightarrow 0$ limit of a Bessel function (at fixed order) is known [23] to be $J_\nu(z) \approx (z/2)^\nu / \Gamma(\nu + 1)$, and thus for very small t we can approximate:

$$\begin{aligned} J_{(\pm ik_x)}(\omega_{yz}t) &\approx \frac{(\omega_{yz}t/2)^{\pm ik_x}}{\Gamma(\pm ik_x + 1)} \propto (\omega_{yz}t/2)^{\pm ik_x} \\ &= \exp \{ \ln [(\omega_{yz}t/2)^{\pm ik_x}] \} = \exp [(\pm ik_x) \ln (\omega_{yz}t/2)] \\ &= \cos [k_x \ln (\omega_{yz}t/2)] \pm \sin [k_x \ln (\omega_{yz}t/2)] , \end{aligned} \quad (32)$$

justifying the self-similar, infinitely oscillatory behavior described above for $t \rightarrow 0$. This result is consistent with the analysis in Sagnotti and Zwiebach [4], where they found an infinite number of phase oscillations to occur as $t \rightarrow 0$; and in our numerical studies, we similarly find oscillations (varying $\sim 20-30\%$ for order-unity k_i choices) in the phase velocity of the fields satisfying Eq. 22 as $t \rightarrow 0$. Zooming in on the small- t regime, those oscillations also seem to be self-similar in character, becoming increasingly frequent as we get to the smallest simulated values of t .

Now considering the overall amplitude envelope of these oscillating P^μ , we find that $T^{0y} \sim T^{0z} \propto t^{-1}$, so that $P^y \sim P^z$ are constant in time, which makes physical sense given that the x - and y -axes are static, and it agrees with Equation 3.9 of Sagnotti and Zwiebach [4]. However, we find that $T^{0x} \propto t^{-3}$ (such that $P^x \propto t^{-2}$), which is not predicted by their Equation 3.9, and reflects the evolving nature of the x -axis, the only non-static direction. A result reminiscent of this can, in fact, be seen in their Equation 3.10, where their T^{00} energy expression contains terms proportional both to t^{-1} and to t^{-3} ; though that expression was obtained for $t \rightarrow \infty$ (where the WKB/geometric optics approximation holds), and ignoring interference terms. From our explicit calculation, at large t – where P^y and P^z dominate – we find that $T^{00} \propto t^{-1}$, just like T^{0y} and T^{0z} , so that the envelope of P^t is constant; but for $t \rightarrow 0$, where P^x dominates, T^{00} makes a sharp turn at some small, critical value of t (determined by the k_i parameters), and its envelope becomes proportional to t^{-2} for smaller

t values, so that $P^t \propto t^{-1}$. (These results also seem generally in line with those of Goorjian [10]; though they restricted their electromagnetic vector potential fields to lie along a specific spatial axis, a condition which we have not imposed here.)

Lastly for this case, considering that $T^{00} \propto t^{-2}$ at small t , there may be concern that the energy density increases so much that the “test field” approximation mentioned in Section II B might break down back towards the initial singularity. However, for Kasner cases with general indices (p_x, p_y, p_z) , one finds that the Ricci tensor components (and the Ricci scalar) due to the vacuum gravitational fields are also proportional to t^{-2} , so that they increase just as fast as this electromagnetic T^{00} does as $t \rightarrow 0$, thus the gravitational feedback by the electromagnetic fields would remain equally unimportant (relative to that of the vacuum gravitational fields) at all times. But before one concludes that the test field assumption should never be a problem for stress-energy terms with this time dependence, one must recall that this particular Kasner $(1, 0, 0)$ case is actually flat spacetime, and thus the Ricci tensor and scalar are always zero, apparently making this case somewhat ambiguous. (In fact, in Appendix B below, we show that this problem transforms exactly to the case of the ordinary fields oscillating on a Minkowski background.) However, this also allows us to draw the obvious conclusion that as long as one does not *start out* with electromagnetic waves of such an intensity that they would create significant curvature even in flat space, then the test field assumption should remain a safe one here.

IV. KASNER SPECIAL CASE $\{(-1/3), (2/3), (2/3)\}$

The other axisymmetric case satisfying the vacuum Kasner conditions is the one where two axes expand as $\sim t^{2/3}$, with the third contracting as $\sim t^{-1/3}$. Sticking to the convention of choosing the x -axis as the one with the “unique” expansion rate, we define $\{p_x, p_y, p_z\} = \{(-1/3), (2/3), (2/3)\}$.

Applying these indices to Eq. 17, the 2nd-order wave equation for $F_x'' \in \{E_x''(t), B_x''(t)\}$ becomes:

$$F_{x,t,t}'' - \left(\frac{1}{3t}\right) F_{x,t}'' + \left(k_x^2 t^{2/3} + \frac{k_y^2 + k_z^2}{t^{4/3}}\right) F_x'' = 0, \quad (33)$$

Alternatively, the temporal functions for the remaining fields, $F_y'' \in \{E_y''(t), B_y''(t)\}$ and $F_z'' \in \{E_z''(t), B_z''(t)\}$, will satisfy 4th-order wave equations derived from appropriate cyclic permutations of Eq. 20 for these p_i values.

As we saw for the Kasner $(1, 0, 0)$ special case, the easiest way to solve the 4th-order equations would be by guessing modified versions of the solutions to the homogeneous 2nd-order equation. Unfortunately for this case, Eq. 33 does not appear to be a simply transformable variation of any standard differential equation that has known analytical solutions. (In general, it does not appear that there are known analytical solutions to this type of equation when one has at least two distinct p_i indices that are each unequal to 1.)

It is known [e.g., 5], as can easily be determined using standard mathematical software, that solutions to equations like Eq. 33 can be expressed as non-resolved integrals of Heun Biconfluent functions; but it is unclear if such expressions are more useful than just obtaining direct numerical solutions to the equation. In any case, this author has not found any convenient analytical solutions to the 2nd-order wave equation for this Kasner case.

This problem has long been a stumbling block for researchers studying this (and more complicated) Kasner cases, so that various simplifications or approximations are necessary to obtain some kind of analytical understanding and solutions. As discussed above in the Introduction, a number of authors have opted either to restrict the wave propagation to be along a single spatial axis [3, 4, 10], to focus on early- and late-time approximations [7–9], or both [11]. Previously, we have obtained solutions for specific cases involving restricted propagation wavevectors in metrics with carefully chosen axis expansion rates [24]; but as we are mainly interested in the most general propagation behaviors, we will primarily employ fully numerical methods for these studies going forward.

One interesting use of numerical methods here is to study the phase velocity of the propagating F_x'' fields obeying Eq. 33. Some of the results can be predicted from the easily obtained early- and late-time solutions, as are given (for example) in Dhurandhar *et al.* [7]; while some other results are discovered numerically. But considering the approximated solutions first can provide insight for interpreting the full numerical solutions.

For late times, where $[(k_y^2 + k_z^2) t^{-4/3}] \ll (k_x^2 t^{2/3})$, we drop the former term from Eq. 33 – essentially equivalent to setting $k_y^2 \approx k_z^2 \approx 0$ – and the “exact” solution for that approximated equation is a linear combination of sinusoids, $F_x'' \propto \cos / \sin[(3/4) k_x t^{4/3}]$. This implies an adiabatically-varying temporal oscillation frequency of $\omega_a \equiv [(3/4) k_x t^{4/3}]_{,t} = (k_x t^{1/3})$, which makes sense given that Eq. 33 becomes $F_{x,t,t}'' \approx -[(k_x t^{1/3})^2] F_x''$ for very large t . Since the solution is sinusoidal, we can read off the (coordinate) phase velocity of this late-time wave as $v \approx v_x \approx \omega_a/k \approx \omega_a/k_x = (k_x t^{1/3})/k_x = t^{1/3}$ (and where we recall that $c \equiv 1$). Thus

the late-time physical speed (aligned almost entirely along the contracting x-axis) is equal to $v^{\text{Phys}} \approx g_{xx}(v_x)^2 = t^{-2/3}(t^{1/3})^2 = 1$. Recalling also that $g_{tt} = -1$, the wave thus behaves now exactly as a light ray would, propagating at null speed and obeying the geometric optics approximation, just as one expects to be true when the frequencies and/or time t are sufficiently large [7].

We cannot, however, assume geometric optics behavior all the way down to $t \rightarrow 0$, where the terms containing negative powers of t and the first time derivative of the field are important. For early times, where $[(k_y^2 + k_z^2) t^{-4/3}] \gg (k_x^2 t^{2/3})$, we drop the latter term, and the “exact” solution to Eq. 33 approximated in this way involves linear combinations of Bessel and Neumann functions of order 2, becoming: $F''_x \propto \{(t^{2/3}) J_2/Y_2[3 \sqrt{(k_y^2 + k_z^2)} t^{1/3}]\}$.

The argument of these solutions looks right, giving an effective (when adiabatically-varying) frequency of $\omega_a \equiv [3 \sqrt{(k_y^2 + k_z^2)} t^{1/3}]_{,t} = [\sqrt{(k_y^2 + k_z^2)} t^{-2/3}]$, which makes sense to the extent that Eq. 33 for early times may *almost* be approximated as $F''_{x,t,t} \approx -\{[\sqrt{(k_y^2 + k_z^2)} t^{-2/3}]^2\} F''_x$. But in this case, the $[F''_{x,t}/(3t)]$ term remains important, and the Bessel/Neumann function solutions here will deviate significantly from sinusoidal propagating behavior at small- t . (Dropping the $[F''_{x,t}/(3t)]$ term as well, of course, would finally result in sinusoidal solutions with argument proportional to $t^{1/3}$; but numerical work confirms that the Bessel/Neumann functions are actually the correct (approximate) solutions down to $t \rightarrow 0$, as will be demonstrated shortly.) Hence we expect (and indeed find) the wave phase velocity to differ significantly from the geometric optics expectation for null rays as $t \rightarrow 0$; namely, failing to evolve as $\{v_y, v_z\} \propto t^{-2/3}$. (Though $v_x \approx 0$ still remains true).

In Section III above, we briefly discussed the construction of purely forward- and backward-propagating waves from the known analytical temporal solutions for the Kasner (1, 0, 0) case, in conjunction with the sinusoidal spatial functions. Using the almost purely sinusoidal behavior of Bessel functions at large- t , we combined the spatial and temporal functions in a way that numerically eliminated nearly all of the standing wave contributions for the forward (or backward) – i.e., rightward (or leftward) – traveling waves. From those constructed unidirectional waves, we obtained the phase velocity of each by following a wavefront of its $\{E'_x(t, x, y, z), B'_x(t, x, y, z)\}$ fields, after which we used the metric to calculate the true physical speed of that wave phase velocity. As noted in that section, our simulations for the Kasner (1, 0, 0) case found phase velocities that oscillated (about the null speed of $v^{\text{Phys}} = 1$), in a presumably infinite series of self-similar oscillations

as $t \rightarrow 0$; results that were in qualitative agreement with findings from prior authors.

For this Kasner $\{(-1/3), (2/3), (2/3)\}$ case, we conduct a similar procedure; but, not having analytical solutions for the F_x'' fields obeying the full Eq. 33, we used a pair of numerical solutions (similarly treated to remove standing-wave contributions) in order to construct the rightward- and leftward-propagating waves. (Note that the waves propagating in either direction behaved indistinguishably from one another, as they should for this spatially homogeneous spacetime.) But what we do find for this metric, is that wave propagation speeds are *inhibited* as the initial singularity is approached. The phase velocity of the waves, though staying very close to the null ray speed of $v^{\text{Phys}} = 1$ for sufficiently large t , begins to decrease for t values close to unity (the transition time depending quantitatively upon the particular k_i values), and as $t \rightarrow 0$ here, $v^{\text{Phys}} \rightarrow 0$.

A plot of this (physical) wave phase velocity is shown in Figure 1, for three sets of wavevector values: $k_x = \{0.5, 1.0, 2.0\}$, where for all of those we set $k_{yz} \equiv \sqrt{k_y^2 + k_z^2} = 1$. As expected, we see that higher frequencies (i.e., larger k_x here) insures better agreement (down to earlier time t) with the $v = c$ light ray speed expectation; but in all cases, as $t \rightarrow 0$, there is a point where the predictions of geometric optics break down, and the phase velocity of the waves (in all propagation directions, as our numerical results show) become inhibited by the anisotropically-contracting nature of this metric. While it is difficult to make conclusions about energy propagation speeds based solely upon phase velocities, a naive conclusion would be that energy propagation may very well get choked off as $t \rightarrow 0$ in this metric; a purely wave-based effect that has no analogous implication from simple null-ray estimations.

Beyond the obvious interest in such findings obtained for these homogeneous vacuum metrics, it is especially intriguing that the Kasner $\{(-1/3), (2/3), (2/3)\}$ case is equivalent to the vacuum-limit of a radiation-filled metric that is not just anisotropic, but also inhomogeneous; yet which remains highly symmetrical nevertheless, and thus amenable for study.

Specifically, what may be called the Kuang-Li-Liang (KLL) metric – i.e., one particular variety of the cases given in Kuang *et al.* [25] – is defined as:

$$ds^2 = \frac{J^2(T \pm X)}{T^{1/2}}(-dT^2 + dX^2) + T(dY^2 + dZ^2) , \quad (34)$$

where J is a general (real) function of the variable $(T + X)$ – or alternatively, of $(T -$

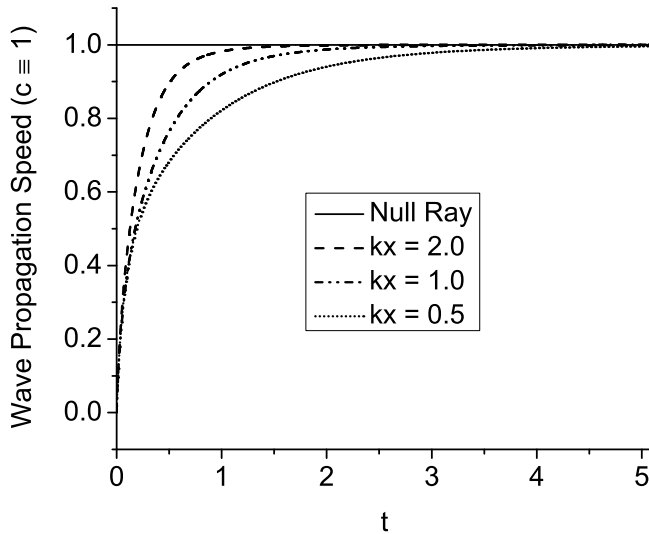


FIG. 1. Total physical speed of the phase velocity for the $\{E'_x, B'_x\}$ fields obeying Eq. 33 in the Kasner $\{(-1/3), (2/3), (2/3)\}$ metric, plotted versus time from the initial singularity. The wavenumber k_x is varied, while holding $(k_y^2 + k_z^2) = 1$. For each numerical simulation, rightward and leftward propagating waves for each case are found to produce indistinguishable results. The line $v = 1$ represents the prediction for null rays according to geometric optics.

X) – which to satisfy reasonable energy conditions must obey $J'/J > 0$. This metric breaks homogeneity but remains planar-symmetric gravitationally, and is filled with pure electromagnetic radiation. (That background radiation is semi-plane-symmetric, in the sense that rotations within the YZ -plane will alter the electromagnetic polarization but not its energy density.)

This non-vacuum, conformally non-flat metric also possesses a cosmological (expanding/contracting) quality, in that it is known [26] that the limit $J \rightarrow 1$ makes this metric exactly equivalent to the Kasner $\{(-1/3), (2/3), (2/3)\}$ case. One can verify this using the substitutions: $t = [(4/3) T^{3/4}]$, $x = [(3/4)^{-1/3} X]$, $y = [(3/4)^{2/3} Y]$, $z = [(3/4)^{2/3} Z]$.

Such properties all make this KLL metric (for various choices of $J(T \pm X)$) an extremely interesting physical system for studying wave propagation, using the formalism, methodology, and numerical tools developed for and demonstrated in this paper. As a generalization of Kasner $\{(-1/3), (2/3), (2/3)\}$, the KLL metric almost certainly needs a numerical treatment; and we intend to use the intuition gained from all of our results from the Kasner cases

discussed here as a stepping stone for interpreting future results that will be obtained for that physically richer (though more mathematically complicated) system.

V. DISCUSSION AND SUMMARY

In this paper, we applied the curved spacetime Maxwell equations to the general class of Kasner metrics, and used the usual vector identities to produce 2nd-order differential equations for the full set of electromagnetic fields.

Unlike the results from previous authors that we have seen, our 2nd-order wave equations were not fully uncoupled (as they are in flat spacetime), but contained nonhomogeneous driving terms indicating a mixing between the electric and magnetic field components, generated directly by the anisotropic nature of the Kasner expansion/contraction rates. To eliminate the coupling in the wave equations, it was necessary to produce 4th-order differential equations; and consequently we derived 4th-order wave equations that are valid for all of the fields in any metric with Kasner-like axis expansion rate coefficients (i.e., whether or not they satisfy the Kasner vacuum conditions).

We then considered two special axisymmetric Kasner cases, for which the wave equations for the fields along the axis of symmetry are greatly simplified. First, for the $(p_x, p_y, p_z) = (1, 0, 0)$ case, we obtained the explicit solutions for all of the fields, for a wave with the most general wavevector components, traveling in a general direction through the three dimensional space. This included full solutions to the 4th-order wave equations, deriving what we believe are additional field solutions that had not been found in previous studies of this Kasner case. We also studied the energetics of the fields, showing them to agree (where comparable) with previous studies, and in general providing us with confidence in the appropriateness of the assumption made by treating these solutions as test fields, propagating upon an essentially unchanged background Kasner metric.

Next, we considered the $\{p_x, p_y, p_z\} = \{(-1/3), (2/3), (2/3)\}$ case, deriving our 2nd-order wave equations for the fields, and pointing out (as noted by previous researchers) that the equations (even without considering the nonhomogeneous driving terms) are analytically unsolvable. Using late-time and early-time approximations, we set expectations for what the temporal oscillation frequencies would be; and we also noted that the geometric optics approximation should be valid at late times, but should be expected to break down at early

times, leading to deviations from null ray behavior that only become apparent as t decreases below (approximately) unity and heads to zero.

Using a numerical program written specifically for this research, designed to compute phase velocity of the wave by following a wavefront through the evolving metric, we find that the above expectations were correct: at late times, the physical speed of the phase velocity is nearly exactly equal to $v = c \equiv 1$; but as $t \rightarrow 0$, the wave propagation is sharply inhibited, causing $v \rightarrow 0$. Again as expected, larger wavenumbers (i.e., larger k_x) allowed the geometric optics approximation to remain valid longer, down to smaller t , forestalling (but not preventing) the ultimate breakdown of the validity of the null ray treatment of the light waves.

Lastly, we noted a very interesting class of non-vacuum, radiation-filled, inhomogeneous metrics with cosmology-like behavior, for which the Kasner $\{(-1/3), (2/3), (2/3)\}$ case is the vacuum limit; and which should be perfectly suited for future study using the analytical formalism and numerical tools developed in this paper.

Appendix A: Derivation of the 2nd-order wave equation

Here we derive the inhomogeneous wave equation for E'_x , Eq. 14 – from which similar equations can be inferred for all of the fields – from the 1st-order curved space Maxwell equations for such fields.

Recall that these renormalized fields obey $\nabla \cdot \mathbf{E}' = \nabla \cdot \mathbf{B}' = 0$ (the dot products being defined here as in flat spacetime); and the three curl equations which we need here (of the six inferred from Eq's. 13), are reproduced here as:

$$E'_{x,t} = [t^{(-p_x - p_y + p_z)} B'_{z,y}] - [t^{(-p_x + p_y - p_z)} B'_{y,z}] , \quad (\text{A1a})$$

$$B'_{y,t} = [t^{(-p_x - p_y + p_z)} E'_{z,x}] - [t^{(p_x - p_y - p_z)} E'_{x,z}] , \quad (\text{A1b})$$

$$B'_{z,t} = [t^{(p_x - p_y - p_z)} E'_{x,y}] - [t^{(-p_x + p_y - p_z)} E'_{y,x}] . \quad (\text{A1c})$$

Next, we take the combination:

$$\partial/\partial y \{ [t^{(-p_x - p_y + p_z)}] \times \text{Eq. A1c} \} - \partial/\partial z \{ [t^{(-p_x + p_y - p_z)}] \times \text{Eq. A1b} \} . \quad (\text{A2})$$

For clarity, we will treat the left hand side (LHS) and right hand side (RHS) of the resulting

equation separately. Working out the combination just mentioned, we get:

$$\text{LHS} = [t^{(-p_x - p_y + p_z)}] B'_{z, t, y} - [t^{(-p_x + p_y - p_z)}] B'_{y, t, z} , \quad (\text{A3a})$$

$$\text{RHS} = [t^{-2p_y} E'_x]_{, y, y} - [t^{-2p_x} E'_y]_{, y, x} - [t^{-2p_x} E'_z]_{, z, x} + [t^{-2p_z} E'_x]_{, z, z} . \quad (\text{A3b})$$

Note that we have used the fact that we can exchange (commute) the spatial partial derivatives freely here.

Then, using $\nabla \cdot \mathbf{E}' = 0$, and thus $[t^{-2p_x} (\nabla \cdot \mathbf{E}')]_{, x} = 0$, we can write:

$$[t^{-2p_x} E'_x]_{, x, x} = -[t^{-2p_x} E'_y]_{, y, x} - [t^{-2p_x} E'_z]_{, z, x} , \quad (\text{A4})$$

and hence we get:

$$\text{RHS} = [t^{-2p_x} E'_x]_{, x, x} + [t^{-2p_y} E'_x]_{, y, y} + [t^{-2p_z} E'_x]_{, z, z} \equiv \{t\nabla^2\} E'_x . \quad (\text{A5})$$

Next, for the LHS, we commute the time derivatives outward in Eq. A3a; though since $\partial/\partial t$ cannot be moved through powers of t without generating extra product rule terms, we add them in as necessary, to get:

$$\begin{aligned} \text{LHS} &= \{[t^{(-p_x - p_y + p_z)}] B'_{z, y} - [t^{(-p_x + p_y - p_z)}] B'_{y, z}\}_{, t} \\ &\quad - \left[\frac{(-p_x - p_y + p_z)}{t} t^{(-p_x - p_y + p_z)} B'_{z, y} \right] + \left[\frac{(-p_x + p_y - p_z)}{t} t^{(-p_x + p_y - p_z)} B'_{y, z} \right] \\ &= \{E'_{x, t}\}_{, t} + \left(\frac{p_x}{t}\right) \{[t^{(-p_x - p_y + p_z)}] B'_{z, y} - [t^{(-p_x + p_y - p_z)}] B'_{y, z}\} \\ &\quad + \frac{(p_y - p_z)}{t} [t^{(-p_x - p_y + p_z)}] B'_{z, y} + t^{(-p_x + p_y - p_z)} B'_{y, z} \\ &= E'_{x, t, t} + \left(\frac{p_x}{t}\right) E'_{x, t} \\ &\quad + \frac{(p_y - p_z)}{t} [t^{(-p_x - p_y + p_z)}] B'_{z, y} + t^{(-p_x + p_y - p_z)} B'_{y, z} , \end{aligned} \quad (\text{A6})$$

where the second and third equalities were obtained using repeated applications of Eq. A1a.

Finally, setting the LHS from Eq. A6 equal to the RHS from Eq. A5, we get:

$$E'_{x, t, t} + \left(\frac{p_x}{t}\right) E'_{x, t} + \frac{(p_y - p_z)}{t} [t^{(-p_x - p_y + p_z)}] B'_{z, y} + t^{(-p_x + p_y - p_z)} B'_{y, z} = \{t\nabla^2\} E'_x , \quad (\text{A7})$$

which is the same as Equations 14,15a, as promised.

Appendix B: Reduction of the Kasner (1, 0, 0) electromagnetic tensor to Minkowski Form

It is known [27] that the metric in Eq. 21 can be transformed to Minkowski space with the substitutions:

$$\tilde{t} \equiv t \cosh x, \quad \tilde{x} = t \sinh x. \quad (\text{B1})$$

The reverse transformations are thus given by:

$$t = (\tilde{t}^2 - \tilde{x}^2)^{1/2}, \quad x = \operatorname{arctanh}(\tilde{x}/\tilde{t}). \quad (\text{B2})$$

The covariant electromagnetic tensor can be transformed to the new coordinate system via [17]:

$$\tilde{F}^{\mu\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} F^{\alpha\beta}, \quad (\text{B3})$$

with:

$$\mathcal{M}_\mathcal{K} \equiv \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} = \begin{bmatrix} \cosh x & t \sinh x & 0 & 0 \\ \sinh x & t \cosh x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{t}(\tilde{t}^2 - \tilde{x}^2)^{-1/2} & \tilde{x} & 0 & 0 \\ \tilde{x}(\tilde{t}^2 - \tilde{x}^2)^{-1/2} & \tilde{t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (\text{B4})$$

As defined in Section II A, and applied to this Kasner (1, 0, 0) metric, we can write the (contravariant) electromagnetic tensor as:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & (t^{-2}E_x) & E_y & E_z \\ -(t^{-2}E_x) & 0 & (t^{-1}B_z) & -(t^{-1}B_y) \\ -E_y & -(t^{-1}B_z) & 0 & (t^{-1}B_x) \\ -E_z & (t^{-1}B_y) & -(t^{-1}B_x) & 0 \end{bmatrix}. \quad (\text{B5})$$

Now, remembering that the new coordinates are Minkowski space – so that we raise and lower indices with $\eta = \operatorname{Diag}(-1, 1, 1, 1)$ – the transformed electromagnetic (covariant) tensor is calculated as:

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \eta \tilde{F}^{\mu\nu}(\eta)^T = \eta \mathcal{M}_\mathcal{K} F^{\alpha\beta} (\eta \mathcal{M}_\mathcal{K})^T \\ &= \frac{1}{(\tilde{t}^2 - \tilde{x}^2)^{1/2}} \begin{bmatrix} 0 & -E_x & -(E_y \tilde{t} + B_z \tilde{x}) & -(E_z \tilde{t} - B_y \tilde{x}) \\ E_x & 0 & (B_z \tilde{t} + E_y \tilde{x}) & -(B_y \tilde{t} - E_z \tilde{x}) \\ (E_y \tilde{t} + B_z \tilde{x}) & -(B_z \tilde{t} + E_y \tilde{x}) & 0 & B_x \\ (E_z \tilde{t} - B_y \tilde{x}) & (B_y \tilde{t} - E_z \tilde{x}) & -B_x & 0 \end{bmatrix}. \quad (\text{B6}) \end{aligned}$$

Comparing this result to the corresponding flat spacetime tensor:

$$\tilde{F}_{\mu\nu}^{(\text{Flat})} = \begin{bmatrix} 0 & -\tilde{E}_x & -\tilde{E}_y & -\tilde{E}_z \\ \tilde{E}_x & 0 & \tilde{B}_z & -\tilde{B}_y \\ \tilde{E}_y & -\tilde{B}_z & 0 & \tilde{B}_x \\ \tilde{E}_z & \tilde{B}_y & -\tilde{B}_x & 0 \end{bmatrix}. \quad (\text{B7})$$

we see that the Kasner $(1, 0, 0)$ electromagnetic fields are exactly the same as in Minkowski space, if we relate the Kasner fields (*before* the separation into non-static variables via Eq's. 16-17) to Minkowski ones, by taking combinations like:

$$\begin{aligned} \tilde{E}_y &\equiv \frac{(E_y \tilde{t} + B_z \tilde{x})}{(\tilde{t}^2 - \tilde{x}^2)^{1/2}} = (E_y \cosh x + B_z \sinh x) , \\ \tilde{B}_z &\equiv \frac{(B_z \tilde{t} + E_y \tilde{x})}{(\tilde{t}^2 - \tilde{x}^2)^{1/2}} = (B_z \cosh x + E_y \sinh x) , \end{aligned} \quad (\text{B8})$$

and so on.

Finally, note that the Kasner $(1, 0, 0)$ metric is not equivalent to the *entire* flat spacetime; but, as is obvious from Eq. B6 (and from the definitions in Eq. B2), the region of Minkowski space bounded by $|\tilde{t}| \geq |\tilde{x}|$, $t \geq 0$ is enough to cover the entire Kasner allowed coordinate range of $t_{\text{Kas}} \geq 0$, $-\infty < \{x_{\text{Kas}}, y_{\text{Kas}}, z_{\text{Kas}}\} < \infty$.

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