Invariant differential operators for the Jacobi algebra \mathcal{G}_2

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Abstract

In the present paper we construct explicitly the intertwining differential operators for the Jacobi algebra \mathcal{G}_2 . For the construction we use the singular vectors of the Verma modules over \mathcal{G}_2 which we have constructued earlier. We construct the function spaces on which the operators act. We find two versions of the left (representation) action and the right action. These actions are combined with the singular vectors to provide the intertwining differential operators.

1 Introduction

Consider a Lie group G, e.g., the Lorentz, Poincaré, conformal groups, and differential equations

$$\mathcal{I} f = j \tag{1.1}$$

which are G-invariant. These play a very important role in the description of physical symmetries - recall, e.g., the early examples of Dirac, Maxwell, d'Allembert, equations and nowadays the latest applications of (super-)differential operators in conformal field theory, supergravity, string theory, see e.g. [1]. Naturally, it is important to construct systematically such invariant equations and operators.

To recall the notions, consider a Lie group G and two representations T, T' acting in the representation spaces C, C', which may be Hilbert, Fréchet, etc. An *invariant* (or intertwining) operator \mathcal{I} for these two representations is a continuous linear map

$$\mathcal{I}: C \longrightarrow C' \tag{1.2}$$

such that

$$T'(g) \circ \mathcal{I} = \mathcal{I} \circ T(g) , \quad \forall g \in G .$$
 (1.3)

Then we say that the equation (1.1) is a G - invariant equation. Note that $\ker \mathcal{I}$, $\operatorname{im} \mathcal{I}$ are invariant subspaces of C, C', resp.

If G is semisimple then there exist canonical ways for the construction of the intertwining differential operators, cf., e.g., [2, 3]. In this method there is a correspondence between invariant differential operators and singular vectors of Verma modules over the (complexified) Lie algebra in consideration.

The procedure may be applied for more general classes of Lie groups. For instance, it was applied to the Schrödinger group [4, 5] in, e.g., [6, 7].

This is what we try to do in the present paper for the case of \mathcal{G}_2 .

2 **Preliminaries**

The procedure that we shall follow requires first that we find the singular vectors of the Verma modules over \mathcal{G}_2 . This task was fulfilled in [8]. Furthermore there are given all necessary details, and thus, we can present the preliminaries in a shorter fashion.

The Jacobi algebra is the semi-direct sum $\mathcal{G}_n := \mathcal{H}_n \ni sp(n,\mathbb{R})_{\mathbb{C}}$ [9, 10]. The Heisenberg algebra \mathcal{H}_n is generated by the boson creation (respectively, annihilation) operators a_i^+ (a_i^-) , $i, j = 1, \ldots, n$, which verify the canonical commutation relations

$$[a_i^-, a_j^+] = \delta_{ij}, \qquad [a_i^-, a_j^-] = [a_i^+, a_j^+] = 0.$$
 (2.1)

 \mathcal{H}_n is an ideal in \mathcal{G}_n , i.e., $[\mathcal{H}_n, \mathcal{G}_n] = \mathcal{H}_n$, determined by the commutation relations (following the notation of [11]):

$$\left[a_k^+, K_{ij}^+\right] = \left[a_k^-, K_{ij}^-\right] = 0,$$
 (2.2a)

$$[a_i^-, K_{kj}^+] = \frac{1}{2}\delta_{ik}a_j^+ + \frac{1}{2}\delta_{ij}a_k^+, \qquad [K_{kj}^-, a_i^+] = \frac{1}{2}\delta_{ik}a_j^- + \frac{1}{2}\delta_{ij}a_k^-, \qquad (2.2b)$$

$$\left[K_{ij}^{0}, a_{k}^{+}\right] = \frac{1}{2}\delta_{jk}a_{i}^{+}, \qquad \left[a_{k}^{-}, K_{ij}^{0}\right] = \frac{1}{2}\delta_{ik}a_{j}^{-}. \tag{2.2c}$$

 $K_{ij}^{\pm,0}$ are the generators of the $\mathcal{S}_n \equiv sp(n,\mathbb{R})_{\mathbb{C}}$ algebra:

$$[K_{ij}^{-}, K_{kl}^{-}] = [K_{ij}^{+}, K_{kl}^{+}] = 0, 2[K_{ij}^{-}, K_{kl}^{0}] = K_{il}^{-} \delta_{kj} + K_{il}^{-} \delta_{ki}, (2.3a)$$

$$2[K_{ii}^{-}, K_{kl}^{+}] = K_{ki}^{0} \delta_{li} + K_{li}^{0} \delta_{ki} + K_{ki}^{0} \delta_{li} + K_{li}^{0} \delta_{ki}$$
(2.3b)

$$2[K_{ij}^+, K_{kl}^0] = -K_{ik}^+ \delta_{jl} - K_{jk}^+ \delta_{li}, \quad 2[K_{ji}^0, K_{kl}^0] = K_{jl}^0 \delta_{ki} - K_{ki}^0 \delta_{lj}.$$
 (2.3c)

First, for simplicity, we introduce the following notations for the basis of S_2 :

$$S^{+}: b_{i}^{+} \equiv K_{ii}^{+}, i = 1, 2; c^{+} \equiv K_{12}^{+}, d^{+} \equiv K_{12}^{0}$$

$$S^{-}: b_{i}^{-} \equiv K_{ii}^{-}, i = 1, 2; c^{-} \equiv K_{12}^{-}, d^{-} \equiv K_{21}^{0}$$

$$(2.4a)$$

$$K: h_{i} \equiv K_{ii}^{0}, i = 1, 2.$$

$$(2.4c)$$

$$S^-: b_i^- \equiv K_{ii}^-, i = 1, 2; c^- \equiv K_{12}^-, d^- \equiv K_{21}^0$$
 (2.4b)

$$\mathcal{K} : h_i \equiv K_{ii}^0, \quad i = 1, 2.$$
 (2.4c)

We need also the triangular decomposition of \mathcal{G}_2 :

$$\mathcal{G}_{2}^{+} := \text{l.s.} \{ a_{i}^{+}, b_{i}^{+}, c^{+}, d^{+} \}, i = 1, 2,
\mathcal{G}_{2}^{-} := \text{l.s.} \{ a_{i}^{-}, b_{i}^{-}, c^{-}, d^{-} \}, i = 1, 2,
\mathcal{K}_{2} := \text{l.s.} \{ h_{i}, 1 \}, i = 1, 2.$$
(2.5)

Clearly, the Abelian subalgebra \mathcal{K} is a Cartan subalgebra of \mathcal{S}_2 . Furthermore, \mathcal{K} plays the role of Cartan subalgebra for the whole algebra. Thus, we may treat the elements of \mathcal{G}_2^{\pm} as root subspaces w.r.t. \mathcal{K} . This may be explicated by the eigenvalues w.r.t. (h_1, h_2) as follows:

$$a_1^{\pm} : \pm (\frac{1}{2}, 0), \quad a_2^{\pm} : \pm (0, \frac{1}{2}),$$
 (2.6)
 $b_1^{\pm} : \pm (1, 0), \quad b_2^{\pm} : \pm (0, 1), \quad c^{\pm} : \pm (\frac{1}{2}, \frac{1}{2}), \quad d^{\pm} : \pm (\frac{1}{2}, -\frac{1}{2})$

We may also introduce the analogs of simple roots α_1, α_2 which would correspond here to the generators d^+, a_2^+ , resp. Then the correspondence generators \longleftrightarrow roots would be:

$$(b_1^+, b_2^+, c^+, d^+, a_1^+, a_2^+) \longleftrightarrow (2(\alpha_1 + \alpha_2), 2\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1, \alpha_1 + \alpha_2, \alpha_2)$$
 (2.7)

We consider the lowest weight Verma modules over \mathcal{G}_2 and found a complete list of singular vectors. There are five types of singular vectors and they exist in Verma modules with a particular value of the lowest weight. Explicit formula of them is found in §4.1 of [8].

For the explicit construction of the intertwining differential operators we need a parameter space. That would be some coset space of the Jacobi group G as generated by \mathcal{G}_2 . Then we need its triangular decomposition $G = G^+KG^-$ and Borel subgroup $B = KG^-$.

Now we can define the space of the right covariant functions:

$$C_{\Lambda} = \{ \mathcal{F} \in C^{\infty}(G) \mid \mathcal{F}(gkg^{-}) = e^{\Lambda(H)}\mathcal{F}(g) \}$$
 (2.8)

where $g \in G$, $k = e^H \in K$, $g^- \in G^-$, $H \in \mathcal{K}$, $\Lambda \in \mathcal{K}^*$. Thus the functions of \mathcal{C}_{Λ} are actually functions on G/B, or locally on G^+ .

Correspondingly we define the right action of \mathcal{G}_2 on \mathcal{C}_{Λ} :

$$(\pi_R(X)\mathcal{F})(g) \doteq \frac{d}{dt}\mathcal{F}(g\exp(tX))\Big|_{t=0}, \qquad X \in \mathcal{G}_2, \ g \in G$$
 (2.9)

and the left action of \mathcal{G}_2

$$(\pi_L(X)\mathcal{F})(g) \doteq \frac{d}{dt}\mathcal{F}(\exp(-tX)g)\Big|_{t=0}, \qquad X \in \mathcal{G}_2, \ g \in G$$
 (2.10)

In the next section we present these construction in explicit detail.

3 Right and left actions on G^+

3.1 Right action

For the elements q^+ of G^+ we write:

$$g^{+} = \exp\left(x_1 a_1^{\dagger} + x_2 a_2^{\dagger}\right) \exp\left(y_1 b_1^{+} + y_2 b_2^{+} + z c^{+} + w d^{+}\right). \tag{3.1}$$

It is important that there are only three non-vanishing relations among the generators of \mathcal{G}_2^+ :

$$[b_2^{\dagger}, d^{\dagger}] = -c^{\dagger}, \qquad [a_2^{\dagger}, d^{\dagger}] = -\frac{1}{2}a_1^{\dagger}, \qquad [c^{\dagger}, d^{\dagger}] = -\frac{1}{2}b_1^{\dagger}.$$
 (3.2)

Using these relations, it is easy to compute the right action of \mathcal{G}_2^+ :

$$\pi_{R}(a_{1}^{\dagger}) = \partial_{x_{1}},$$

$$\pi_{R}(a_{2}^{\dagger}) = \partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}},$$

$$\pi_{R}(b_{1}^{\dagger}) = \partial_{y_{1}},$$

$$\pi_{R}(b_{2}^{\dagger}) = \partial_{y_{2}} + \frac{w^{2}}{24}\partial_{y_{1}} + \frac{w}{2}\partial_{z},$$

$$\pi_{R}(c^{\dagger}) = \partial_{z} + \frac{w}{4}\partial_{y_{1}},$$

$$\pi_{R}(d^{\dagger}) = \partial_{w} - \frac{1}{4}\left(z + \frac{y_{2}w}{6}\right)\partial_{y_{1}} - \frac{y_{2}}{2}\partial_{z}.$$
(3.3)

As an example, we show the computation of $\pi_R(a_2^{\dagger})$. First, noting (3.2) one may have

$$g^{+}e^{ta_{2}^{\dagger}} = e^{A} e^{ta_{2}^{\dagger}} e^{-ta_{2}^{\dagger}} e^{B} e^{ta_{2}^{\dagger}}$$

$$= \exp(A + ta_{2}^{\dagger}) \exp\left(B + \frac{tw}{2}a_{1}^{\dagger}\right)$$

$$= \exp\left(A + t\left(a_{2}^{\dagger} + \frac{w}{2}a_{1}^{\dagger}\right)\right) \exp(B)$$
(3.4)

where $A := x_1 a_1^{\dagger} + x_2 a_2^{\dagger}, B := y_1 b_1^{\dagger} + y_2 b_2^{\dagger} + z c^{\dagger} + w d^{\dagger}$. It follows that

$$\left(\pi_R(a_2^{\dagger})\mathcal{F})(g^+) = \frac{d}{dt}\mathcal{F}(g^+e^{ta_2^{\dagger}})\right|_{t=0} = \left(\partial_{x_2} + \frac{w}{2}\partial_{x_1}\right)\mathcal{F}(g^+). \tag{3.5}$$

3.2 Left action

The left action is computed by using the Baker-Campbell-Hausdorff formula:

$$\ln e^{X} e^{Y} = X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ((\operatorname{ad} X)^{2} (Y) + (\operatorname{ad} Y)^{2} (X))$$
$$- \frac{1}{24} [Y, [X, [X, Y]]] - \frac{1}{720} ((\operatorname{ad} Y)^{4} (X) + (\operatorname{ad} X)^{4} (Y)) + \cdots$$
(3.6)

where adX(Y) := [X, Y]. We here present the final results and omit the computational details.

Left action of \mathcal{G}_2^+ :

$$\pi_{L}(a_{1}^{\dagger}) = -\partial_{x_{1}},
\pi_{L}(a_{2}^{\dagger}) = -\partial_{x_{2}},
\pi_{L}(b_{1}^{\dagger}) = -\partial_{y_{1}},
\pi_{L}(b_{2}^{\dagger}) = -\partial_{y_{2}} - \frac{w^{2}}{24}\partial_{y_{1}} + \frac{w}{2}\partial_{z},
\pi_{L}(c^{\dagger}) = -\partial_{z} + \frac{w}{4}\partial_{y_{1}},
\pi_{L}(d^{\dagger}) = -\partial_{w} - \frac{x_{2}}{2}\partial_{x_{1}} - \frac{1}{4}\left(z - \frac{y_{2}w}{6}\right)\partial_{y_{1}} - \frac{y_{2}}{2}\partial_{z}.$$
(3.7)

Left action of \mathcal{K}_2 :

$$\pi_{L}(h_{1}) = -\frac{x_{1}}{2}\partial_{x_{1}} - y_{1}\partial_{y_{1}} - \frac{z}{2}\partial_{z} - \frac{w}{2}\partial_{w} - \Lambda(h_{1}),$$

$$\pi_{L}(h_{2}) = -\frac{x_{2}}{2}\partial_{x_{2}} - y_{2}\partial_{y_{2}} - \frac{z}{2}\partial_{z} + \frac{w}{2}\partial_{w} - \Lambda(h_{2}),$$

$$\pi_{L}(1) = -\hat{\Lambda}$$
(3.8)

where $\hat{\Lambda}$ is the value of the central element of the Heisenberg algebra: $[a_i^-, a_j^+] = \delta_{ij} 1$.

Left action of \mathcal{G}_2^- :

$$\pi_{L}(a_{1}^{-}) = -\left(y_{1} + \frac{wz}{4} + \frac{y_{2}w^{2}}{12}\right) \partial_{x_{1}} - \frac{1}{2}\left(z + \frac{y_{2}w}{2}\right) \partial_{x_{2}} - x_{1}\hat{\Lambda},$$

$$\pi_{L}(a_{2}^{-}) = -\frac{1}{2}\left(z + \frac{y_{2}w}{2}\right) \partial_{x_{1}} - y_{2} \partial_{x_{2}} - x_{2}\hat{\Lambda},$$

$$\pi_{L}(b_{1}^{-}) = x_{1}\pi_{L}(a_{1}^{-}) - \left(y_{1}^{2} + \frac{w^{2}}{96}\left(z^{2} + y_{2}wz + \frac{y_{2}^{2}w^{2}}{12}\right)\right) \partial_{y_{1}} - \frac{1}{4}\left(z + \frac{y_{2}w}{2}\right)^{2} \partial_{y_{2}}$$

$$- \left(y_{1}z + \frac{w}{8}\left(z^{2} + \frac{2y_{2}wz}{3} + \frac{y_{2}^{2}w^{2}}{4}\right)\right) \partial_{z} - w\left(y_{1} - \frac{y_{2}w^{2}}{24}\right) \partial_{w}$$

$$+ \frac{x_{1}^{2}}{2}\hat{\Lambda} - 2\left(y_{1} - \frac{y_{2}w^{2}}{24}\right) \Lambda(h_{1}) - \frac{w}{2}\left(z + \frac{y_{2}w}{2}\right) \Lambda(h_{2}),$$

$$\pi_{L}(b_{2}^{-}) = x_{2}\pi_{L}(a_{2}^{-}) - \frac{y_{2}wz}{12} \partial_{y_{1}} - y_{2}^{2} \partial_{y_{2}}$$

$$- \frac{y_{2}}{2}\left(z + \frac{y_{2}w}{2}\right) \partial_{z} - \left(z - \frac{y_{2}w}{2}\right) \partial_{w} + \frac{x_{2}^{2}}{2}\hat{\Lambda} - 2y_{2}\Lambda(h_{2}),$$

$$\pi_{C}^{-}) = \frac{x_{2}}{2}\pi_{L}(a_{1}^{-}) + \frac{x_{1}}{2}\pi_{L}(a_{2}^{-})$$

$$- \frac{1}{4}\left(y_{1}\left(z - \frac{y_{2}w}{6}\right) + \frac{y_{2}w}{8}\left(wz + \frac{y_{2}w^{2}}{18}\right)\right) \partial_{y_{1}} - \frac{y_{2}}{2}\left(z + \frac{y_{2}w}{2}\right) \partial_{y_{2}}$$

$$- \frac{1}{2}\left(y_{2}\left(y_{1} + \frac{wz}{4} + \frac{5y_{2}w^{2}}{24}\right) + \frac{z^{2}}{2}\right) \partial_{z} - \left(y_{1} + \frac{wz}{4} - \frac{y_{2}w^{2}}{6}\right) \partial_{w}$$

$$+ \frac{x_{1}x_{2}}{2}\hat{\Lambda} - \frac{1}{2}\left(z - \frac{y_{2}w}{2}\right)\Lambda(h_{1}) - \frac{1}{2}\left(z + \frac{3y_{2}w}{2}\right)\Lambda(h_{2}),$$

$$\pi_{L}(d^{-}) = -\frac{x_{1}}{2}\partial_{x_{2}} + \frac{w}{4}\left(y_{1} - \frac{wz}{12}\right)\partial_{y_{1}} - \frac{1}{2}\left(z + \frac{y_{2}w}{2}\right)\partial_{y_{2}}$$

$$- \left(y_{1} + \frac{y_{2}w^{2}}{12}\right)\partial_{z} + \frac{w^{2}}{4}\partial_{w} + \frac{w}{2}(\Lambda(h_{1}) - \Lambda(h_{2})).$$
(3.9)

It has been verified by direct computation (with MAPLE) that the left action given above is compatible with the defining commutation relations of \mathcal{G}_2 .

4 Invariant differential operators: first version

First we give the list of singular vectors that were found in [8]. We denote the lowest weight vector of the Verma module by $|0\rangle$ and the lowest weight by $\Lambda_k = \Lambda(h_k)$. The parameters p^k and q^3 take a positive integer and the weight of the singular vector is denoted by Λ' .

(i)
$$\Lambda_1 - \Lambda_2 = \frac{1}{2}(1 - p^1)$$

$$\left| v_s^{\Lambda'} \right\rangle = (d^+)^{p^1} \left| 0 \right\rangle, \quad \Lambda' = \Lambda + p^1(\delta_1 - \delta_2). \tag{4.1}$$

(ii) for all
$$\Lambda_1$$
, $\Lambda_2 = \frac{3}{4} - \frac{p^2}{2}$

$$\left| v_s^{\Lambda'} \right\rangle = (\hat{b}_2^+)^{p^2} \left| 0 \right\rangle, \quad \Lambda' = \Lambda + 2p^2 \delta_2. \tag{4.2}$$

(iii)
$$\Lambda_1 = \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \ \Lambda_2 = \frac{3}{4} - \frac{1}{2}q^3, \ (p^3 \neq q^3, p^3 \neq 2q^3)$$

(a) $p^3 < q^3$

$$\left|v_s^{\Lambda'}\right\rangle = \sum_{k=0}^{\left\lfloor p^3/2\right\rfloor} \sum_{n=0}^{p^3-2k} c(k,n) \left|k, q^3 - k - n, n, p^3 - 2k - n\right\rangle \tag{4.3}$$

(b) $q^3 < p^3 < 2q^3$

$$\left|v_{s}^{\Lambda'}\right\rangle = \left(\sum_{k=0}^{p^{3}-q^{3}} \sum_{n=0}^{q^{3}-k} + \sum_{k=p^{3}-q^{3}+1}^{\lfloor p^{3}/2 \rfloor} \sum_{n=0}^{p^{3}-2k} \right) c(k,n) \left|k, q^{3}-k-n, n, p^{3}-2k-n\right\rangle$$
(4.4)

(c) $2q^3 < p^3$

$$\left|v_s^{\Lambda'}\right\rangle = \sum_{k=0}^{q^3} \sum_{n=0}^{q^3-k} c(k,n) \left|k, q^3 - k - n, n, p^3 - 2k - n\right\rangle$$
 (4.5)

where Λ' and c(k, n) are common for (a) (b) (c) and given by

$$\Lambda' = \Lambda + p^3 \delta_1 + (2q^3 - p^3) \delta_2, \tag{4.6}$$

$$c(k,n) = \frac{p^3!q^3!}{4^k k! n! (p^3 - 2k - n)! (q^3 - k - n)!}.$$
(4.7)

(iv)
$$\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2}$$

$$\left|v_s^{\Lambda'}\right\rangle = \sum_{k=0}^{\left\lfloor p^4/2\right\rfloor} \sum_{n=0}^{p^4-2k} c(k,n) \left|k, p^4 - k - n, n, p^4 - 2k - n\right\rangle, \quad \Lambda' = \Lambda + p^4(\delta_1 + \delta_2),$$
(4.8)

$$c(k,n) = \frac{p^4!}{4^k k! n! (p^4 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2})}{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2} - k - n)}.$$
 (4.9)

(v) $\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}$, for all Λ_2

$$\left|v_s^{\Lambda'}\right\rangle = \sum_{k=0}^{p^5} \sum_{n=0}^{p^5-k} c(k,n) \left|k, p^5 - k - n, n, 2p^5 - 2k - n\right\rangle, \quad \Lambda' = \Lambda + 2p^5 \delta_1, \quad (4.10)$$

$$c(k,n) = \frac{(-1)^n}{4^k k! n!} \frac{p^5!}{(p^5 - k - n)!} \frac{\Gamma(2\Lambda_2 - p^5 - \frac{3}{2} + 2k + n)}{\Gamma(2\Lambda_2 - p^5 - \frac{3}{2})}.$$
 (4.11)

Note that there is a change of basis w.r.t. [8], namely:

$$|k, \ell, n, m\rangle := (\hat{b}_1^+)^k (\hat{b}_2^+)^\ell (\hat{c}^+)^n (d^+)^m |0\rangle$$
 (4.12)

and

$$\hat{b}_k^+ := b_k^+ - \frac{1}{2} (a_k^+)^2, \qquad \hat{c}^+ := c^+ - \frac{1}{2} a_1^+ a_2^+.$$
 (4.13)

Then using the right action obtained in $\S 3$ we have using (4.13):

$$\pi_{R}(\hat{b}_{1}^{+}) = \partial_{y_{1}} - \frac{1}{2}\partial_{x_{1}}^{2},$$

$$\pi_{R}(\hat{b}_{2}^{+}) = \partial_{y_{2}} + \frac{w^{2}}{24}\partial_{y_{1}} + \frac{w}{2}\partial_{z} - \frac{1}{2}\left(\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}\right)^{2},$$

$$\pi_{R}(\hat{c}^{+}) = \partial_{z} + \frac{w}{4}\partial_{y_{1}} - \frac{1}{2}\left(\partial_{x_{1}}\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}^{2}\right).$$

$$(4.14)$$

Thus the invariant differential operators are given by substituting the above right action in the singular vectors given above:

(i)
$$\Lambda_1 - \Lambda_2 = \frac{1}{2}(1 - p^1)$$

$$\tilde{\mathcal{D}}_{(i)} = \left(\partial_w - \frac{1}{4}\left(z + \frac{y_2w}{6}\right)\partial_{y_1} - \frac{y_2}{2}\partial_z\right)^{p^1} \tag{4.15}$$

(ii) for all Λ_1 , $\Lambda_2 = \frac{3}{4} - \frac{p^2}{2}$

$$\tilde{\mathcal{D}}_{(ii)} = \left(\partial_{y_2} + \frac{w^2}{24}\partial_{y_1} + \frac{w}{2}\partial_z - \frac{1}{2}\left(\partial_{x_2} + \frac{w}{2}\partial_{x_1}\right)^2\right)^{p^2}$$
(4.16)

(iii)
$$\Lambda_1 = \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \ \Lambda_2 = \frac{3}{4} - \frac{1}{2}q^3, \ (p^3 \neq q^3, p^3 \neq 2q^3)$$

(a)
$$p^3 < q^3$$

$$\tilde{\mathcal{D}}_{(iii,a)} = \sum_{k=0}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3-2k} c(k,n) \, \mathcal{P}(p^3, q^3, k, n)$$
(4.17)

(b)
$$q^3 < p^3 < 2q^3$$

$$\tilde{\mathcal{D}}_{(iii,b)} = \left(\sum_{k=0}^{p^3 - q^3} \sum_{n=0}^{q^3 - k} + \sum_{k=p^3 - q^3 + 1}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} c(k,n) \, \mathcal{P}(p^3, q^3, k, n) \right)$$
(4.18)

(c)
$$2q^3 < p^3$$

$$\tilde{\mathcal{D}}_{(iii,c)} = \sum_{k=1}^{q^3} \sum_{n=1}^{q^3-k} c(k,n) \, \mathcal{P}(p^3, q^3, k, n)$$
(4.19)

where $\mathcal{P}(p^3, q^3, k, n)$ and c(k, n) are common for (a) (b) (c) and given by

$$\mathcal{P}(p^{3}, q^{3}, k, n) = \left(\partial_{y_{1}} - \frac{1}{2}\partial_{x_{1}}^{2}\right)^{k} \left(\partial_{y_{2}} + \frac{w^{2}}{24}\partial_{y_{1}} + \frac{w}{2}\partial_{z} - \frac{1}{2}\left(\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}\right)^{2}\right)^{q^{3}-k-n} \times \left(\partial_{z} + \frac{w}{4}\partial_{y_{1}} - \frac{1}{2}\left(\partial_{x_{1}}\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}^{2}\right)\right)^{n} \times \left(\partial_{w} - \frac{1}{4}\left(z + \frac{y_{2}w}{6}\right)\partial_{y_{1}} - \frac{y_{2}}{2}\partial_{z}\right)^{p^{3}-2k-n},$$

$$c(k, n) = \frac{p^{3}!q^{3}!}{4^{k}k!n!(p^{3} - 2k - n)!(q^{3} - k - n)!}.$$

$$(4.21)$$

(iv) $\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2}$

$$\tilde{\mathcal{D}}_{(iv)} = \sum_{k=0}^{\lfloor p^4/2 \rfloor} \sum_{n=0}^{p^4-2k} \frac{p^4!}{4^k k! n! (p^4 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2})}{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2} - k - n)} \\
\times \left(\partial_{y_1} - \frac{1}{2} \partial_{x_1}^2\right)^k \left(\partial_{y_2} + \frac{w^2}{24} \partial_{y_1} + \frac{w}{2} \partial_z - \frac{1}{2} \left(\partial_{x_2} + \frac{w}{2} \partial_{x_1}\right)^2\right)^{p^4 - k - n} \\
\times \left(\partial_z + \frac{w}{4} \partial_{y_1} - \frac{1}{2} \left(\partial_{x_1} \partial_{x_2} + \frac{w}{2} \partial_{x_1}^2\right)\right)^n \\
\times \left(\partial_w - \frac{1}{4} \left(z + \frac{y_2 w}{6}\right) \partial_{y_1} - \frac{y_2}{2} \partial_z\right)^{p^4 - 2k - n} \tag{4.22}$$

(v) $\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}$, for all Λ_2

$$\tilde{\mathcal{D}}_{(v)} = \sum_{k=0}^{p^{5}} \sum_{n=0}^{p^{5}-k} \frac{(-1)^{n}}{4^{k}k!n!} \frac{p^{5}!}{(p^{5}-k-n)!} \frac{\Gamma(2\Lambda_{2}-p^{5}-\frac{3}{2}+2k+n)}{\Gamma(2\Lambda_{2}-p^{5}-\frac{3}{2})} \times \left(\partial_{y_{1}} - \frac{1}{2}\partial_{x_{1}}^{2}\right)^{k} \left(\partial_{y_{2}} + \frac{w^{2}}{24}\partial_{y_{1}} + \frac{w}{2}\partial_{z} - \frac{1}{2}\left(\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}\right)^{2}\right)^{p^{5}-k-n} \times \left(\partial_{z} + \frac{w}{4}\partial_{y_{1}} - \frac{1}{2}\left(\partial_{x_{1}}\partial_{x_{2}} + \frac{w}{2}\partial_{x_{1}}^{2}\right)\right)^{n} \times \left(\partial_{w} - \frac{1}{4}\left(z + \frac{y_{2}w}{6}\right)\partial_{y_{1}} - \frac{y_{2}}{2}\partial_{z}\right)^{2p^{5}-2k-n}$$

$$(4.23)$$

5 Final expressions for the intertwining differential operators

To simplify our results we make the following change of parameters: $(x_1,x_2,y_1,y_2,z,w) \to (\xi_1,\xi_2,\eta_1,\eta_2,\zeta,\omega)$, namely:

$$\xi_1 = x_1 - \frac{w}{2}x_2,$$
 $\xi_2 = x_2,$

$$\eta_1 = y_1 + \frac{w^2}{12}y_2 - \frac{w}{4}z, \qquad \eta_2 = y_2, \qquad \zeta = z - \frac{w}{2}y_2, \qquad \omega = w.$$
 (5.1)

Inverse transform is given by

$$x_1 = \xi_1 + \frac{\omega}{2}\xi_2,$$
 $x_2 = \xi_2,$
 $y_1 = \eta_1 + \frac{\zeta\omega}{4} + \frac{\eta_2\omega^2}{24},$ $y_2 = \eta_2,$ $z = \zeta + \frac{\eta_2\omega}{2},$ $w = \omega.$ (5.2)

It follows that

$$\partial_{x_1} = \partial_{\xi_1}, \qquad \partial_{x_2} = \partial_{\xi_2} - \frac{\omega}{2} \partial_{\xi_1},
\partial_{y_1} = \partial_{\eta_1}, \qquad \partial_{y_2} = \partial_{\eta_2} + \frac{\omega^2}{12} \partial_{\eta_1} - \frac{\omega}{2} \partial_{\zeta}, \qquad \partial_z = \partial_{\zeta} - \frac{\omega}{4} \partial_{\eta_1},
\partial_w = \partial_{\omega} - \frac{1}{4} \left(\zeta - \frac{\eta_2 \omega}{6} \right) \partial_{\eta_1} - \frac{\eta_2}{2} \partial_{\zeta} - \frac{\xi_2}{2} \partial_{\xi_1}.$$
(5.3)

Then, the right action becomes:

$$\pi_R(a_k^+) = \partial_{\xi_k}, \qquad \pi_R(b_k^+) = \partial_{\eta_k}, \qquad \pi_R(c^+) = \partial_{\zeta},$$

$$\pi_R(d^+) = \partial_{\omega} - \frac{\zeta}{2} \partial_{\eta_1} - \eta_2 \partial_{\zeta} - \frac{\xi_2}{2} \partial_{\xi_1}.$$
 (5.4)

The left action is also simplified and now reads as follows:

$$\pi_{L}(a_{1}^{+}) = -\partial_{\xi_{1}}, \qquad \pi_{L}(a_{2}^{+}) = -\partial_{\xi_{2}} + \frac{\omega}{2}\partial_{\xi_{1}},
\pi_{L}(b_{1}^{+}) = -\partial_{\eta_{1}}, \qquad \pi_{L}(b_{2}^{+}) = -\partial_{\eta_{2}} - \frac{\omega^{2}}{4}\partial_{\eta_{1}} + \omega\partial_{\zeta},
\pi_{L}(c^{+}) = -\partial_{\zeta} + \frac{\omega}{2}\partial_{\eta_{1}}, \qquad \pi_{L}(d^{+}) = -\partial_{\omega},$$
(5.5)

$$\pi_{L}(h_{1}) = -\frac{\xi_{1}}{2}\partial_{\xi_{1}} - \eta_{1}\partial_{\eta_{1}} - \frac{\zeta}{2}\partial_{\zeta} - \frac{\omega}{2}\partial_{\omega} - \Lambda(h_{1}),$$

$$\pi_{L}(h_{2}) = -\frac{\xi_{2}}{2}\partial_{\xi_{2}} - \eta_{2}\partial_{\eta_{2}} - \frac{\zeta}{2}\partial_{\zeta} + \frac{\omega}{2}\partial_{\omega} - \Lambda(h_{2}),$$

$$\pi_{L}(1) = -\hat{\Lambda},$$
(5.6)

$$\pi_{L}(a_{1}^{-}) = -\left(\eta_{1} + \frac{\zeta\omega}{4}\right) \partial_{\xi_{1}} - \frac{1}{2}(\zeta + \eta_{2}\omega) \partial_{\xi_{2}} - \left(\xi_{1} + \frac{\xi_{2}\omega}{2}\right) \hat{\Lambda},$$

$$\pi_{L}(a_{2}^{-}) = -\frac{\zeta}{2} \partial_{\xi_{1}} - \eta_{2} \partial_{\xi_{2}} - \xi_{2} \hat{\Lambda},$$

$$\pi_{L}(b_{1}^{-}) = \left(\xi_{1} + \frac{\xi_{2}\omega}{2}\right) \pi_{L}(a_{1}^{-}) + \frac{\xi_{2}\omega}{2} \left(\eta_{1} + \frac{\zeta\omega}{4}\right) \partial_{\xi_{1}} - \left(\eta_{1}^{2} - \frac{\zeta^{2}\omega^{2}}{16}\right) \partial_{\eta_{1}}$$

$$-\frac{1}{4}(\zeta + \eta_{2}\omega)^{2} \partial_{\eta_{2}} - \left(\eta_{1} + \frac{\zeta\omega}{4}\right) (\zeta \partial_{\zeta} + \omega \partial_{\omega} - 2\Lambda(h_{1}))$$

$$+\frac{1}{2} \left(\xi_{1} + \frac{\xi_{2}\omega}{2}\right)^{2} \hat{\Lambda} - \frac{\omega}{2}(\zeta + \eta_{2}\omega)\Lambda(h_{2}),$$

$$\pi_{L}(b_{2}^{-}) = \xi_{2}\pi_{L}(a_{2}^{-}) + \frac{\xi_{2}\zeta}{2} \partial_{\xi_{1}} + \frac{\zeta^{2}}{4} \partial_{\eta_{1}} - \eta_{2}^{2} \partial_{\eta_{2}} - \zeta \partial_{\omega} + \frac{\xi_{2}^{2}}{2} \hat{\Lambda} - 2\eta_{2}\Lambda(h_{2}),$$

$$\pi_{L}(c^{-}) = \frac{\xi_{2}}{2}\pi_{L}(a_{1}^{-}) + \frac{1}{2} \left(\xi_{1} + \frac{\xi_{2}\omega}{2}\right) \pi_{L}(a_{2}^{-}) + \left(\eta_{1} + \frac{\zeta\omega}{2}\right) \left(\frac{\xi_{2}}{2} \partial_{\xi_{1}} - \partial_{\omega}\right)$$

$$+ \frac{\zeta^{2}\omega}{8} \partial_{\eta_{1}} - \frac{1}{2}(\zeta + \eta_{2}\omega)(\eta_{2}\partial_{\eta_{2}} + \Lambda(h_{2})) - \frac{\zeta^{2}}{4} \partial_{\zeta}$$

$$+ \frac{\xi_{2}}{2} \left(\xi_{1} + \frac{\xi_{2}\omega}{2}\right) \hat{\Lambda} - \frac{\zeta}{2}\Lambda(h_{1}),$$

$$\pi_{L}(d^{-}) = \frac{\omega\xi_{1}}{4} \partial_{\xi_{1}} - \frac{1}{2} \left(\xi_{1} + \frac{\omega\xi_{2}}{2}\right) \partial_{\xi_{2}} + \frac{\omega\eta_{1}}{2} \partial_{\eta_{1}} - \frac{1}{2}(\zeta + \omega\eta_{2})\partial_{\eta_{2}}$$

$$- \eta_{1}\partial_{\zeta} + \frac{\omega^{2}}{4} \partial_{\omega} + \frac{\omega}{2}(\Lambda(h_{1}) - \Lambda(h_{2})).$$
(5.7)

 $\pi_L(b_k^-), \pi_L(c^-), \pi_L(d^-)$ have the following simpler expressions:

$$\pi_{L}(b_{1}^{-}) = \xi_{1}\pi_{L}(a_{1}^{-}) - \frac{\xi_{1}\omega}{2}\pi_{L}(a_{2}^{-}) + \omega\pi_{L}(c^{-}) - \frac{\omega^{2}}{4}\pi_{L}(b_{2}^{-})
- \eta_{1}^{2}\partial_{\eta_{1}} - \frac{\zeta^{2}}{4}\partial_{\eta_{2}} - \eta_{1}\zeta\partial_{\zeta} + \frac{\xi_{1}^{2}}{2}\hat{\Lambda} - 2\eta_{1}\Lambda(h_{1}),
\pi_{L}(b_{2}^{-}) = -\xi_{2}\eta_{2}\partial_{\xi_{2}} + \frac{\zeta^{2}}{4}\partial_{\eta_{1}} - \eta_{2}^{2}\partial_{\eta_{2}} - \zeta\partial_{\omega} - \frac{\xi_{2}^{2}}{2}\hat{\Lambda} - 2\eta_{2}\Lambda(h_{2}),
\pi_{L}(c^{-}) = \frac{\xi_{2}}{2}\pi_{L}(a_{1}^{-}) + \frac{1}{2}\left(\xi_{1} - \frac{\xi_{2}\omega}{2}\right)\pi_{L}(a_{2}^{-}) + \frac{\omega}{2}\pi_{L}(b_{2}^{-})
+ \frac{\xi_{2}\eta_{1}}{2}\partial_{\xi_{1}} - \frac{\zeta\eta_{2}}{2}\partial_{\eta_{2}} - \frac{\zeta^{2}}{4}\partial_{\zeta} - \eta_{1}\partial_{\omega} + \frac{\xi_{1}\xi_{2}}{2}\hat{\Lambda} - \frac{\zeta}{2}(\Lambda(h_{1}) + \Lambda(h_{2})),
\pi_{L}(d^{-}) = -\frac{\omega}{2}\left(\pi_{L}(h_{1}) - \pi_{L}(h_{2})\right) - \frac{\xi_{1}}{2}\partial_{\xi_{2}} - \frac{\zeta}{2}\partial_{\eta_{2}} - \eta_{1}\partial_{\zeta} - \frac{\omega^{2}}{4}\partial_{\omega}. \tag{5.8}$$

Finally, we pass to the "hat" basis:

$$\pi_R(\hat{b}_k^+) = \partial_{\eta_k} - \frac{1}{2} \partial_{\xi_k}^2, \qquad \pi_R(\hat{c}^+) = \partial_{\zeta} - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2}. \tag{5.9}$$

Then the **final** expressions for the invariant differential operators are:

(i)
$$\Lambda_1 - \Lambda_2 = \frac{1}{2}(1 - p^1)$$

$$\mathcal{D}_{(i)} = \left(\partial_{\omega} - \frac{\xi_2}{2}\partial_{\xi_1} - \frac{\zeta}{2}\partial_{\eta_1} - \eta_2\partial_{\zeta}\right)^{p^1}$$
(5.10)

(ii) for all Λ_1 , $\Lambda_2 = \frac{3}{4} - \frac{p^2}{2}$

$$\mathcal{D}_{(ii)} = \left(\partial_{\eta_2} - \frac{1}{2}\partial_{\xi_2}^2\right)^{p^2} \tag{5.11}$$

(iii)
$$\Lambda_1 = \frac{5}{4} - \frac{1}{2}(p^3 - q^3), \ \Lambda_2 = \frac{3}{4} - \frac{1}{2}q^3, \ (p^3 \neq q^3, p^3 \neq 2q^3)$$

(a) $p^3 < q^3$

$$\mathcal{D}_{(iii,a)} = \sum_{k=0}^{\lfloor p^{3}/2 \rfloor} \sum_{n=0}^{p^{3}-2k} \frac{p^{3}! q^{3}!}{4^{k} k! n! (p^{3}-2k-n)! (q^{3}-k-n)!} \left(\partial_{\eta_{1}} - \frac{1}{2} \partial_{\xi_{1}}^{2} \right)^{k} \times \left(\partial_{\eta_{2}} - \frac{1}{2} \partial_{\xi_{2}}^{2} \right)^{q^{3}-k-n} \left(\partial_{\zeta} - \frac{1}{2} \partial_{\xi_{1}} \partial_{\xi_{2}} \right)^{n} \times \left(\partial_{\omega} - \frac{\xi_{2}}{2} \partial_{\xi_{1}} - \frac{\zeta}{2} \partial_{\eta_{1}} - \eta_{2} \partial_{\zeta} \right)^{p^{3}-2k-n}$$
(5.12)

(b) $q^3 < p^3 < 2q^3$

$$\mathcal{D}_{(iii,b)} = \left(\sum_{k=0}^{p^3 - q^3} \sum_{n=0}^{q^3 - k} + \sum_{k=p^3 - q^3 + 1}^{\lfloor p^3/2 \rfloor} \sum_{n=0}^{p^3 - 2k} \right) \mathcal{P}(p^3, q^3, k, n)$$
 (5.13)

(c)
$$2q^3 < p^3$$

$$\mathcal{D}_{(iii,c)} = \sum_{k=0}^{q^3} \sum_{n=0}^{q^3-k} \mathcal{P}(p^3, q^3, k, n)$$
 (5.14)

where the summand $\mathcal{P}(p^3, q^3, k, n)$ for (b) (c) is same as (a).

(iv)
$$\Lambda_1 + \Lambda_2 = 2 - \frac{p^4}{2}$$

$$\mathcal{D}_{(iv)} = \sum_{k=0}^{\lfloor p^4/2 \rfloor} \sum_{n=0}^{p^4-2k} \frac{p^4!}{4^k k! n! (p^4 - 2k - n)!} \frac{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2})}{\Gamma(2\Lambda_1 + p^4 - \frac{3}{2} - k - n)} \times \left(\partial_{\eta_1} - \frac{1}{2} \partial_{\xi_1}^2\right)^k \left(\partial_{\eta_2} - \frac{1}{2} \partial_{\xi_2}^2\right)^{p^4 - k - n} \left(\partial_{\zeta} - \frac{1}{2} \partial_{\xi_1} \partial_{\xi_2}\right)^n \times \left(\partial_{\omega} - \frac{\xi_2}{2} \partial_{\xi_1} - \frac{\zeta}{2} \partial_{\eta_1} - \eta_2 \partial_{\zeta}\right)^{p^4 - 2k - n}$$
(5.15)

(v)
$$\Lambda_1 = \frac{5}{4} - \frac{p^5}{2}$$
, for all Λ_2

$$\mathcal{D}_{(v)} = \sum_{k=0}^{p^{5}} \sum_{n=0}^{p^{5}-k} \frac{(-1)^{n}}{4^{k}k!n!} \frac{p^{5}!}{(p^{5}-k-n)!} \frac{\Gamma(2\Lambda_{2}-p^{5}-\frac{3}{2}+2k+n)}{\Gamma(2\Lambda_{2}-p^{5}-\frac{3}{2})} \times \left(\partial_{\eta_{1}} - \frac{1}{2}\partial_{\xi_{1}}^{2}\right)^{k} \left(\partial_{\eta_{2}} - \frac{1}{2}\partial_{\xi_{2}}^{2}\right)^{p^{5}-k-n} \left(\partial_{\zeta} - \frac{1}{2}\partial_{\xi_{1}}\partial_{\xi_{2}}\right)^{n} \times \left(\partial_{\omega} - \frac{\xi_{2}}{2}\partial_{\xi_{1}} - \frac{\zeta}{2}\partial_{\eta_{1}} - \eta_{2}\partial_{\zeta}\right)^{2p^{5}-2k-n}$$
(5.16)

6 Conclusions and Outlook

In this paper we have presented explicit expressions for the intertwining differential operators related to the Jacobi algebra \mathcal{G}_2 . These results are the first explicit example for \mathcal{G}_n with $n \geq 2$ and elucidate that there are more variety of intertwining differential operators than \mathcal{G}_1 [6]. It is, therefore, natural to extend the present and that of [8] to \mathcal{G}_n with $n \geq 3$. However, recall that the theory of parabolic Verma modules for non-semisimple Lie algebras is not developed yet. This implies that the search for intertwining differential operators for \mathcal{G}_n , even for n = 2, is a highly non-trivial problem.

The results of this paper would be useful to anyone who would like to study the explicit implications of \mathcal{G}_2 invariance. Naturally, the first possible applications would be to find explicit solutions of the equations $\mathcal{DF} = 0$, where \mathcal{D} would be any of the new explicit operators from Section 5. Next, following the applications of the conformal and Schrödinger groups one may look for explicit expressions of correlation functions invariant under \mathcal{G}_2 . Altogether, there are many possible applications.

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