Gravitational Collapse in the EGB Gravity

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The Einstein- Gauss- Bonnet (EGB) gravity is an important modification of the Einstein theory of gravity and, for many gravitational phenomena, the Gauss- Bonnet (GB) correction term leads to drastic differences. In this paper, we study gravitational collapse in the 5-dimensional EGB theory. We construct the spherical marginally trapped surfaces and determine the evolution of marginally trapped surfaces when the infalling matter admits a wide variety of initial density distribution. We show that the location of black hole horizon depends crucially on the initial density and velocity profile of the inflating matter as well as on the GB coupling constant. A detailed comparison is made with the results of Einstein's theory.

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I. INTRODUCTION

The study of gravitational collapse of a self- gravitating isolated system remains a matter of great physical importance in understanding large scale structures in the universe, as well as towards discerning the formation of black hole horizons, spacetime singularities and the cosmic censorship conjecture [1–6]. In general relativity (GR), the spherical gravitational collapse and the singularity theorems have been studied at length. Although several important aspects including the cosmic censorship, non-symmetrical collapse remain to be understood completely, the progress in this direction has been remarkable.

The models of gravitational collapse in alternate theories of gravity, including higher dimensional ones, are also being studied with interest since it is believed that one (or some) of these theories may solve problems affecting GR, including spacetime singularities [7–13]. Among these, modified gravity theories with higher curvature corrections arise naturally. Indeed, GR is viewed as an effective field theory in which the Einstein-Hilbert action is only a low energy contribution and higher curvature terms consistent with the diffeomorphism invariance may become relevant as one goes to higher energies [14-21]. Such higher curvature terms have been explicitly obtained in string theories [22–26]. These higher curvature corrections should leave imprints at low energy scales which become important for low energy physics too, affecting the horizon structure of large black holes. The Einstein-Gauss-Bonnet (EGB) theory is possibly the simplest diffeomorphism invariant modification of GR whose equations of motion contain no more than second order in time derivatives [15–17, 27–29]. This generalization is also known to be the unique lowest order correction in the Lovelock action. Furthermore, since the EGB gravity is free from ghosts (if the coupling constant has the same sign as the GR term) and leads to a well-defined initial value problem, it is a respectable theory of gravity in higher dimensions, and its solutions have also been a matter of interest. In particular, black hole solutions in the EGB theory are well known. They include the Boulware- Deser, and other spherically symmetric solutions [24, 30–32]. Black holes in EGB theory are also testbeds to gain fundamental insights into various quantum aspects of gravity like the horizon entropy [33–35].

Thus, because of importance of the EGB theory as a natural higher dimensional theory, effect of the GB correction term on the spherically symmetric gravitational collapse and singularity structure have received attention. Naturally, particular emphasis has been placed on the inhomogenous dust collapse models of Lemaitre- Tolman- Bondi (LTB) type [7–11]. In particular, [7] has carried out a complete study of the singularity structure of all the collapse models for spacetime dimensions $n \geq 5$. It arises from this study that (i) all naked singularities for $n \geq 6$ are massless, and (ii) for n = 5, all singularities with mass $> 2\lambda$, with λ being the GB coupling constant, are censored. This

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feature was also studied in the context of a marginally bound LTB spacetime by directly solving for the singularity curves, and the apparent horizon, for a simple matter model [10]. Although, some features of [7] were borne out in [10], in particular that the central (as well as non-central) singularity is naked, and this untrapped region increases with coupling constant $\lambda > 0$, it remains a possibility that these structures of local naked singularity may well wash out if a more complicated or realistic matter profile is considered. This expectation is not unwarranted since the occurrence of naked singularities break the Censorship conjecture [6], and the Seifert conjecture [36], which essentially states that massive singularities must be censored inside a trapped region. Of course, it remains a possibility that these conjectures themselves need modifications in higher dimensions, just like the Hoop conjecture [37]. Hence, it is essential that gravitational collapse in the 5- dimensional EGB model be studied in the full generality, using a large class of models where the matter admits a wide variety of initial density and velocity profiles. This study shall, therefore be useful to identify the region of the parameter space where such singular structures arise.

Here, we develop the formalism of gravitational collapse in the EGB theory further to the scenarios where, (a) the collapse is bounded (or, for that matter, unbounded), and that (b) density function of the collapsing matter has a realistic initial density distribution profile, and (c) use this formalism to locate spherical (marginal) trapped surfaces developing during the collapse of matter fields. This shall be carried out by directly solving the equations of motion arising in the EGB theory of gravity. This, to our knowledge, are significant improvements since direct study using explicit solutions have been carried out only for marginally bound collapse models (see for example [10]). Additionally, in the literature, the density of the collapsing matter profiles are restricted to simple power law models (including those carried out in [10]) and hence, these studies exclude possible realistic scenarios in which matter admits wider class of density distributions. Such density distributions include for example, a gaussian, or a matter profile with more complicated dependence on space, including the angular coordinates (although, here we shall only concern ourselves with matter profiles depending on radial coordinates). Indeed, the formation of spacetime singularity, the apparent horizon (AH), the event horizon (EH) and their time development, depend not only on the theory, or the initial velocity profile, but are also intimately connected with the density distribution of the collapsing matter. For example in GR, the formation and dynamics of the AH changes drastically with variations in the density profile [38, 39], and it is natural to expect that such time-development of horizons will also be observed for the GB modification too.

In this paper, we study these issues in the context of the inhomogeneous LTB collapse models in the EGB theory, by carefully addressing them with examples. We track the motion of the collapsing shells, and simultaneously follow the time development of horizon in relation to this collapsing matter. In particular, we consider the horizon to be foliated by closed spherical 3- dimensional surfaces, such that the expansion scalar of the outgoing null normal vanishes $\theta_{(\ell)} = 0$, while that of the ingoing null normal is negative $\theta_{(n)} < 0$. This formulation of the black hole horizon is called Marginally Trapped Tube (MTT) and has found use in analytical and numerical studies of black holes, in particular in understanding their classical nature, quantum behaviour, as well as their stability under various geometric and physical variations [38, 40–57]. Note that since MTT is not associated with a particular signature, it can describe various states of a horizon. For example, a black hole horizon in equilibrium is a null MTT and is referred to as an isolated horizon (IH) (see [41, 42, 45, 52, 57]. A growing black hole admits a spacelike MTT, and is called a dynamical horizon (DH) (see [43–45, 54, 55] for these horizons as well as their variations). Further, it is useful to describe a MTT with timelike signature, which admits matter flow in both directions, and is called a timelike tube. Thus MTTs provide an unified framework to study time evolution of black holes through different phases. The nature of spherical MTTs during gravitational collapse in GR has been studied in detail for various class of matter fields [38, 39]. However, spherical MTTs in the EGB theory remains to be studied in the context of gravitational collapse of inhomogeneous matter fields (the LTB models), and here we fill this gap by making a detail study of these matter collapse models. We carry out, (i) study the collapse end state with special emphasis on the formation of horizons, and in particular, track the location of spherical marginally trapped tubes with variation of matter profile, and (ii) for the mass profiles considered here, identify the regions of the parameter space where the MTT evolves as a DH (spacelike), where it might be timelike, and when it reaches equilibrium and become a null IH. This shall also help us to (iii) correctly locate the spherical outermost trapped surface developing during gravitational collapse. We must stress that although MTTs in 4- dimensions have been studied [38, 39], their behaviour is drastically different in the EGB models, even for *large* coupling constants.

The paper is arranged as follows: In the next section, we briefly discuss the equations of motions for the EGB theory and it's reduction in the context of spherically symmetric spacetimes, in the $(t, r, \theta, \phi, \psi)$ coordinates. We shall also discuss the matter contributions to these equations and the way to determine the spherically symmetric MTTs for these spacetimes. In section III, we solve the equations of motion directly for the marginally bounded and bounded cases. The solution for the unbounded case is similar, and so we shall not repeat it here. We conclude in section IV with discussions.

II. MARGINALLY TRAPPED TUBES IN THE EGB THEORY

The formalism of MTT as a quasilocal description of black hole horizons was developed in [46]. In the following, we present a brief discussion on this formalism, and set up the basic notations for our later use. Let us consider a 5- dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ with signature (-, +, +, +, +). Let Δ be a hypersurface in \mathcal{M} which may be spacelike, timelike or even null. Δ is taken to be topologically $S^3 \times \mathbb{R}$. At each point of the spacetime, we shall have 2 null vectors and three spacelike vectors. The null vector fields ℓ^{μ} and n^{μ} are respectively the outgoing and the ingoing vector fields orthogonal to the 3- sphere cross-sections of Δ , with $\ell \cdot n = -1$. The three normalised spacelike vectors tangential to the 3- sphere are called $\hat{\theta}$, $\hat{\phi}$, and $\hat{\psi}$ respectively, and are orthogonal to the null vectors ℓ^{μ} and n^{μ} . If ℓ^{μ} is a vector field tangential to Δ and normal to the S^3 foliations, then $\ell^{\mu} = \ell^{\mu} - C n^{\nu}$. Now, assume that the S^3 foliations are such that its null normals satisfy the following conditions: (i) $\theta_{(\ell)} = 0$, and (ii) $\theta_{(n)} < 0$. The hypersurface Δ foliated by such surfaces is called a MTT. Note that MTT does not carry a specific signature. Since $\ell \cdot \ell = 2C$, the constant C determines the signature of Δ . When C = 0, Δ is null, foliated by ℓ^{μ} and it describes a black hole in equilibrium (an IH). it describes a black hole in equilibrium (an IH), a DH when it is spacelike (C > 0), or simply a timelike membrane when C < 0 and Δ is timelike. Thus, MTT is an unified formalism for horizon evolution. The value of C can be determined for various gravitational collapse processes, and for a wide class of energy momentum tensors. Hence, the entire evolution of the MTT can be unambiguously determined throughout the evolution process, if the signature of C is known.

As t^{μ} is orthogonal to the foliations and tangential to Δ , it generates a foliation preserving flow so that on Δ , the following condition holds:

$$\mathcal{L}_t \, \theta_{(\ell)} \triangleq 0. \tag{1}$$

This equation implies that $C = [\mathcal{L}_{\ell} \theta_{(\ell)}/\mathcal{L}_n \theta_{(\ell)}]$. To determine the value of the constant C, we use the geometrical equations of 3-surface geometry given in the appendix (VB). These equations imply that the constant C which determines the nature of the MTT is given by:

$$C = \frac{G_{\mu\nu} \,\ell^{\mu} \ell^{\nu}}{3(2\pi^2/\mathcal{A})^{2/3} - G_{\mu\nu} \,\ell^{\mu} n^{\nu}},\tag{2}$$

where we have used the relation between area of the round 3-sphere \mathcal{A} , and the scalar curvature: $\mathcal{R} = 6(2\pi^2/\mathcal{A})^{2/3}$. We shall also assume that the Einstein- Gauss- Bonnet field equations $G_{\mu\nu} \equiv R_{\mu\nu} - (1/2)R g_{\mu\nu} = T_{\mu\nu}^{-1}$, holds on Δ . The signature of C in eqn. (2) is a quantity of utmost importance since it decides the nature and stability of horizon [47, 51], and, as may be observed from the above equation, this value is regulated by the null components of the energy-momentum tensor as well as area of the cross-sections of the MTT. However, in the following sections where we shall treat a wide class of energy-momentum tensors for collapse models of the LTB type, we shall observe that details like the initial velocity profile, initial density profile of the collapsing matter, and the dimension of the spactime play important role as well. Indeed, in several cases, simple changes in the density profile alters the nature and time of formation of the spacetime singularity, and that of the MTT quite drastically. For example, in 4- dimensions, if the matter profile is smooth, the MTT begins as a spacelike hypersurface from the center of the cloud as soon as matter begins to fall, and asymptotes to the null event horizons as infall of matter is discontinued. Trapped surfaces in 4-dimensions are discussed in [38, 49, 58–68]. However, in the 5- dimensional EGB theory, even for the collapse of marginally bound matter with density admitting a Gaussian distribution, the central singularity forms earlier than the corresponding MTT. This happens because the EGB equations allow the formation of MTT only at the later shell coordinates, and hence, the collapse of the first few shells leads to an untrapped singularity.

In the following section, we shall discuss the EGB equations of motion for the spherical collapse of matter fields, and determine the requirements for formation of trapped surfaces in the 5- dimensions.

A. The equations of motion

The action for the 5- dimensional EGB theory is given by

$$S = \int d^5x \sqrt{-g} \left(R + \lambda L_{GB} \right) + S_{matter}, \tag{3}$$

¹ We use the units of c=1 and $8\pi G=1$, or equivalently, we scale the components of the energy-momentum tensor by $8\pi G$. In case of the EGB theory too, we shall write the Einstein equations in the similar manner, $G_{\mu\nu}=T_{\mu\nu}$. In that case, $T_{\mu\nu}$ shall imply a sum of terms, due to matter variables $T_{\mu\nu}$ and, due to extra geometric variables arising out of the GB correction.

where R is the Ricci scalar, g denotes determinant of the metric $g_{\mu\nu}$ and, λ is coupling constant of the Gauss-Bonnet term. The Gauss-Bonnet Lagrangian (L_{GB}) is given by

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\delta}R^{\mu\nu\sigma\delta}.$$
 (4)

The action eqn. (3) leads to the following field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu} - \lambda H_{\mu\nu},$$
 (5)

where the term $G_{\mu\nu}$ is the usual Einstein tensor as in GR, $T_{\mu\nu}$ is the energy momentum tensor, and $H_{\mu\nu}$ is the contribution due to the Gauss-Bonnet term. In the above equation (5), the term $H_{\mu\nu}$ signifies the following

$$H_{\mu\nu} = H'_{\mu\nu} - \frac{1}{2} g_{\mu\nu} L_{GB}$$

$$= 2 \left[RR_{\mu\nu} - 2R_{\mu\lambda} R^{\lambda}_{\ \nu} - 2R^{\lambda\sigma} R_{\mu\lambda\nu\sigma} + R_{\mu}^{\ \lambda\sigma\delta} R_{\nu\lambda\sigma\delta} \right] - \frac{1}{2} g_{\mu\nu} L_{GB}. \tag{6}$$

Note that $H_{\mu\nu}$ may be considered as an effective energy momentum tensor adding to the usual matter tensor.

Now, we consider a general spherically symmetric collapsing cloud of fluid bounded by a spherical surface. In the comoving coordinates, the line element of a 5 dimensional spherically symmetric spacetime geometry can be written as

$$ds^{2} = -e^{2\alpha(r,t)}dt^{2} + e^{2\beta(r,t)}dr^{2} + R(r,t)^{2} \left[d\theta^{2} + \sin^{2}\theta \, d\phi^{2} + \sin^{2}\theta \, \sin^{2}\phi \, d\psi^{2} \right], \tag{7}$$

where $\alpha(r,t)$, $\beta(r,t)$ and $R(r,t)^2$ are metric functions to be determined. R(r,t) is radius of the collapsing matter cloud whereas, θ , ϕ , ψ are the angular coordinates of that 3-sphere. The energy momentum tensor for the fluid is taken to be

$$T_{\mu\nu} = (p_t + \rho)u_{\mu}u_{\nu} + p_t g_{\mu\nu} + (p_r - p_t)X_{\mu}X_{\nu}$$
(8)

where $\rho(r,t)$ is density, whereas $p_r(r,t)$ and $p_t(r,t)$ are the radial and tangential components of pressure. The u^{μ} and X^{μ} are unit time-like and space-like vectors satisfying $u_{\mu}u^{\mu} = -X_{\mu}X^{\mu} = -1$ In the comoving co-ordinates the four velocity and the unit space-like vector of the fluid as $u^{\mu} = e^{-\alpha}(\partial_t)^{\mu}$ and $X^{\mu} = e^{-\beta}(\partial_r)^{\mu}$.

The equation of motion for this metric in the EGB theory are given by

$$\rho(r,t) = \frac{3}{2} \frac{F'(r,t)}{R^3 R'},\tag{9}$$

$$p_r(r,t) = -\frac{3}{2} \frac{\dot{F}(r,t)}{R^2 \dot{R}},\tag{10}$$

$$\dot{R}' = \dot{R}\alpha' + R'\dot{\beta},\tag{11}$$

$$\alpha' = \frac{3R'}{R} \frac{p_t - p_r}{\rho + p_r} - \frac{p_r'}{\rho + p_r},\tag{12}$$

$$F(r,t) = R^{2} (1 - G + H) + 2\lambda (1 - G + H)^{2},$$
(13)

where the superscripts primes (') and dots (·) represent partial derivatives with respect to r and t respectively. The quantity R(r,t) is physical radius for matter configuration and F(r,t) is the Misner-Sharp mass function. The first and the second equations, (9) and (10), are the G_{00} and the G_{11} equations. The third is the R_{01} equation. The fourth equation is the Bianchi identity $\nabla_{\mu}T^{\mu r}=0$, which for the pressureless matter implies that the metric variable $\alpha'=0$. The equation (13), is the equation for the mass function with the functions H(r,t) and G(r,t) defined as $H=e^{-2\alpha}\dot{R}^2$ and $G=e^{-2\beta}R'^2$.

Several points are to be noted regarding the abovementioned equations of motion. First, the relation between the matter variables and the geometric variables in the above equations (7)- (13) are modified in comparison to the 4-dimensional Einstein theory. The changes in the numerical factors are due to dimensionality of the spacetime as well as due to change in the theory itself, see for example equation (13).

 $^{^{2}}$ The symbol R is used to denote both Ricci scalar and the radius of the matter configuration. We deliberately kept the same symbol since they will not appear simultaneously to cause any confusion.

Second, the number of independent equations are five in number. The unknown functions in this problem are the three metric variables $\alpha(t,r)$, $\beta(t,r)$, R(t,r), three matter variables $p_r(r,t)$, $p_t(t,r)$, $\rho(t,r)$, and the mass-function F(t,r). This combination allows two freely specifiable functions. Since the equations give dynamical evolution of the functions, it is natural to specify these functions at an initial time $t=t_i$, and allow the Einstein equations to evolve the dynamical functions. Since we shall be dealing with pressureless (dust) collapse, it is useful to point out that for dust collapse, p_r and p_t are taken to vanish at t_i , and this fixes the function $\alpha(t,r)=\alpha(t)$. We shall show below that this effectively implies $\alpha=0$, since we can rescale the time coordinate. The remaining freely specifiable functions are the density $\rho(r,t_i)$, and $\beta(r,t_i)$ which, as we shall show below, implies the specification of initial density and velocity profiles of the collapsing matter. We shall also assume that $R(r,t_i)=r$. This requirement is consistent with the regularity conditions discussed below. By choosing different values of r at the initial surface gives the time evolution of the various shells of matter.

Thirdly, few regularity conditions on the metric functions must also be enforced during the collapse process. The positivity and regularity of the density $\rho(t,r)$, and equation (9) imply that the mass function F(t,r) must smoothly vanish at the center of the matter configuration at r=0. The condition R(t,r)=0 is the genuine spacetime singularity where the density and the curvature scalars blow up. Note that the density also blows up for R'=0, although this is not a genuine spacetime singularity and can be removed. This condition in fact implies shell-crossings, when shells of fixed r cross each other. The sufficient condition which guarantees no shell-crossing is R'>0, which ensures that shells maintain their ordering. Note that $R(r,t_i)=r$, and any other index on r leads either to shell-crossing or affects differentiability of metric functions at center of the matter configuration. An important requirement for gravitational collapse is to require $\dot{R}(r,t)<0$. Finally, we shall ensure in our study that no trapped surface is present at the initial data, by checking that the value of r at the initial surface is greater than the condition of formation of trapped surface at that coordinate.

Now, with the metric given in equation (7), the outgoing and the incoming null normals to the 3-sphere are given by³:

$$\ell^{\mu} = (\partial_t)^{\mu} + e^{-\beta(t,r)} (\partial_r)^{\mu} \tag{14}$$

$$n^{\mu} = (1/2)(\partial_t)^{\mu} - (1/2)e^{-\beta(t,r)}(\partial_r)^{\mu}. \tag{15}$$

This leads to the following expressions for the expansion scalars:

$$\theta_{(\ell)} = \frac{3}{R(r,t)} [\dot{R} + R' \exp(-\beta)] = \frac{3}{R(r,t)} [\dot{R} + \sqrt{1 - k(r)}], \tag{16}$$

$$\theta_{(n)} = \frac{3}{R(r,t)} [\dot{R} - \sqrt{1 - k(r)}], \tag{17}$$

where we have used the relation $R' = e^{\beta(r,t)} \sqrt{1 - k(r)}$. This relation is obtained as follows: For the case of pressureless matter, eqn. (12) gives $\alpha' = 0$ which along with eqn. (11) implies:

$$G(t,r) = e^{-2\beta(r,t)} R'^2 \equiv E(r) \equiv 1 - k(r),$$
 (18)

where k(r), E(r) are the integration functions. From the equation (13), the equation of motion of collapsing configuration gives the following expression for $\dot{R}(t,r)$:

$$\dot{R}(r,t) = -\left[(1/4\lambda) \left\{ \sqrt{R^4 + 8\lambda F(r,t)} - R^2 \right\} - k(r) \right]^{1/2}, \tag{19}$$

where we have used the –ve sign, as required for gravitational collapse. It follows from this equation (19), and the equation (16) that the condition for $\theta_{(\ell)} = 0$ requires:

$$R_M(r,t) = \sqrt{F(r,t) - 2\lambda},\tag{20}$$

which at the same time is also the condition for $\theta_{(n)} < 0$. Thus, for the spacetimes we are studying, all the three spheres which satisfy eqn. (20) are marginally trapped spheres.

³ These expressions are valid for dust collapse. In general, one has $e^{-\alpha(r,t)}(\partial_t)^{\mu}$ in place of $(\partial_t)^{\mu}$ in equations (14) and (15).

As discussed earlier following eqn (2), the dynamics of the marginally trapped surfaces (whether they are timelike, spacelike or null), depends upon sign of the expansion parameter C. On the trapped surface, it is defined by

$$C = \frac{T_{\mu\nu}\ell^{\mu}\ell^{\nu} - \lambda H_{\mu\nu}\ell^{\mu}\ell^{\nu}}{3/R(r,t)^{2} - T_{\mu\nu}\ell^{\mu}n^{\nu} + \lambda H_{\mu\nu}\ell^{\mu}n^{\nu}},$$
(21)

where eqn.(2) and eqn.(5) have been used. Now, the task is to write down all the components in $T_{\mu\nu}$ as well as in $H_{\mu\nu}$ in terms of the matter variables. The expression for $T_{\mu\nu}$ is already given in equation (8). The details of the calculation for $H_{\mu\nu}$ is carried out in the appendix (V A). The quantity $H_{\mu\nu} \ell^{\mu} \ell^{\nu}$ in equation (21) is given by:

$$H_{\mu\nu}\ell^{\mu}\ell^{\nu} = 2\left[\frac{6F(\rho + p_r)}{(F - 2\lambda)^2} + 2p_{\theta}^2 - 4p_r p_{\theta} - \frac{2}{3}p_{\theta}(\rho + p_r)\right],\tag{22}$$

Similarly, the expressions for $H_{\mu\nu} \ell^{\mu} n^{\nu}$ involves two terms, which are given by:

$$H'_{\mu\nu} \ell^{\mu} n^{\nu} = 2 \left[4p_{\theta} \left(p_{\theta} + \frac{2}{9} \rho - \frac{4}{3} p_{r} \right) - \frac{2}{(F - 2\lambda)^{2}} \{ 6F p_{t} + (F + 4\lambda) (\rho - p_{r}) \} \right]$$

$$- 6 \left\{ p_{t} + \frac{2}{3} (\rho - p_{r}) - \frac{3F}{(F - 2\lambda)^{2}} \right\}^{2} + \frac{16}{9} \left(\rho^{2} + p_{r}^{2} \right) - 72 \frac{\lambda^{2}}{(F - 2\lambda)^{4}} \right].$$

$$(23)$$

The term involving the L_{GB} gives the following expression in terms of the matter variables:

$$L_{GB} = R^{2} - 4R_{tt}R^{tt} - 4R_{rr}R^{rr} - 12R_{\theta\theta}R^{\theta\theta} + 6R_{trtr}R^{trtr} + 18R_{t\theta t\theta}R^{t\theta t\theta} + 18R_{r\theta r\theta}R^{r\theta r\theta} + 18R_{\theta\phi\theta\phi}R^{\theta\phi\theta\phi}$$

$$= \left[\frac{2}{3}\left(\rho - p_{r}\right) - 2p_{t}\right]^{2} + 18\left[\frac{F^{2} + 32\lambda^{2}}{\left(F - 2\lambda\right)^{4}}\right] + 6\left[p_{t} + \frac{2}{3}\left(\rho - p_{r}\right) - \frac{3F}{\left(F - 2\lambda\right)^{2}}\right]^{2}$$

$$- \frac{12}{9}\left(\rho - p_{r}\right)^{2} - 4\left[\frac{2}{3}\left(\rho + p_{r}\right) + p_{t}\right]^{2} - 4\left[\frac{2}{3}\left(\rho + p_{r}\right) - p_{t}\right]^{2}. \tag{24}$$

Using these expressions in equation (21), we shall understand the evolution of spherical MTTs for various collapse scenarios.

III. GRAVITATIONAL COLLAPSE FOR PRESSURELESS MATTER

Let us use the equations derived above to understand the dynamics of collapse process for pressureless matter configuration. In the absence of pressure, the EGB equation (10) implies that F = F(r), whereas eqn. (12) gives $\alpha' = 0$. The metric function $\alpha(t,r)$ is a function of t only. This allows the rescaling of the time coordinate so that effectively $\alpha(t,r) = 0$. The metric function $\beta(t,r)$ follows from eqn. (18). This two solutions implies that the metric is given by:

$$ds^{2} = -dt^{2} + \frac{R'^{2}}{1 - k(r)} dr^{2} + R(r, t)^{2} d\Omega_{3},$$
(25)

where $d\Omega_3$ is the metric of an unit round 3-sphere, and R(t,r) is obtained from the equation (13), which gives the equation of motion of the collapsing matter configuration in 5D-EGB theory:

$$\dot{R}^{2}(r,t) = -k(r) - \frac{R^{2}}{4\lambda} + \frac{R^{2}}{4\lambda} \left[1 + \frac{8\lambda F}{R^{4}} \right]^{1/2}, \tag{26}$$

where we have used eqn.(18). The function k(r) can take either signatures or zero. The situation where k(r) remains vanishing during the collapse process is called a marginally bound collapse, whereas the one in which k(r) admits a positive signature is called a bounded collapse. We shall deal with these two cases only. The behaviour for unbounded gravitational collapse in EGB theory is similar and shall not be carried out here.

Now, one has to ensure that this metric existing inside the collapsing matter cloud must be matched to an exterior static spherically symmetric metric. Such a metric is already well known as the Boulware- Deser- Wheeler solution

[24, 30–32]. We shall always ensure that metric of the collapsing matter cloud remains matched to an external Boulware- Deser- Wheeler solution of mass M, across a timelike hypersurface r_b . As we show in the appendix (VC), such a matching leads to the condition that $F(r_b) = M$.

In the following, we shall consider a wide variety of density profiles for matter fields and note the formation of singularity and spherically symmetric trapped surfaces and horizons.

A. Marginally bound collapse

For the marginally bound collapse, we have k=0. From the equations (13) and (26), the equation of motion is

$$\dot{R}^{2}(r,t) = -\frac{R^{2}}{4\lambda} + \frac{R^{2}}{4\lambda} \left[1 + \frac{8\lambda F}{R^{4}} \right]^{1/2}.$$
 (27)

Using some simple substitutions and algebra we get the equation for matter shells corresponding to values of R(r,t) (see also [10])

$$t_{sh} = t_s - \left[\frac{\lambda R^2}{\sqrt{R^4 - 8\lambda F} - R^2} \right]^{1/2} - \frac{\sqrt{\lambda}}{2\sqrt{2}} \tan^{-1} \left[\frac{3R^2 - \sqrt{R^4 - 8\lambda F}}{2\sqrt{2} \left\{ \sqrt{R^4 - 8\lambda F} - R^2 \right\}^{1/2}} \right], \tag{28}$$

where t_s is the time of the formation of singularity, and is given by:

$$t_s = \frac{\sqrt{\lambda}}{2\sqrt{2}} \tan^{-1} \left[\frac{3r^2 - \sqrt{r^4 - 8\lambda F}}{2\sqrt{2} \left\{ \sqrt{r^4 - 8\lambda F} - r^2 \right\}^{1/2}} \right] + \left[\frac{\lambda r^2}{\sqrt{r^4 - 8\lambda F} - r^2} \right]^{1/2}.$$
 (29)

The expression of the time for shells reach the Boulware- Deser-Wheeler horizon or the MTT, obtained for $R(r,t) = \sqrt{F(r,t) - 2\lambda}$ is given by t_{AH} :

$$t_{AH} = t_s - \left[\frac{\lambda (F - 2\lambda)}{\sqrt{(F - 2\lambda)^2 - 8\lambda F} - (F - 2\lambda)} \right]^{1/2} - \frac{\sqrt{\lambda}}{2\sqrt{2}} \tan^{-1} \left[\frac{3 (F - 2\lambda) - \sqrt{(F - 2\lambda)^2 - 8\lambda F}}{2\sqrt{2} \left\{ \sqrt{(F - 2\lambda)^2 - 8\lambda F} - (F - 2\lambda) \right\}^{1/2}} \right] (30)$$

Given these expressions we now proceeds to understand the nature of MTTs for some realistic mass profiles.

Examples

1. Let us first consider a collapsing matter profile which admits a variation in the density distribution according to the choice of two parameters ς and r_0 . The density distribution is of the following form:

$$\rho(r) = \frac{m_0 \mathcal{E}(\varsigma)}{r_0^4} \left[1 - \operatorname{Erf} \left\{ \varsigma \left(\frac{r}{r_0} - 1 \right) \right\} \right], \tag{31}$$

where $m_0 = m(r \to \infty)$ is the total mass of the cloud, r_0 is the label on the matter shell coordinate where the variation of the density with the radial coordinate is largest, $i.e - (d\rho/dr)$ is highest. We shall choose the value of $r_0 = 2$. The parameter ς in equation (31) controls the variation of density function. A similar density profile was also studied for LTB models in 4-d GR [38, 39]. As seen from the plot in figure (1)(a), a larger value of ς implies a step- function- type distribution of the density, whereas, for a lower value of ς , the density varies slowly with r. So, ς is a control parameter for approach towards the OSD model- larger the value of ς , closer is the density to isotropy, and smaller values of ς implies inhomogeneities. The function $\mathcal{E}(\sigma)$ has the following form:

$$\mathcal{E}(\varsigma) = 3\varsigma^3 \left[2\pi\varsigma (2\varsigma^2 + 3)(1 + \operatorname{Erf} \varsigma) + 4\sqrt{\pi} \exp(-\varsigma)(1 + \varsigma^2) \right]^{-1}, \tag{32}$$

and Erf is the usual error function. We consider the cases where $\varsigma = 5$ and 15. The graphs are given in figure (1).

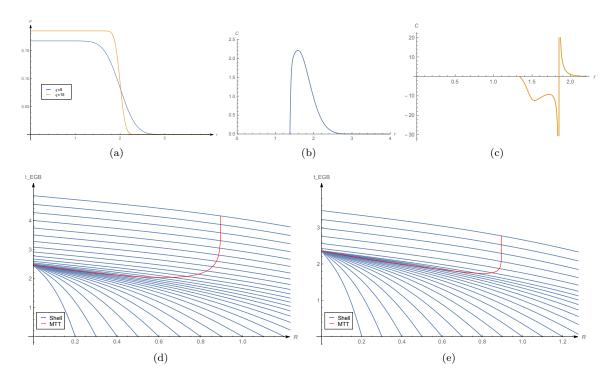


FIG. 1: These figures give the gravitational collapse for the density profile of eqn. (31). For the plot we have used the EGB coupling constant $\lambda=0.1$. The figure (a) gives the density fall-off for two choices of the control parameter ς , (b) is the plot of the function C for $\varsigma=5$. The signature of C shows that the MTT in this case is spacelike. (c) is the plot of the function C for $\varsigma=15$. The signature of C shows that the MTT in this case is timelike. The figure (d) is R(r,t) vs t graph for $\varsigma=5$, (e) gives the time development of MTT for $\varsigma=15$ along with the collapse of each shell. In the R-t graphs, the shells are denoted by blue lines whereas the red lines are the MTT. The straight vertical red lines in (d) and (e) represents the isolated horizon phase of the MTT and is reached when no more matter falls in.

From figure (1)(d), we note that as shells begin to collapse, the MTT begins to form, and grows with the fall of the shells, until the growth stops when all the shells upto r=2 has fallen in. This happens since the matter density is almost zero after r=2. After all the matter goes in, the MTT becomes null, as seen by the straight line in figure (1)(d). The MTT becomes null at R=0.89 since the total mass of the cloud is unity, and hence for $\lambda=0.1$, the MTT is obtained from eqn. (20) to be $\sqrt{0.8}=0.894$.In this region, the MTT has reached the IH phase.

Two further points need to be noticed. First, for $\varsigma = 5$, the MTT are spacelike. This may be seen from the values of C in figure (1)(b). However, if we look at the R(r,t)-t graph in figure (1)(d), it seems that the MTT may have become timelike in certain regions. This apparent contradiction was also noted earlier in [38, 47] and happens due to non-trivial ways in which the MTT crosses the chosen folations. For $\varsigma = 15$, the MTT is surely timelike, as may be noted from figures (1)(c) and (1)(e). The MTT begins to form earlier at r = 1.7 at t = 1.8 and then begins to grow on either side to match with the MTT at the center R = 0, and also towards the IH at R = 0.89. This possible points towards an unstable MTT, as was pointed out in the case of GR in [38, 39, 47]

Secondly, as can be noted from the graph in figure (1)(e), all the shells, denoted by the blue lines reach the singularity at R=0 at the same time, which is a distinctive feature of the OSD process. As the value of ς is lowered, the example of (1)(d) shows that the shells begin to deviate marginally from this feature since the deviation in the density profile remains small. This also points to the fact that this collapse process is similar to that in GR, at least in this particular case of isotropic collapse.

2. For the next example, we take the mass density to have following form [10, 39]:

$$\rho(r) = m_0 [1 - (r/r_0)] \Theta(100 - r) \tag{33}$$

where $\Theta(x)$ denotes the Heaviside theta function, and $r_0 = 100 \, m_0$. The graph of ρ , C and R(r,t)-t are given in the figure (2)(a), (2)(b), and (2)(c) respectively. Note that the MTT begin around t = 2900 when the shell at r = 50 has already fallen in. After this growth, it remains a dynamical horizon throughout and becomes an

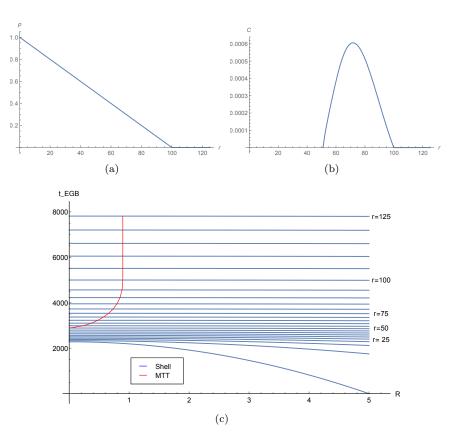


FIG. 2: The graphs show the (a) density distribution ρ , for equation (33), (b) values of C, and (c) formation of MTT along with the shells. For the plot we have used $\lambda = 0.1$. Note that the MTT begins to form only after some shells have fallen in the singularity. This is a direct consequence of the fact that eqn. (20) requires the mass function F(r) to exceed 2λ for a real valued $R_M(r,t)$ in the equation of MTS.

isolated horizon only when the matter shells stops falling at r = 100 and all the matter has collapsed. This behaviour in the R - t plot is reflected in the graph of C quite faithfully. Indeed, the signature of C indicates that MTT is spacelike, beginning at r = 50 and continues until the shell at r = 100 falls, after which it becomes null.

Note however that MTT does not begin to form immediately, but only after some shells have fallen in. This is because of a simple reason but leads to some important consequences, and is discussed below: The MTT forms only when the condition in eqn. (20) is satisfied. Indeed, for the early shells, the value of F(r) for these shells, i.e. the amount of matter contained inside the sphere of radius r at the initial time, is smaller than the value of λ , which here is taken to be 0.1. For that reason, $R_M(r,t)$ does not admit real values. It is only after sufficient number of shells have fallen in, that condition of trapped surface can be evaluated to obtain a real value. Until that time, the central singularity remains naked for a trapped surface. Our study reveals this feature in a direct manner since we have been able to probe each and every matter shells quite elaborately.

3. Let us now consider a Gaussian density profile with the density given by the following form:

$$\rho(r) = \frac{3m_0}{r_0^4} \exp(-r^2/r_0^2),\tag{34}$$

where m_0 is the total mass of the matter cloud, r_0 is a parameter which indicates the distance where the density of the cloud decreases to $[\rho(0)/e]$. In our example, we have chosen $r_0 = 100 \, m_0$ and the EGB coupling constant $\lambda = 0.1$. Note that the MTT begins only after the shell at r = 90 has fallen in. As explained in the previous subsection, this is a direct consequence of the relation eqn.(20). The MTT in Fig. (3)(b) and (3)(c) clearly shows that the MTT is spacelike, and attains the IH phase when the shells at r = 300 has fallen in.

4. Let us consider a density profile given by the following form for $r \in [0, \pi r_0]$:

$$\rho(r) = (\gamma/r_0^2) \left[\pi - (r/5r_0) \{ 3 + 2\cos^2(5r/r_0) \} \right]$$
(35)

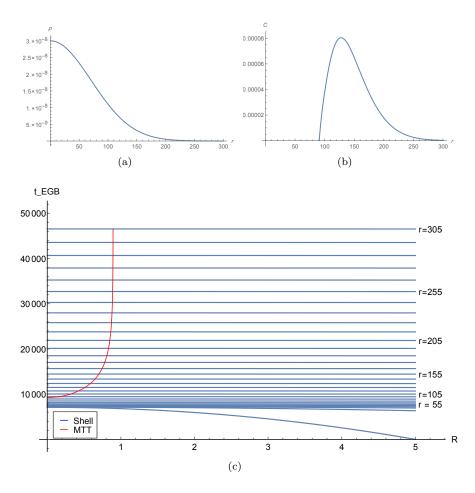


FIG. 3: The graphs show the (a) density distribution for eqn. (34), (b) values of C, and (c) formation of MTT along with the shells. The straight lines of MTT in (c), after the shell r = 300, represents the isolated horizon phase.

where γ is a dimensionless constant. This example constitutes a situation where the MTTs are a series of timelike membranes interspaced with dynamical horizons. A similar profile was used to study gravitational collapse in 4d GR [38, 39]. In our example, we have chosen $r_0 = 1$, $\gamma = 1/120$, and the EGB coupling constant $\lambda = 0.1$. Note the peculiar dynamics of the MTT from figure (4)(c). The MTT first forms for the shell at r = 1.5, and then evolves in a timelike manner to reach towards the MTT formed after the shells at r = 1.35 have fallen in. Again note that during the initial period, the central singularity is not covered by the MTT and remains naked, as expected due to equation (20). During the period the shells from r = 1.7 to r = 2.0 collapse, the MTT is a dynamical horizon, as may also be confirmed from the graph of C in figure (4)(b). This behaviour is repeated until matter stops falling at r = 3.0, when the MTT reaches the equilibrium state of an IH.

5. Two shells falling consecutively on a black hole: Let us assume that a black hole of mass M exists, upon which a density profile of the following form falls:

$$\rho(r) = \frac{12 (m_0/r_0^4) [(r/r_0) - \varsigma]^2}{[2 (4 + \varsigma^2) + (9 + 2\varsigma^2) \sqrt{\pi} e^{\varsigma^2} \{1 + \varsigma \operatorname{Erf}(\varsigma)\}]} \exp[(2r/r_0)\varsigma - (r/r_0)^2], \tag{36}$$

where $m_0 = M/2$, (M = 1) is the mass of the shell, $2r_0$ is the width of each shell, and $\zeta = 10m_0$. The graphs corresponding to this case is given in (5). Note that these graphs constitute the case where two mass profiles fall on a black hole one after the other. The spacetime singularity already exists into which these shells fall in. Note that as the fist profile falls, the MTT begins from the already existing horizon at R = 0.89 and develops until the shells corresponding to r = 23 to r = 25 fall in carrying no mass with them. At these times, the MTT reaches an equilibrium state, and becomes dynamical only after the second mass profile begins to fall. So, the MTT passes through multiple stages of dynamical horizon, interspaced with isolated horizons when no matter is infalling. This behaviour is easily verifiable from figures (5)(b), and (5)(c).

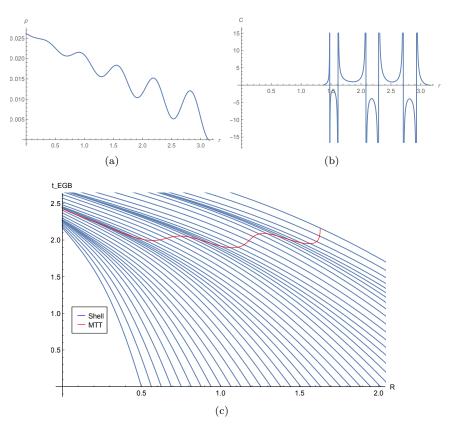


FIG. 4: The graphs show the (a) density distribution, (b) values of C, and (c) formation of MTT along with the shells for the density in equation (35).

B. Bounded collapse

For bounded collapse, we again have $\alpha' = 0$ and $G(r,t) = e^{-2\beta(r,t)}R'^2 = 1 - k(r) = E(r)$, where E(r) is the integration function. For this case k > 0, the equation of motion is given by (26)

$$\dot{R}^{2}(r,t) = -k(r) - \frac{R^{2}(r,t)}{4\lambda} + \frac{R^{2}(r,t)}{4\lambda} \left[1 + \frac{8\lambda F(r,t)}{R^{4}(r,t)} \right]^{1/2}.$$
 (37)

This equation of motion (37) can be rewritten in the following form:

$$dt = -\frac{2\sqrt{\lambda} dR}{\sqrt{-R^2 - 4\lambda k + \sqrt{R^4 + 8\lambda F}}}$$
(38)

To integrate this equation of motion (38), we consider a parametric choice of R(r,t) of the following form:

$$x = -R^{2}(r,t) - 4\lambda k(r) + \sqrt{R^{4}(r,t) + 8\lambda F(r)}$$
(39)

A simple calculation of squaring both sides leads to the following expression:

$$R(r,t) = \frac{1}{\sqrt{2}} \left[\frac{8\lambda F}{(x+4\lambda k)} - (x+4\lambda k) \right]^{1/2}$$
(40)

Using this expression of eqn. (40), a simple calculation leads to modification of (38):

$$dt = \frac{\sqrt{\lambda} \left((x + 4\lambda k)^2 + 8\lambda F \right) dx}{\sqrt{2}\sqrt{x}(x + 4\lambda k)^{3/2} \sqrt{8\lambda F - (x + 4\lambda k)^2}}$$

$$\tag{41}$$

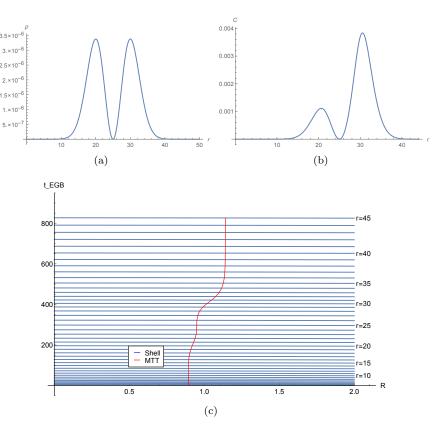


FIG. 5: The graphs show the (a) the density profile from eqn. (36), (b) values of C, and (b) formation of MTT along with the shells which fall consecutively on a black hole.

The integration of the above equation gives the equation of the collapsing shell to be:

$$t_{sh} = A_1 \left[\left(8\lambda F - 2\sqrt{2\lambda F}(x + 4\lambda k) \right) + A_2 \left\{ \left(\sqrt{2F} + 2\sqrt{\lambda}k \right) \text{ EllipticE}[N_1, 2N_2] \right. \\ \left. - \left(\sqrt{2F} - 2\sqrt{\lambda}k \right) \text{ EllipticF}[N_1, 2N_2] - 2\sqrt{\lambda}k \text{ EllipticPi}[N_2, N_1, 2N_2] \right\} \right] \\ \left. - \left(A_1 \right)_0 \left[\left(8\lambda F - 2\sqrt{2\lambda F}(x_0 + 4\lambda k) \right) + \left(A_2 \right)_0 \left\{ \left(\sqrt{2F} + 2\sqrt{\lambda}k \right) \text{ EllipticE}[(N_1)_0, 2N_2] \right. \\ \left. - \left(\sqrt{2F} - 2\sqrt{\lambda}k \right) \text{ EllipticF}[(N_1)_0, 2N_2] - 2\sqrt{\lambda}k \text{ EllipticPi}[N_2, (N_1)_0, 2N_2] \right\} \right]$$

$$(42)$$

The equation for the spherical MTTs, are obtained for $R(r,t) = \sqrt{F(r) - 2\lambda}$ and gives:

$$t_{AH} = (A_1)_{2M} \left[\left(8\lambda F - 2\sqrt{2\lambda F}(x_{2m} + 4\lambda k) \right) + (A_2)_{2M} \left\{ \left(\sqrt{2F} + 2\sqrt{\lambda}k \right) \text{ EllipticE}[(N_1)_{2M}, 2N_2] \right. \right. \\ \left. - \left(\sqrt{2F} - 2\sqrt{\lambda}k \right) \text{ EllipticF}[(N_1)_{2M}, 2N_2] - 2\sqrt{\lambda}k \text{ EllipticPi}[N_2, (N_1)_{2M}, 2N_2] \right\} \right] \\ \left. - (A_1)_0 \left[\left(8\lambda F - 2\sqrt{2\lambda F}(x_0 + 4\lambda k) \right) + (A_2)_0 \left\{ \left(\sqrt{2F} + 2\sqrt{\lambda}k \right) \text{ EllipticE}[(N_1)_0, 2N_2] \right. \right. \\ \left. - \left(\sqrt{2F} - 2\sqrt{\lambda}k \right) \text{ EllipticF}[(N_1)_0, 2N_2] - 2\sqrt{\lambda}k \text{ EllipticPi}[N_2, (N_1)_0, 2N_2] \right\} \right], \tag{43}$$

where the coefficients A_1 , A_2 , and the arguments N_1 , N_2 are given by:

$$A_{1} = \frac{\sqrt{x}}{k(x+4\lambda k)^{3/2}\sqrt{8\lambda F} - (x+4\lambda k)^{2}}, \quad A_{2} = \frac{(2^{5}\lambda^{2}F)^{1/4}\sqrt{x+4\lambda k}\sqrt{-8\lambda F} + (x+4\lambda k)^{2}}{\sqrt{x\sqrt{2F}} + 2\sqrt{\lambda}k}$$

$$N_{1} = \sin^{-1}\left[\frac{(\sqrt{2F} + 2\sqrt{\lambda}k)(x+4\lambda k)}{2\lambda k\{2\sqrt{2\lambda F} + (x+4\lambda k)\}}\right]^{1/2}, \quad N_{2} = \frac{2\sqrt{\lambda}k}{\sqrt{2F} + 2\sqrt{\lambda}k}.$$

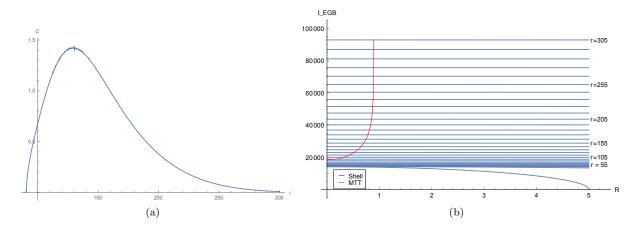


FIG. 6: The graphs show the (a) values of C, and (b) formation of MTT along with the shells for the Gaussian distribution of eqn. (34).

The terms with subscript 0 and 2M represents its value at the initial shells at $r=r_0$ and at the formation of MTT with $R=R=\sqrt{F-2\lambda}$. For example, $x=-R^2-4\lambda k+\sqrt{R^4+8\lambda F}$, whereas, its value at 0 represents $x_0=-r^2-4\lambda k+\sqrt{r^4+8\lambda F}$. The EllipticF[ϕ , z] represents an incomplete elliptic integral of the first kind, EllipticE[ϕ , z] is elliptic integral of the second kind, whereas EllipticPi[y; ϕ , z] is an elliptic integral of the third kind [69].

In the following we shall take several examples to show how a MTT develops during the bounded gravitational collapse in EGB theory.

Example

- 1. Let us consider a Gaussian profile with the density given by eqn. (34). The form of the density is same as in figure (3)(a). In our example, we have chosen $r_0 = 100 \, m_0$. The behaviour of MTT is similar to that discussed for the marginally bound case, see figure (6). However, the time of formation of MTT and the value of the C has changed in comparison. Again note that the MTT forms only after shells at r = 90 collapse. Before that shell falls in, the singularity remains naked. The time of formation of MTT changes in comparison to the marginally bound case of figure (3). The straight lines of MTT in (b), after the shell r = 300, represents the isolated horizon phase.
- 2. Let us consider a density profile given by the following form:

$$\rho(r) = (m_0/8\pi r_0^4) \exp(-r/r_0), \tag{44}$$

where m_0 is the total mass of the matter cloud, r_0 is a parameter which indicates the distance where the density of the cloud decreases to $[\rho(0)/e]$. The MTT begins after the shells at r=30 have fallen into the singularity. The MTT remains spacelike through out its time evolution, and reaches an equilibrium state only after the density reaches negligible values. These conclusions are easily be verified from the graphs in figure (7). Note again that the MTT begins only after sufficient number of shells have collapsed to the singularity in accordance to the choice of $\lambda=0.1$ in eqn. (20).

- 3. Two shells falling consecutively on a black hole: The graphs corresponding to this case is given in figure (8). Note that these graphs have a similar behaviour to those in figure (5), except that the times for formation of MTTs have changed.
- 4. Let us again consider the density profile given by eqn. (35), given in figure (4)(a). The behaviour is of the MTTs and the shells, for the bounded collapse as given in figure (9) is similar to the graphs in figure (4), with the exception that the time of formation of MTTs, and as well as those of the shells reaching the singularity has changed.

Similar study may be carried out for more complicated matter profiles and other matter sources. These studies can be made using the techniques developed above.

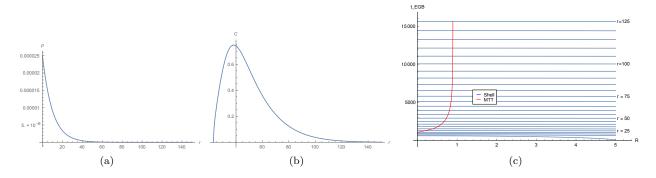


FIG. 7: The graphs show the (a) values of C, and (b) formation of MTT along with the shells for the matter profile with exponentially falling density distribution given in eqn. (44). The MTT is spacelike.

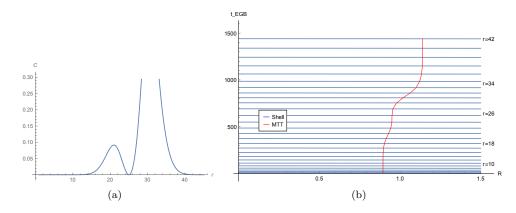


FIG. 8: The graphs show the (a) values of C, and (b) formation of MTT along with the shells which fall consecutively on a black hole. The value of C remains positive and large, and for that reason it is not plotted here. As a consequence MTT remains spacelike.

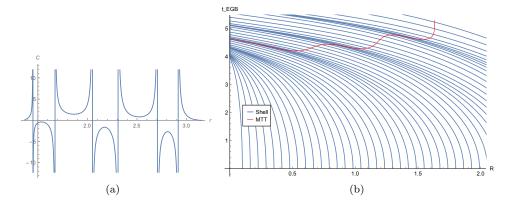


FIG. 9: The graphs show the (a) density distribution, (b) values of C, and (c) formation of MTT along with the shells for the bounded collapse of the density profile discussed in eqn. (35). The MTT is quite complicated and goes through various modulations.

IV. DISCUSSIONS

This paper deals with the study of gravitational collapse in EGB gravity in 5- dimensions. The Gauss-Bonnet modification of the Einstein gravity changes the geometry of the spacetime, and the structure of the horizon and singularity quite drastically. We developed techniques to analyse these effects in the phenomena of gravitational collapse in this theory. In this context, several questions arise naturally regarding the process of the collapse phe-

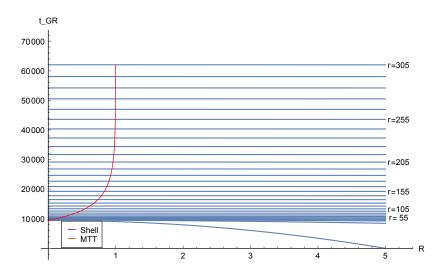


FIG. 10: The graphs show formation of MTT along with the shells for marginally bound collapse of pressureless matter with a Gaussian density profile of eqn. (34), in 5- dimensional GR.

nomenon itself as well as the outcome of gravitational collapse of matter. To understand these details, we have, in this paper, developed a set of analytical and numerical techniques to locate spherical marginally trapped surfaces in the spacetime, when the collapse is in progress. We locate these MTTs for a large class of matter profiles and initial velocity profiles. This study helps us to address several questions regarding gravitational collapse in the EGB theory:

(i) Role of the GB term and the coupling constant λ : The GB term introduces several changes in the equation of motion of the gravitational field. The most drastic is the change in the form of the mass function F(r,t) given in eqn. (13). In fact, this equation shows that the GB term leads to quadratic effects involving R(r,t) and R'(r,t). As a result of this quadratic contribution of R, the equation of motion of the radius of the dust cloud is altered significantly, see eqn. (19). Naturally, this change in the equation of motion of the spherically symmetric matter configuration implies that the collapsing matter spheres will get trapped at different times. A direct reflection of this fact is in the expressions for the expansion of the outward and the inward null normals $\theta_{(\ell)}$ and $\theta_{(n)}$ in eqns. (14) and (15) It follows as a direct result of (15) that the equation defining a marginally trapped surface is dependent on the GB coupling constant λ , see eqn. (20). The marginally trapped surface (MTS) forms at $R_M(r,t) = F(r,t)^{1/2}$ in the 5- dimensional Einstein theory, whereas it forms at $\sqrt{F(r,t)-2\lambda}$ in the EGB theory. In this paper, we have kept the value of $\lambda = 0.1$, and so, the equation for MTS, eqn. (20) implies that real values of $R_M(r,t)$ is only possible only if sufficient number of shells have fallen in so that the cloud if massive enough to overcome the effect of the GB coupling constant λ . This effect on the formation of a MTS and the MTT is directly visible in the graphs in fig. (2), fig. (3) as well as in the fig. (6). The coupling constant results in the delay in the formation of MTT, and as can be noticed from these figures, begins to form quite later than the formation of central singularity due collapsing shells. This effect is not visible in fig. (5), since the system already has a spacetime singularity, and so, this initial black hole horizon censors all the singularities arising out of shell collapse.

It is also instructive to compare this same study of MTTs for the Gaussian profile in eqn. (34) in the 5- dimensional Einstein theory. As expected, the MTT begins just as the first shells start to collapse and the MTT equilibriates at R = 1, since the total mass of the profile is unity, and the MTT is $R_M(r,t) = F(r,t)^{1/2}$. This is given in fig. (10).

(ii) Nature of the central singularity: Since many of these configurations lead to shell collapsing naked singularities due to gravitational collapse of the initial shells, and that MTTs do not cover them, it becomes essential to characterise them, and make a clear classification. We have explicitly verified that, in each of the cases where the central singularity is naked initially, satisfy the following relation: The weak cosmic censorship is violated for each of these collapse processes until the mass function $F(r) > 2\lambda$ (see also [7]). The fact that the curvature strength of the singularity is a weak is obtained as follows: Note that the singularity is defined to be strong if the spacetime volume contained within Jacobi vector fields is reduced to zero at the singularity. The singularity is weak otherwise. According to the standard characterizations of singularities in 4- dimensions [4], a sufficient condition for a strong singularity is that at least one causal geodesic t^{μ} , with affine parameter v must satisfy the following condition:

$$\lim_{v \to v_0} (v \to v_0)^2 R_{\mu\nu} t^{\mu} t^{\nu} > 0 \tag{45}$$

For our spacetime, and a radial timelike vector field, a simple calculation shows that $\lim_{v\to v_0} (v\to v_0)^2 R_{\mu\nu} t^{\mu} t^{\nu} = 0$. Here too, the role of the Gauss- Bonnet coupling becomes crucial, and plays an important role in weakening the singularity. So, although the singularities are naked at the beginning of the collapse process, the singularity is harmless since they are weakly naked.

(iii) Are MTT true black hole boundaries? The actual extent of a black hole region is a matter of great debate. Over the years, global as well as quasilocal considerations have led to several formulations of horizon. Out of them, event horizon, Killing horizons have been quite useful to study physical phenomena of black holes. The quasilocal formulations based on trapped surfaces, and in particular the definitions of trapping horizons and MTTs [46, 60] have been extensively used to prove classical and quantum laws of black hole dynamics. Although, it must also be pointed out that the formulation of MTT as a black hole boundary may need modifications, in particular in respect to the conditions on $\theta_{(n)}$, they may be quite useful for this purpose. However, the main issue lies in locating the non-spherically symmetric MTTs as well, and in the context of 4- dimensional spacetimes, they are yet to be completely specified [63–65]. Furthermore, for some spacetimes, the black hole boundary is identical with the event horizon [70, 71]. Our study using spherical MTTs in 5- dimensions show that they may indeed be used as a boundary of a black hole region, although a non-spherical MTTs and their location is equally important to be understood in this context. We must also point out that our study needs to be extended for more general matter fields and geometries, so that such questions may be included in our discussions.

To conclude, we have explicitly shown, with a wide range of examples, that the nature of trapped surface, its formation and time development, is intimately related to the initial velocity and the initial density profile of the matter fields. Additionally, due to the presence of the EGB coupling constant λ , the formation of MTT gets delayed further, depending on the amount of matter a particular matter shell encloses within its boundaries. All these effects have been conclusively demonstrated through the examples considered in the main part of the paper. We must however admit that a full understanding of these phenomenon of gravitational collapse and the censorship conjecture shall require the methods of non-spherical gravitational collapse.

Acknowledgements

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V. APPENDIX

A. Expressions for curvature using matter variables

In the following, we collect the expressions of the various curvature components for the metric (7). These components have been used in the main part of the paper to determine the evolution of MTT, and in determining the signature of the MTT in eqn. (21). The quantities like the Ricci scalar (R_s) , Ricci tensors and the Riemann tensors in terms of the energy density, radial and tangential pressure and mass function.

First, the Riemann tensors are obtained using the metric functions and the matter variables:

$$\begin{split} R_{\theta\phi\theta\phi} &= F(r,t)\sin^2\theta\,, \qquad R_{\theta\psi\theta\psi} = \sin^2\theta R_{\theta\phi\theta\phi}\,, \qquad R_{\phi\psi\phi\psi} = \sin^2\theta \sin^2\phi \, R_{\theta\phi\theta\phi}, \\ R_{t\phi t\phi} &= \sin^2\theta R_{t\theta t\theta}\,, \qquad R_{t\psi t\psi} = \sin^2\theta \sin^2\phi \, R_{t\theta t\theta}, \qquad R_{t\theta r\theta} = 0 \\ R_{r\phi r\phi} &= \sin^2\theta \, R_{r\theta r\theta}\,, \qquad R_{r\psi r\psi} = \sin^2\theta \sin^2\phi \, R_{r\theta r\theta}, \\ R_{t\theta t\theta} &= -(1/2)e^{2\alpha}\frac{R}{\dot{R}}\,\frac{d}{dt}\left[\frac{F(r,t)}{R^2(r,t)} - 1\right] \\ R_{r\theta r\theta} &= \left(1/2\right)e^{2\beta}\frac{R}{R'}\,\frac{d}{dr}\left[\frac{F(t,r)}{R^2(t,r)} - 1\right]\,, \\ R_{rtrt} &= \left[p_t - (2/3)\left(\rho + p_r\right) - (3F/R^4)\right]e^{2(\alpha + \beta)}. \end{split}$$

The Ricci tensors are obtained similarly using the metric in eqn. (7).

$$R_{tt} = (-2\rho/3 + p_r/3 + p_t)e^{2\alpha}, R_{rr} = (-\rho/3 + 2p_r/3 - p_t)e^{2\beta}, R_{\theta\theta} = -(R^2/3)(\rho + p_r), R_{\phi\phi} = \sin^2\theta R_{\theta\theta}, R_{\psi\psi} = \sin^2\theta \sin^2\phi R_{\theta\theta}, R_{rt} = 0,$$

The Ricci scalar is given by $R_s = -(2/3)(\rho + p_r) - 2p_t$. Using these expressions, and the expressions for null normals in eqns. (14) and (15), it can be shown easily that:

$$\begin{split} H_{ab}\ell^{a}\ell^{b} &= 2\left[\frac{6F\left(\rho+p_{r}\right)}{\left(F-2\lambda\right)^{2}}+2p_{\theta}^{2}-4p_{r}p_{\theta}-\frac{2}{3}p_{\theta}\left(\rho+p_{r}\right)\right],\\ H_{ab}\ell^{a}n^{b} &= 2\left[4p_{\theta}\left(p_{\theta}+\frac{2}{9}\rho-\frac{4}{3}p_{r}\right)-\frac{2}{\left(F-2\lambda\right)^{2}}\left\{6Fp_{t}+\left(F+4\lambda\right)\left(\rho-p_{r}\right)\right\}\right.\\ &\left.-6\left\{p_{t}+\frac{2}{3}\left(\rho-p_{r}\right)-\frac{3F}{\left(F-2\lambda\right)^{2}}\right\}^{2}+\frac{16}{9}\left(\rho^{2}+p_{r}^{2}\right)-72\frac{\lambda^{2}}{\left(F-2\lambda\right)^{4}}\right]. \end{split}$$

We can also similarly determine an expression for L_{GB} in terms of matter variables and the mass function.

$$\begin{split} L_{GB} &= R^2 - 4R_{tt}R^{tt} - 4R_{rr}R^{rr} - 12R_{\theta\theta}R^{\theta\theta} + 6R_{trtr}R^{trtr} + 18R_{t\theta t\theta}R^{t\theta t\theta} + 18R_{r\theta r\theta}R^{r\theta r\theta} + 18R_{\theta\phi\theta\phi}R^{\theta\phi\theta\phi} \\ &= \left[\frac{2}{3}\left(\rho - p_r\right) - 2p_t\right]^2 + 18\left[\frac{F^2 + 32\lambda^2}{(F - 2\lambda)^4}\right] + 6\left[p_t + \frac{2}{3}\left(\rho - p_r\right) - \frac{3F}{(F - 2\lambda)^2}\right]^2 \\ &- \frac{12}{9}\left(\rho - p_r\right)^2 - 4\left[\frac{2}{3}\left(\rho + p_r\right) + p_t\right]^2 - 4\left[\frac{2}{3}\left(\rho + p_r\right) - p_t\right]^2. \end{split}$$

B. Three- surface geometry

The subspace in our problem is a three dimensional sphere. To understand the geometry of this subspace, we shall present a general formulation of subspaces. Let $(\mathcal{M}, g_{\mu\nu}, \nabla_{\mu})$ be a 5- dimensional time- oriented spacetime with a metric compatible covariant derivative $\nabla_{\mu} g_{\nu\lambda} = 0$. Let us assume that \mathcal{S} be a closed, orientable, spacelike 3-surface embedded in \mathcal{M} . Let us denote the two future pointing null vectors by ℓ^{μ} (outward pointing) and n^{μ} (inward pointing), such that $\ell \cdot n = -1$.

The induced metric h_{ab} on the 3- surface S is given by:

$$h_{ab} = e^{\mu}_{\ a} e^{\nu}_{\ b} \, q_{\mu\nu},\tag{46}$$

where $e^{\mu}{}_{a}$ denotes the pullback map, and a, b, \ldots indicate indices on S. The functions $e^{\mu}{}_{a}$ are orthogonal to ℓ^{μ} and n^{μ} . This implies that the pushforward of the inverse two- metric h^{ab} is given by:

$$g^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}h^{ab} - \ell^{\mu}n^{\nu} - \ell^{\nu}n^{\mu}. \tag{47}$$

The second important quantities of importance are the extrinsic curvatures. This is vector on the normal bundle N(S) of S, and it has two components.

$$k^{(\ell)}{}_{ab} = e^{\mu}{}_{a}e^{\nu}{}_{b} \nabla_{\mu} \ell_{\nu}, \qquad k^{(n)}{}_{ab} = e^{\mu}{}_{a}e^{\nu}{}_{b} \nabla_{\mu} n_{\nu},$$
 (48)

where the extrinsic curvature itself may be written as:

$$k^{\mu}{}_{ab} = k^{(n)}{}_{ab} \ell^{\mu} + k^{(\ell)}{}_{ab} n^{\mu}. \tag{49}$$

The Riemann tensor on \mathcal{M} and on \mathcal{S} are given respectively by:

$$(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})Z_{\lambda} = R_{\mu\nu\lambda\sigma}Z^{\sigma} \tag{50}$$

$$(\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a) z_c = \mathcal{R}_{abcd} z^d, \tag{51}$$

where \mathcal{D} is the metric compatible derivative operator on \mathcal{S} , so that $\mathcal{D}_a h_{bc} = 0$. The Gauss equation for the spacetime and submanifold gives the following equation:

$$e^{\mu}{}_{a}e^{\nu}{}_{b}e^{\lambda}{}_{c}e^{\sigma}{}_{d}R_{\mu\nu\lambda\sigma} = \mathcal{R}_{abcd} - (k^{(\ell)}{}_{ac}k^{(n)}{}_{bd} + k^{(n)}{}_{ac}k^{(\ell)}{}_{bd}) + (k^{(\ell)}{}_{ad}k^{(n)}{}_{bc} + k^{(n)}{}_{ad}k^{(\ell)}{}_{bc}), \tag{52}$$

and the Codazzi equations may be written in the following forms corresponding to each of the two normals:

$$e^{\mu}{}_{a}e^{\nu}{}_{b}e^{\lambda}{}_{c}\ell^{\sigma}R_{\mu\nu\lambda\sigma} = (D_{b} - \omega_{b})k^{(\ell)}{}_{ac} - (D_{a} - \omega_{a})k^{(\ell)}{}_{bc}$$
(53)

$$e^{\mu}{}_{a}e^{\nu}{}_{b}e^{\lambda}{}_{c}n^{\sigma}R_{\mu\nu\lambda\sigma} = (D_{b} - \omega_{b})k^{(n)}{}_{ac} - (D_{a} - \omega_{a})k^{(n)}{}_{bc},$$
 (54)

where $\omega_{\underline{a}} \equiv e^{\mu}{}_{a} \omega_{\mu}$ is the pullback of the connection on the normal bundle $\mathcal{N}(\mathcal{S})$, and is defined using the equation for the Shape operator to get: $\omega_{a} = -n_{\sigma} e^{\lambda_{a}} \nabla_{\lambda} \ell^{\sigma}$.

The variation of the submanifold S in the normal direction $N^{\mu} = A\ell^{\mu} - Bn^{\mu}$, A and B being constants, is given by the variation of the abovementioned spacetime variables. The variation in the induced metric is:

$$\nabla_N h_{ab} = 2Ak_{ab}^{(\ell)} - 2Bk_{ab}^{(n)} \tag{55}$$

whereas, the variation of the area element $\sqrt{h} = \sqrt{\det h_{ab}}$ is given by:

$$\nabla_N \sqrt{h} = (1/2)\sqrt{h} \, h^{ab} \, \nabla_N h_{ab} = (A\theta_{(\ell)} - B\theta_{(n)})\sqrt{h}. \tag{56}$$

The extrinsic curvatures are written in terms of the expansion scalar and the shear tensors of the two null normals:

$$k^{(\ell)}{}_{ab} = \frac{1}{(D-2)} \theta_{(\ell)} h_{ab} + \sigma^{(\ell)}_{(ab)}, \quad k^{(n)}{}_{ab} = \frac{1}{(D-2)} \theta_{(n)} h_{ab} + \sigma^{(n)}_{(ab)}$$
(57)

where the expansion scalar and the shear tensors are defined as:

$$\theta_{(\ell)} = \nabla_{\mu} \ell^{\mu} - \kappa_{(\ell)} \tag{58}$$

$$\sigma_{ab}^{(\ell)} = \left[e^{\mu}{}_{a} e^{\nu}{}_{b} - \frac{h_{ab}}{(D-2)} g^{\mu\nu} \right] \nabla_{\mu} \ell_{\nu} + \kappa_{(\ell)} h_{ab}, \tag{59}$$

where $\kappa_{(\ell)} = -n_{\nu}\ell^{\mu} \nabla_{\mu}\ell^{\nu}$ is the measure of affinity of the null normal. These equations for the other null-normal n^{μ} is obtained by $\ell^{\mu} \leftrightarrow n^{\mu}$.

Let us now consider how the foliation is evolved along N^{μ} . Since ℓ_{μ} and n_{μ} are normal to \mathcal{S} , their pullback on \mathcal{S} vanish. Thus, $e^{\mu}{}_{a}\ell_{\mu} \equiv \ell_{\underline{\mu}} = 0$, and also the same is true for $n_{\underline{\mu}}$. This foliation is assumed to be preserved in the evolution under N^{μ} , so that $(\pounds_{N}\ell)_{\underline{\mu}} = 0$, and $(\pounds_{N}n)_{\underline{\mu}} = 0$ is assumed to hold true. These equations imply that:

$$N^{\mu}\nabla_{\mu}\ell_{\underline{a}} = \kappa_{(N)}\ell_{\underline{a}} - (D_{\underline{a}} - \omega_{\underline{a}})B, \tag{60}$$

$$N^{\mu}\nabla_{\mu}n_{\underline{\mu}} = -\kappa_{(N)}n_{\underline{\mu}} + (D_{\underline{\mu}} + \omega_{\underline{\mu}})B, \tag{61}$$

where $\kappa_{(N)} = -n_{\mu}N^{\nu}\nabla_{\nu}\ell^{\mu}$ is called the surface gravity corresponding to the vector field N^{μ} . A direct calculation leads to the following results on the variation of $\theta_{(\ell)}$ [50]

$$\nabla_N \theta_{(\ell)} - \kappa_N \theta_{(\ell)} = -d^2 B + 2\omega^{\mu} d_{\mu} B - B[\omega^{\mu} \omega_{\mu} - d_{\mu} \omega^{\mu} - (R/2) - G_{\mu\nu} \ell^{\mu} n^{\nu} - \theta_{(\ell)} \theta_n] - A[\sigma_{(\ell)}^2 + G_{\mu\nu} \ell^{\mu} \ell^{\nu} + (1/2) \theta_{(\ell)}^2].$$
(62)

C. Matching conditions at shell boundary

In the following, we present the junction condition of a LTB metric, formed due to collapse of a spherically symmetric matter configuration, with the spherically symmetric due to a body of mass M. The interior LTB metric of the spacetime \mathcal{M}^- is given by eqn. (25):

$$ds_{-}^{2} = -dt^{2} + \frac{R'^{2}}{1 - k(r)} dr^{2} + R(r, t)^{2} d\Omega_{3},$$
(63)

where $d\Omega_3$ is the metric of an unit round 3-sphere, and R(t,r) is obtained from the equation (13). The metric of the external spacetime \mathcal{M}_+ is the Boulware-Deser-Wheeler solution [24, 30–32], which for 5- dimensions is given by:

$$ds_{+}^{2} = -F(\bar{R}) dT^{2} + F(\bar{R})^{-1} d\bar{R}^{2} + \bar{R}^{2} d\Omega_{3}, \tag{64}$$

where T and \bar{R} are the time and radial coordinates in \mathcal{M}_+ , and the metric function $F(\bar{R})$ is:

$$F(\bar{R}) = 1 + \frac{\bar{R}^2}{4\lambda} \left[1 \mp \sqrt{1 + \frac{8\lambda M}{\bar{R}^4}} \right] \tag{65}$$

gives the external vacuum solution for a spherical body of mass M when the -ve sign is chosen.

The matching is to be carried out at the timelike hypersurface Σ given by r_b . Let us denote the coordinates on this surface Σ to be $(\tau, \theta, \phi, \psi)$. From \mathcal{M}^- , we can write down the surface Σ as $f_-(r,t) = r - r_b = 0$, and hence, the induced metric on Σ is

$$ds_{-}^{2} = -d\tau^{2} + r_{b}^{2} d\Omega_{3}. {(66)}$$

From the point of view of the exterior spacetime, the hypersurface may be described by $r = \bar{R}_{\Sigma}(\tau)$ and $t = T_{\Sigma}(\tau)$, with no change in the angular variables. The line element of the hypersurface is then given by

$$ds_{+}^{2} = -\left[F(\bar{R}_{\Sigma})\dot{T}_{\Sigma}^{2} - F(\bar{R}_{\Sigma})^{-1}\dot{\bar{R}}_{\Sigma}^{2}\right]d\tau^{2} + \bar{R}_{\Sigma}(\tau)^{2}d\Omega_{3},$$
(67)

where the dots imply derivative with respect to τ .

The induced metric in equations in (66) and (67) must have matched metric functions. This implies that:

$$F(\bar{R}_{\Sigma})\dot{T}_{\Sigma}^{2} - F(\bar{R}_{\Sigma})^{-1}\dot{\bar{R}}_{\Sigma}^{2} = 1 \tag{68}$$

Now, let u^{μ} and n^{μ} denote the velocity of the matter variables and the normal to the Σ respectively. They must satisfy the conditions $u^{\mu}u_{\mu} = -1$, $n^{\mu}n_{\mu} = 1$, whereas, $u^{\mu}n_{\mu} = 0$. From the interior spacetime, the expressions of these vectors is easily obtained:

$$u^{\mu} = \delta_0^{\mu} \equiv (\partial_{\tau})^{\mu}, \qquad n_{\mu} = \frac{R'}{\sqrt{1 - k(r)}} (dr)_{\mu}.$$
 (69)

From the exterior spacetime, these vectors are also obtained similarly to give:

$$u^{\mu} = \dot{T}_{\Sigma} (\partial_{\tau})^{\mu} + \dot{\bar{R}}_{\Sigma} (\partial_{r})^{\mu}, \qquad n_{\mu} = -\dot{\bar{R}}_{\Sigma} (d\tau)_{\mu} + \dot{T}_{\Sigma} (dr)_{\mu}. \tag{70}$$

The extrinsic curvatures are easily determined from these normals for the exterior as well the interior spacetimes:

$$K_{\tau\tau}^{-} = 0, \qquad K_{\theta\theta}^{-} = \bar{R}_{\Sigma} \sqrt{1 - k(r_b)}$$
 (71)

$$K_{\tau\tau}^{+} = \dot{\bar{R}}_{\Sigma}^{-1} [\dot{F}(\bar{R}_{\Sigma}) \dot{T}_{\Sigma} + F(\bar{R}_{\Sigma}) \ddot{T}_{\Sigma}] \qquad K_{\theta\theta}^{+} = \bar{R}_{\Sigma} F(\bar{R}_{\Sigma}) \dot{T}_{\Sigma}. \tag{72}$$

The $K_{\theta\theta}$ equations imply the following relation:

$$\frac{dT_{\Sigma}}{d\tau} = \frac{\sqrt{1 - k(r_b)}}{F(\bar{R}_{\Sigma})},\tag{73}$$

whereas the equation (68) gives the following equation for the function \bar{R}_{Σ} :

$$\frac{d\bar{R}_{\Sigma}}{d\tau} = [1 - k(r_b) - F(\bar{R}_{\Sigma})]^{1/2}.$$
 (74)

This implies that the following relation hold good:

$$\left(\frac{d\bar{R}_{\Sigma}}{d\tau}\right)^{2} = -k(r_{b}) + \frac{\bar{R}_{\Sigma}^{2}}{4\lambda} \left[1 \mp \sqrt{1 + \frac{8\lambda M}{\bar{R}_{\Sigma}^{4}}} \right].$$
(75)

A simple comparison with equation (26) implies that the condition $M = F(r_b)$ must be satisfied at the boundary.

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